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An implementation of the Vickrey outcome with gross-substitutes

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Abstract: We consider a market with only one seller and many buyers. The seller owns several indivisible objects on sale. Each buyer can receive many objects and has a gross-substitutes valuation for every package of objects. The gross-substitutes condition guarantees the non-emptiness of the core of the market (Ausubel and Milgrom, 2002). Moreover, the Vickrey outcome (Vickrey, 1961) of the market leads to a core payoff in which each buyer gets his maximum core payoff. The aim of this paper is to analyze the following mechanism. Simultaneously, each buyer requests a package by announcing how much he would pay for it. After all buyers' requests, the seller decides the final assignment of packages and the prices. If a buyer gets a package of objects, it must be his request or an allocation at least as good as his request. The subgame perfect equilibrium outcomes of the mechanism correspond to the Vickrey outcome of the market.

JEL Codes: C71, C72.

Keywords: assignment model, mechanism, implementation, Vickrey outcome.

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1 Introduction

In this paper, we consider a market in which many buyers and only one seller meet. The seller owns many indivisible and heterogeneous objects on sale. On the other side of the market, each buyer is interested in packages of objects and has a non-negative valuation for each of them. Buyers' valuations satisfy the gross-substitutes condition.[1] An outcome for this market specifies an assignment of the objects to a group of buyers and the payment each buyer makes for his assigned package of objects.

The problem concerns the efficient assignment of packages of objects to buyers. An outstanding outcome for this market, the *Vickrey outcome*, is given by the *Vickrey (allocation) rule*.[2] The Vickrey outcome has the following interesting properties: the assignment of the objects is efficient and; if a buyer gets a package, he pays the social opportunity cost of allocating to him that package. In spite of its properties, the Vickrey outcome may generate a low revenue for the seller. To deal with this fact, it has been considered in the literature,[3] as a competitive standard, the belonging of the Vickrey outcome to the core of the associated coalitional game. Ausubel & Milgrom (2002) shows that: if the gross-substitutes condition holds, then the Vickrey outcome belongs to the core. Even more, it is the best core allocation for the buyers. In a recent paper, Goeree & Lien (2016) shows an impossibility result for core-selecting auctions: if the Vickrey outcome does not belong to the core, then no core-selecting auction exists.

In this paper, we study whether the strategic interaction of all agents leads to core allocations. In particular, we introduce a simple mechanism which resembles a bidding procedure. While in standard auctions only buyers play, a key feature of our mechanism is that all buyers and the seller interact. The mechanism works as follows. First, each buyer requests (for instance, bidding in a sealed envelope) a package he would like to buy and how much he would pay for it. Then, the seller decides the final allocation and the prices. In more detail, she chooses a group of buyers, and she sells a package at a price to each of these buyers in such a way that no buyer is worse off than with his initial request.

A usual requirement for allocating objects is efficiency. When buyers request packages of objects simultaneously, an overlapping problem may arise. Then, this may produce a loss of efficiency in the allocation. In particular, we show that if the seller is restricted to choose only among requested packages, the outcome of a subgame perfect equilibrium (SPE) in pure strategies is not efficient due to the coordination problem among buyers' requests. As a consequence, the outcome does not belong to the core. In order to avoid this problem, the seller is allowed to allocate non requested packages as long as this does not make any buyer worse off. We prove then that in any SPE, the final allocation of the objects is efficient for the whole market. In a second result, we prove that every SPE outcome of the game coincides with the Vickrey outcome of the market.

If each buyer can acquire at most one object, Demange *et al.* (1986) proposes the following allocation mechanism. Selling prices start at reservation prices; then every

[1] Condition introduced by Kelso & Crawford (1982).

[2] In fact, VCG mechanisms (Vickrey (1961), Clarke (1971) and Groves (1973)). See Milgrom (2004) for details. For a characterization of the Vickrey (allocation) rule, see Chew & Serizawa (2007).

[3] See for instance Day & Raghavan (2007) and Day & Milgrom (2008).

buyer requests the objects he would like to buy at the announced prices; if it is possible to allocate each object to a buyer who requests it, the procedure is done; otherwise, the price of the overdemanded objects is increased and the procedure is iterated with new prices. The mechanism leads to the Vickrey outcome.

When each object belongs to a single seller and each agent can make at most one partnership, we are in the setting of the *assignment game* (Shapley & Shubik, 1972). In this market, the multi-item auction (Demange *et al.*, 1986) produces the best core element for the buyers. For the same market, Pérez-Castrillo & Sotomayor (2002) considers a buying and selling procedure to implement in SPE the best core element for the sellers (which is supported by the maximum competitive equilibrium price vector). The mechanism works as follows. Simultaneously, each seller puts the price of her object. Then, given an order, each buyer reports his preferred matchings, taking into account what the previous buyer has reported. If buyers play a dominant strategy consisting of truly reporting their indifferences, then the SPE outcomes correspond to the best core element for the sellers.

Our paper is also related to the model introduced in Wilson (1978) for an exchange economy. All but one agent play as bidders, the remaining agent plays as an auctioneer. First, all bidders play simultaneously by requesting a set of feasible trades to the auctioneer. In the second stage, the auctioneer chooses for each bidder at most one trade. If a trade was chosen from a bidder, then he will participate in the exchange. Otherwise, he will stay with his initial resources. The author shows that there exists a (principal) Nash equilibrium which leads a core allocation. It is shown that if the market, is replicated, the outcome given by a (principal) Nash equilibrium is a competitive equilibrium outcome.

The mechanism introduced in our paper resembles the game considered in Wilson (1978). Notwithstanding, we are in a different setting. The seller is the owner of all objects and they are all indivisible. When buyers request, they only choose one package to buy. Even more, the seller can choose among not requested packages. Moreover, if we replicate our market, similar to Wilson (1978), the SPE outcome yields the minimum competitive equilibrium. This result follows from Gul & Stacchetti (1999).

This paper considers the implementation problem as in Pérez-Castrillo & Sotomayor (2002). We provide a mechanism which tries to capture a natural bidding procedure in which all agents play in complete information. It produces efficient allocations in SPE. Moreover, it implements in SPE, the Vickrey outcome. Since the gross substitutes condition is satisfied, this outcome is in the core, that is, no coalition of players can improve its payoff by trading only among themselves.

The paper is divided as follows. Next section is devoted to an introduction of the market and the cooperative game associated to it. In section 3, the mechanism is presented and we characterize its set of SPE outcomes. Section 4 is devoted to some concluding remarks. Finally, an Appendix contains some technical lemmas needed to establish the implementation result.

