Ruin problems for a discrete time risk model with non-homogeneous conditions

ANNA CASTAÑER [a], M. MERCÈ CLARAMUNT [a], MAUDE GATHY [b], CLAUDE LEFÈVRE [b, 1] and MAITE MÁRMOL [a]

[a] Universitat de Barcelona, Departament de Matemàtica Econòmica, Financera i Actuarial, Barcelona

[b] Université Libre de Bruxelles, Département de Mathématique, Bruxelles

This paper is concerned with a non-homogeneous discrete time risk model where premiums are fixed but non-uniform, and claim amounts are independent but non-stationary. It allows one to account for the influence of inflation and interest and the effect of variability in the claims. Our main purpose is to develop an algorithm for calculating the finite time ruin probabilities and the associated ruin severity distributions. The ruin probabilities are shown to rely on an underlying algebraic structure of Appell type. That property makes the computational method proposed quite simple and efficient. Its application is illustrated through some numerical examples of ruin problems. The well-known Lundberg bound for ultimate ruin probabilities is also reexamined within such a non-homogeneous framework.

Keywords: Discrete time risk model; Non-uniform premiums; Non-stationary claims; Rates of interest; Finite time ruin probability; Ruin severity distribution; Computational methods; Lundberg bound

1 A non-homogeneous discrete time risk model

The classical compound Poisson and binomial risk models assume that the premium income is constant over time and the claim amounts form a sequence of independent identically distributed (i.i.d.) random variables. These assumptions of homogeneity of premiums and claim amounts can be too restrictive in reality, especially because of the influence of the economic environment. For instance, inflation and interest can affect, sometimes drastically, the evolution of the reserves of the company. Claim amounts and premiums have often a tendency to increase for various socio-economic reasons (e.g. higher loss levels and larger compensations or coverages).

In this section, we generalize the compound binomial risk model in order to account for such factors of non-homogeneity. For that, it will be necessary to specify, inter alia, the time when premiums are collected and how they are evaluated. To begin with, we are going to consider a particular model which incorporates arbitrary fixed interest rates.

The influence of interest force. Risk theory with interest income has received an increasing attention in the literature. A number of works are devoted to models

[1]Corresponding author. E-mail: clefevre@ulb.ac.be

in continuous-time; see the books by Rolski et al. (1999) and Asmussen (2000) and the references therein. In comparison, papers on models in discrete time are much less numerous; see e.g. Cheng et al. (2000). A discrete time formulation, however, may be more appropriate for many applications in insurance.

Recently, different extensions have been proposed to include interest in the compound binomial model. A case with fixed interest rates was studied by de Kok (2003) and Lefèvre and Picard (2006). A variant with i.i.d. premiums and a given interest rate was examined by Yang (1999) and Sun and Yang (2003). An extended version with i.i.d. interest rates was investigated by Cai (2002a), Yang and Zhang (2006) and Wei and Hu (2008). A model with heavy tailed risks was analyzed e.g. by Tang (2004). Cases with dependent interest rates were discussed e.g. by Cai (2002b) and Yao and Wang (2010).

The discrete time model considered here assumes arbitrary fixed interest rates, i.i.d. claim amounts and a non-uniform deterministic premium income. Let $t \in \mathbb{N}$ denote the time scale. The initial reserves of the company are of amount $u \geq 0$. The rates of interest per period $(t-1, t]$, $t \in \mathbb{N}_0$, are given constants $i_t \geq 0$, non-necessarily identical. The total claim amounts that occur in the periods $(t-1, t]$ are i.i.d. non-negative random variables Y_t. Of course, these claim amounts are registered at the end of each period. The premiums received for the periods $(t-1, t]$ are given by $p_t \geq 0$. These premiums are collected in many cases at the beginning of each period, but not always (see below).

Let $U(t)$, $t \in \mathbb{N}_0$, be the discounted value, at time 0, of the surplus of the company up to the end of the t-th period; put $U(0) = u$. If the premiums are received at the beginning of the periods, the dynamic of the surplus is then given by

$$U(t) = u + p_1 - \frac{Y_1}{1+i_1} + \left[p_2 - \frac{Y_2}{1+i_2}\right]\frac{1}{1+i_1} + \ldots + \left[p_t - \frac{Y_t}{1+i_t}\right]\prod_{j=1}^{t-1}\frac{1}{1+i_j}. \quad (1.1)$$

In the rare cases where premiums are collected at the end of the periods,

$$U(t) = u + (p_1 - Y_1)\frac{1}{1+i_1} + (p_2 - Y_2)\frac{1}{(1+i_1)(1+i_2)} + \ldots + (p_t - Y_t)\prod_{j=1}^{t}\frac{1}{1+i_j}. \quad (1.2)$$

For premiums that are received at a uniform rate per time unit, one might consider that they are registered at the middle of the periods, so that

$$U(t) = u + \left[p_1 - \frac{Y_1}{(1+i_1)^{1/2}}\right]\frac{1}{(1+i_1)^{1/2}} + \left[p_2 - \frac{Y_2}{(1+i_2)^{1/2}}\right]\frac{1}{(1+i_2)^{1/2}(1+i_1)} + \ldots + \left[p_t - \frac{Y_t}{(1+i_t)^{1/2}}\right]\frac{1}{(1+i_t)^{1/2}}\prod_{j=1}^{t-1}\frac{1}{1+i_j}, \quad t \in \mathbb{N}_0. \quad (1.3)$$

The premiums p_t may be calculated through standard actuarial rules (e.g. Kaas et al. (2008)). Being determined in function of the real risk to cover, they will thus depend on the rates of interest and the exact time when they are received in the period. This situation, although natural, is not always the one described in various works on the topic. So, a frequent assumption is that the premiums form a sequence of given constants (e.g. de Kok (2003)) or a sequence of i.i.d. random variables, independently of the claim amounts (e.g. Cai (2002a)).

Let us first consider the expected value principle, with a safety loading factor θ_t. By definition, one has $p_t = (1+\theta_t)E[Y_t/(1+i_t)]$ for (1.1), $p_t = (1+\theta_t)E(Y_t)$ for (1.2) and $p_t = (1+\theta_t)E[Y_t/(1+i_t)^{1/2}]$ for (1.3). Inserting these premiums then yields the same recurrence for the surplus in the three cases:

$$U(t) = u + \sum_{j=1}^{t} [(1+\theta_j)E(Y_j) - Y_j] \prod_{k=1}^{j} \frac{1}{1+i_k}. \tag{1.4}$$

The reason why the three models are identical is that the expected value satisfies the scale invariance property. Thus, this is also true when using the standard deviation principle; the model (1.4) then holds in which the term $[\ldots]$ is replaced by $E(Y_j) + \theta_j\sigma(Y_j) - Y_j$.

With the variance, exponential or Esscher principles, for instance, the three models become distinct. So, in the former case, (1.1) becomes

$$U(t) = u + \sum_{j=1}^{t} \left[E(Y_j) + \frac{\theta_j}{1+i_j}\sigma^2(Y_j) - Y_j\right] \prod_{k=1}^{j} \frac{1}{1+i_k}. \tag{1.5}$$

For (1.2), the term $[\ldots]$ in (1.5) is replaced by $E(Y_j) + \theta_j\sigma^2(Y_j) - Y_j$, and for (1.3), by $E(Y_j) + [\theta_j/(1+i_j)^{1/2}]\sigma^2(Y_j) - Y_j$. The differences are evident too with the exponential principle.

A general non-homogeneous environment. Under interest force, both premiums and claim amounts are time-dependent through the discount factors. Let us now introduce a more general non-homogeneous discrete time model in which premiums are non-uniform and claim amounts have non-stationary independent distributions. This extension allows us to cover other causes of non-homogeneity, in particular the time dependency of the cost or dangerousness of the risks.

The surplus process under concern is defined by $U(0) = u \geq 0$ and

$$U(t) = u + c(t) - S(t), \quad t \in \mathbb{N}_0, \tag{1.6}$$

where $c(t)$ and $S(t)$ denote the cumulated premiums and claim severities during the first t periods, respectively. In a context of interest rates, $c(t)$ and $S(t)$ correspond to the discounted values, at time 0, of these amounts.

By construction, $c(t) = c_1 + \ldots + c_t$, where the premiums c_t in period t are fixed non-negative (discounted) amounts. They are collected at some given time in the period, for example at the beginning, the end or the middle of the period. Thus, for the previous models with interest, putting

$$a(0) = 1 \quad \text{and} \quad a(t) = \prod_{j=1}^{t}(1+i_j), \quad t \in \mathbb{N}_0, \tag{1.7}$$

these premiums are given by $c_t = p_t/a(t-1)$ for the model (1.1), by $c_t = p_t/a(t)$ for (1.2) and by $c_t = p_t/a(t-1)(1+i_t)^{1/2}$ for (1.3).

