Cooperative assignment games with the inverse Monge property

F. Javier Martínez-de-Albéniz[a,1], Carles Rafels[a]

[a] *Dep. de Matemàtica Econòmica, Financera i Actuarial*
Universitat de Barcelona
Av. Diagonal 690, 08034 Barcelona, Spain

Abstract

We study inverse-Monge assignment games, namely cooperative assignment games in which the assignment matrix satisfies the inverse Monge property. For square inverse-Monge assignment games, we describe their cores and we obtain a closed formula for the buyers-optimal and the sellers-optimal core allocations. We also apply the above results to solve the non-square case.

Keywords:
assignment game, core, Monge matrix, buyers-optimal core allocation, sellers-optimal core allocation

1. Introduction

The Monge property of a matrix was named by Hoffman (1963) in recognition of the work of the 18th-century French mathematician Gaspard Monge, who used the property to solve a soil-transport problem. The property has been subsequently applied to a variety of areas - for specific references, applications and properties see the surveys in Burkard (2007) or Burkard et al. (1996).

Here, we show that by using Monge matrices, the analysis of cooperative assignment games is simplified thanks to the Monge conditions and properties. Our main interest lies in describing the core and two specific allocations

Email addresses: `javier.martinezdealbeniz@ub.edu` (F. Javier Martínez-de-Albéniz), `crafels@ub.edu` (Carles Rafels)

[1]Tel: +34934020415, Fax: +34934034892

[2]The authors acknowledge the support from research grant ECO2011-22765 (Ministerio de Ciencia e Innovación and FEDER).

- the buyers-optimal and the sellers-optimal core allocations, when dealing with inverse-Monge assignment games.

The optimal (linear sum) assignment problem is that of finding an optimal matching, given a matrix that collects the potential profit of each pair of agents of opposite sectors. Examples include the placement of workers in jobs, students in colleges, or physicians in hospitals. Once an optimal matching has been made, the question arises as how to share the output between partners. This question was first considered in Shapley and Shubik (1972). They associate each assignment problem with a cooperative game, or game in coalitional form. In the assignment game, the worth of each coalition of agents is taken as the maximum profit they can attain.

The main solution concept in cooperative games is the core. The core of a game consists of allocations of the optimal profit (the worth of the grand coalition) in such a way that no subcoalition can further improve upon it. Thus, if the parties agree to share the profit of cooperation by means of a core allocation, no coalition has any incentive to depart from the grand coalition and act on its own. Shapley and Shubik prove that the core of the assignment game is a nonempty polyhedral convex set and that this coincides with the set of solutions of the dual linear program related to the linear sum optimal assignment problem.

In this paper we study assignment games in which the matrix satisfies the inverse Monge property, called inverse-Monge assignment games. This property is equivalent to the supermodularity of the matrix, interpreted as a function on the product of the set of indices with the usual order.

We show that for square inverse-Monge assignment games, the central tridiagonal band of the matrix, that is the main diagonal, the upper diagonal and the lower diagonal, is sufficient to determine the core. As a result, and unlike the general case, not all inequalities are necessary to describe the core explicitly, and in this case the buyer-seller exact representative of the matrix (Núñez and Rafels, 2002b) can be computed by a closed formula. Two extreme points in the core, the buyers-optimal and the sellers-optimal core allocations, are computed with the aid of the previous representation.

The paper is organized as follows. In Section 2 we describe the assignment game and the pertinent results. In Section 3, inverse-Monge assignment games are defined and we describe the core of a square inverse-Monge assignment game. In Section 4 we give an explicit formula to compute the buyers-optimal and the sellers-optimal core allocations. In Section 5, the non-square case is considered. We conclude in Section 6 with some remarks.

2. The assignment game

An assignment problem (M, M', A) is defined to be a nonempty finite set M of agents, usually named buyers, a nonempty finite set M' of another type of agents, usually named sellers, and a nonnegative matrix $A = (a_{ij})_{(i,j)\in M\times M'}$. Entry a_{ij} represents the profit obtained by the mixed-pair $(i, j) \in M \times M'$ if they trade. Let us assume there are $|M| = m$ buyers and $|M'| = m'$ sellers. If $m = m'$, the assignment problem is said to be square. Let us denote by $\mathrm{M}^+_{m\times m'}$ the set of nonnegative matrices with m rows and m' columns.

A *matching* $\mu \subseteq M \times M'$ between M and M' is a bijection from $M_0 \subseteq M$ to $M'_0 \subseteq M'$, such that $|M_0| = |M'_0| = \min\{|M|, |M'|\}$. We write $(i, j) \in \mu$ as well as $j = \mu(i)$ or $i = \mu^{-1}(j)$. The set of all matchings is denoted by $\mathcal{M}(M, M')$. A buyer $i \in M$ is unmatched by μ if there is no $j \in M'$ such that $(i, j) \in \mu$. Similarly, $j \in M'$ is unmatched by μ if there is no $i \in M$ such that $(i, j) \in \mu$.

A *matching* $\mu \in \mathcal{M}(M, M')$ is *optimal* for the assignment problem (M, M', A) if for all $\mu' \in \mathcal{M}(M, M')$ we have $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$, and we denote the set of optimal matchings by $\mathcal{M}^*_A(M, M')$.