2 The market and some preliminaries

Consider a market with m buyers and only one seller. The finite set of buyers is denoted by $M = \{1, 2, ..., m\}$ and the seller is denoted by 0. She owns a finite set of indivisible objects on sale, denoted by Q. The set of objects Q includes copies of a dummy object q_0, as many as the number of buyers. Each buyer i has a *valuation* for each package of objects,[4] $w_i : 2^Q \to R_+$ such that[5] $w_i(\emptyset) = 0$. Moreover, each agent has a quasi-linear utility function. If a buyer i buys package R, we interpret $w_i(R)$ as the gain[6] that can be splitted between buyer i and the seller. Given a price vector $\rho \in \mathbb{R}^Q_+$, the demand set of buyer i consists of

$$D_i(\rho) = \{R \subseteq Q | w_i(R) - \sum_{j \in R} \rho_j \geq w_i(R') - \sum_{j \in R'} \rho_j \text{ for all } R' \subseteq Q\}.$$

Definition 2.1. *Buyer i's valuation w_i satisfies*

i. Monotonicity: $w_i(S) \geq w_i(T)$ for all $T \subseteq S \subseteq Q$.

ii. Gross-substitutes condition: for any two price vectors $\rho, \rho' \in \mathbb{R}^Q_+$ such that $\rho' \geq \rho$, and any $R \in D_i(\rho)$, there exists $R' \in D_i(\rho')$ such that $\{j \in R | r_j = r'_j\} \subseteq R'$.

Monotonicity says that for any buyer, the more objects in a package, the better. The gross-substitutes condition was introduced by Kelso & Crawford (1982). This property has been also widely studied in Gul & Stacchetti (1999). When buyers' valuations do not satisfy it, market clearing prices may not exist. Valuation functions that satisfy the two above properties can be found in Gul & Stacchetti (1999). Take for instance, a buyer i with a k_i-satiation valuation, that is, i values packages up to a given capacity $k_i \in \mathbb{N}$. More precisely, buyer i values every package $Q' \subseteq Q$ at

$$w_i(Q') = \max_{\substack{Q'' \subseteq Q': \\ |Q''| \leq k_i}} \{w_i(Q'')\}.$$

Note that when $k_i = 1$ for every $i \in M$, we are in the setting of Demange *et al.* (1986).

Therefore our market is described by $(M, \{0\}, Q, w)$ where w stands for buyers' valuations, $w = (w_i)_{i \in M}$. An allocation of the set of objects Q assigns all objects to a group of buyers S such that each object is assigned only to a buyer. That is, an allocation of Q to S consists of $A = (A_i)_{i \in S}$ such that $A_i \neq \emptyset$ for each $i \in S$, $\bigcup_{i \in S} A_i = Q$ and $A_i \cap A_{i'} = \emptyset$ if $i \neq i'$. We denote by $\mathcal{A}(S)$ the set of all allocations of Q to S. We say that an allocation $A \in \mathcal{A}(S)$ is efficient for S if

$$\sum_{i \in S} w_i(A_i) \geq \sum_{i \in S} w_i(A'_i) \text{ for all } A' \in \mathcal{A}(S).$$

[4]For each set S, we will denote by $|S|$ the cardinality of S and by 2^Q the power set of S.

[5]We assume that for each buyer i and for each dummy object j_0, $w_i(R \cup \{j_0\}) = w_i(R)$ for all $R \subseteq Q \setminus \{j_0\}$.

[6]The reservation price of each package is assumed to be zero.

We denote by $\mathcal{A}^*(S)$ the set of efficient allocations for S.

An outstanding allocation mechanism is the the Vickrey rule. Given a market $(M, \{0\}, Q, w)$, the Vickrey outcome produces an efficient allocation of the objects together with a payoff vector $u^* \in \mathbb{R}^{M\cup\{0\}}$, where

$$u_i^* = \max_{A\in\mathcal{A}(M)} \left\{ \sum_{t\in M} w_t(A_t) \right\} - \max_{A\in\mathcal{A}(M\setminus\{i\})} \left\{ \sum_{t\in M\setminus\{i\}} w_t(A_t) \right\}, \tag{1}$$

for each buyer i. By efficiency, the seller's payoff is $u_0^* = \max_{A\in\mathcal{A}(M)} \left\{\sum_{t\in M} w_t(A_t)\right\} - \sum_{t\in M} u_t^*$. In spite of its interesting properties, the Vickrey auction may generate a low revenue for the seller. To determine if the seller's revenue is unacceptably low in the Vickrey outcome, we will consider the criteria used in Ausubel & Milgrom (2002), Day & Raghavan (2007) and Day & Milgrom (2008). The Vickrey outcome must belong to the core of an associated coalitional game. In order to introduce the core, let us consider the coalitional game[7] as in Ausubel & Milgrom (2002). This game is denoted by $(M \cup \{0\}, v)$. The worth of any coalition formed by only one type of agents is zero because in these cases there is no trade. When a coalition is formed by a group of buyers $S \subseteq M$ and the seller, the worth is given by

$$v(S \cup \{0\}) = \max_{A\in\mathcal{A}(S)} \left\{ \sum_{i\in S} w_i(A_i) \right\}.$$

Ausubel & Milgrom (2002) shows that the Vickrey outcome belongs to the core of the game $(M \cup \{0\}, v)$ if the gross-substitutability condition is satisfied. In that case, the coalitional game is *bidders-submodular*. This means that the marginal contribution of any buyer to any coalition containing the seller decreases as the coalition grows larger. More precisely, a game $(M \cup \{0\}, v)$ is bidders-submodular if for all $i \in M$ and all $T \subseteq S \subseteq M \setminus \{i\}$, it holds that

$$v((T \cup \{0\}) \cup \{i\}) - v(T \cup \{0\}) \geq v((S \cup \{0\}) \cup \{i\}) - v(S \cup \{0\}). \tag{2}$$

The following expression is equivalent to (2)

$$v(S \cup \{0\}) - v(T \cup \{0\}) \geq \sum_{i\in S\setminus T} \Big(v(S \cup \{0\}) - v((S \setminus \{i\}) \cup \{0\}) \Big), \tag{3}$$

for all $T \subseteq S \subseteq M$.

3 A Mechanism to implement the Vickrey outcome

In this section, we introduce a mechanism to implement the Vickrey outcome in our market with m buyers and only one seller. This mechanism will be denoted by Γ and

[7] A *game in coalitional form with transferable utility* is a pair (N, v) formed by a finite set of players N and a characteristic function v that assigns a real number $v(S)$ to each coalition $S \subseteq N$, with $v(\emptyset) = 0$. The core of a game (N, v) is $C(v) = \{x \in \mathbb{R}^N \mid \sum_{i\in N} x_i = v(N), \sum_{i\in S} x_i \geq v(S) \text{ for all } S \subseteq N\}$.

it has the following two phases. First, only buyers play. Each buyer announces the package of objects he wants and the price he would pay for it. All these requests are made simultaneously. In the second phase, the final allocation and the prices are determined: with the information of buyers' requests, the seller chooses a coalition of buyers and assigns to each of these buyers a package at a price. The seller is allowed to allocate the requested package to a buyer at his proposed price or a different package at a price that makes this buyer not worse off than with his initial request.

A key point of this mechanism is that the seller plays an active role. Notice that, once buyers have made requests, the seller could be restricted to choose which package she will allocate only among those requested that do not overlap. The following example shows that under this constraint, there is a SPE in which the allocation of the objects is not efficient.