Moreover, $S(t) = X_1 + \ldots + X_t$, where the claim sizes X_t in period t are non-negative independent and non-stationary (discounted) random variables. So, $X_t = Y_t/a(t)$ for the three models with interest (1.1), (1.2) and (1.3). Each X_t has a continuous distribution function F_t on $\mathbb{R}_0^+$ with, in addition, a positive probability mass at 0 (i.e. $F_t(0) > 0$). The assumption of continuity seems to be reasonable, and the inclusion of a mass at 0 translates the possibility of no claim at all (as it is in the classical risk models). As usual, the net profit condition and the no ripoff condition are expected to hold, i.e.

$$c_t > E(X_t), \quad \text{and} \quad P(X_t > c_t) > 0, \quad t \in \mathbb{N}_0. \tag{1.8}$$

Our purpose is to investigate some questions raised by the eventual ruin of the company. Ruin occurs at time T when the surplus becomes negative, i.e.

$$T = \inf\{t \in \mathbb{N}_0 : \mathrm{U(t)} < 0\}.$$

The paper is organized as follows. In Section 2, an algorithm is developed to evaluate the ruin probabilities and the associated ruin severity distributions, on any finite time horizon. Its application is illustrated, in Section 3, by means of several numerical examples. In Section 4, the well-known Lundberg bound for ultimate ruin probabilities is reexamined within a non-homogeneous framework.

2 An algorithm for finite time ruin probabilities

In this Section, we present an algorithm for calculating, in the model (1.6), the finite time ruin probabilities and the associated ruin severity distribution. The ruin probabilities are shown to rely on an underlying algebraic structure of Appell type. This makes the computational method proposed quite simple and efficient. In particular, it will be more easily applicable than an alternative method proposed by De Vylder and Goovaerts (1988) for a simplified model.

Appell polynomials are standard in mathematics (e.g. Mullin and Rota (1970)). Pseudoplynomials of Appell form can be defined in an analogous way (Picard and Lefèvre (1996)). These (pseudo)polynomials may be exhibit in the study of various first crossing problems in probability and statistics. This is especially true in ruin theory for some risk models, in discrete or continuous time; see e.g. Picard and Lefèvre (1997), Ignatov and Kaishev (2000), (2004), Ignatov et al. (2004), Lefèvre and Picard (2006), Lefèvre and Loisel (2009) and Dimitrova and Kaishev (2010). The assumptions made here of continuous claims will lead us to point out, using a simple approach, the existence of an algebraic structure of similar type with, this time, a continuous index.

2.1 An integral representation

The probability under study is

$$\phi(t, x) \equiv P[T > t \text{ and } U(t) \geq x], \quad t \in \mathbb{N} \text{ and } 0 \leq \mathrm{x} \leq \mathrm{u} + \mathrm{c(t)},$$

i.e. the probability that at time t, the company is not ruined and its reserves are at least equal to x. Evidently, the non-ruin probability until t is simply $\phi(t) \equiv \phi(t, 0) = P(T > t)$.

Let us introduce the following sequence of (integer) times: for any $s \in \mathbb{R}^+$,

$$v_s = \sup\{t \in \mathbb{N} : \mathrm{u} + \mathrm{c(t)} < \mathrm{s}\} \quad \text{if s > u},$$

where $c(0) \equiv 0$, and let $v_s = 0$ if $s \leq u$. Thus, v_s represents the last time where a global claim amount equal to s would lead to ruin, if ever; otherwise, $v_s = 0$. Observe that after v_s, the first time when a total claim amount of $s > u$ would not cause ruin, depends on the time when premiums are collected. Indeed, if premiums are received at the beginning of the period, ruin would not arise at time $v_s + 0$, while if premiums are received at the end or the middle of the period, ruin would not occur at time $v_s + 1$. Figure 1 shows the graph of $u + c(t)$ (indicated by black points for $t \in \mathbb{N}_0$) and the value of v_s for some given s, when the premiums are received at the beginning or the end of the period; Figure 2 gives the graph of v_s in function of s (this is the same in both cases).

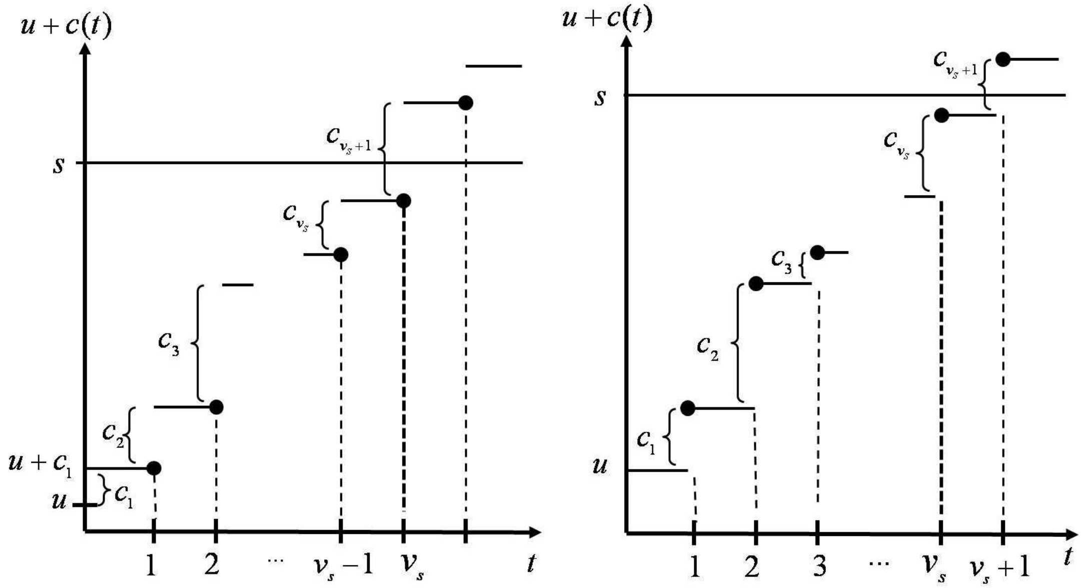

Figure 1: Cumulated premium function and some couple (s, v_s) when premiums are received at the beginning (left side) or the end (right side) of the period.

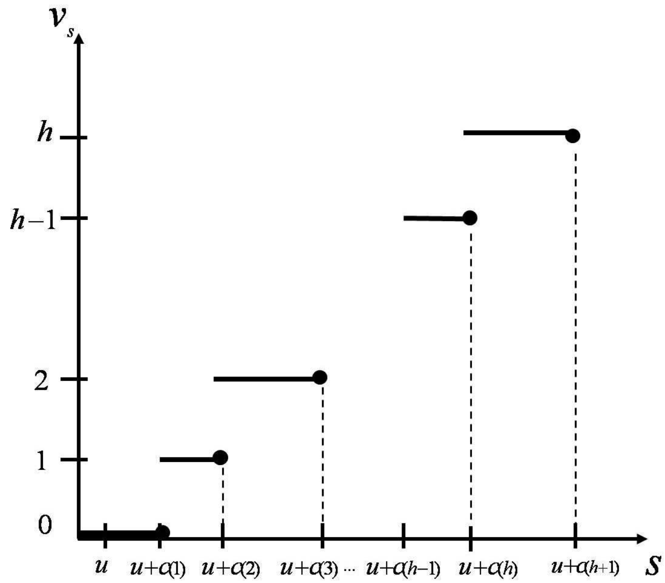

Figure 2: Graph of v_s in function of s (in both cases).

Proposition 2.1 *$\phi(t,x)$ can be represented as*

$$\phi(t,x) = F_{S(t)}(0) + \int_{w=0}^{u+c(t)-x} b_w \, F_{S(t)}(u+c(t)-x-w)\, dw, \tag{2.1}$$

where b_w, $w \in \mathbb{R}^+$, is a real function satisfying the equations

$$0 = \int_{w=0}^{s} b_{s-w}\, dF_{S(v_s)}(w), \quad s \in \mathbb{R}^+. \tag{2.2}$$

Proof. By definition,

$$\begin{aligned} \phi(t,x) &= P[T > t \text{ and } S(t) \leq u + c(t) - x] \\ &= P[S(t) = 0] + \int_{s=0}^{u+c(t)-x} \phi_s(t)\, ds, \end{aligned} \tag{2.3}$$

where

$$\phi_s(t)\, ds \equiv P[T > t \text{ and } s < S(t) \leq s + ds], \quad s \in \mathbb{R}^+. \tag{2.4}$$

Looking at the instant v_s, one observes that $\phi_s(t) = 0$ when $t \leq v_s$. On another hand, for $t > v_s$, the event $[T > t, S(t) \sim s]$ (using an obvious notation $\sim$) is equivalent to both events $[T > v_s, S(v_s) \sim s - w]$ and $[S(v_s, t) \equiv S(t) - S(v_s) \sim w]$. The reason here is that ruin in the time interval (v_s, t) is then impossible. In other words,

$$\phi_s(t) = \int_{w=0}^{s} \phi_{s-w}(v_s)\, dF_{S(v_s,t)}(w), \quad t > v_s. \tag{2.5}$$

Formula (2.5) provides a remarkable expansion of $\phi_s(t)$ in which the coefficients are given by the previous functions ϕ_{s-w} evaluated at a same point v_s. In fact, (2.5) looks like a generalized Taylor expansion for a function $\phi_s(t)$ that has an algebraic structure of Appell type (so that ϕ_{s-w} is the w-th generalized derivative of ϕ_s); see e.g. Lefèvre and Picard (2006), formula (4.9) and Lefèvre (2007), formula (5.12).