Shapley and Shubik (1972) associate any assignment problem with a game in coalitional form (*assignment game*) with player set $N = M \cup M'$ and characteristic function w_A defined by A in the following way: for $S \subseteq M$ and $T \subseteq M'$,

$$w_A(S \cup T) = \max_{\mu\in\mathcal{M}(S,T)} \left\{ \sum_{(i,j)\in\mu} a_{ij} \right\}, \tag{1}$$

where $\mathcal{M}(S, T)$ is the set of matchings from S to T and $w_A(S \cup T) = 0$ if $\mathcal{M}(S, T) = \varnothing$.

The *core* of the assignment game[3],

$$C(w_A) = \left\{ (x, y) \in \mathbb{R}^M_+ \times \mathbb{R}^{M'}_+ \;\middle|\; \begin{array}{l} x(S) + y(T) \geq w_A(S \cup T), \\ \text{for all } S \subseteq M \text{ and } T \subseteq M', \text{ and} \\ x(M) + y(M') = w_A(M \cup M') \end{array} \right\},$$

is always nonempty and, if $\mu \in \mathcal{M}^*_A(M, M')$ is an arbitrary optimal matching, the core is the set of nonnegative payoff vectors $(u, v) \in \mathbb{R}^M_+ \times \mathbb{R}^{M'}_+$ such

[3]For any vector $z \in \mathbb{R}^N$, with $N = \{1, \ldots, n\}$ and any coalition $R \subseteq N$ we denote by $z(R) = \sum_{i\in R} z_i$. As usual, the sum over the empty set is zero.

that

$$u_i + v_j \geq a_{ij} \text{ for all } (i,j) \in M \times M', \tag{2}$$

$$u_i + v_j = a_{ij} \text{ for all } (i,j) \in \mu, \tag{3}$$

and the payoff to unmatched agents by μ is null. This coincides (see Shapley and Shubik, 1972) with the set of solutions of the dual of the linear program related to the linear sum assignment problem.

Among the core allocations of an assignment game, there are two specific extreme core points: the *buyers-optimal core allocation* $(\overline{u}^A, \underline{v}^A)$ where each buyer attains her maximum core payoff and each seller his minimum, and the *sellers-optimal core allocation* $(\underline{u}^A, \overline{v}^A)$ where each seller attains his maximum core payoff and each buyer her minimum. From Roth and Sotomayor (1990) we know that the maximum payoff of an agent is his/her marginal contribution, and that this can be attained for all agents on the same side at the same core allocation. For any assignment game $(M \cup M', w_A)$, we have

$$\begin{array}{l} \overline{u}_i^A = w_A(M \cup M') - w_A(M \cup M' \setminus \{i\}) \text{ for all } i \in M, \text{ and} \\ \overline{v}_j^A = w_A(M \cup M') - w_A(M \cup M' \setminus \{j\}) \text{ for all } j \in M'. \end{array} \tag{4}$$

Notice that, if μ is an arbitrary optimal matching of (M, M', A), we obtain from the description of the core that $\underline{u}_i^A = a_{i\mu(i)} - \overline{v}_{\mu(i)}^A$ for all $i \in M$ assigned by μ and $\underline{v}_j^A = a_{\mu^{-1}(j)j} - \overline{u}_{\mu^{-1}(j)}^A$ for all $j \in M'$ assigned by μ, while agents unmatched by μ have a fixed null core payoff. Therefore, the minimum core payoffs for a sector are determined by knowing an optimal matching and the maximum core payoffs of the other sector.

Moreover, the minimum payoff that a mixed-pair $(i,j) \in M \times M'$ obtains in the core of a square assignment game $(M \cup M', w_A)$ (see Núñez and Rafels, 2002b, Theorem 2) is given by:

$$\min_{(x,y) \in C(w_A)} \{x_i + y_j\} = a_{i\mu(i)} + a_{\mu^{-1}(j)j} - w_A(N) + w_A(N \setminus \{\mu(i), \mu^{-1}(j)\}), \tag{5}$$

where $\mu \in \mathcal{M}_A^*(M, M')$ is an arbitrary optimal matching.

3. Inverse-Monge assignment games

In this section we introduce assignment games where the assignment matrix satisfies the inverse Monge property, also known in the literature as anti-Monge, contra-Monge, or supermodular.

Definition 3.1. *An assignment problem* (M, M', A) *is an* inverse Monge assignment problem *if*

$$a_{ij} + a_{kl} \geq a_{il} + a_{kj} \quad \textit{for all} \;\; 1 \leq i < k \leq m, \quad \textit{and} \;\; 1 \leq j < l \leq m'. \quad (6)$$

This definition is equivalent to saying that any 2×2 subproblem has an optimal matching in its main diagonal. The inverse Monge property only has to be checked for consecutive 2×2 subproblems (adjacent rows and columns). That is, a matrix $A \in \mathrm{M}^{+}_{m \times m'}$ satisfies the inverse Monge property if and only if

$$a_{ij} + a_{i+1\,j+1} \geq a_{i\,j+1} + a_{i+1\,j} \;\text{ for all } 1 \leq i \leq m-1, \;\text{ and } 1 \leq j \leq m'-1, \; (7)$$

and then it can be easily tested.

