# Internal Josephson effect in ultracold atoms

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**Abstract:** We have studied the internal Josephson effect between two hyperfine components of a Bose-Einstein condensate within the non-linear Gross-Pitaevskii equation framework. We have derived the sine-Gordon equation for the relative phase between the two components under the assumption of small variations of the density. In order to check the validity of this approximation, we have performed numerical simulations of the 1D Gross-Pitaevskii equation and we have found a good agreement with the theoretical predictions.

## I. INTRODUCTION

In nature we can find two big types of elementary particles: bosons and fermions. The difference between the two families is the value of the intrinsic angular momentum (spin): it is an integer number for bosons and a half-integer number for fermions. Compared to fermions, bosons do not verify the Pauli exclusion principle and it is possible to have some bosons in the same quantum state. Let us suppose that we have a gas of $N$ bosons that we can restrict spatially through a confinement potential $V(\mathbf{r})$. Quantum mechanics establishes that there will be a discretization of the energy values that the confined particles could adquire. If we decrease the temperature, the lowest energy states keep increasing their occupation number. When $T \to 0$ the system starts to condense (we have a high percentage of particles in the lowest energy monoparticle state) and we get a Bose-Einstein condensate (BEC). The De Broglie wavelength of the particles depends inversely on the temperature $\lambda \propto T^{-1/2}$. If $d$ is the average distance between the gas particles, when $T \to 0$ then $\lambda \approx d$ and the wavelengths of the particles overlap losing their particle identity and generating a matter wave that can be described by a macroscopic wave function. A BEC is a manifestation of quantum mechanics at a macroscopic scale. It is a coherent quantum system that can be described in an exact way through a single-particle wave function [1].

The quantum tunneling effect is one of the most relevant phenomena that are predicted by quantum mechanics. When these phenomena can be observed at a macroscopic scale it receives the name of "Josephson effect". The theory of the Josephson effect was developed in the context of superconductors. Heike Kamerlingh Onnes in 1911 "measuring the electric resistance of mercury" observed that the electrical resistance vanished below a critical temperature ($T_c \approx 4K$). But this phenomena was not understood until 1957 when J. Bardeen, L. Cooper and R. Schrieffer enunciated the BCS theory that postulated that the current carriers in a superconductors were pairs of electrons (Cooper pairs) that presented bosonic character due to the fact that they couple in the spin state $S = 0$. A superconductor is a BEC of Cooper pairs [2].

In BEC systems, we can distinguish between two types of Josephson effects: the internal (IJE) and the external (EJE). On the one hand the external Josephson effect is the tunneling of particles from one BEC to another through a physical potential barrier that separates them. An example of the EJE is a Josephson Junction, which consists of two superconductors coupled by a weak link (a thin insulator). In this system we can observe a direct current crossing the insulator in the absence of an external electromagnetic field. This Josephson current is proportional to the sine of the phase difference across the insulator between the two superconductors. On the other hand the internal Josephson effect is the tunneling of particles between two internal hyperfine states of the atom $|1\rangle$ and $|2\rangle$. The two condensates live together in the same trap and they see the same external potential. A typical example of experimental relevance of the IJE is an ultracold gas of $^{87}$Rb in which two hyperfine states live together $|F, M\rangle$ and the transit of particles from one state to another is possible [5].

In this project we study the internal Josephson effect in a two component BEC. This system allows us to obtain the sine-Gordon equation for the relative phase and to make the subsequent analysis of this equation, which is the final goal of this project.

The text is organized as follows: In section II we present the theoretical framework. In section III the hydrodynamical Josephson equations are obtained for two coupled condensates. Section IV presents the well-known sine-Gordon equation for the homogeneous system. The dispersion relation is obtained in section V. The numerical results are presented in section VI and finally the conclusions are commented.