Example 3.1. *The seller owns the set of objects $Q = \{q_1, q_2\}$ and there are two buyers. The valuations are $w_1(\{q_1\}) = 3$, $w_1(\{q_2\}) = 4$, $w_1(\{q_1, q_2\}) = 6$, $w_2(\{q_1\}) = 5$, $w_2(\{q_2\}) = 4$ and $w_2(\{q_1, q_2\}) = 5$. Suppose that both buyers request package $\{q_1, q_2\}$ at price 5. Assume that the seller can make the final allocation only among requested packages. There is a SPE where the seller allocates the requested package to buyer 1 at the proposed price and nothing to buyer 2. However, the final allocation is not efficient and hence, does not lead to any core element.*

The previous example shows that simultaneous requests, in general, could generate a coordination problem which may damage the efficiency of the final allocation. Notwithstanding, efficiency may be improved by allowing the seller to allocate packages that have been not requested, at a price that makes the buyers who receive them no worse off than their initial request.

In more detail, the two phases of the mechanism Γ are:

1. Buyers play simultaneously. Each buyer i announces a tentative package and how much he would pay for it, $(B_i, x_i) \in 2^Q \times \mathbb{R}_+$.

   We denote by (B, x) the requests of all buyers, where $B = (B_i)_{i \in M}$ and $x = (x_i)_{i \in M}$.

2. The seller chooses the triple (S, A, p) where: a) $S \subseteq M$ is a coalition of buyers; b) $A \in \mathcal{A}(S)$ is an allocation of Q to S; and c) $p = (p_i)_{i \in S} \in \mathbb{R}^S_+$ determines the payment each buyer $i \in S$ makes for package A_i, such that

$$w_i(A_i) - p_i \geq w_i(B_i) - x_i \text{ for each } i \in S. \tag{4}$$

Therefore, we denote by $((B, x), \mathcal{S})$ a strategy profile, where $\mathcal{S}$ stands for the seller's strategy. After the seller has played, the allocation $A \in \mathcal{A}(S)$ assigns a package to each buyer in S. Buyer $i \in S$ receives package A_i, he pays p_i and his payoff is $w_i(A_i) - p_i$. If a buyer i does not receive a package, that is $i \in M \setminus S$, he pays nothing and his payoff is zero. The seller's payoff is $\sum_{i \in S} p_i$.

Now, we start the analysis of the mechanism Γ. We are interested in the SPE of this mechanism. The following result contains the description of a SPE strategy profile in which the payoff vector is the Vickrey outcome.

Proposition 3.2. *The Vickrey outcome of* $(M, \{0\}, Q, w)$ *is attained in a SPE of* Γ.

Proof. Let us denote by M_i^v the marginal contribution of $i \in M$ in $(M \cup \{0\}, v)$, that is, $M_i^v = v(M \cup \{0\}) - v((M \setminus \{i\}) \cup \{0\})$, recall (1). Assume that each $i \in M$ announces (B_i, x_i) such that $w_i(B_i) - x_i = M_i^v$. Given (B, x), the seller chooses any $A \in \mathcal{A}^*(M)$ and $p = (p_i)_{i \in M} \in \mathbb{R}_+^M$ such that $w_i(A_i) - p_i = M_i^v$. We must see that such payments exist. Take any $A \in \mathcal{A}^*(M)$ and for any $i' \in M$ notice that

$$w_{i'}(A_{i'}) = \sum_{i \in M} w_i(A_i) - \sum_{i \in M \setminus \{i'\}} w_i(A_i) \geq v(M \cup \{0\}) - v((M \setminus \{i'\}) \cup \{0\}) = M_{i'}^v.$$

Therefore, for each $i \in M$, we have $w_i(A_i) \geq M_i^v \geq 0$. Then, let $p = (p_i)_{i \in M} \in \mathbb{R}_+^M$ be such that $w_i(A_i) - p_i = M_i^v$. Now, we will see that this triple (M, A, p) is a seller's best reply to the buyers' strategies. It is obvious that the outcome of the above strategies will be the Vickrey outcome.

Consider any (S', A', p') that satisfies (4) for each $i \in S'$. Since $(M \cup \{0\}, v)$ satisfies bidders-submodularity (3), then

$$\begin{aligned}\sum_{i \in M} p_i = v(M \cup \{0\}) - \sum_{i \in M} M_i^v &\geq v(S' \cup \{0\}) - \sum_{i \in S'} M_i^v \\ &\geq \sum_{i \in S'} \Big(w_i(A'_i) - M_i^v \Big) \geq \sum_{i \in S'} p'_i.\end{aligned}$$

Now, we see that any buyer $i' \in M$, by requesting $(B_{i'}, x_{i'})$ with $w_{i'}(B_{i'}) - x_{i'} = M_{i'}^v$, is playing a best reply to the other agents' strategies. First, consider any $A' \in \mathcal{A}^*(M \setminus \{i'\})$. For all $i^* \in M \setminus \{i'\}$, we have

$$M_{i^*}^v \leq v((M \setminus \{i'\}) \cup \{0\}) - v((M \setminus \{i', i^*\}) \cup \{0\}) \leq w_i(A'_{i^*}),$$

where the first inequality is due to bidders-submodularity (2) and the second one because of $\sum_{i \in M \setminus \{i', i^*\}} w_i(A'_i) \leq v((M \setminus \{i', i^*\}) \cup \{0\})$. Therefore, there is a vector $p' = (p'_i)_{i \in M \setminus \{i'\}}$ such that $p'_i = w_i(A'_i) - M_i^v \geq 0$ for all $i \in M \setminus \{i'\}$. Then, we have

$$\sum_{i \in M \setminus \{i'\}} p'_i = v((M \setminus \{i'\}) \cup \{0\}) - \sum_{i \in M \setminus \{i'\}} M_i^v \; = v(M \cup \{0\}) - \sum_{i \in M} M_i^v = \sum_{i \in M} p_i. \quad (5)$$

On one hand, assume that buyer i' unilaterally modifies his request to $(B'_{i'}, x'_{i'})$ such that $w_{i'}(B'_{i'}) - x'_{i'} > w_{i'}(B_{i'}) - x_{i'}$. This means that all his current acceptable packages require a lower price. Because of (5), the seller will maximize her payoff with the triple $(M \setminus \{i'\}, A', p')$. Therefore, buyer i' will not be better off. On the other hand, if buyer i' requests $(B'_{i'}, x'_{i'})$ such that $w_{i'}(B'_{i'}) - x'_t < w_t(B_{i'}) - x_{i'}$, then he may acquire some package at a price that makes him worse off.

This concludes the proof that there is a SPE that yields the Vickrey outcome. $\square$

Our aim is to prove that in fact, in any SPE, each buyer gets his marginal contribution. To this end, let us remark that in any SPE, the seller will price packages as high as possible given constraint (4), that is, inequality in (4) is satisfied as an equality.

The next proposition proves that in any SPE, the final allocation of the goods is efficient for the whole market.