The associated cooperative game $(M \cup M', w_A)$, see (1), to an inverse Monge assignment problem (M, M', A) is called a *inverse-Monge assignment game.* Moreover, if $m = m'$, we have a square inverse-Monge assignment game.

For square inverse Monge assignment problems (see Burkard et al., 1996) an optimal matching, which may not be unique, is placed on the main diagonal. Then the worth of the grand coalition for the associated coalitional game is given by

$$w_A\left(M \cup M'\right) = \sum_{k=1}^{m} a_{kk}.$$

We analyze the core of an inverse-Monge assignment game and in the next section we characterize the buyers-optimal and the sellers-optimal core allocations. To obtain these we have to compute the marginal contribution of a player, see (4), and therefore it is crucial to know how to compute an optimal matching for a non-square inverse Monge assignment problem.

The next proposition provides some indications as to where to look for an optimal matching. Its proof can be found in Aggarwal et al. (1992).

Proposition 3.1. *For any inverse Monge assignment problem* (M, M', A), *with* $|M| \leq |M'|$, *at least one optimal matching* $\mu \in \mathcal{M}^*_A(M, M')$ *is monotone, i.e.*

$$\textit{for all} \;\; i_1, i_2 \in M, \quad \textit{with} \;\; i_1 < i_2, \quad \textit{then} \;\; \mu(i_1) < \mu(i_2).$$

Monotone matchings can be seen as generalized main diagonal matchings, since they coincide with the matchings given by the main diagonal entries of the square submatrices of maximal order. When the inverse Monge assignment problem is square, there is only one monotone matching, placed on the main diagonal. So, henceforth, to distinguish the agents on the two sides of the problem we will denote by $k' \in M'$ the partner of player $k \in M$ by this optimal matching.

Now we are in position to give a description of the core of a square Monge assignment game.

Theorem 3.1. *Let $(M \cup M', w_A)$ be a square inverse-Monge assignment game. Then $(u, v) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$ belongs to the core of the game, $C(w_A)$, if and only if*

$$u_i + v_i = a_{ii} \qquad \textit{for} \quad i = 1, 2, \ldots, m, \tag{8}$$

$$u_i + v_{i+1} \geq a_{i\,i+1} \qquad \textit{for} \quad i = 1, 2, \ldots, m-1, \tag{9}$$

$$u_{i+1} + v_i \geq a_{i+1\,i} \qquad \textit{for} \quad i = 1, 2, \ldots, m-1. \tag{10}$$

Proof. The 'only if' part is obvious from the definition of the core of an assignment game, see (2) and (3), and the fact that one optimal matching is placed on the main diagonal.

Now to prove the 'if' part, consider for $i+1 < j$, the square subproblem formed by $\{i, i+1, \ldots, j-1\} \times \{(i+1)', \ldots, j'\}$. One optimal matching in the square inverse Monge subproblem is placed on the main diagonal of the submatrix, that is, $\mu = \{(i, i+1), (i+1, i+2), \ldots, (j-1, j)\}$, and then:

$$\sum_{k=i}^{j-1} a_{k\,k+1} \geq a_{ij} + \sum_{k=i+1}^{j-1} a_{kk}.$$

Now, considering (8) and (9), we obtain

$$u_i + v_j = \sum_{k=i}^{j-1} u_k + \sum_{k=i+1}^{j} v_k - \sum_{k=i+1}^{j-1} a_{kk} \geq \sum_{k=i}^{j-1} a_{k\,k+1} - \sum_{k=i+1}^{j-1} a_{kk} \geq a_{ij}.$$

If $j+1 < i$, just take the subproblem $\{j+1, \ldots, i\} \times \{j', \ldots, (i-1)'\}$, and repeat a similar argument. □

The above result reduces the number of inequalities needed to obtain the core of a square inverse-Monge assignment game, see (2) and (3). In fact, the core of a square inverse-Monge assignment game is the same as that of the assignment game in which we preserve the main diagonal and the upper and lower diagonals, and reduce to zero all the remaining entries.

In the following proposition we show an important consequence of the above theorem; two square inverse-Monge assignment games have the same core if and only if their matrices have the same principal band, that is, the elements of the main diagonal and the upper and lower diagonals.

Proposition 3.2. *Let $(M \cup M', w_A)$ and $(M \cup M', w_B)$ be two square inverse-Monge assignment games. The following statements are equivalent:*

1. $C(w_A) = C(w_B)$,
2. $a_{ij} = b_{ij}$ *for all* $(i, j) \in M \times M'$ *such that* $|i - j| \leq 1$.

Proof. 1. $\Longrightarrow$ 2. Since both square matrices satisfy the inverse Monge property, each has one optimal matching on its main diagonal. Therefore, from the non-emptiness of the core and using (8), $a_{ii} = b_{ii}$ for all $i = 1, \ldots, m$. Moreover from (5), and taking into account that the main diagonal is an optimal matching, for $i = 1, \ldots, m-1$,

$$\min_{(u,v)\in C(w_A)} \{u_i + v_{i+1}\} = a_{ii} + a_{i+1\,i+1} - w_A(M \cup M') + w_A(M \cup M' \setminus \{i', i+1\}).$$

We compute $w_A(M \cup M' \setminus \{i', i+1\})$, for $i = 1, 2, ..., m-1$. Since the subproblem $(M \setminus \{i+1\}) \times (M' \setminus \{i'\})$ is square, its main diagonal is optimal and then:

$$w_A(M \cup M' \setminus \{i', i+1\}) = \sum_{k=1}^{i-1} a_{kk} + a_{i\,i+1} + \sum_{k=i+2}^{m} a_{kk}.$$

Therefore, we obtain $\min_{(u,v)\in C(w_A)} \{u_i + v_{i+1}\} = a_{i\,i+1}$.