## II. THE GROSS-PITAEVSKII EQUATION

The theoretical framework of this project is the Gross-Pitaevskii (GP) equation. It is a non-linear Schrödinger-like equation that describes a dilute system of $N$ weakly interacting bosons that are confined in an external potential $V(\mathbf{r})$ in the limit of low temperature $T \to 0$. The time-dependent GP equation is written as [1]:

$$i\hbar\frac{\partial\Psi}{\partial t} = \left[\frac{-\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) + g\left|\Psi\right|^2\right]\Psi, \tag{1}$$

where $\Psi(\mathbf{r},t)$ is the wavefunction of the condensate normalized to the number of particles $\int \left|\Psi\right|^2 d\mathbf{r} = N$, $m$ is the atomic mass and $g = 4\pi\hbar^2 a/m$ is the coupling constant that depends on the $s$-wave scattering length $a$, which is related to the effective cross section $\sigma$ of the interaction through $\sigma = 4\pi a^2$. Separating the temporal and spatial part $\Psi(\mathbf{r},t) = \psi(\mathbf{r})\exp(-i\mu t/\hbar)$, and substituting in (1) one obtains the time-independent GP equation:

$$\left[\mathcal{H}_0 + g\left|\psi\right|^2\right]\psi = \mu\psi, \tag{2}$$

where $\mathcal{H}_0 = \frac{-\hbar^2}{2m}\nabla^2 + V(\mathbf{r})$ and $\mu$ is the chemical potential of the system for the stationary state $\psi$.

Since the system is very dilute ($\left|\psi\right|^2\left|a\right|^3 \ll 1$) the collisions between two particles are more common than the collisions between three particles (which are very unlikely). On the other hand, since the temperature is so low, the dispersed particle is not able to see the details of the internal structure of the disperser particle. We have considered that the interaction potential is a contact potential represented by a Dirac delta function as $V(\mathbf{r}) = g\,\delta(\mathbf{r})$.

It is worth remarking that the coherence of the quantum system allows us to describe it through the single-particle wave function $\psi$. Each one of these particles is subjected to an average potential generated by the $N-1$ remaining particles distributed in density according to $\left|\psi(\mathbf{r})\right|^2$. Therefore the GP equation is a mean field equation.

## III. COUPLED CONDENSATES AND JOSEPHSON EQUATIONS

We are interested in obtaining the Josephson equations that describe the dynamics between two coherently coupled hyperfine states. We are going to consider the IJE between two states $\Psi_1$ and $\Psi_2$ of the same atom species confined in the same trap. We can write the total density of the system as $\left|\Psi_1\right|^2 + \left|\Psi_2\right|^2$ normalized to the number of particles $\int d\mathbf{r}(\left|\Psi_1\right|^2 + \left|\Psi_2\right|^2) = N_1 + N_2 = N$. The total number of atoms $N$ is conserved. The coupled GP equations are [4]:

$$i\hbar\frac{\partial\Psi_1}{\partial t} = \left[\mathcal{H}_0 + g_{11}\left|\Psi_1\right|^2 + g_{12}\left|\Psi_2\right|^2\right]\Psi_1 - \frac{\hbar}{2}\Omega\Psi_2, \tag{3a}$$

$$i\hbar\frac{\partial\Psi_2}{\partial t} = \left[\mathcal{H}_0 + g_{22}\left|\Psi_2\right|^2 + g_{21}\left|\Psi_1\right|^2\right]\Psi_2 - \frac{\hbar}{2}\Omega\Psi_1, \tag{3b}$$

where $g_{ij}$ is the intensity of the interaction between the particles of the condensate $i$ and the condensate $j$ with $i$,$j$=1,2 and $\Omega$ is the coherent coupling constant (Rabi frequency) allowing the exchange of the population between both components. We can see that $g_{12} = g_{21}$ but in general $g_{11} \neq g_{22}$. However, because of its simplicity we are going to consider $g_{11} = g_{22} = g$, given the fact that in the systems of experimental interest (for example $^{87}$Rb) it has been observed that $g_{11} \simeq g_{22}$ [6].