Proposition 3.3. *Let $((B,x),\mathcal{S})$ be any SPE of Γ and let (S,A,p) be the choice of the seller. Then*

$$\sum_{i\in S} w_i(A_i) = v(S\cup\{0\}) = v(M\cup\{0\}).$$

Proof. First, we prove $\sum_{i\in S} w_i(A_i) = v(S\cup\{0\})$. Assume on the contrary that (S,A,p) is the choice of the seller in a given SPE and $\sum_{i\in S} w_i(A_i) < v(S\cup\{0\})$.

Take any $A' \in \mathcal{A}^*(S)$. If $w_i(A'_i) \geq w_i(B_i) - x_i$ for all $i\in S$, then define $p'_i = w_i(A'_i) - (w_i(B_i) - x_i)$ for each $i\in S$. We have

$$\begin{aligned}\sum_{i\in S} p'_i &= \sum_{i\in S}\Big(w_i(A'_i) - (w_i(B_i)-x_i)\Big) = v(S\cup\{0\}) - \sum_{i\in S}\Big(w_i(B_i)-x_i\Big)\\ &> \sum_{i\in S}\Big(w_i(A_i) - (w_i(B_i)-x_i)\Big) = \sum_{i\in S} p_i.\end{aligned}$$

This contradicts the fact that (S,A,p) maximizes the seller's payoff. Therefore, there is some $i\in S$ such that $w_i(A'_i) < w_i(B_i) - x_i$.

By applying Lemma A.1, in Appendix A, taking $\bar{S} = S$, there exist $\emptyset \neq T \subsetneq S$ and an allocation $\bar{A}\in\mathcal{A}^*(T)$ such that $w_i(\bar{A}_i)\geq w_i(B_i)-x_i$ for all $i\in T$ and

$$\sum_{i\in S\setminus T}\Big(w_i(B_i)-x_i\Big) > v(S\cup\{0\}) - v(T\cup\{0\}). \tag{6}$$

Define $\bar{p}_i = w_i(\bar{A}_i) - (w_i(B_i)-x_i)$ for all $i\in T$. Since (S,A,p) maximizes the seller's payoff, we obtain

$$\begin{aligned}\sum_{i\in S}\Big(w_i(A_i) - (w_i(B_i)-x_i)\Big) &= \sum_{i\in S} p_i \geq \sum_{i\in T}\bar{p}_i = \sum_{i\in T}\Big(w_i(\bar{A}_i) - (w_i(B_i)-x_i)\Big)\\ &= v(T\cup\{0\}) - \sum_{i\in T}\Big(w_i(B_i)-x_i\Big).\end{aligned}$$

Since $T\subseteq S$, then

$$\sum_{i\in S} w_i(A_i) - v(T\cup\{0\}) \geq \sum_{i\in S\setminus T}\Big(w_i(B_i)-x_i\Big),$$

which contradicts (6). Hence $\sum_{i\in S} w_i(A_i) = v(S\cup\{0\})$.

Now, we prove $v(S\cup\{0\}) = v(M\cup\{0\})$. Assume on the contrary that $v(S\cup\{0\}) < v(M\cup\{0\})$. Then, there is some $i'\in M\setminus S$ such that $v((S\cup\{i'\})\cup\{0\}) > v(S\cup\{0\})$. By bidders-submodularity (2), we have that for all $S'\subseteq S$,

$$v((S'\cup\{i'\})\cup\{0\}) - v(S'\cup\{0\}) \geq v((S\cup\{i'\})\cup\{0\}) - v(S\cup\{0\}) > 0. \tag{7}$$

Take any $A' \in \mathcal{A}^*(S \cup \{i'\})$. If $w_i(A'_i) \geq w_i(B_i) - x_i$ for all $i \in S$, then define $p'_i = w_i(A'_i) - (w_i(B_i) - x_i)$ for each $i \in S$. Notice that

$$\sum_{i \in S} p_i = v(S \cup \{0\}) - \sum_{i \in S} \Big(w_i(B_i) - x_i \Big) < v((S \cup \{i'\}) \cup \{0\}) - \sum_{i \in S} \Big(w_i(B_i) - x_i \Big)$$
$$= \sum_{i \in S \cup \{i'\}} w_i(A'_i) - \sum_{i \in S} \Big(w_i(B_i) - x_i \Big) = \sum_{i \in S} p'_i + w_{i'}(A'_{i'}).$$

Notice that $w_{i'}(A'_{i'}) > 0$, since otherwise there would be another triple (S, A^*, p^*) better than (S, A, p) for the seller, which contradicts the assuption of SPE. Moreover, buyer $i' \notin S$, hence his payoff is zero. Since $w_{i'}(A'_{i'}) > 0$, he has incentives to deviate by requesting $(A'_{i'}, x'_{i'})$ such that

$$w_{i'}(A'_{i'}) - x'_{i'} > 0 \quad \text{and} \quad \sum_{i \in S} p_i < \sum_{i \in S} p'_i + x'_{i'}, \tag{8}$$

in order to receive the package $A'_{i'}$ at the price $p'_{i'} = x'_{i'}$, which gives him a positive payoff. This contradicts that $((B, x), \mathcal{S})$ forms a SPE. Therefore, there is some $i \in S$ such that $w_i(A'_i) < w_i(B_i) - x_i$.

By applying Lemma A.2, in Appendix A, to $\bar{S} = S$, there exist $T \subseteq \bar{S}$ and $\bar{A} \in \mathcal{A}^*(T \cup \{i'\})$ such that $w_i(\bar{A}_i) \geq w_i(B_i) - x_i$ for all $i \in T$ and

$$\sum_{i \in S \setminus T} \Big(w_i(B_i) - x_i \Big) > v(S \cup \{0\}) - v(T \cup \{0\}) \geq v(S \cup \{0\}) - v((T \cup \{i'\}) \cup \{0\}), \tag{9}$$

where the last inequality comes from the monotonicity of v. Define $\bar{p}_i = w_i(\bar{A}_i) - (w_i(B_i) - x_i)$ for each $i \in T$. Taking (9) into account, we get

$$\sum_{i \in S} p_i = v(S \cup \{0\}) - \sum_{i \in S} \Big(w_i(B_i) - x_i \Big)$$
$$< v((T \cup \{i'\}) \cup \{0\}) - \sum_{i \in T} \Big(w_i(B_i) - x_i \Big) = \sum_{i \in T} \bar{p}_i + w_{i'}(A'_{i'}),$$

and then buyer i' has incentives to deviate.[8] This completes the proof and hence $v(S \cup \{0\}) = v(M \cup \{0\})$. $\square$

The following theorem is the main result of the paper. It shows that the game Γ implements in SPE the Vickrey outcome.

Theorem 3.4. *The outcome of any SPE of Γ is the Vickrey outcome of the market $(M, \{0\}, Q, w)$.*

[8] See argument below (8).