This equality implies the existence of an allocation $(u, v) \in C(w_A)$ such that $u_i + v_{i+1} = a_{i\,i+1}$. By the hypothesis of the equality of the cores we obtain that $a_{i\,i+1} \geq b_{i\,i+1}$. A symmetric argument leads to $a_{i\,i+1} = b_{i\,i+1}$ for $i = 1, 2, ..., m-1$. The equality between $a_{i+1\,i}$ and $b_{i+1\,i}$ is proved analogously.

2. $\Longrightarrow$ 1. It is straightforward from Theorem 3.1. $\square$

4. Buyers-optimal and sellers-optimal core allocations

In this section we use the previous results to simplify the calculation of the buyers-optimal and the sellers-optimal core allocations for an arbitrary square inverse-Monge assignment game.

First we compute what is called the buyer-seller exact representative matrix of the original problem (M, M', A). The buyer-seller exact representative matrix A^r was introduced in Núñez and Rafels (2002b) as the only matrix to have two important properties: (1) it has the same core as the original game, i.e. $C(w_A) = C(w_{A^r})$, and (2) all its entries are attainable from a core element, i.e. for each $(i,j) \in M \times M'$ there exists $(u,v) \in C(w_{A^r})$ such that $u_i + v_j = a^r_{ij}$.

Moreover, entries in this matrix can be defined by using the core as:

$$a^r_{ij} = \min_{(u,v)\in C(w_A)} \{u_i + v_j\},$$

or by using the characteristic function when matrix A is square, see (5), as:

$$a^r_{ij} = a_{i\mu(i)} + a_{\mu^{-1}(j)j} - w_A(M \cup M') + w_A(M \cup M' \setminus \{\mu(i), \mu^{-1}(j)\}),$$

for any optimal matching $\mu \in \mathcal{M}^*_A(M, M')$.

Now we define an auxiliary matrix for any square assignment matrix. We prove later that this matrix is the buyer-seller exact representative matrix, when the original matrix satisfies the inverse Monge property.

Let $A \in \mathrm{M}^+_{m\times m}$. Define $\widetilde{A} = (\widetilde{a}_{ij})_{(i,j)\in M\times M'}$ in the following way:

$$\widetilde{a}_{ij} = \begin{cases} \sum_{k=i}^{j-1} a_{k\,k+1} - \sum_{k=i+1}^{j-1} a_{kk} & \text{for} \quad 1 \le i < j \le m, \\ a_{ii} & \text{for} \quad 1 \le i = j \le m, \\ \sum_{k=j}^{i-1} a_{k+1\,k} - \sum_{k=j+1}^{i-1} a_{kk} & \text{for} \quad 1 \le j < i \le m, \end{cases} \tag{11}$$

where the summation over an empty set of indices is zero. Notice that entries in the principal band do not change, that is, $\widetilde{a}_{ij} = a_{ij}$ for $|i-j| \le 1$. It is easy to see that $\widetilde{a}_{i\,j} + \widetilde{a}_{i+1\,j+1} = \widetilde{a}_{i\,j+1} + \widetilde{a}_{i+1\,j}$ for $|i-j| \ge 1$.

Let (M, M', A) be a square inverse Monge assignment problem, and let $\widetilde{A}$ be defined as in (11). Then, $\widetilde{a}_{ij} \ge a_{ij}$ for all $(i,j) \in M \times M'$. To see this, consider for $i < j$ the square subproblem formed by $\{i, i+1, \ldots, j-1\} \times \{(i+1)', \ldots, j'\}$. One optimal matching is placed on its main diagonal and then $\sum_{k=i}^{j-1} a_{k\,k+1} \ge a_{ij} + \sum_{k=i+1}^{j-1} a_{kk}$. The inequality follows. The case $j < i$ is similar. It is easy to see that this matrix $\widetilde{A}$ satisfies the inverse Monge property, from its definition.

Proposition 4.1. *For any square inverse-Monge assignment game $(M \cup M', w_A)$, matrix $\widetilde{A}$ defined by (11) is the buyer-seller exact representative of matrix A, $A^r = \widetilde{A}$, that is,*

1. $C(w_A) = C(w_{\widetilde{A}})$,
2. *For each pair $(i,j) \in M \times M'$ there exists $(u,v) \in C(w_A)$ such that $u_i + v_j = \widetilde{a}_{ij}$.*

Proof. Since $\widetilde{A}$ has the same principal band as the original matrix A, we obtain that both associated games give rise to the same core (see Proposition 3.2). Then $C(w_A) = C(w_{\widetilde{A}})$. Let us check that any entry of the matrix $\widetilde{A}$ is attainable by a point of the core of the game.