We are going to transform the coupled equations (3a) and (3b) into the hydrodynamical picture changing to polar coordinates $\Psi_j(\mathbf{r},t) = \sqrt{n_j}\exp(\theta_j)$ where $n_j$ and $\theta_j$ are the density and phase, respectively, of the $j$ component. Substituting in the coupled equations (3a) and (3b), after some algebraic calculations and splitting them into imaginary and real parts we obtain the Josephson equations for component "1",

$$\frac{\partial n_1}{\partial t} + \nabla(n_1\mathbf{v}_1) = -\Omega\sqrt{n_1 n_2}\,\sin(\theta_2 - \theta_1), \tag{4}$$

$$-\hbar\frac{\partial\theta_1}{\partial t} = \mathcal{Q} + \mathcal{P} + \frac{m\mathbf{v}_1^2}{2} + V(\mathbf{r}) - \frac{\hbar}{2}\sqrt{\frac{n_2}{n_1}}\Omega\cos(\theta_2 - \theta_1), \tag{5}$$

where $\mathcal{P} = gn_1 + g_{12}n_2$ , $\mathcal{Q} = \frac{-\hbar^2}{2m\sqrt{n_1}}\nabla^2\sqrt{n_1}$ is the quantum pressure term, and $\mathbf{v}_1 = \hbar\nabla\theta_1/m$ is the superfluid velocity. The corresponding equations for component "2" can be obtained by swapping indexes and changing the sign of the second term in the upper equation.

The equation (4) is a continuity equation but with an additional sinusoidal, source term. The amplitude of the oscillation is the critical current $J_c = \Omega\sqrt{n_1 n_2}$. This term locally injects particles that come from the state "2" into the state "1" (there is a flux of particles from the state $|2\rangle$ to the state $|1\rangle$ because of the phase difference $\theta_2 - \theta_1$ between the two states) [2].

The equation (5) is the momentum conservation equation. It can be seen by applying the gradient $\nabla$ and dividing by the mass $m$. We obtain $m\frac{\partial\mathbf{v}_1}{\partial t} = -\nabla\phi$ where the right side of the equation plays the role of a force (gradient of a potential $\phi$).

Both equations (4) and (5) can be identified as the Euler equations for an irrotational ($\nabla \times \mathbf{v} = 0$) and non-viscous fluid.

## IV. HOMOGENEUS SYSTEM AND SINE-GORDON EQUATION

We consider an homogeneous two-component system with $V(\mathbf{r}) = 0$. The ground state for each component is $\Psi_j(\mathbf{r}, t) = \sqrt{n_j}\exp(-i\mu t/\hbar)$ that corresponds to a uniform density. In order to simplify, we consider that $n_1 = n_2 = n/2$ where $n$ is the total density. We are going to introduce a small perturbation $[\delta n(\mathbf{r}, t), \theta(\mathbf{r}, t)]$ around the uniform density and uniform phase corresponding to the equilibrium state. That is $[\sqrt{n_1}, \sqrt{n_2}] \longrightarrow [\sqrt{n_1 + \delta n_1}\exp(i\theta_1), \sqrt{n_2 + \delta n_2}\exp(i\theta_2)]$ where we have considered that $\theta_j$ are the phase-perturbations directly. Introducing the perturbed states in (4) and (5) and expanding to first order we obtain:

$$\frac{\partial \delta n_1}{\partial t} + \nabla(n_1 \frac{\hbar}{m}\nabla\theta_1) = -\Omega\sqrt{n_1 n_2}\,\sin(\theta_2 - \theta_1), \tag{6}$$

$$-\hbar\frac{\partial \theta_1}{\partial t} = \frac{-\hbar^2}{2m\sqrt{n_1}}\nabla^2\sqrt{\delta n_1} + g\delta n_1 + g_{12}\delta n_2 - \hbar\Omega\frac{\delta n_2}{n}, \tag{7}$$