As a corollary, we are going to show that $\phi_s(t)$ can be reexpressed more simply as

$$\phi_s(t) = \int_{w=0}^{s} b_{s-w}\, dF_{S(t)}(w), \quad t \geq v_s, \tag{2.6}$$

for appropriate coefficients b_w, $0 \leq w \leq s$. First, we note that since $\phi_s(v_s) = 0$, (2.6) with $t = v_s$ implies that b_w, $w \in \mathbb{R}^+$, satisfies the integral relations (2.2). Now, consider $t > v_s$. The claim amounts per periods being independent, one has

$$P[S(t) \sim w] = \int_{y=0^-}^{w} P[S(v_s) \sim w-y]\, dF_{S(v_s,t)}(y). \tag{2.7}$$

Substituting (2.7) in the right-hand side of (2.6) and permuting the two summations with $z \equiv w - y$ then yields

$$\begin{aligned} \phi_s(t) &= \int_{w=0}^{s} b_{s-w} \left\{ \int_{y=0^-}^{w} P[S(v_s) \sim w-y]\, dF_{S(v_s,t)}(y) \right\} dw \\ &= \int_{y=0^-}^{s} \left\{ \int_{z=0}^{s-y} b_{s-y-z}\, P[S(v_s) \sim z]\, dz \right\} dF_{S(v_s,t)}(y). \end{aligned}$$

As $v_s \geq v_{s-y}$, (2.6) is applicable to $\{\ldots\}$ and gives $\phi_{s-y}(v_s)$. Therefore, $\phi_s(t)$ admits the announced representation (2.5).

Finally, let us insert (2.6) (where w is substituted for $s-w$) inside (2.3) (note that $t \geq v_s$). One then gets

$$\begin{aligned} \phi(t,x) &= F_{S(t)}(0) + \int_{s=0}^{u+c(t)-x} \left\{ \int_{s-w=0}^{s} b_w\, dF_{S(t)}(s-w) \right\} dw \\ &= F_{S(t)}(0) + \int_{w=0}^{u+c(t)-x} b_w \left\{ \int_{s=w}^{u+c(t)-x} dF_{S(t)}(s-w) \right\} dw, \end{aligned}$$

which gives the other stated formula (2.1). $\diamond$

Note that the coefficients b_w in (2.1), (2.2) are independent of t, which can provide a computational advantage. For instance, suppose that $\phi(t,x)$ has already been calculated. If now $\phi(t+\tau,x)$ is needed, it suffices to determine, using (2.2), new b_w's for $w \in (u + c(t) - x, u + c(t+\tau) - x)$ and then to apply (2.1) with $F_{S(t+\tau)}(\ldots)$. For $\phi(t-\tau,x)$, the required b_w's are already known and it thus suffices to evaluate $F_{S(t-\tau)}(\ldots)$.

Remarks. De Vylder and Goovaerts (1988) developed a different algorithm to compute finite time non-ruin probabilities in a classical discrete time risk model (i.e. with no interest and i.i.d. claim amounts). The method mainly consists in conditioning on the claim amount during the first period $(0,1)$. It was applied in several recent papers including Cai (2002b), Sun and Yang (2003) and Yang and Zhang (2006).

Let us first show how this algorithm can be easily extended to the present model. For that, we introduce the shifted non-ruin probabilities

$$\phi_j(t|u) = P(X_j \leq u+c_j, X_j+X_{j+1} \leq u+c_j+c_{j+1}, \ldots, X_j+\ldots+X_t \leq u+c_j+\ldots+c_t),$$

for $j = 1, \ldots, t$. Evidently, $j = 1$ gives $\phi_1(t|u) = P(T > t) = \phi(t)$. Conditioning on X_j, we get

$$\phi_j(t|u) = \int_{y=0^-}^{u+c_j} \phi_{j+1}(t|u+c_j-y)\, dF_j(y), \quad j = 1, \ldots, t-1, \tag{2.8}$$

together with

$$\phi_t(t|u) = F_t(u+c_t). \tag{2.9}$$

The relations (2.8), (2.9) allow the recursive computation of the probabilities ϕ_j for $j = t, \ldots, 1$, hence $\phi(t)$. A large number of calculations, however, is involved because the initial reserves increases at each step j. Note also that changing t in τ yields a different starting point in the recursion, which implies new calculations from the beginning.

Now, $P(T = t) = \phi(t-1) - \phi(t)$ follows from (2.1) (with $\phi(-1) \equiv 1$). One can also obtain the joint distribution with the ruin severity defined as $|U(T)|$. Let

$$\chi(t,x) = P(T = t \text{ and } |U(T)| \leq x), \quad t \in \mathbb{N}_0 \text{ and } \mathrm{x} \in \mathbb{R}_0^+,$$

i.e. the probability that ruin occurs at time t and the ruin severity stays under a level x.

Corollary 2.2

$$\chi(t,x) = \chi_0(t,x)\, F_{S(t-1)}(0) + \int_{w=0}^{u+c(t-1)} b_w \left\{ \int_{s=0}^{u+c(t-1)-w} \chi_{w+s}(t,x)\, dF_{S(t-1)}(s) \right\} dw, \tag{2.10}$$

where

$$\chi_w(t,x) = F_t(u+c(t)+x-w) - F_t(u+c(t)-w), \quad w \in \mathbb{R}^+.$$

Proof. One writes

$$\begin{aligned} \chi(t,x) &= P[S(t-1)=0]\, P[u+c(t) < X_t \leq u+c(t)+x] + \\ &\quad \int_{s=0}^{u+c(t-1)} P[T > t-1,\, S(t-1) \sim s]\, P[u+c(t) < s+X_t \leq u+c(t)+x]\, ds \\ &= F_{S(t-1)}(0)\, \chi_0(t,x) + \int_{s=0}^{u+c(t-1)} \phi_s(t-1)\, \chi_s(t,x)\, ds, \end{aligned}$$

where $\phi_s(t-1)$ is defined as in (2.4). Using (2.6) and permuting the two integrals then leads to (2.10). $\diamond$

2.2 A recursive method

In practice, the previous integrals (2.1), (2.2) and (2.10) are computed by discretizing the continuous claim amount distributions. So, let us choose a span h. Using a suitable procedure (see below), each claim X_j, $j \in \mathbb{N}_0$, will be approximated by a discrete version $X_j^{(h)}$ with values nh, probabilities $f_j^{(h)}(n) = P(X_j^{(h)} = nh)$ and distribution function $F_j^{(h)}(n)$, $n \in \mathbb{N}$. By construction, all the $f_j^{(h)}(n)$'s will be positive (especially for $j = 0$). Then, every $S(j) = X_1 + \ldots + X_j$ is approximated by a discrete $S^{(h)}(j)$ with values nh, probabilities $f_{S(j)}^{(h)}(n) = P(S^{(h)}(j) = nh)$ and distribution function $F_{S(j)}^{(h)}(n)$, $n \in \mathbb{N}$. Introduce also $j = 0$ and put $f_{S(0)}^{(h)}(n) = \delta_{n,0}$, $n \in \mathbb{N}$.

Let $T^{(h)}$ be the ruin time in this discretized risk model, i.e.

$$T^{(h)} = \inf\{t \in \mathbb{N}_0 : \mathrm{U^{(h)}(t) \equiv u + c(t) - S^{(h)}(t) < 0}\}.$$

The probability to be evaluated is now $\phi^{(h)}(t,x) = P[T^{(h)} > t,\, U^{(h)}(t) \geq x]$, for $t \in \mathbb{N}$, $0 \leq x \leq u + c(t)$. Proposition 2.1 can be easily adapted as follows. Define the instants

$$v_n^{(h)} = \sup\{t \in \mathbb{N} : \mathrm{u + c(t) < nh}\} \quad \text{for all } \mathrm{n} \in \mathbb{N} : \mathrm{nh > u},$$

with $v_n^{(h)} = 0$ when $nh \leq u$. Let $\lfloor y \rfloor$ denote the integer part of y.