To this end, we introduce a matrix $\overline{A}$, which is no more than the matrix generated only by the main diagonal and the upper diagonal by equality in (7) for the consecutive 2×2 submatrices. Formally,

$$\overline{a}_{ij} = \begin{cases} \sum_{k=i}^{j-1} a_{k\,k+1} - \sum_{k=i+1}^{j-1} a_{kk} & \text{for} \quad 1 \leq i < j \leq m, \\ a_{ii} & \text{for} \quad 1 \leq i = j \leq m, \\ \sum_{k=j}^{i} a_{kk} - \sum_{k=j}^{i-1} a_{k\,k+1} & \text{for} \quad 1 \leq j < i \leq m. \end{cases}$$

It is a simple calculation to show that matrix $\overline{A}$ satisfies $\overline{a}_{ij} + \overline{a}_{kl} = \overline{a}_{il} + \overline{a}_{kj}$ for all $1 \leq i < k \leq m$, and $1 \leq j < l \leq m$. A crucial consequence is that any matching is optimal in $\overline{A}$.

Moreover, notice that[4] $\overline{A} \geq A$, because (a) the main diagonal entries have been preserved, (b) for $1 \leq i < j \leq m$, we have $\overline{a}_{ij} = a^r_{ij} \geq a_{ij}$, and (c) for $1 \leq j < i \leq m$, we have $\overline{a}_{ij} = \sum_{k=j}^{i} a_{kk} - \sum_{k=j}^{i-1} a_{k\,k+1} \geq a_{ij}$ since the square subproblem $\{j, j+1, \ldots, i\} \times \{j', j'+1, \ldots, i'\}$ has an optimal matching on the main diagonal of the restriction of A.

Summarizing, matrices A and $\overline{A}$ have the same main diagonal which is optimal, and from the above comments, we have $C(w_{\overline{A}}) \subseteq C(w_A)$. Moreover, each of the entries in $\overline{A}$ is in some optimal matching (all matchings in $\overline{A}$ are optimal). This implies that for any $(i, j) \in M \times M'$ with $i \leq j$ there exists $(u, v) \in C(w_{\overline{A}})$ such that $u_i + v_j = \overline{a}_{ij} = a^r_{ij}$. This finishes the proof of statement 2 for the elements of the upper triangle.

The proof for the lower triangle is similar, but using matrix $\underline{A}$, whose entries are defined as:

$$\underline{a}_{ij} = \begin{cases} \sum_{k=i}^{j} a_{kk} - \sum_{k=i}^{j-1} a_{k+1\,k} & \text{for} \quad 1 \leq i < j \leq m, \\ a_{ii} & \text{for} \quad 1 \leq i = j \leq m, \\ \sum_{k=j}^{i-1} a_{k+1\,k} - \sum_{k=j+1}^{i-1} a_{kk} & \text{for} \quad 1 \leq j < i \leq m. \end{cases}$$

Matrix $\underline{A}$ is generated by the main diagonal and the lower diagonal of matrix A. Thus, combining statements 1 and 2 we have obtained that matrix $\widetilde{A}$ is the buyer-seller exact representative of matrix A. □

There is a practical, recursive way to compute matrix $\widetilde{A}$, given in (11). The idea is to compute the diagonals parallel to the principal band, starting

[4] Let $A, B \in \mathrm{M}^+_{m \times m'}$, then $A \geq B$ if $a_{ij} \geq b_{ij}$ for all $(i, j) \in M \times M'$.

with the closest. To obtain entries $a^r_{i\,i+2}$ for $i = 1, \ldots, m-2$, we compute them by using formula (11):

$$\widetilde{a}_{i\,i+2} = a_{i\,i+1} + a_{i+1\,i+2} - a_{i+1\,i+1} \quad \text{for} \quad i = 1, \ldots, m-2.$$

Now we continue with the elements of the next parallel diagonal:

$$\widetilde{a}_{i\,i+3} = \widetilde{a}_{i\,i+2} + \widetilde{a}_{i+1\,i+3} - \widetilde{a}_{i+1\,i+2} \quad \text{for} \quad i = 1, \ldots, m-3.$$

The process is repeated until we complete the entries in the upper triangle. Similarly, we can compute the entries in the lower triangle by the recursive method,

$$\begin{aligned} \widetilde{a}_{i+2\,i} &= a_{i+1\,i} + a_{i+2\,i+1} - a_{i+1\,i+1} \quad \text{for} \quad i = 1, \ldots, m-2, \text{ and} \\ \widetilde{a}_{i+k\,i} &= \widetilde{a}_{i+k-1\,i} + \widetilde{a}_{i+k\,i+1} - \widetilde{a}_{i+k-1\,i+1} \quad \text{for} \quad i = 1, \ldots, m-k, \end{aligned}$$

and all $k = 3, \ldots, m-1$.

In the following theorem we provide an explicit formula to compute the agents' maximum and minimum core payoffs. In this way we compute the buyers-optimal and the sellers-optimal core allocation, whenever dealing with square inverse-Monge assignment games. The formula uses the entries of the above buyer-seller exact representative.