Proof. Let $((B,x),\mathcal{S})$ be any SPE of Γ and denote by (S,A,p) the choice of the seller. First, take any $i' \in S$. Define $D \subsetneq M$ by $D = S \cup S^{i'}$ where $S^{i'}$ is as stated in Lemma A.3. We will show that for any $\tilde{A} \in \mathcal{A}^*(D \setminus \{i'\})$, we have $w_i(\tilde{A}_i) \geq w_i(B_i) - x_i$ for all $i \in D \setminus \{i'\}$. To this end, assume on the contrary there is some $i^* \in D \setminus \{i'\}$ such that $w_{i^*}(B_{i^*}) - x_{i^*} > w_{i^*}(\tilde{A}_{i^*})$. Notice that

$$\begin{aligned} w_{i^*}(B_{i^*}) - x_{i^*} > w_{i^*}(\tilde{A}_{i^*}) &\geq v((D \setminus \{i'\}) \cup \{0\}) - v((D \setminus \{i', i^*\}) \cup \{0\}) \\ &\geq v(D \cup \{0\}) - v((D \setminus \{i^*\}) \cup \{0\}) \geq 0, \end{aligned} \tag{10}$$

where the second inequality comes from the fact that $\tilde{A} \in \mathcal{A}^*(D \setminus \{i'\})$ and the third one follows from bidders-submodularity of v.

Take any $A' \in \mathcal{A}^*(D \setminus \{i^*\})$. If $w_i(A'_i) \geq w_i(B_i) - x_i$ for all $i \in D \setminus \{i^*\}$, define $p'_i = w_i(A'_i) - (w_i(B_i) - x_i)$ for all $i \in D \setminus \{i^*\}$. Therefore

$$\begin{aligned} \sum_{i \in S} p_i &= v(D \cup \{0\}) - \sum_{i \in D} \Big(w_i(B_i) - x_i \Big) \\ &< v((D \setminus \{i^*\}) \cup \{0\}) - \sum_{i \in D \setminus \{i^*\}} \Big(w_i(B_i) - x_i \Big) = \sum_{i \in D \setminus \{i^*\}} p'_i, \end{aligned}$$

where the first equality follows from Proposition 3.3, monotonicity of v and $(w_i(B_i) - x_i) = 0$ for all $i \in S^{i'} \setminus S$ and the inequality from (10). This contradicts the fact that (S,A,p) maximizes the seller's payoff. Then there is a buyer $i \in D \setminus \{i^*\}$ such that $w_i(A'_i) < w_i(B_i) - x_i$.

By applying Lemma A.1, in Appendix A, taking $\bar{S} = D \setminus \{i^*\}$, there exist $\emptyset \neq T \subsetneq D \setminus \{i^*\}$ and $\bar{A} \in \mathcal{A}^*(T)$ such that $w_i(\bar{A}_i) \geq w_i(B_i) - x_i$ for all $i \in T$ and

$$\sum_{i \in (D \setminus \{i^*\}) \setminus T} \Big(w_i(B_i) - x_i \Big) > v((D \setminus \{i^*\}) \cup \{0\}) - v(T \cup \{0\}).$$

Making use of (10), notice that,

$$\begin{aligned} \sum_{i \in D \setminus T} \Big(w_i(B_i) - x_i \Big) &> v((D \setminus \{i^*\}) \cup \{0\}) - v(T \cup \{0\}) \\ &\quad + v(D \cup \{0\}) - v((D \setminus \{i^*\}) \cup \{0\}) = v(D \cup \{0\}) - v(T \cup \{0\}). \end{aligned} \tag{11}$$

Define $\bar{p}_i = w_i(\bar{A}_i) - (w_i(B_i) - x_i)$ for each $i \in T$. We have

$$v(D \cup \{0\}) - \sum_{i \in D} \Big(w_i(B_i) - x_i \Big) = \sum_{i \in S} p_i \geq \sum_{i \in T} \bar{p}_i = v(T \cup \{0\}) - \sum_{i \in T} \Big(w_i(B_i) - x_i \Big),$$

where the first equality comes from Proposition 3.3, monotonicity of v, $(w_i(B_i) - x_i) = 0$ for all $i \in S^{i'} \setminus S$ and the inequality comes from the fact that (S,A,p) maximizes the seller's payoff. Then,

$$v(D \cup \{0\}) - v(T \cup \{0\}) \geq \sum_{i \in D \setminus T} \Big(w_i(B_i) - x_i \Big).$$

This contradicts (11). Hence for every $i' \in S$, there is an allocation $\tilde{A} \in \mathcal{A}^*(D \setminus \{i'\})$ such that $w_i(\tilde{A}_i) \geq w_i(B_i) - x_i$ for all $i \in D \setminus \{i'\}$.

Now, we prove that the outcome of any SPE is the Vickrey outcome. For any $i' \in S$, take $\tilde{A} \in \mathcal{A}^*(D \setminus \{i'\})$. Now, define a price vector $\tilde{p} = (\tilde{p}_i)_{i \in D\setminus\{i'\}} \in \mathbb{R}_+^{D\setminus\{i'\}}$ such that $\tilde{p}_i = w_i(\tilde{A}_i) - (w_i(B_i) - x_i)$ for all $i \in D \setminus \{i'\}$. We have

$$\begin{aligned} v(M \cup \{0\}) - \sum_{i\in S}\Big(w_i(B_i) - x_i\Big) &= \sum_{i\in S} p_i \\ &\geq \sum_{i\in D\setminus\{i'\}} \tilde{p}_i = v((M \setminus \{i'\}) \cup \{0\}) - \sum_{i\in D\setminus\{i'\}}\Big(w_i(B_i) - x_i\Big), \end{aligned}$$

where the first equality follows from Proposition 3.3, the inequality since (S, A, p) maximizes the seller's payoff and the last equality from Lemma A.6 (in Appendix A). Then,

$$v(M \cup \{0\}) - v((M \setminus \{i'\}) \cup \{0\}) \geq \sum_{i\in S}\Big(w_i(B_i) - x_i\Big) - \sum_{i\in D\setminus\{i'\}}\Big(w_i(B_i) - x_i\Big).$$

Since $D = S \cup S^{i'}$ and $w_i(B_i) - x_i = 0$ for all $i \in S^{i'} \setminus S$, we obtain

$$v(M \cup \{0\}) - v((M \setminus \{i'\}) \cup \{0\}) \geq w_{i'}(B_{i'}) - x_{i'}. \tag{12}$$

Then $M_i^v \geq w_i(B_i) - x_i$ for all $i \in S$. By Proposition 3.3, we deduce that $M_i^v = 0$ for all $i \in M \setminus S$. Hence, we only must see that $w_i(B_i) - x_i \geq M_i^v$ for all $i \in S$.