Corollary 2.3

$$\phi^{(h)}(t,x) = \sum_{n=0}^{\lfloor (u+c(t)-x)/h \rfloor} b_n^{(h)}\, F_{S(t)}^{(h)}(\lfloor (u + c(t) - x)/h \rfloor - n), \tag{2.11}$$

where the coefficients $b_n^{(h)}$, $n \in \mathbb{N}$, *are determined recursively from*

$$b_0^{(h)} = 1, \quad \text{and} \quad 0 = \sum_{k=0}^{n} b_{n-k}^{(h)}\, f_{S(v_n^{(h)})}^{(h)}(k), \quad n \in \mathbb{N}_0. \tag{2.12}$$

Proof. By a similar argument, one first writes

$$\phi^{(h)}(t,x) = \sum_{n=0}^{\lfloor (u+c(t)-x)/h \rfloor} \phi_n^{(h)}(t), \quad t \in \mathbb{N}, \tag{2.13}$$

where $\phi_n^{(h)}(t) = P[T^{(h)} > t,\, S^{(h)}(t) = nh]$, $n \in \mathbb{N}$. Obviously, $\phi_0^{(h)}(t) = P(S^{(h)}(t) = 0)$. For $n \geq 1$, one sees that $\phi_n^{(h)}(t) = 0$ if $t \leq v_n^{(h)}$, and otherwise,

$$\phi_n^{(h)}(t) = \sum_{k=0}^{n} \phi_{n-k}^{(h)}(v_n^{(h)})\, f_{S(v_n^{(h)},t)}^{(h)}(k), \quad t > v_n^{(h)}, \tag{2.14}$$

as in (2.5). For any $n \in \mathbb{N}$, a simpler expression is found to be

$$\phi_n^{(h)}(t) = \sum_{k=0}^{n} b_{n-k}^{(h)}\, f_{S(t)}^{(h)}(k), \quad t \geq v_n^{(h)}. \tag{2.15}$$

When $n = 0$, (2.15) gives $P[S^{(h)}(t) = 0] = b_0^{(h)} f_{S(t)}^{(h)}(0)$, i.e. $b_0^{(h)} = 1$ as $f_j^{(h)}(0) > 0$ by assumption. Moreover, for $n \geq 1$, taking $t = v_n^{(h)}$ yields (2.12) as $\phi_n^{(h)}(v_n^{(h)}) = 0$. Finally, substituting (2.15) in (2.13) then leads to (2.11). $\diamond$

Let us underline that (2.12) will provide univoquely the $b_n^{(h)}$'s for $n = 0, 1, 2, \ldots$, because all the probabilities $f_j^{(h)}(0)$ are positive. Note that for all n satisfying $nh \leq u + c_1$, then $v_n^{(h)} = 0$ so that (2.12) yields $b_n^{(h)} = 0$.

Concerning the discretization itself, an appropriate method proposed e.g. by Panjer (1986) (for a related problem) consists in approximating any claim amount X_j by both lower and upper discrete bounds, $\underline{X}_j^{(h)}$ and $\overline{X}_j^{(h)}$, whose probabilities are positive and defined by

$$
\begin{aligned}
P(\underline{X}_j^{(h)} = nh) &= P[nh \leq X_j \leq (n+1)h], \quad n \in \mathbb{N}, & (2.16)\\
P(\overline{X}_j^{(h)} = nh) &= P[(n-1)h < X_j \leq nh], \quad n \in \mathbb{N}, & (2.17)
\end{aligned}
$$

with thus $P(\overline{X}_j^{(h)} = 0) = P(X_j = 0)$. Let $\underline{\phi}^{(h)}(t, x)$ and $\overline{\phi}^{(h)}(t, x)$ be the corresponding probabilities of interest. As $\underline{X}_j^{(h)} \leq X_j \leq \overline{X}_j^{(h)}$ in the stochastic sense, one knows that $\overline{\phi}^{(h)}(t, x) \leq \phi^{(h)}(t, x) \leq \underline{\phi}^{(h)}(t, x)$.

Remarks. If $f_j^{(h)}(0) = 0$ for some values of j, then $b_0^{(h)}$ might remain indetermined, so that the system (2.12) will not yield the $b_n^{(h)}$'s required in (2.11). For instance, if X_j has a continuous distribution with no mass at 0, then the approximation (2.17) gives $P(\overline{X}_j^{(h)} = 0) = 0$. On the contrary, the approximation (2.16) gives $P(\underline{X}_j^{(h)} = 0) > 0$; this could be small, however, especially for very small h, hence a possible numerical instability (met in certain applications).

Let us point out that even if $f_j^{(h)}(0) = 0$ for some j, the probability $\phi^{(h)}(t, x)$ can still be computed by using the formulas (2.13) and (2.14). In fact, these relations are valid in all cases, but their structure is more complicated than (2.11) and (2.12). This was confirmed numerically through several examples treated by both methods.

To close, the discretized version of Corollary 2.2 allows us to determine the distribution of the ruin severity. Let $\chi^{(h)}(t, x) = P[T^{(h)} = t, |U^{(h)}(T^{(h)})| \leq x]$, for $t \in \mathbb{N}_0$, $x \in \mathbb{R}_0^+$.

Corollary 2.4

$$\chi^{(h)}(t, x) = \sum_{n=0}^{\lfloor (u+c(t-1))/h \rfloor} b_n^{(h)} \sum_{k=0}^{\lfloor (u+c(t-1))/h \rfloor - n} \chi_{n+k}^{(h)}(t, x)\, f_{S(t-1)}^{(h)}(k), \tag{2.18}$$

where

$$\chi_n^{(h)}(t, x) = F_t^{(h)}(\lfloor (u + c(t) + x)/h \rfloor - n) - F_t^{(h)}(\lfloor (u + c(t))/h \rfloor - n), \quad n \in \mathbb{N}.$$

3 Numerical application to some ruin problems

For illustration, we now use the previous algorithm to treat some specific examples and problems. The discrete time risk process considered below is the model with interest (1.4) where the premiums are calculated under the expected value principle. We recall that this model is independent of the moment when premiums are collected in the period.

Ruin probabilities. Let $\psi(t) = 1 - \phi(t)$ be the ruin probability until time t. Obviously, its lower and upper bounds are $\underline{\psi}^{(h)}(t) = 1 - \underline{\phi}^{(h)}(t, 0)$ and $\overline{\psi}^{(h)}(t) = 1 - \overline{\phi}^{(h)}(t, 0)$.

(1) Suppose that the claim amounts per period, Y_t, have a compound Poisson distribution with Poisson parameter $\lambda_t = 1$ and i.i.d. exponential individual claim amounts $Z_{t,j}$ with mean 1. Wikstad (1971) provided values for finite time ruin probabilities in the continuous (not discrete) time model, assuming a fixed premium rate equal to $1 + \theta$ for different values of the safety loading factor θ.

First, let us incorporate a fixed interest rate i per period. For the recursive procedure, we choose to discretize the discounted individual claims $Z_{t,j}/(1+i)^t$, which are exponentially distributed with mean $1/(1+i)^t$. The (discrete) distribution of the discounted total claim amount per period, $X_t = Y_t/(1+i)^t$, is then evaluated by means of the Panjer recursion. Now, applying the algorithm of Section 2.2 yields the desired lower and upper bounds for the ruin probabilities. Table 1 gives the results obtained when $i = 0$ and those provided by Wikstad (1971). Our discrete time probabilities are smaller, of course, but not so much. The bounds, here and later, can be rather distant because the discretization is made on the individual (not total) claims per period. In most cases, however, only the total claim amounts will be taken into account for discretization. Nevertheless, closer bounds can still be obtained by taking the span h small enough.

θ	Wikstad (1971)	$(\underline{\psi}^{(h)}(10), \overline{\psi}^{(h)}(10))$
0.05	0.0367	$(0.0209659, 0.0319452)$
0.15	0.02770	$(0.0144029, 0.0236300)$
0.25	0.02090	$(0.0097953, 0.0174395)$

Table 1: Ruin probabilities until $t = 10$ for $u = 10$, $i = 0$ and different values of θ, when using $h = 0.01$.

The influence of the interest rate and the time horizon is illustrated in Table 2. As expected, ruin probabilities increase quickly with the horizon. They decrease (rather slowly) with the interest rate: intuitively, a higher interest yields lower discounted premiums and claim amounts, with a stronger effect on the claim (because of its random nature).

t	$i=0$	$i=0.01$	$i=0.05$	$i=0.1$
1	$(0.000250, 0.000257)$	$(0.000234, 0.000240)$	$(0.000170, 0.000175)$	$(0.000115, 0.000119)$
5	$(0.006517, 0.008443)$	$(0.005618, 0.007310)$	$(0.003040, 0.004017)$	$(0.001410, 0.001902)$
10	$(0.020966, 0.031945)$	$(0.017651, 0.026139)$	$(0.007194, 0.011316)$	$(0.002472, 0.004000)$
15	$(0.036574, 0.060984)$	$(0.028563, 0.047949)$	$(0.009966, 0.017308)$	$(0.002833, 0.005044)$
20	$(0.050951, 0.090313)$	$(0.038450, 0.068756)$	$(0.011492, 0.021428)$	$(0.002907, 0.005489)$

Table 2: Ruin probabilities until $t = 1, \ldots, 20$ for $u = 10$, $\theta = 0.05$ and different values of i, when using $h = 0.01$.