and equivalent equations for $\delta n_2$ and $\theta_2$. We are going to consider low energy excitations (that is $k \to 0$). In this limit, the first term on the right hand side of the equation (7) can be dropped ($\nabla^2\sqrt{\delta n_1} \to 0$). Also, the second term in (6) contains $\nabla n_1$ which is negligible in the limit of equal constant densities $n_1 = n_2 = n/2$. As we can observe, the relative phase $\theta = \theta_2 - \theta_1$ appears explicitly in (6) and it is better to change the variables from $[\theta_1, \theta_2] \longrightarrow [\theta, \delta\theta]$ where $\theta$ is the relative phase and $\delta\theta = \theta_1 + \theta_2$ is the "total" phase. Replacing in (6) and (7), eliminating the terms that we mentioned before and doing some algebra with the two equations, one obtains the next two equations for the phases $[\theta, \delta\theta]$:

$$\frac{2m}{(g + g_{12})n}\frac{\partial^2\delta\theta}{\partial t^2} - \nabla^2\delta\theta = 0, \tag{8}$$

$$\frac{2m}{(g - g_{12})n + \hbar\Omega}\frac{\partial^2\theta}{\partial t^2} - \nabla^2\theta = -\frac{2m\Omega}{\hbar}\sin\theta. \tag{9}$$

The equation (8) can be identified as a wave equation for the "total" phase $\delta\theta$ with a propagation speed of sound $c_s = \sqrt{(g + g_{12})n/2m}$. We note that $c_s$ depends on the interaction between the particles ($g$ and $g_{12}$) and the total density of the medium $n$.

The equation (9) is the famous sine-Gordon equation for the relative phase $\theta$. We can identify a characteristic speed $c_J = \sqrt{((g - g_{12})n + \hbar\Omega)/2m}$ and dividing the expression by $c_J^2$, it can be rewritten as [2]:

$$\frac{1}{c_J^2}\frac{\partial^2\theta}{\partial t^2} - \nabla^2\theta = -\frac{1}{\xi^2}\sin\theta, \tag{10}$$

where $\xi = \sqrt{\hbar/2m\Omega}$ represents a characteristic length.

In order to understand the sine-Gordon equation, let us elaborate a simple mechanic analogy through a system of coupled pendula in one dimension. We start from the Lagrangian of a single pendulum:

$$L = \frac{1}{2}m\ell^2\Big(\frac{\partial\theta}{\partial t}\Big)^2 - mg\ell(1 - \cos\theta), \tag{11}$$

where $m$ is the mass of the pendulum, $g$ is the gravity acceleration, $\ell$ is the length of the pendulum and $\theta$ is the displaced angle from the vertical. If we consider a row of $N$ pendula coupled through a harmonic potencial with effective constant $K$ and separated by $\Delta x$, the relative angular displacement between two consecutive pendula can be written as $\ell(\theta_i - \theta_{i-1})$ with $i$=1,2,.... The lagrangian for the system of the $N$ pendula is:

$$L_N = \sum_i \Big[\frac{1}{2}m\ell^2\Big(\frac{\partial\theta_i}{\partial t}\Big)^2 - \frac{1}{2}K\ell^2(\theta_i - \theta_{i-1})^2 - mg\ell(1 - \cos\theta_i)\Big]. \tag{12}$$

Taking the limit $\Delta x \to 0$ in (12) we obtain a lagrangian density:

$$\mathcal{L} = \lim_{\Delta x \to 0} L_N = \frac{1}{2}m\ell^2\Big(\frac{\partial\theta}{\partial t}\Big)^2 - \frac{1}{2}\kappa^2\ell^2\Big(\frac{\partial\theta}{\partial x}\Big)^2 - mg\ell(1 - \cos\theta), \tag{13}$$

where $\kappa = K(\Delta x)$ is the coupling energy for unit length between two pendula. From (13) we can obtain the equation of motion using the Euler-Lagrange equations:

$$\frac{m}{\kappa}\frac{\partial^2\theta}{\partial t^2} - \frac{\partial^2\theta}{\partial x^2} = -\frac{mg}{\kappa\ell}\sin\theta. \tag{14}$$

We can identify (14) with the sine-Gordon equation (10) where $c_J^2 = \kappa/m$ and $\xi^2 = \kappa\ell/mg$. If $\theta$ does not depend on the position, $\partial_x^2\theta \approx 0$ and considering small oscillations ($\sin\theta \approx \theta$) we obtain the classical equation of the pendulum $\ddot{\theta} + \omega^2\theta = 0$ where $\omega^2 = g/\ell$ is the angular frequency of the oscillations.