Take a buyer $i' \in S$, let $(S^{i'}, A^{i'}, p^{i'})$ be as in the statement of Lemma A.3, then

$$v(M \cup \{0\}) - \sum_{i\in S}\Big(w_i(B_i) - x_i\Big) = \sum_{i\in S} p_i = \sum_{i\in S^{i'}} p_i^{i'} = v(S^{i'} \cup \{0\}) - \sum_{i\in S^{i'}}\Big(w_i(B_i) - x_i\Big),$$

where the last equality follows from Proposition A.5, in Appendix A. Then,

$$v(M \cup \{0\}) - v(S^t \cup \{0\}) = \sum_{i\in S\setminus S^t}\Big(w_i(B_i) - x_i\Big) - \sum_{i\in S^t\setminus S}\Big(w_i(B_i) - x_i\Big).$$

By (18), we know that $(w_i(B_i) - x_i) = 0$ for all $i \in S^t \setminus S$. Therefore,

$$v(M \cup \{0\}) - v(S^t \cup \{0\}) = \sum_{i\in S\setminus S^t}\Big(w_i(B_i) - x_i\Big). \tag{13}$$

By bidders-submodularity (3), we have

$$v(M \cup \{0\}) - v(S^t \cup \{0\}) \geq \sum_{i\in M\setminus S^t} M_i^v \geq \sum_{i\in S\setminus S^t} M_i^v.$$

Making use of (13)

$$\sum_{i\in S\setminus S^t}\Big(w_i(B_i) - x_i\Big) \geq \sum_{i\in S\setminus S^t} M_i^v.$$

Together with (12), we have that $w_i(B_i) - x_i = M_i^v$ for all $i \in S$. This completes the proof. $\square$

We have shown that all SPE yield the best core element for buyers. A natural question related to this result is whether competitive prices may support this core outcome. In general, the answer is negative. Notwithstanding, when the gross-substitutes condition is satisfied, the existence of competitive equilibria is guaranteed.[9] Even more, if we replicate the market à la Gul and Stacchetti (see Section 5 in (Gul & Stacchetti, 1999)), similar to the result in Wilson (1978), the best core element for buyers becomes competitive and hence, the outcome of any SPE of our mechanism are supported by some competitive equilibria.

4 Concluding remarks

The paper provides a simple mechanism in which all buyers and the seller play. Initially, buyers submit requests or bids. Then the seller decides the allocation of the objects and the final prices. This mechanism relates the SPE outcomes with the core. A key point of this mechanism is the role played by the seller, which may improve the efficiency of the final allocation. There are two remarks to note from this mechanism. First, although the seller has the final decision, in any outcome of this mechanism, if a buyer gets a package not requested by him, he will get at least the same utility provided by his request. In particular, in any SPE outcome, in spite of the market power of the seller, every buyer gets his maximum core payoff. This means that even in the case in which only some buyers get a package, there is no coalition of agents that can improve it trading only by themselves. Finally, as a consequence of Gul & Stacchetti (1999), when the market is large, the mechanism implements in SPE the minimum competitive equilibrium of the market. That is, in the replicated market, the selling prices of packages given by the mechanism are supported by competitive equilibrium prices for the objects.

A Appendix

The following lemmas are used in the main result, Theorem 3.4.

The first lemma says the following. In equilibrium, if it is not possible to make an efficient allocation to a coalition $\bar{S} \subseteq M$ under constraint (4), then there exists a subcoalition $T \subseteq \bar{S}$ to which it is possible to make an efficient allocation satisfying (4).

Lemma A.1. *Let $((B,x),\mathcal{S})$ be any SPE of Γ and denote by (S,A,p) the choice of the seller. For any coalition of buyers $\emptyset \neq \bar{S} \subseteq M$ there exist $\emptyset \neq T \subseteq \bar{S}$ and an allocation $\bar{A} \in \mathcal{A}^*(T)$ such that $w_i(\bar{A}_i) \geq w_i(B_i) - x_i$ for all $i \in T$. Moreover, if $T \neq \bar{S}$ then*

$$\sum_{i \in \bar{S} \setminus T} \Big(w_i(B_i) - x_i \Big) > v(\bar{S} \cup \{0\}) - v(T \cup \{0\}).$$

Proof. Take any $\emptyset \neq \bar{S} \subseteq M$. First, consider $T_1 = \bar{S}$ and take any allocation $A^1 \in \mathcal{A}^*(T_1)$. If $w_i(A^1_i) \geq w_i(B_i) - x_i$ for all $i \in \bar{S}$, we are done taking $T = \bar{S}$. Otherwise,

[9] In fact, the set of competitive equilibrium price vectors has a complete lattice structure (Gul & Stacchetti, 1999)

there is some $i_1 \in T_1$ such that

$$w_{i_1}(A^1_{i_1}) < w_{i_1}(B_{i_1}) - x_{i_1}.$$

Denote $T_2 = T_1 \setminus \{i_1\}$. By the efficiency of A^1, we have

$$w_{i_1}(B_{i_1}) - x_{i_1} > w_{i_1}(A^1_{i_1}) \geq v(\bar{S} \cup \{0\}) - v(T_2 \cup \{0\}). \tag{14}$$

Take now any allocation $A^2 \in \mathcal{A}^*(T_2)$. If $w_i(A^2_i) \geq w_i(B_i) - x_i$ for all $i \in T \setminus \{i_1\}$, we are done taking $T = T_2$. Otherwise, there is some $i_2 \in T_2$ such that

$$w_{i_2}(A^2_{i_2}) < w_{i_2}(B_{i_2}) - x_{i_2}.$$

Denote $T_3 = T_2 \setminus \{i_2\}$. By the efficiency of A^2, we have

$$w_{i_2}(B_{i_2}) - x_{i_2} > w_{i_2}(A^2_{i_2}) \geq v(T_2 \cup \{0\}) - v((T_2 \setminus \{i_2\}) \cup \{0\}). \tag{15}$$

By adding (14) and (15), we get

$$\sum_{i \in S \setminus T_3} \Big(w_i(B_i) - x_i \Big) > v(\bar{S} \cup \{0\}) - v(T_3 \cup \{0\}).$$

By proceeding recursively, we construct a sequence $\{i_1, ..., i_k\} \subseteq \bar{S}$ such that for any $l \in \{1, ..., k\}$ there are: a coalition $T_{l+1} = \bar{S} \setminus \{i_1, ..., i_l\}$ and an allocation $A^{l+1} \in \mathcal{A}^*(T_{l+1})$ such that $w_{i_l}(A^{l+1}_{i_l}) < w_{i_l}(B_{i_l}) - x_{i_l}$. Moreover,

$$\sum_{i \in \bar{S} \setminus T_{l+1}} \Big(w_i(B_i) - x_i \Big) > v(\bar{S} \cup \{0\}) - v(T_{l+1} \cup \{0\}).$$

Now take any efficient allocation $A^{k+1} \in \mathcal{A}^*(T_{k+1})$. If $w_i(A^{k+1}_i) < w_i(B_i) - x_i$ for all $i \in T_{k+1}$, we are done taking $T = T_{k+1}$. Otherwise, we continue the procedure one more step. Notice that, since $\bar{S}$ is finite, we will eventually reach T_r with $|T_r| = 1$. In that case, let us write $T_r = \{i\}$. If $A^r \in \mathcal{A}^*(\{i\})$, then $w_i(A^r_i) \geq w_i(B_i) - x_i$, since $w_i(A^r_i) \geq w_i(R)$ for all $R \subseteq Q$. Hence $A^r \in \mathcal{A}^*(\{i\})$ satisfies the requirements and we are done with $T = T_r$. □

The following lemma proceeds similarly as the previous one. Then, we state it without a proof.