Next, let us consider time varying interest rates i_t. Suppose, for instance, that i_t increases linearly of 0.01 per period during the first ten periods, and then decreases linearly at the same speed for the next ten periods. In other words, $i_t = 0.01t$ for $t = 1, \ldots, 10$ and $i_t = 0.1 - 0.01(t-10)$ for $t = 11, \ldots, 20$. A straigthforward adaptation of the previous method leads to the results presented in Table 3. Note that the bounds are almost equal. By comparing with Table 2, column 2, we see that this interest function decreases the ruin probabilities in a significant way on a longer time horizon.

t	$(\underline{\psi}^{(h)}(t), \overline{\psi}^{(h)}(t))$
1	$(0.000234, 0.000241)$
5	$(0.005836, 0.005901)$
10	$(0.014603, 0.014722)$
15	$(0.019354, 0.019498)$
20	$(0.021932, 0.022697)$

Table 3: Ruin probabilities until $t = 1, \ldots, 20$ for $u = 10$, $\theta = 0.05$ and a special interest function, when using $h = 0.01$.

(2) A standard method to approximate a compound distribution is to have recouse to a translated gamma distribution (see e.g. Bowers et al. (1997). Shortly, suppose that the total claim amount in a given period, Y, has a compound distribution with i.i.d. individual claims distributed as Z, say. Let G be a gamma random variable with (positive) parameters (α, β); its distribution function is given by

$$P(G \leq x) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^x y^{\alpha-1} e^{-\beta y} dy, \quad x > 0.$$

Then, the total claim amount Y is approximated by a translated gamma random variable $G + x_0$ such that the first three central moments coincide. This implies that

$$\alpha = 4\sigma^6(Z)/\{E\left[Z - E(Z)\right]^3\}^2,$$

$$\beta = 2\sigma^2(Z)/E\left[Z - E(Z)\right]^3,$$

$$x_0 = E(Z) - \alpha/\beta.$$

Recently, Afonso et al. (2009) used that approximation to estimate finite time ruin probabilities in the continuous time model considered by Widstad (1971). We are going to proceed similarly with the previous discrete time model, in the case of a constant interest rate i. Here, the total claim amount in period t, Y_t, has a compound Poisson distribution with parameter $\lambda_t = 1$ and exponential individual claims $Z_{t,j}$ of mean 1. So, one directly sees that Y_t is approximated by a gamma random variable with parameters $(\alpha = 8/9, \beta = 2/3, x_0 = -1/3)$. Note that if Y is a translated gamma with parameters (α, β, x_0), aY is a translated gamma with parameters $(\alpha, \beta/a, ax_0)$. Thus, the discounted total claim amount $X_t = Y_t/(1+i)^t$ is approximated by a translated gamma random variable with parameters $(\alpha = 8/9, \beta = 2(1+i)^t/3, x_0 = -1/3(1+i)^t)$. Now, to evaluate the ruin probabilities, it suffices to discretize these gamma distributions and then apply the algorithm of Section 2.2. Table 4 gives the two bounds obtained when $i = 0$ and the approximated values provided in Afonso et al. (2009). Here too, the results are not so different, although derived through discrete and continuous time scales respectively.

θ	Afonso et al. (2009)	$(\underline{\psi}^{(h)}(10), \overline{\psi}^{(h)}(10))$
0.05	0.03487	$(0.0326677, 0.0335324)$
0.15	0.02832	$(0.0242231, 0.0248709)$
0.25	0.02011	$(0.0179559, 0.0184383)$

Table 4: Approximated ruin probabilities until $t = 10$ for $u = 10$, $i = 0$ and different values of θ, when using $h = 0.01$.

Table 5 illustrates how ruin probabilities are affected by the interest rate and the time horizon. Observe that the lower and upper bounds are very close in that example (because the discretization is made on the total claim amounts per period). Note also that the bounds can be rather different from those obtained in Table 2, i.e. without using the approximation.

t	$i = 0$	$i = 0.01$	$i = 0.05$	$i = 0.1$
1	$(0.000367, 0.000370)$	$(0.000346, 0.000348)$	$(0.000262, 0.000264)$	$(0.000188, 0.000189)$
5	$(0.008695, 0.008862)$	$(0.007557, 0.007707)$	$(0.004234, 0.004328)$	$(0.002071, 0.002122)$
10	$(0.032668, 0.033532)$	$(0.026718, 0.027473)$	$(0.011579, 0.011982)$	$(0.004158, 0.004327)$
15	$(0.063583, 0.065521)$	$(0.049845, 0.051521)$	$(0.017767, 0.018569)$	$(0.005190, 0.005466)$
20	$(0.095925, 0.099095)$	$(0.072704, 0.075442)$	$(0.022114, 0.023312)$	$(0.005619, 0.005972)$

Table 5: Approximated ruin probabilities until $t = 1, \ldots, 20$ for $u = 10$, $\theta = 0.05$ and different values of i, when using $h = 0.01$.

Finally, Figure 3 points out that the choice of the span h is crucial to get reasonable bounds on the ruin probabilities.

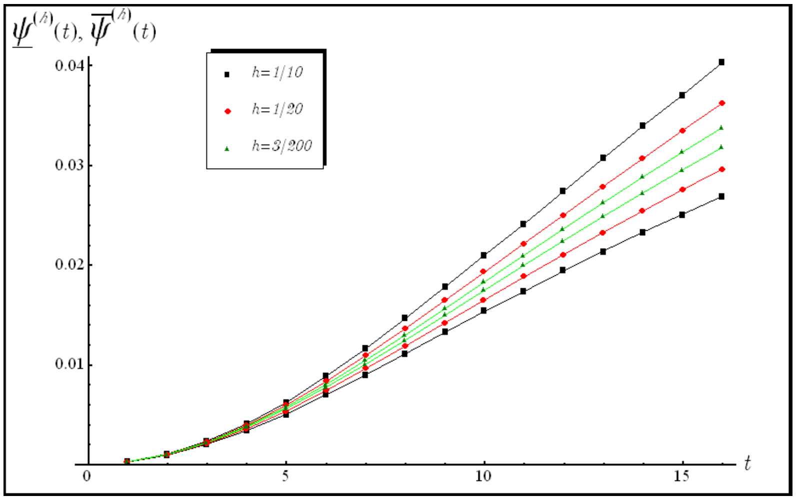

Figure 3: Approximated ruin probabilities until t for $u = 10$, $\theta = 0.05$, $i = 0.03$ and different values of h.

Conditional surplus distribution in case of non-ruin. We now want to illustrate the behaviour of the probability that the discounted reserves of the company at time t are at least equal to x, $x \in \mathbb{R}_0^+$, given that ruin has not occurred until t. In the notation of Proposition 2.1, this probability corresponds to the ratio $\phi(t,x)/\phi(t)$.

Let us consider the latter example where the discounted claim amounts per period are translated gamma distributed with the same parameters. The algorithm of Section 2.2 is of application and allow us to draw the three graphs of Figure 4, when $t = 1, 4, 10$. Although continuous, the distributions are only given at points equidistant of 0.2. The choice of $u = 0$ is just to have a better representation. Note that the reserves at t are at most equal to the total premiums $u + c(t)$. So, for the given parameters, $\phi(1,x) > 0$ if $x \leq (1+\theta)/(1+i) = 1.02$ and $\phi(1,x) = 0$ otherwise; $\phi(4,x) > 0$ if $x \leq 3.90$ and $\phi(10,x) > 0$ if $x \leq 8.95$. As the lower and upper bounds for the discretized claim amounts lead to close results, only the latter, i.e. $\underline{\phi}^{(h)}(t,x)/\underline{\phi}^{(h)}(t)$, have been represented. It is worth observing that the conditional surplus distribution tends to be stable after a few periods only.

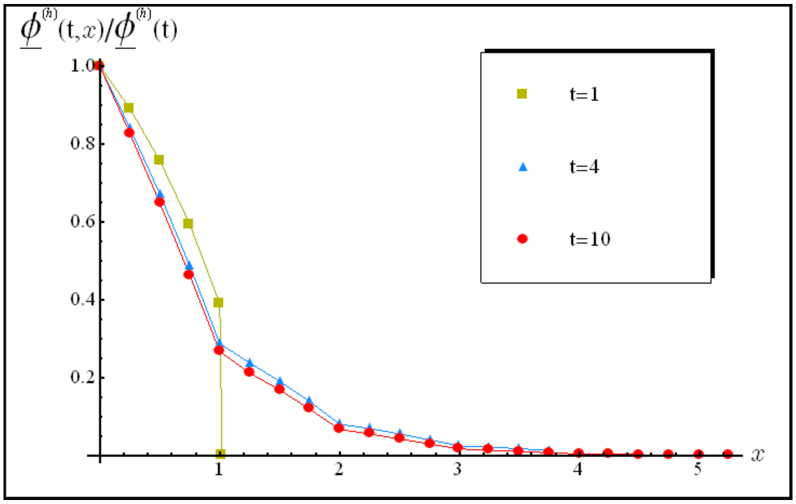

Figure 4: Conditional surplus distributions for $u = 0$, $\theta = 0.05$, $i = 0.03$ and $t = 1, 4, 10$, when using $h = 0.01$.