## V. DISPERSION RELATION $\omega(\mathbf{k})$

We consider the 1D sine-Gordon equation. Nowadays it is possible to confine the condensate in a single dimension using different experimental techniques. For example, the condensate can be tightly confined in the transversal coordinates $(x, y)$ but loosely in the axial coordinate $z$ (that is so that $\omega_\perp/\omega_z >> 1$). Thus, the system behaves effectively as one-dimensional along a single

coordinate $z$ and $V(z) = 0$. In particular, in ring geometries the dynamics can be restricted to the angular coordinate and the assumption of $V = 0$ is exact [7]. If we suppose that the perturbation of the relative phase $\theta$ is small we can make the approximation $\sin\theta \approx \theta$. Expanding the phase in plane waves $\theta = e^{i(\omega t - kx)}$ and substituting in (10) we obtain the dispersion relation:

$$\omega = c_J\sqrt{\frac{2m\Omega}{\hbar} + k^2}, \tag{15}$$

which can be rewritten introducing the characteristic length $\xi$ as:

$$\omega = \omega_0\sqrt{1 + (k\xi)^2}, \tag{16}$$

where $\omega_0 = c_J/\xi$.

The relation $\omega(k)$ is represented in the Fig. 1. The group speed of the moving waves is given by:

$$v_g = \frac{\partial\omega(k)}{\partial k} = \frac{\omega_0\xi^2}{\sqrt{1 + (k\xi)^2}}k. \tag{17}$$

We are going to analyze the limits of this expression, which takes two limiting values:

- For $k\xi \ll 1$, in this limit $v_g \to 0$, and the perturbation can not spread into the medium when $k = 0$. We can obtain the gap energy $E_0 = \hbar\omega_0 = \hbar c_J/\xi$ which is the minimum energy that we have to supply to the system to generate an excitation. This limit provides the angular frequency $\omega_0$ of the plasma oscillations described by the equation $\partial^2_{tt}\theta = -\omega_0^2\theta$.

- For $k\xi \gg 1$, $v_g = c_J$, and the dispersion relation is linear $\omega = c_J k$ (the medium is not dispersive). The perturbations with different $k$ propagate at the same speed $c_J$ which corresponds to the maximum speed in the medium.

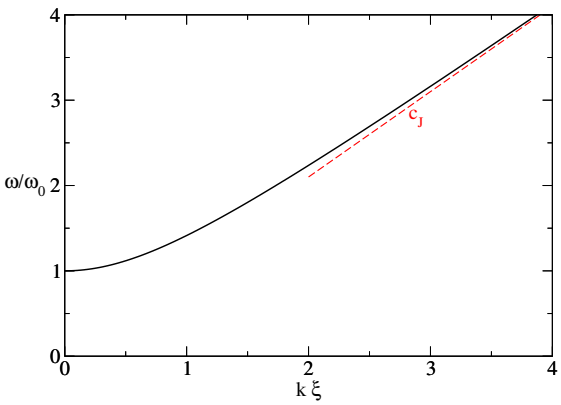


FIG. 1: Dispersion relation in the limit of small perturbations in the relative phase $\theta$ for positive wave vectors. The relation is symmetric for negative wave vectors. The maximum speed of propagation in the medium is also represented (red).