Theorem 4.1. *Let $(M \cup M', w_A)$ be a square inverse-Monge assignment game, and matrix A^r its buyer-seller exact representative, given by (11). The buyers-optimal core allocation $(\overline{u}^A, \underline{v}^A)$ and the sellers-optimal core allocation $(\underline{u}^A, \overline{v}^A)$ are given by:*

$$\begin{aligned} \overline{u}^A_i &= a_{ii} - \underline{v}^A_i, & \underline{v}^A_i &= \max_{t=1,\ldots,m} \{a^r_{ti} - a_{tt}\}, \\ \underline{u}^A_i &= \max_{t=1,\ldots,m} \{a^r_{it} - a_{tt}\}, & \overline{v}^A_i &= a_{ii} - \underline{u}^A_i, \end{aligned}$$

for $i = 1, \ldots, m$.

Proof. Since matrix A is square and satisfies the inverse Monge property, we know that $w_A(M \cup M') = \sum_{k=1}^m a_{kk}$. To obtain $\overline{u}^A_i$ using (4), for all $i \in M$, we need to compute the worth of $w_A(M \cup M' \setminus \{i\})$. A similar reasoning is applied to compute $\overline{v}^A_j$ for any player $j \in M'$.

Let $i \in M$ and denote by A_{-i} the matrix that results from A when we remove row i. We know that matrix A_{-i} satisfies the inverse Monge property. By Proposition 3.1 at least one optimal matching of the subproblem $(M \setminus \{i\}, M', A_{-i})$ has to be monotone, and since matrix A_{-i} has $m-1$ rows and

m columns, the monotone matchings can be described by $\mu_1, \ldots, \mu_m$, where, for $1 \leq t < i$,

$$\begin{aligned}
\mu_t &= \{(1,1), \ldots, (t-1,t-1), (t,t+1), \ldots, (i-1,i), (i+1,i+1), \ldots, (m,m)\}\,, \\
\mu_i &= \{(1,1), \ldots, (i-1,i-1), (i+1,i+1), \ldots, (m,m)\}\,, \quad \text{and for} \quad i < t \leq m, \\
\mu_t &= \{(1,1), \ldots, (i-1,i-1), (i+1,i), \ldots, (t,t-1), (t+1,t+1), \ldots, (m,m)\}\,.
\end{aligned}$$

Therefore, for any $i \in M$,

$$\begin{aligned}
\overline{u}_i^A &= w_A(M \cup M') - w_A(M \cup M' \setminus \{i\}) \\
&= \sum_{k=1}^{m} a_{kk} - \max_{t=1,\ldots,m} \left\{ \sum_{k \in M \setminus \{i\}} a_{k\mu_t(k)} \right\} \\
&= a_{ii} - \max_{t=1,\ldots,m} \left\{ \sum_{k \in M \setminus \{i\}} \left(a_{k\mu_t(k)} - a_{kk} \right) \right\}. \qquad (12)
\end{aligned}$$

Recall expression (11) of the buyer-seller exact matrix and notice now that in expression (12), $\sum_{k \in M \setminus \{i\}} \left(a_{k\mu_t(k)} - a_{kk} \right)$ becomes

$$\begin{aligned}
\sum_{k=t}^{i-1} (a_{k\,k+1} - a_{kk}) = a^r_{ti} - a_{tt}, &\qquad \text{for } 1 \leq t < i, \\
0 = a^r_{tt} - a_{tt}, &\qquad \text{for } t = i, \\
\sum_{k=i+1}^{t} (a_{k\,k-1} - a_{kk}) = a^r_{ti} - a_{tt}, &\qquad \text{for } i < t \leq m.
\end{aligned}$$

The result follows. For the rest of the proof just notice that $\overline{u}_i^A + \underline{v}_i^A = a_{ii}$ and $\underline{u}_i^A + \overline{v}_i^A = a_{ii}$ for $i = 1, \ldots, m$. □

As a consequence we can compute the *fair-division point* (Thompson, 1981) the midpoint between the optimal allocation for the buyers and the optimal allocation for the sellers. Núñez and Rafels (2002a) prove that this point coincides with the τ-value of the assignment game.

5. Non-square inverse-Monge assignment games

In this section we analyze the core, the buyers-optimal and the sellers-optimal core allocations of a non-square inverse-Monge assignment game $(M \cup M', w_A)$, where $|M| < |M'|$.

Fix a monotone optimal matching $\mu \in \mathcal{M}^*_A(M, M')$. The non-optimally assigned sellers receive a zero payoff in any core allocation, but they introduce significant bounds for the buyers' payoffs, since any core allocation $(u, v) \in C(w_A)$ must satisfy $u_i + v_j \geq a_{ij}$, for all $(i, j) \in M \times M'$. This implies $u_i \geq \max_{j \notin \mu(M)}\{a_{ij}\} = \overline{a}^{\mu}_i$, for all $i \in M$.

Therefore we associate any non-square inverse-Monge assignment game $(M \cup M', w_A)$ where $|M| < |M'|$, with a $m \times (m+1)$ assignment game in which buyers and sellers that are assigned under the monotone optimal matching μ have been placed together in the first positions, and the last $m+1$ column is $(\overline{a}^{\mu}_i)_{i \in M}$ defined above, in which we have condensed all unmatched sellers' columns. Notice that we are not assuming that this "condensed" matrix satisfies the inverse Monge property, but the square submatrix of assigned players does.

By using this procedure, we can find the buyers-optimal core allocation and the sellers-optimal core allocation of an arbitrary non-square inverse-Monge assignment game. The following theorem provides the formulas to compute them, similar to the formulas derived in Theorem 4.1.