Conditional deficit distribution in case of ruin. Let us pursue by examining the distribution function of the discounted ruin severity at time t, given that ruin occurs at t. This function is given by $P(|U(t)| \leq x \,|\, T = t]$, $x \in \mathbb{R}_0^+$. In the notation of Corollary 2.2, it corresponds to the ratio $\chi(t, x)/P(T = t)$, rewritten as $\chi(x|t)$.

The computations are made for the same example with translated gamma claim amounts per period. Applying again the algorithm (see Corollary 2.2) provides the three graphs, when $t = 2, 5, 10$, drawn in Figure 5, for points equidistant of 0.5. Only the approximations based on the lower bounds, $\underline{\chi}^{(h)}(x|t)$ say, are represented. Notice that the conditional deficit at ruin is stochastically smaller on a longer time horizon.

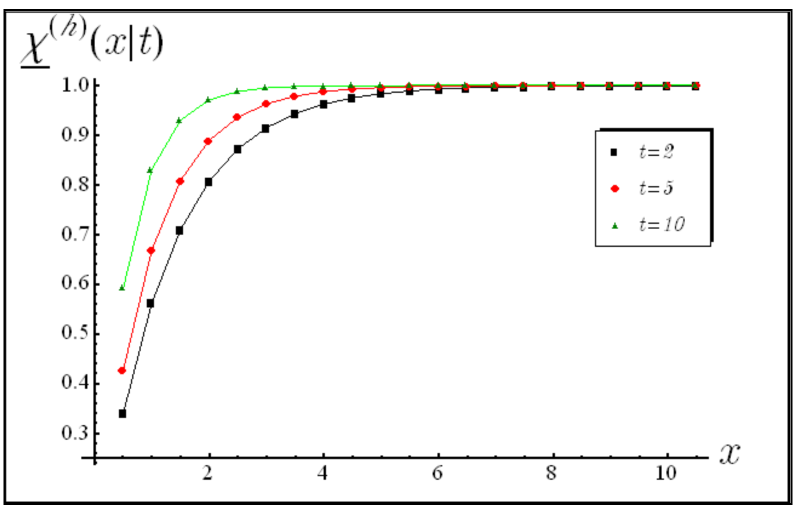

Figure 5: Conditional deficit distributions for $u = 10$, $\theta = 0.05$, $i = 0.1$ and $t = 2, 5, 10$, when using $h = 0.01$.

Table 6 gives the values of the 0.95 quantile of $\overline{\chi}^{(h)}(x|t)$, denoted by $\overline{\chi}_{0.95}^{(h)}$, for different time horizons and interest rates. It also includes the probability $\overline{\chi}^{(h)}(10|t)$, which may be of special interest as it represents the probability that, given ruin occurs at t, the company looses less than the double of its initial investment $u = 10$. We see that $\overline{\chi}_{0.95}^{(h)}$ decreases for longer horizons or higher interest rates (this effect being more important for a larger t). In all cases here, $\overline{\chi}^{(h)}(10|t)$ is almost equal to 1 (a good news for the company!).

t	i	$\overline{\chi}_{0.95}^{(h)}$	$\overline{\chi}^{(h)}(10\|t)$
1	0	4.45	0.998807
	0.05	4.24	0.999145
	0.1	4.05	0.999717
5	0	4.41	0.998866
	0.05	3.45	0.999428
	0.1	2.73	0.999979
10	0	4.39	0.998885
	0.05	2.70	0.999977
	0.1	1.70	0.999997
15	0	4.39	0.998872
	0.05	2.11	0.999999
	0.1	1.05	1

Table 6: Quantiles $\overline{\chi}_{0.95}^{(h)}$ and probabilities $\overline{\chi}^{(h)}(10|t)$ for $u = 10$, $\theta = 0.05$ and different values of t and i, when using $h = 0.01$.

4 Approximation by a Lundberg type bound

In the classical risk models, the Lundberg bound provides a first approximation (an upper bound) to the ultimate ruin probabilities when the claim amounts have a light-tailed distribution. Our purpose in this Section is to derive a Lundberg type bound for the non-homogeneous model under concern.

To establish Proposition 4.1 below, we mainly adapt a method of proof followed in Yang (1999) and Cai (2002a). Let $\psi(t;u)$ ($\equiv 1-\phi(t)$) be the probability that ruin occurs during the period $(0,t]$, $t \in \mathbb{N}_0$, given the initial reserves u. For each t, define the function

$$f_t(r) = E[e^{r(X_t - c_t)}], \quad r \in \mathbb{R}^+, \tag{4.1}$$

and suppose that $r_{t,\infty} \equiv \sup\{r \in \mathbb{R}^+ : \mathrm{E}(\mathrm{e}^{\mathrm{r}\mathrm{X}_\mathrm{t}}) < \infty\} > 0$.

Proposition 4.1 *Define $R(t) = \min\{\rho_1, \ldots, \rho_t\}$, where ρ_t is the (positive) root, if it exists, of the equation $f_t(r) = 1$, and otherwise, $\rho_t = r_{t,\infty}$. Then,*

$$\psi(t;u) \leq e^{-R(t)\,u}, \quad t \in \mathbb{N}_0. \tag{4.2}$$

Proof. For each t, $f_t(r)$ is a convex function with $f_t(0) = 1$ and $f_t'(0) = E(X_t) - c_t < 0$ by (1.8). Thus, there exists some $r(t) > 0$ satisfying

$$f_j[r(t)] \leq 1 \quad \text{for} \quad j = 1, \ldots, t. \tag{4.3}$$

Given such a $r(t)$, one sees that by the property (4.3), the finite sequence

$$V_0 = 1 \quad \text{and} \quad V_j = e^{r(t)[S(j)-c(j)]}, \quad j = 1, \dots, t,$$

is a supermartingale with respect to $\mathcal{F}_j = \sigma\{X_1, \dots, X_j\}$. This is also the case for the finite sequence $W_j = V_{T\wedge j}$, $j = 0, \dots, t$. Applying the optional stopping theorem then yields $E(W_0) = 1 \geq E(W_t)$. Now, one has

$$E(W_t) \geq E[V_{T\wedge t}\, I(T \leq t)] = E\{e^{r(t)[S(T)-c(T)]}\, I(T \leq t)\} > e^{r(t)u}\, P(T \leq t),$$

which implies $\psi(t; u) \leq e^{-r(t)\,u}$. Evidently, a larger value for $r(t)$ is preferable, hence the choice of $R(t)$. $\diamond$

Notice that the inequality (4.2) is given for any finite time horizon. So, the adjustment coefficient, i.e. $R(t)$, depends here on t. In practice, however, all the examples tested show that the approximation may be useful only if t is extremely large (as with the classical models). In such cases, computing the true ruin probability by an algorithmic method is almost impossible, hence the interest of the bound.

We are now going to determine the explicit expression of the roots ρ_t in two special situations. We will then close by commenting on a few numerical illustrations.

Poisson compound of exponential claims. Suppose that X_t, $t \in \mathbb{N}_0$, are independent compound Poisson distributed with Poisson parameter λ_t and i.i.d. individual claim amounts $D_{t,j}$, $j \in \mathbb{N}_0$, with finite moment generating function $E(e^{rD_t})$. Then, ρ_t is the positive solution, if it exists, of the equation

$$\lambda_t + rc_t = \lambda_t E(e^{rD_t}). \tag{4.4}$$

In particular, if the $D_{t,j}$'s are exponentially distributed with mean μ_{D_t}, (4.4) gives

$$\rho_t = 1/\mu_{D_t} - \lambda_t/c_t. \tag{4.5}$$

Let us come back to a model of Section 1(i), with interest rates i_t, premiums p_t and independent claim amounts Y_t, here possibly non-stationary. As seen in Section 1(ii), this is a special case of the model (1.6) where $X_t = Y_t/a(t)$, $a(t)$ being given by (1.7), and c_t is defined accordingly in function of p_t. Suppose that the Y_t's are independent compound Poisson variables with parameter λ_t and i.i.d. exponential claims $Z_{t,j}$. Of course, the X_t's are then independent compound Poisson random variables as above, with exponential claims $D_{t,j} = Z_{t,j}/a(t)$.