Lemma A.2. *Let $((B, x), \mathcal{S})$ be any SPE of Γ and denote by (S, A, p) the choice of the seller in this equilibrium. For any coalition of buyers $\emptyset \neq \bar{S} \subsetneq M$ and any $i' \in M \setminus \bar{S}$, there exist $T \subseteq \bar{S}$ and an allocation $\bar{A} \in \mathcal{A}^*(T \cup \{i'\})$ such that $w_i(\bar{A}_i) \geq w_i(B_i) - x_i$ for all $i \in T$. Moreover if $T \neq \bar{S}$ then*

$$\sum_{i \in \bar{S} \setminus T} \Big(w_i(B_i) - x_i \Big) > v((\bar{S} \cup \{i'\}) \cup \{0\}) - v(T \cup \{0\}).$$

The next lemma shows that in any SPE, and for any buyer who receives a package, there is an alternative choice that makes the seller indifferent.

Lemma A.3. *Let $((B,x),\mathcal{S})$ be any SPE of Γ and denote by (S,A,p) the choice of the seller. For each buyer $i^* \in S$, there is a triple (S^*,A^*,p^*) such that $S^* \subseteq M \setminus \{i^*\}$, $A^* \in \mathcal{A}(S^*)$, $p^* = (p_i^*)_{i\in S^*} \in \mathbb{R}_+^{S^*}$ and $w_i(A_i^*) - p_i^* \geq w_i(B_i) - x_i$ for all $i \in S^*$ and*

$$\sum_{i\in S^t} p_i^t = \sum_{i\in S} p_i. \tag{16}$$

Proof. Assume on the contrary that (S,A,p) is the choice of the seller in a SPE, and there exists $t \in S$ and for all $S^t \subseteq M \setminus \{t\}$, all $A^t \in \mathcal{A}(S^t)$ and all $(p_i^t)_{i\in S^t} \in \mathbb{R}_+^{S^t}$ such that $w_i(A_i^t) - p_i^t \geq w_i(B_i) - x_i$ for all $i \in S^t$, it holds

$$\sum_{i\in S^t} p_i^t < \sum_{i\in S} p_i. \tag{17}$$

Notice that $p_t > 0$, otherwise it is straightforward to find a triple $(S\setminus\{t\}, A', p')$ that satisfies equality (16), in contradiction with our assumption. Then, since (17) holds for all $S^t \subseteq M \setminus \{t\}$, all $A^t \in \mathcal{A}(S^t)$ and all $(p_i^t)_{i\in S^t} \in \mathbb{R}_+^{S^t}$ such that $w_i(A_i^t) - p_i^t \geq w_i(B_i) - x_i$ for all $i \in S^t$, buyer t has incentives to deviate by slightly decreasing the price he proposed to pay for the package B_t in such a way that the inequality (17) is still maintained. This contradicts that $((B,x),\mathcal{S})$ is a SPE. □

The next remark easily follows from Lemma A.3.

Remark A.4. *Let (S,A,p) be the choice of the seller in any SPE of Γ. For any $t \in S$, let (S^t,A^t,p^t) be as in the statement of Lemma A.3. Then*

$$w_i(B_i) - x_i = 0 \quad \text{for all } i \in S^t \setminus S. \tag{18}$$

Otherwise, if for some $i \in S^t \setminus S$, $w_i(B_i) - x_i > 0$, buyer i has incentives to increase a bit x_i, to make the seller choose (S^t,A^t,p^t) instead of (S,A,p), so that the buyer gets a positive payoff.

Lemma A.5 is related with the previous lemma. It says that for each buyer t who gets a package in equilibrium, if we consider (S^t,A^t,p^t) as stated in Lemma A.3, then A^t is efficient for S^t.

Lemma A.5. *Let $((B,x),\mathcal{S})$ be any SPE of Γ and denote by (S,A,p) the choice of the seller in this equilibrium. For each buyer $t \in S$, let (S^t,A^t,p^t) be as in the statement of Lemma A.3. Then*

$$\sum_{i\in S^t} w_i(A_i^t) = v(S^t \cup \{0\}).$$

Proof. Assume on the contrary that (S,A,p) is the choice of the seller under a SPE and $\sum_{i\in S^t} w_i(A_i^t) < v(S^t \cup \{0\})$.

Take any allocation $A' \in \mathcal{A}^*(S^t)$. If $w_i(A_i') \geq w_i(B_i) - x_i$ for all $i \in S^t$, then define $p_i' = w_i(A_i') - (w_i(B_i) - x_i)$ for each $i \in S^t$. We have

$$\begin{aligned}\sum_{i\in S} p_i = \sum_{i\in S^t} p_i^t &= \sum_{i\in S^t} w_i(A_i^t) - \sum_{i\in S^t}\Big(w_i(B_i) - x_i\Big)\\ &< v(S^t\cup\{0\}) - \sum_{i\in S^t}\Big(w_i(B_i) - x_i\Big) = \sum_{i\in S^t} p_i',\end{aligned}$$

where the first equality comes from Lemma A.3. This contradicts the fact that (S, A, p) maximizes the seller's payoff. Therefore, there is some $i \in S^t$ such that $w_i(A'_i) < w_i(B_i) - x_i$.

By applying Lemma A.1 to $\bar{S} = S^t$, there exist $\emptyset \neq T \subsetneq S^t$ and an efficient allocation $\bar{A} \in \mathcal{A}^*(T)$ such that $w_i(\bar{A}_i) \geq w_i(B_i) - x_i$ for all $i \in T$ and

$$\sum_{i \in S^t \setminus T} \Big(w_i(B_i) - x_i \Big) > v(S^t \cup \{0\}) - v(T \cup \{0\}). \tag{19}$$

Define $\bar{p}_i = w_i(\bar{A}_i) - (w_i(B_i) - x_i)$ for all $i \in T$. Since (S, A, p) maximizes the seller's payoff, we obtain

$$\begin{aligned}\sum_{i \in S^t} \Big(w_i(A^t_i) - (w_i(B_i) - x_i) \Big) &= \sum_{i \in S^t} p^t_i = \sum_{i \in S} p_i \\ &\geq \sum_{i \in T} \bar{p}_i = v(T \cup \{0\}) - \sum_{i \in T} \Big(w_i(B_i) - x_i \Big).\end{aligned}$$

Since $T \subseteq S^t$

$$\sum_{i \in S^t} w_i(A^t_i) - v(T \cup \{0\}) \geq \sum_{i \in S^t \setminus T} \Big(w_i(B_i) - x_i \Big),$$

This contradicts (19). Hence $\sum_{i \in S^t} w_i(B_i) = v(S^t \cup \{0\})$. □

The next lemma relates Lemma A.3 and A.5. For any equilibrium, let S be the set of buyers who gets a package, $t \in S$ and S^t be as stated in Lemma A.3. Then the worth attained by $(S \setminus \{t\}) \cup S^t$ and $M \setminus \{t\}$ is the same.