Theorem 5.1. *Let $A \in \mathrm{M}^+_{m \times m}$ be a square matrix satisfying the inverse Monge property, and A^r its buyer-seller exact representative, given by (11). Let*

$$B = \left(\begin{array}{c:c} & a_{1\,m+1} \\ A & \vdots \\ & a_{m\,m+1} \end{array} \right) \in \mathrm{M}^+_{m \times (m+1)}, \quad \textit{with} \quad w_B\left(M \cup M'\right) = \sum_{k=1}^{m} a_{kk},$$

where $M = \{1, \ldots, m\}$ and $M' = \{1', \ldots, (m+1)'\}$.

Then,the buyers-optimal core allocation $(\overline{u}^B, \underline{v}^B)$ and the sellers-optimal core allocation $(\underline{u}^B, \overline{v}^B)$ are given by:

$$\begin{array}{rclrcl} \overline{u}^B_i & = & \overline{u}^A_i = a_{ii} - \underline{v}^A_i, & \underline{v}^B_i & = & \underline{v}^A_i = \max\limits_{t=1,\ldots,m} \{a^r_{ti} - a_{tt}\}, \\ \underline{u}^B_i & = & \max\limits_{t=1,\ldots,m} \{a^r_{it} - a_{tt} + a_{t\,m+1}\}, & \overline{v}^B_i & = & a_{ii} - \underline{u}^B_i, \end{array}$$

for $i = 1, \ldots, m$, and $\overline{v}^B_{m+1} = \underline{v}^B_{m+1} = 0$.

Proof. Let us prove first that $\overline{u}^B_i = \overline{u}^A_i$ for $i = 1, \ldots, m$. Consider matrix $B_0 \in \mathrm{M}^+_{m \times (m+1)}$, in which the $m+1$ column of B has been filled with zeros. It is obvious that $\overline{u}^A_i = \overline{u}^{B_0}_i$ for $i = 1, \ldots, m$, and, since some entries have been lowered to zero, $\overline{u}^{B_0}_i \geq \overline{u}^B_i$ for $i = 1, \ldots, m$. Then $\overline{u}^{B_0}_i \geq \overline{u}^B_i \geq a_{i\,m+1}$ for $i = 1, \ldots, m$. Notice that $(\overline{u}^{B_0}, \underline{v}^{B_0}) \in C(w_B)$ and then the buyers-optimal core allocation for w_{B_0} coincides with the buyers-optimal core allocation for w_B, which proves $\overline{u}_i^B = \overline{u}_i^A$ for $i = 1, \ldots, m$. Since $\overline{u}_i^A + \underline{v}_i^A = \overline{u}_i^B + \underline{v}_i^B = a_{ii}$ for $i = 1, \ldots, m$, we have $\underline{v}_i^A = \underline{v}_i^B$ for $i = 1, \ldots, m$.

Now, to compute $\overline{v}_i^B$ for $i = 1, \ldots, m$, we use expression (4) and thus we need to obtain the worth of $w_B(M \cup M' \setminus \{i'\})$. This worth is computed as the maximum of all matchings obtained by taking entry $a_{t\,m+1}$ for $t = 1, \ldots, m$ and a matching in the subproblem $M \setminus \{t\} \times M' \setminus \{i', (m+1)'\}$:

$$w_B(M \cup M' \setminus \{i'\}) = \max_{t=1,\ldots,m} \left\{ a_{t\,m+1} + w_B((M \setminus \{t\}) \cup M' \setminus \{i', (m+1)'\}) \right\}.$$

This subproblem $M \setminus \{t\} \times M' \setminus \{i', (m+1)'\}$ is square and since matrix A satisfies the inverse Monge property, we know that the optimal matching is in its main diagonal. Therefore, and using expression (11), $w_B((M \setminus \{t\}) \cup M' \setminus \{i', (m+1)'\})$ equals:

$$\begin{aligned}
\text{a)}\quad \text{if } t < i, \quad & \sum_{k=1}^{t-1} a_{kk} + \sum_{k=t}^{i-1} a_{k+1\,k} + \sum_{k=i+1}^{m} a_{kk} = \\
& \sum_{k=1}^{m} a_{kk} - a_{tt} - a_{ii} + \sum_{k=t}^{i-1} a_{k+1\,k} - \sum_{k=t+1}^{i-1} a_{kk} = \\
& \sum_{k=1}^{m} a_{kk} - a_{tt} - a_{ii} + a_{it}^r, \\
\text{b)}\quad \text{if } t = i, \quad & \sum_{k=1}^{t-1} a_{kk} + \sum_{k=t+1}^{m} a_{kk} = \sum_{k=1}^{m} a_{kk} - a_{ii}, \\
\text{c)}\quad \text{if } t > i, \quad & \sum_{k=1}^{i-1} a_{kk} + \sum_{k=i}^{t-1} a_{k\,k+1} + \sum_{k=t+1}^{m} a_{kk} = \\
& \sum_{k=1}^{m} a_{kk} - a_{ii} - a_{tt} + \sum_{k=i}^{t-1} a_{k\,k+1} - \sum_{k=i+1}^{t-1} a_{kk} = \\
& \sum_{k=1}^{m} a_{kk} - a_{ii} - a_{tt} + a_{it}^r.
\end{aligned}$$