Consider the case where premiums are collected at the beginning of the period. This implies that $c_t = p_t/a(t-1)$ (see again Section 1(ii)). If the premiums are calculated under the expected value principle, then $p_t = (1+\theta_t)E[Y_t/(1+i_t)]$. As $E(Y_t) = \lambda_t \mu_{Z_t}$, one gets

$$c_t = (1+\theta_t)E(Y_t)/a(t) = \lambda_t(1+\theta_t)\mu_{Z_t}/a(t).$$

From (4.5) and since $\mu_{D_t} = \mu_{Z_t}/a(t)$, we deduce that

$$\rho_t = \frac{\theta_t}{1+\theta_t}\,\frac{1}{\mu_{Z_t}}\,a(t). \tag{4.6}$$

As expected, a more dangerous exponential claim amount (i.e. with larger μ_{Z_t}) decreases ρ_t, and thus increases the (approximated) ruin probabilities. On the contrary, higher interest rates yield lower ruin probabilities, meaning that they tend to decrease more strongly the claim amount than the premium (this was already observed in Section 3). Suppose, for instance, that the company desires to adjust the safety loading factors to counterbalance the effects of the interest rates until time t. Thus, the objective is to have $\rho_j \equiv \rho$ for all $j = 1, \ldots, t$, which will hold by choosing $\theta_j = \rho\mu_{Z_j}/[a(j) - \rho\mu_{Z_j}]$ (if positive).

In the same situation but under the standard deviation principle, $p_t = E[Y_t/(1+i_t)] + \theta_t\sigma[Y_t/(1+i_t)]$. Since $\sigma^2(Y_t) = \lambda_t E(Z_t^2) = 2\lambda_t\mu_{Z_t}^2$,

$$c_t = [E(Y_t) + \theta_t\sigma(Y_t)]/a(t) = (\lambda_t\mu_{Z_t} + \sqrt{2\lambda_t}\theta_t\mu_{Z_t})/a(t),$$

yielding by (4.5),

$$\rho_t = \frac{\sqrt{2\lambda_t}\theta_t}{\lambda_t + \sqrt{2\lambda_t}\theta_t}\frac{1}{\mu_{Z_t}}a(t). \tag{4.7}$$

Note that ρ_t here is decreasing in the claim arrival rate λ_t. It is larger than (4.6) (expected value principle) when the standard deviation principle provides a higher premium, i.e. when $\sigma(Y_t) > E(Y_t)$, or equivalently, $2 > \lambda_t$. Under the variance principle, $p_t = E[Y_t/(1+i_t)] + \theta_t\sigma^2[Y_t/(1+i_t)]$, so that

$$\rho_t = \frac{2\mu_{Z_t}\theta_t}{1 + i_t + 2\mu_{Z_t}\theta_t}\frac{1}{\mu_{Z_t}}a(t). \tag{4.8}$$

This time, ρ_t becomes independent of λ_t, as in (4.6) (expected value principle). It is larger than (4.7) (standard deviation principle) when the variance principle provides a higher premium, i.e. when $\sigma(Y_t)/(1+i_t) > 1$, or equivalently, $\sqrt{2\lambda_t}\mu_{Z_t} > 1 + i_t$.

By (4.6), (4.7) or (4.8), when all claim amounts, claim arrival rates and safety loading factors are constant, $\rho(t)$ will increase over time because of the interest rates. In such a case, the adjustment coefficient $R(t) = \min\{\rho_1, \ldots, \rho_t\}$ is just given by ρ_1, i.e. only the root of $f_1(r) = 1$ (for the first period) has to be retained. A root for a subsequent period value may become relevant if at some time, ρ_t decreases and goes below the value ρ_1. For instance, suppose that by some effects of inflation, the average claim amount μ_{Z_t} increases over time. Denote by l_t, $t \geq 2$, the rate of inflation during the period $(t-1, t]$; thus, $\mu_{Z_t} = \mu_{Z_{t-1}}(1 + l_t)$, $t \geq 2$. The relation (4.6) then becomes

$$\rho_t = \frac{\theta_t}{1 + \theta_t}\frac{1}{\mu_{Z_1}}\frac{a(t)}{b(t-1)}, \tag{4.9}$$

where $b(t-1) = \prod_{j=2}^{t}(1 + l_j)$. When, for instance, $\theta_t \equiv \theta$, $i_t \equiv i$ and $l_t \equiv l$ for all t, (4.9) gives $\rho_t = [\theta/(1+\theta)\mu_{Z_1}](1+i)^t/(1+l)^{t-1}$. Thus, if $i < l$, ρ_t is now a decreasing function, hence $R(t) = \rho_t$, i.e. the root of $f_t(r) = 1$ (for the last period). Of course, other scenarios are possible.

If premiums are received at the end of the period, $c_t = p_t/a(t)$ where $p_t = (1+\theta_t)E(Y_t)$ under the expected value principle, $p_t = E(Y_t) + \theta_t\sigma(Y_t)$ under the standard deviation principle and $p_t = E(Y_t) + \theta_t\sigma^2(Y_t)$ under the variance principle. One then finds that ρ_t is still given by the formulas (4.6) and (4.7) in the two first cases, and in the third case, by a formula similar to (4.8) where the term $1+i_t$ is replaced by 1. With premiums received at the middle of the periods, (4.6) and (4.7) remain true, and for (4.8), the term $1+i_t$ in $a(t)$ is replaced by $(1+i_t)^{1/2}$.

Approximated normal claims. Suppose that $X_t = \max(0, D_t)$, $t \in \mathbb{N}_0$, where the D_t's have independent continuous distributions on $\mathbb{R}$ with finite moment generating function. Then, ρ_t exists and is the positive root of the equation

$$e^{rc_t} = P(D_t \leq 0) + E[e^{rD_t} I(D_t > 0)]. \tag{4.10}$$

In particular, let each D_t be normally distributed with finite mean μ_{D_t} and variance $\sigma^2_{D_t}$. Simple approximations to $\rho(t)$ can be obtained by bounding the right-hand side of (4.10). Indeed, it is clear that

$$E(e^{rD_t}) \leq P(D_t \leq 0) + E[e^{rD_t} I(D_t > 0)] \leq P(D_t \leq 0) + E(e^{rD_t}),$$

and remember that $E(e^{rD_t}) = \exp(r\mu_{D_t} + r^2\sigma^2_{D_t}/2)$. Thus, $\rho_t^{min} \leq \rho_t \leq \rho_t^{max}$, where ρ_t^{min} is solution of $e^{rc_t} = E(e^{rD_t})$ i.e.

$$\rho_t^{min} = 2(c_t - \mu_{D_t})/\sigma^2_{D_t}, \tag{4.11}$$

and ρ_t^{max} is solution of

$$e^{rc_t} - e^{r\mu_{D_t} + r^2\sigma^2_{D_t}/2} = P(D_t \leq 0).$$

Clearly, ρ_t^{min} will provide a very accurate approximation to ρ_t when $P(D_t \leq 0)$ is small enough (if, for instance, $\mu_{D_t} \geq 3\sigma^2_{D_t}$).

Let us reexamine the model with interest of Section 1(i). Thus, as before, one writes $X_t = Y_t/a(t)$. Suppose that $Y_t = \max(0, Z_t)$, where the Z_t's are independent normal random variables. The X_t's then are independent random variables of the above type, with $D_t = Z_t/a(t)$. For the sequel, we focus on the case where $P(D_t \leq 0) \approx 0$, meaning that X_t and D_t are almost equidistributed (so, Y_t and Z_t too). Then, $\rho_t \approx \rho_t^{min}$.

Consider premiums that are collected at the beginning of the period. Under the expected value principle, we obtain

$$\rho_t^{min} = 2\theta_t \frac{\mu_{Z_t}}{\sigma^2_{Z_t}} a(t). \tag{4.12}$$

Note that ρ_t decreases with the variance-to-mean ratio of Z_t. This translates the intuitive idea that a larger dispersion of the claim amounts increases the ruin probabilities. Under the standard deviation principle, $\rho_t^{min} = 2\theta_t a(t)/\sigma_{Z_t}$, which does not depend on μ_{Z_t}. Under the variance principle, $\rho_t^{min} = 2\theta_t a(t-1)$, independently of μ_{Z_t}, σ_{Z_t} and i_t.

Let $R^{min}(t) = \min(\rho_1^{min}, \ldots, \rho_t^{min})$. If ρ_t^{min} is increasing over time, then $R^{min}(t) = \rho_1^{min}$. Different situations could arise, however. A typical example is when the claim amounts are subjected to inflation (as for (4.9)). Also, the relative dispersion of the claim amounts can vary during the period $(0, t]$. Suppose, for instance, that it reaches a maximum at some time τ; then, $R^{min}(t) = \rho_\tau^{min}$, in absence of interest.

If premiums are collected at the end or middle of the period, the value of ρ_t^{min} is not modified under the expected value or standard deviation principles. With the variance principle, $\rho_t^{min} = 2\theta_t a(t)$ for premiums at the end, and $\rho_t^{min} = 2\theta_t a(t-1)(1+i_t)^{1/2}$ for premiums at the middle.

