

# Multiple edges of a quantum Hall system in a strong electric field

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In this paper we show that if the electrons in a quantum Hall sample are subjected to a constant electric field in the plane of the material, comparable in magnitude to the background magnetic field on the system of electrons, a multiplicity of edge states localized at different regions of space is produced in the sample. The actions governing the dynamics of these edge states are obtained starting from the well-known Schrödinger field theory for a system of nonrelativistic electrons, where on top of the constant background electric and magnetic fields, the electrons are further subject to slowly varying weak electromagnetic fields. In the regions between the edges, dubbed as the "bulk," the fermions can be integrated out entirely and the dynamics expressed in terms of a local effective action involving the slowly varying electromagnetic potentials. It is further shown how the bulk action is gauge noninvariant in a particular way, and how the edge states conspire to restore the  $U(1)$  electromagnetic gauge invariance of the system. In the edge action we obtain a heretofore unnoticed gauge-invariant term that depends on the particular edge. We argue that this term may be detected experimentally as different edges respond differently to a monochromatic probe due to this term. [S0163-1829(96)03835-0]

## I. INTRODUCTION

Field theories of planar systems have enjoyed a considerable measure of popularity in recent years, particularly in the sphere of condensed-matter physics. Specifically, the field theory of the quantum Hall effect (QHE), in its various incarnations, is the focus of a great deal of vigorous activity. As it was noted by several authors,<sup>1-4</sup> a finite Hall sample supports gapless excitations at the edge (edge states) which have been experimentally measured. The action governing the edge excitations plays a crucial role in maintaining the electromagnetic  $U(1)$  gauge invariance,<sup>2-4</sup> and it would be a worthwhile endeavor to extract this action in a clean fashion from the underlying microscopic action describing the generic QHE.

In the sequel, we have developed a technique which allows us to calculate the effective action for strong magnetic and electric fields in a QH sample. A key ingredient is the proper splitting of the microscopic action into two pieces, one leading to a local effective action in the bulk and the other to the edge states. Perturbation theory must be organized in a different fashion for the two pieces. The maintenance of the  $U(1)$  gauge invariance emerges in a rather non-trivial fashion.

We begin with a system of planar nonrelativistic electrons in a constant strong magnetic field perpendicular to the plane, and a constant electric field in the plane. The electric field need not be small. In fact we shall see that if the electric field is comparable to the magnetic field, an infinite number of edges are created even when the chemical potential is small. On the other hand, if, for a given chemical potential, we reduce the strength of the electric field, the distance between edges becomes very large, and hence only one edge need be considered in a realistic situation.<sup>5</sup>

The appearance of an edge state every time the chemical potential crosses a Landau level can already be inferred from

Ref. 1. However, it is also clear from Ref. 1 that for any finite value of the chemical potential, only a finite number of edge states are encountered. This can be traced back to the fact that the single-particle wave functions are required to vanish at the sample boundary. We have addressed a somewhat different situation here. We have considered an infinite sample with a constant electric field. Our single-particle wave functions are not required to vanish at any spatial point. The presence of the electric field ensures that the chemical potential, no matter what its value is, intersects all the Landau levels, thereby producing an infinite number of edges.

An interesting feature of this work is that we have developed an operator technique which permits a gauge-invariant treatment of the strong magnetic and electric fields. The Landau gauge and the symmetric gauge, for instance, which are among the popular choices for a strong magnetic field, can be easily seen to lead to the same operator algebra. This is also true for any interpolating gauge choice between them. It is important to notice that the splitting of the electromagnetic field between the constant strong magnetic and electric components and the slowly varying (weak) background depends on the gauge choice for the former. Although it is indeed technically possible to work within a given gauge with its explicit single-particle wave functions, the gauge invariance of the slowly varying background will be lost in the effective action unless a great deal of caution is exercised. The gauge-independent operator algebra obviates this problem. It further avoids using explicit single-particle electronic wave functions, which simplifies the calculations immeasurably.

Another interesting discussion included here concerns the role of the electron mass. Since we are working in a nonrelativistic theory for the electrons, it is implicit that  $m^2 \gg B$ , and hence any term suppressed by  $1/m$  should be considered smaller than the same term suppressed by  $1/\sqrt{B}$ . However, it is quite easy to realize the scenario  $B \gg m^2$ . Although a proper treatment of this case would require the full relativistic-

tic theory, our calculation exhibits the flexibility required to adapt to this relationship between the external parameters. That is, finite-mass effects can be incorporated systematically.