Lemma A.6. *Let $((B, x), \mathcal{S})$ be any SPE of Γ and denote by (S, A, p) the choice of the seller in this equilibrium. For each buyer $t \in S$, let (S^t, A^t, p^t) be as in the statement of Lemma A.3 and let $D = S \cup S^t$. Then*

$$v((D \setminus \{t\}) \cup \{0\}) = v((M \setminus \{t\}) \cup \{0\}).$$

Proof. First, we show that $v((S^t \cup \{i'\}) \cup \{0\}) = v(S^t \cup \{0\})$ for any $i' \in M \setminus D$. Assume on the contrary that (S, A, p) is the choice of the seller in a SPE and there is some $i' \in M \setminus D$ such that $v((S^t \cup \{i'\}) \cup \{0\}) > v(S^t \cup \{0\})$.

Take any allocation $A' \in \mathcal{A}^*(S^t \cup \{i'\})$. If $w_i(A'_i) \geq w_i(B_i) - x_i$ for all $i \in S^t$, then define $p'_i = w_i(A'_i) - (w_i(B_i) - x_i)$ for each $i \in S^t$. Since (S, A, p) maximizes the seller's payoff and because of Lemma A.3, we have

$$\begin{aligned}\sum_{i \in S} p_i = \sum_{i \in S^t} p^t_i &= v(S^t \cup \{0\}) - \sum_{i \in S^t} \Big(w_i(B_i) - x_i \Big) \\ &< v((S^t \cup \{i'\}) \cup \{0\}) - \sum_{i \in S^t} \Big(w_i(B_i) - x_i \Big) = \sum_{i \in S^t} p'_i + w_{i'}(A'_{i'}),\end{aligned}$$

where the second equality comes from Lemma A.5. Therefore, buyer i' has incentives to deviate which contradicts that $((B,x),\mathcal{S})$ is a SPE. Therefore, there is some $i \in S^t$ such that $w_i(A'_i) < w_i(B_i) - x_i$.

By applying Lemma A.2 to $\bar{S} = S^t$ and $t = i'$, there exist $\emptyset \neq T \subseteq \bar{S}$ and $\bar{A} \in \mathcal{A}^*(T)$ such that $w_i(\bar{A}_i) \geq w_i(B_i) - x_i$ for all $i \in T$ and

$$\sum_{i\in(S^t\cup\{i'\})\setminus T} \Big(w_i(B_i) - x_i\Big) > v((S^t \cup \{i'\}) \cup \{0\}) - v(T \cup \{0\}) \\ > v(S^t \cup \{0\}) - v(T \cup \{0\}), \tag{20}$$

where the second inequality comes from the assumption. Define $\bar{p}_i = w_i(\bar{A}_i) - (w_i(B_i) - x_i)$ for each $i \in T$. Taking (20) into account, we obtain

$$\sum_{i\in S^t} p_i = v(S^t \cup \{0\}) - \sum_{i\in S^t} \Big(w_i(B_i) - x_i\Big) \\ < v((T \cup \{i'\}) \cup \{0\}) - \sum_{i\in T} \Big(w_i(B_i) - x_i\Big) = \sum_{i\in T} \bar{p}_i + w_{i'}(\bar{A}_t).$$

Then, buyer i' has incentives to deviate.[10] Hence $v((S^t \cup \{i'\}) \cup \{0\}) = v(S^t \cup \{0\})$.

Now, we prove $v((D \setminus \{t\}) \cup \{0\}) = v((M \setminus \{t\}) \cup \{0\})$. Assume on the contrary that $v((D \setminus \{t\}) \cup \{0\}) < v((M \setminus \{t\}) \cup \{0\})$. Then, there is some $i' \in M \setminus D$ such that

$$v((D \cup \{i'\}) \setminus \{t\}) \cup \{0\}) > v((D \setminus \{t\}) \cup \{0\}). \tag{21}$$

Notwithstanding, since $v((S^t \cup \{i\}) \cup \{0\}) = v(S^t \cup \{0\})$ for all $i \in M \setminus D$

$$0 = v((S^t \cup \{i'\}) \cup \{0\}) - v(S^t \cup \{0\}) \\ \geq v(((D \cup T) \setminus \{t\}) \cup \{0\}) - v(((D \cup T) \setminus \{i', t\}) \cup \{0\}) > 0,$$

where the inequality comes from (3) and the strict inequality from (21). This is a contradiction. Hence, $v((D \setminus \{t\}) \cup \{0\}) = v((M \setminus \{t\}) \cup \{0\})$. □

References

Ausubel, L., & Milgrom, P. 2002. Ascending auctions with package bidding. *Frontiers of Theoretical Economics*, **1(1), Article 1**, 489–509.

Āzacis, H. 2008. Double implementation in a market for indivisible goods with a price constraint. *Games and Economic Behavior*, **62**, 140–154.

Chew, S., & Serizawa, S. 2007. Characterizing the Vickrey combinatorial auction by induction. *Economic Theory*, **33**, 393–406.

Clarke, E. 1971. Multipart pricing of public goods. *Public Choice*, **11**, 17–33.

[10]See the argument in (8).

Day, R., & Milgrom, P. 2008. Core-selecting package auctions. *International Journal of Game Theory*, **36**, 393–407.

Day, R., & Raghavan, Subramanian. 2007. Fair payments for efficient allocations in public sector combinatorial auctions. *Management Science*, **53**, 1389–1406.

Demange, G., Gale, D., & Sotomayor, M. 1986. Multi-item auctions. *Journal of Political Economy*, **94**, 863–872.

Goeree, J., & Lien, Y. 2016. On the impossibility of core-selecting auctions. *Theoretical Economics*, **11**, 41–52.

Groves, T. 1973. Incentives in teams. *Econometrica*, **41**, 617–631.

Gul, F., & Stacchetti, E. 1999. Walrasian equilibrium with gross substitutes. *Journal of Economic Theory*, **87**, 95–124.

Kelso, A., & Crawford, V. 1982. Job matching, coalition formation and gross substitutes. *Econometrica*, **50**, 1483–1504.

Milgrom, P. 2004. Putting auction theory to work. *Cambridge University Press*, **New York**.

Pérez-Castrillo, D., & Sotomayor, M. 2002. A simple selling and buying procedure. *Journal of Economic Theory*, **103**, 461–474.

Shapley, L., & Shubik, M. 1972. The assignment game I: the core. *International Journal of Game Theory*, **1**, 111–130.

Vickrey, W. 1961. Counterspeculation, auctions and competitive sealed tenders. *Journal of Finance*, **16**, 8–37.

Wilson, R. 1978. Competitive exchange. *Econometrica*, **46**, 577–585.