Numerical illustrations. Let us consider the model with interest (1.1) where the premiums are collected at the beginning of the period. The interest rates per period are assumed to be constant, i say, and all the loading factors are fixed to $\theta = 0.05$. Initially, $u = 10$. As in case (i) above (and example (1) in Section 3), the claim amounts per period Y_t have a compound Poisson distribution, with $\lambda_t = 1$ and i.i.d. exponential claims $Z_{t,j}$ of mean 1.

Let us calculate the premiums under the expected value, standard deviation and variance principles. Then, the roots ρ_t are given by (4.6), (4.7) and (4.8), respectively. In all cases, ρ_t increases with t (through the factor $a(t) = (1+i)^t$), so that $R(t) = \rho_1$ for all t. With (4.6), $\rho_1 = 0.05(1+i)/1.05$; with (4.7), $\rho_1 = 0.05\sqrt{2}(1+i)/(1+0.05\sqrt{2})$; with (4.8), $\rho_1 = 0.1(1+i)/(1+i+0.1)$. Here thus, the corresponding Lundberg type bounds remain constant over time and are given by $e^{-\rho_1 u}$. Table 7 provides these bounds for different interest rates. As indicated earlier, they decrease with i. Note that the bounds under the expected value principle are larger than under the standard deviation principle (because $2 > \lambda_t = 1$; see the explanation given just after (4.7)), and the latter are larger than under the variance principle (because $\sqrt{2} > 1+i$; see above after (4.8)). Roughly speaking, in the present situation, the variance principle yields the smallest approximated ruin probabilities (as the premiums are the highest); in that sense, it is the safest principle for the company.

i	expected val. pr.	std. deviation pr.	variance principle
0	0.621145	0.516640	0.402890
0.01	0.618194	0.513239	0.402560
0.05	0.606531	0.499858	0.401301
0.1	0.592260	0.483622	0.399849

Table 7: Lundberg type bounds under the expected value, standard deviation and variance principles, for $u = 10$, $\theta = 0.05$ and different values of i.

Consider now the expected value principle, and let us suppose that the average claim amount increases over time with a fixed inflation rate $l = 0.025$ per period. Then, by (4.9), $\rho(t) = (0.05/1.05)(1+i)^t/(1.025)^{t-1}$. Thus, if $i < l = 0.025$, ρ_t is decreasing with t, hence $R(t) = \rho_t$; if $i \geq 0.025$, ρ_t is nondecreasing with t, hence $R(t) = \rho_1$. Table 8 gives the Lundberg type bounds for different horizons and interest rates. As the first two interest rates are smaller than the inflation rate, these bounds increase over time and are given by $e^{-\rho_t u}$; the last two interest rates being greater than the inflation rate, the bounds here are constant and equal to $e^{-\rho_1 u}$.

t	$i = 0$	$i = 0.01$	$i = 0.05$	$i = 0.1$
5	0.649596	0.635451	0.606531	0.592260
10	0.682973	0.656264	0.606531	0.592260
15	0.713899	0.676203	0.606531	0.592260
20	0.742397	0.695268	0.606531	0.592260

Table 8: Lundberg type bounds under the expected value principle, for $u = 10$, $\theta = 0.05$, $l = 0.025$ and different values of t and i.

Acknowledgments

This work was conducted while C. Lefèvre was visiting the Departament de Matemàtica Econòmica, Financera i Actuarial, Universitat de Barcelona. His thanks go to all the members of the Departament for their warm welcome and many interesting discussions. C. Lefèvre is grateful to the Belgian FNRS for its partial support. A. Castañer, M. M. Claramunt and M. Mármol thank the Spanish Ministerio de Educación y Ciencia for its support (grant ECO2010-22065-C03-03).

References

[1] Afonso, L. B., Egídio dos Reis, A. D. and Waters, H. R. (2009). Calculating continuous time ruin probabilities for a large portfolio with varying premiums. *ASTIN Bulletin* **39**, 117-136.

[2] Asmussen, S., 2000. *Ruin Probabilities*. World Scientific, Singapore.

[3] Bowers, N. L., Gerber, H. U., Hickman, J. C., Jones, D. A., Nesbitt, C. J. (1997). *Actuarial Mathematics*, 2nd edition. The Society of Actuaries, Schaumburg.

[4] Cai, J. (2002a). Discrete time risk models under rates of interest. *Probability in the Engineering and Informational Sciences* **16**, 309-324.

[5] Cai, J. (2002b). Ruin probabilities with dependent rates of interest. *Journal of Applied Probability* **39**, 312-323.

[6] Cheng, S., Gerber, H. U. and Shiu, E. S. W. (2000). Discounted probabilities and ruin theory in the compound binomial model. *Insurance: Mathematics and Economics*, **26**, 239-250.

[7] de Kok, T. G. (2003). Ruin probabilities with compounding assets for discrete time finite horizon problems, independent period claim sizes and general premium structure. *Insurance: Mathematics and Economics* **33**, 645-658.

[8] Denuit, M., Dhaene, J., Goovaerts, M., Kaas, R. and Laeven, R. (2006). Risk measurement with equivalent utility principles. Statistics and Decisions **24**, 1-26.

[9] De Vylder, F. and Goovaerts, M. J. (1988). Recursive calculation of finite-time ruin probabilities. *Insurance: Mathematics and Economics* **7**, 1-7.

[10] Dimitrova, D. S. and Kaishev, V. K. (2010). Optimal joint survival reinsurance: An efficient frontier approach. *Insurance: Mathematics and Economics* **47**, 27-35.

[11] Ignatov, Z. G. and Kaishev, V. K. (2000). Two-sided bounds for the finite time probability of ruin. *Scandinavian Actuarial Journal* **1**, 46-62.

[12] Ignatov, Z. G. and Kaishev, V. K. (2004). A finite-time ruin probability formula for continuous claim severities. *Journal of Applied Probability* **41**, 570-578.

[13] Ignatov, Z. G., Kaishev, V. K. and Krachunov, R. S. (2004). Optimal retention levels, given the joint survival of cedent and reinsurer. *Scandinavian Actuarial Journal* **6**, 401-430.

[14] Kaas, R., Goovaerts, M., Dhaene, J. and Denuit, M. (2008). *Modern Actuarial Risk Theory Using R*. Springer, New York.

[15] Lefèvre, C. (2007). Discrete compound Poisson process with curved boundaries: Polynomial structures and recursions. *Methodology and Computing in Applied Probability* **9**, 243-262.

[16] Lefèvre, C. and Picard, P. (2006). A nonhomogeneous risk model for insurance. *Computers and Mathematics with Applications* **51**, 325-334.

[17] Lefèvre, C. and Loisel, S. (2009). Finite-time ruin probabilities for discrete, possibly dependent, claim severities. *Methodology and Computing in Applied Probability* **11**, 425-441.

[18] Mullin, R. and Rota, G.-C. (1970). On the foundations of combinatorial theory - III: Theory of binomial enumeration. In Harris, B. (ed.), *Graph Theory and its Applications*. Academic, New York, pp. 167-213.

[19] Panjer, H. H. (1986). Direct calculation of ruin prababilities. *The Journal of Risk and Insurance* **53**, 521-529.

[20] Picard, P. and Lefèvre, C. (1996). First crossing of basic counting processes with lower non-linear boundaries: A unified approach through pseudopolynomials (I). *Advances in Applied Probability* **28**, 853-876.

[21] Picard, P. and Lefèvre, C. (1997). The probability of ruin in finite time with discrete claim size distribution. *Scandinavian Actuarial Journal* **1**, 58-69.

[22] Rolski, T., Schmidli, H., Schmidt, V. and Teugels, J. (1999). *Stochastic Processes for Insurance and Finance*. Wiley, Chichester.

[23] Sun, L. and Yang, H. (2003). Ruin theory in a discrete time risk model with interest incomes. *British Actuarial Journal* **9**, 637-652.

[24] Tang, Q. (2004). The ruin probability of a discrete time risk model under constant interest with heavy tails. *Scandinavian Actuarial Journal* **3**, 229-240.

[25] Wei, X. and Hu, Y. (2008). Ruin probabilities for discrete time risk models with stochastic rates of interest. *Insurance: Mathematics and Economics* **78**, 707-715.

[26] Wikstad, N. (1971). Exemplification of ruin probabilities. *ASTIN Bulletin* **6**, 147-152.

[27] Yang, H. (1999). Non-exponential bounds for ruin probability with interest effect included. *Scandinavian Actuarial Journal* **1**, 66-79.

[28] Yang, H. and Zhang, L. (2006). Ruin problems for a discrete time risk model with random interest rate. *Mathematical Methods of Operations Research* **63**, 287-299.

[29] Yao, D. and Wang, R. (2009). Upper bounds for ruin probabilities in two dependent risk models under rates of interest. *Applied Stochastic Models in Business and Industry* **26**, 362-373.

Address for correspondence:
Claude Lefèvre
Département de Mathématique
Université Libre de Bruxelles
Boulevard du Triomphe
Campus de la Plaine, C.P. 210
B-1050 Bruxelles
Email: clefevre@ulb.ac.be
Tel: 32 2 6505894