Thus we obtain:

$$w_B(M \cup M' \setminus \{i'\}) = \sum_{k=1}^{m} a_{kk} - a_{ii} + \max_{t=1,\ldots,m} \{a_{it}^r - a_{tt} + a_{t\,m+1}\}, \text{ and}$$

$$\overline{v}_i^B = w_B\left(M \cup M'\right) - w_B(M \cup M' \setminus \{i'\}) = a_{ii} - \max_{t=1,\ldots,m} \{a_{it}^r - a_{tt} + a_{t\,m+1}\}.$$

Finally, $\underline{u}_i^B = a_{ii} - \overline{v}_i^B = \max_{t=1,\ldots,m}\{a_{it}^r - a_{tt} + a_{t\,m+1}\}$, for $i = 1, \ldots, m$, and since $(m+1)'$ is unmatched, $\overline{v}_{m+1}^B = \underline{v}_{m+1}^B = 0$. □

The following example illustrates the previous results.

Example 5.1. *Let C be the following 4×7 assignment matrix, which satisfies the inverse Monge property:*

$$C = \begin{pmatrix} \mathbf{12} & 11 & 2 & 8 & 33 & 0 & 9 \\ 6 & 8 & 3 & \mathbf{10} & 40 & 9 & 30 \\ 11 & 13 & 13 & 21 & \mathbf{52} & 22 & 44 \\ 1 & 4 & 5 & 13 & 45 & 26 & \mathbf{60} \end{pmatrix}.$$

We want to compute the buyers-optimal and the sellers-optimal core allocations, $(\overline{u}^C, \underline{v}^C)$ and $(\underline{u}^C, \overline{v}^C)$.

A monotone optimal matching μ is marked in bold. In order to apply Theorem 5.1 we rearrange optimally matched players to the first positions and condense entries for all non-optimally assigned sellers in the last column. Therefore:

$$B = \left(\begin{array}{c:c} & \overline{a}_1^\mu \\ A & \vdots \\ & \overline{a}_4^\mu \end{array}\right) = \left(\begin{array}{cccc:c} \mathbf{12} & 8 & 33 & 9 & 11 \\ 6 & \mathbf{10} & 40 & 30 & 9 \\ 11 & 21 & \mathbf{52} & 44 & 22 \\ 1 & 13 & 45 & \mathbf{60} & 26 \end{array}\right).$$

Notice that B does not satisfy the inverse Monge property, but submatrix A with assigned players does.

First we compute A^r, the buyer-seller exact representative of the first part of matrix B using (11), and we write it inside matrix $\widehat{B}$.

$$\widehat{B} = \left(\begin{array}{cccc:c} \mathbf{12} & 8 & 38 & 30 & 11 \\ 6 & \mathbf{10} & 40 & 32 & 9 \\ 17 & 21 & \mathbf{52} & 44 & 22 \\ 10 & 14 & 45 & \mathbf{60} & 26 \end{array}\right).$$

Now, using Theorem 4.1 and Theorem 5.1, we compute for buyer 1 and seller 1 their extreme core payoffs:

$$\underline{v}_1^C = \underline{v}_1^B = \underline{v}_1^A = \max_{t=1,\ldots,4}\{a_{t1}^r - a_{tt}\} = \max\{0, 6-10, 17-52, 10-60\} = 0,$$

$$\overline{u}_1^C = \overline{u}_1^B = a_{11} - \underline{v}_1^C = 12 - 0 = 12,$$

$$\begin{aligned} \underline{u}_1^C = \underline{u}_1^B &= \max_{t=1,\ldots,4}(a_{1t}^r - a_{tt} + a_{t5}) \\ &= \max\{12-12+11, 8-10+9, 38-52+22, 30-60+26\} = 11, \end{aligned}$$

$$\overline{v}_1^C = \overline{v}_1^B = a_{11} - \underline{u}_1^C = 12 - 11 = 1, \quad \textit{where } a_{t5} \textit{ denotes } \overline{a}_t^\mu,\ t = 1, \ldots, 4.$$

Proceeding in the same way with the remaining agents, we obtain the buyers-optimal and sellers-optimal core allocations:

$$(\overline{u}^C, \underline{v}^C) = (12, 10, 22, 38; 0, 0, 0, 0, 30, 0, 22),$$
$$(\underline{u}^C, \overline{v}^C) = (11, 10, 22, 26; 1, 0, 0, 0, 30, 0, 34).$$

Note that we do not need that the original matrix of a non-square assignment problem satisfies the inverse Monge property. It is enough that the restriction to the square subproblem with assigned players does.

6. Concluding remarks

We have studied assignment games in which the valuation matrix satisfies the inverse Monge property. This class is a large set and forms a full-dimensional convex cone in the space of assignment matrices (see Burkard et al., 1996). We characterize here the buyers-optimal and sellers-optimal core allocations.

Interestingly, our results can be applied to a more general class of matrices too. Indeed, notice that a permutation of rows and/or columns may give rise to an inverse Monge matrix. They are called permuted Monge matrices (Burkard et al., 1996). For instance, consider a matrix that satisfies the standard Monge property, i.e., one where the inequalities sign in (6) is reversed. By reordering the columns (or rows) from the last to the first, this matrix can be transformed into an inverse Monge matrix. So our results can be adapted to the original Monge matrix.