## VI. NUMERICAL RESULTS

In order to verify the dispersion relation obtained theoretically, it is necessary to perform numerical simulations of the 1D Gross-Pitaevskii equation (3a). We have employed a numerical method based in a finite difference approach for the spacial discretization, and a norm preserving scheme for the time evolution.
We consider an homogeneous system ($V(\mathbf{r}) = 0$). The non-perturbed state $\psi_0$ corresponds to uniform densities $n_1 = n_2$ and phases $\theta_1 = \theta_2 = 0$. Now, as a general procedure [5], we introduce a localized gaussian perturbation in the relative phase $\theta$ with a specific amplitude $A$ and width $w$, that is $\psi_j = \psi_0 \to \psi_0 \pm \delta\psi(x)$, where $\delta\psi(x) = A\exp(-x^2/2w^2)$. If we change to the momentum space $\psi(x) \to \psi(k)$ by Fourier transforming, we can control the wavelengths of the introduced perturbation which is related to $w$.
In particular, we consider that the perturbation width is larger than the condensate dimensions (in practice $w \to \infty$ and $k \to 0$). If we allow the temporal evolution of the system, we observe the plasma oscillation in the density of the condensate components, and we can calculate the oscillation frequency which corresponds to $\omega_0$. Fig. 2. shows the real time evolution of the system for this type of perturbation. The amplitude of the plasma oscillation is represented against the dimensionless time $\Omega t$, and we can measure the oscillation period $T$. The oscillation frequency is obtained as $\omega_0 = 2\pi/T$.
We have calculated the plasma oscillation changing the interaction strengths and we have observed that when $g_{12} \to g$ then $\omega_0 \to \Omega$ according to the predictions of the analytical model. Fig. 2. represents the plasma oscillations for two cases, $g/g_{12} = 2$ and $g = g_{12}$, for the former the relative error $\epsilon = (\omega_{GP} - \omega_0)/\omega_{GP}$ is $\epsilon = 0.34\%$, whereas for the latter is $\epsilon = 0.01\%$.

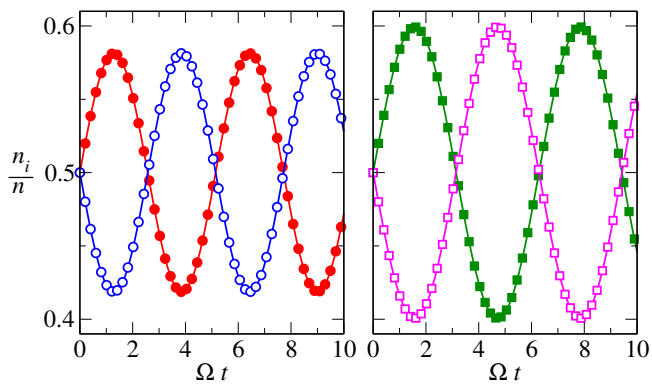


FIG. 2: Plasma oscillation for two components in two cases: on the left, $g/g_{12} = 2$, and on the right, $g = g_{12}$ .

We can observe that the oscillation amplitude decreases when $g \to g_{12}$, and reaches the maximum at $g = g_{12}$. The energy barrier for producing spin exci-

tations between the two hiperfine states (the gap) is proportional to $(g - g_{12})n + \hbar\Omega$ and it is minimum when $g = g_{12}$; in this case, the Josephson current is maximum.

## VII. CONCLUSIONS

In this work we point out the importance of the superfluid phase in order to explain the Josephson effect. We have studied the dynamics of the relative phase between the two coherently coupled hyperfine states of a BEC in a homogeneous background. We have demonstrated that the relative phase, in the limit of small perturbations of the ground state, follows a sine-Gordon equation, as occurs in superconducting long Josephson junctions. After making an analytical derivation of the equation of motion for the relative phase, and by identifying the relevant parameters (the propagation speed of the perturbations and the characteristic length of the coupling), we have compared them with the numerical results obtained by solving the 1D Gross-Pitaevskii equation. We have focused on the plasma oscillations, and we have obtained a good agreement between the numerics and the analytical expressions.
Many interesting questions deserve a further research which is beyond the scope of this work; for example, a more throrough analysis of the dynamics of the relative phase against different type of perturbations, or the study of the solitonic solutions of the sine-Gordon equation. On the other hand, the generalization of the used formalism allows to study quantum systems of different nature with the same analytical techniques, as it is the case of a superconductor or an ultracold rubidium gas. Finally it is noteworthy that the research on spinor condensates is currently a very active field in the ultracold gases community, and there is a fruitful interplay between theoretical predictions and experiments.

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