Last but not least, we have provided what we believe to be a very clear derivation of the edge actions. We also discuss in detail the ambiguities inherent to anomalous gauge theories, and how they can be accommodated to obtain a perfectly gauge-invariant effective action once the contributions from the bulk have been taken into account.

Having established the salient features of this work, we go on to its organizational details. In Sec. II, we establish the notation, and discuss the operator algebra for the unperturbed system together with the classification of the perturbations in powers of the large background magnetic field. We further discuss the role of the chemical potential (which specifies the ground state of the system) in defining the edge of the system in the presence of a constant electric field. We note that, near the edges, the derivative expansion that we have envisioned is bound to fail. We indicate how to excise small neighborhoods of the edges and reserve them for a separate treatment, thereby establishing a clear domain of validity for the derivative expansion. In Sec. III, we provide a detailed example of the derivative expansion we have performed, and indicate how interesting terms emerge with a minimum effort. Section IV contains the final results of the derivative expansion in the bulk, where we establish how potentially damaging terms cancel out in a clean fashion. The net effect of this expansion is to provide a local electromagnetic effective action governing the bulk of the system. This effective action is not gauge invariant in itself, but all the gauge noninvariance is pushed to the edges. In Sec. V, we show how to deal with the neighborhoods of the edges in a direct fashion. This results in the emergence of 1+1 dimensional anomalous chiral gauge theories at the edges. We show that local counterterms can be added to the basic anomaly to cancel fully the gauge dependence of the bulk action obtained in Sec. IV. Section VI is reserved for indicating further avenues of work along these lines. Nontrivial manipulations required in the calculation are displayed in the Appendix.

## II. NOTATION AND FORMULATION

The system we wish to discuss is described by the Lagrangian density

$$\mathcal{L} = \hat{\psi}(\mathbf{x}, t) \left[ iD_0 + \frac{1}{2m} \mathbf{D}^2 \right] \hat{\psi}(\mathbf{x}, t). \quad (2.1)$$

Here  $\hat{\psi}(\mathbf{x}, t)$  is the second quantized fermion field operator satisfying the standard anticommutation relation  $\{\hat{\psi}(\mathbf{x}, t), \hat{\psi}(\mathbf{y}, t)\} = \delta^{(3)}(\mathbf{x} - \mathbf{y})$ . Here

$$iD_0 \equiv i\partial_t - a^0(\mathbf{x}, t) - A^0(\mathbf{x}, t) \quad (2.2)$$

and

$$-i\mathbf{D} \equiv -i\mathbf{\nabla} - \mathbf{A}(\mathbf{x}, t) - \mathbf{a}(\mathbf{x}, t). \quad (2.3)$$

The uniform background fields  $E$  and  $B$  arise from  $A_0$  and  $\mathbf{A}$ .  $E$  is in the plane and taken to be along the  $x$  direction, and the magnetic field  $B$  is perpendicular to the plane. They are

taken to be of comparable magnitude. The action corresponding to (2.1) has the obvious  $U(1)$  gauge invariance

$$\hat{\psi}(\mathbf{x}, t) \rightarrow \exp \theta(\mathbf{x}, t) \hat{\psi}(\mathbf{x}, t),$$

$$\delta[a^\mu(\mathbf{x}, t) + A^\mu(\mathbf{x}, t)] = \partial^\mu \theta(\mathbf{x}, t).$$

In the sequel, we shall not display the dependence of the functions on  $\mathbf{x}$  and  $t$  explicitly. This action may be written in a more transparent form

$$S = \int d\mathbf{x} dt \hat{\psi}(\mathbf{x}, t) [\hat{\Pi}^0 - h] \hat{\psi}(\mathbf{x}, t), \quad (2.4)$$

where  $h$  is the corresponding single-particle Hamiltonian

$$h = \frac{1}{2m} (-i\mathbf{\nabla} - \mathbf{A} - \mathbf{a})^2 + Ex + a^0, \quad (2.5)$$

and

$$\hat{\Pi}^0 = i\partial_t - A^0 + Ex.$$

Integrating the fermions out, one obtains an effective action in terms of the  $a^\mu$ ,

$$S_{\text{eff}} = -i \text{Tr} \ln[\hat{\Pi}^0 - h]. \quad (2.6)$$

To evaluate  $S_{\text{eff}}$  exactly, we need to know the exact spectrum of  $h$ , which is not possible as the  $a^\mu$  are arbitrary. However,

$$h_0 \equiv \frac{1}{2m} (-i\mathbf{\nabla} - \mathbf{A})^2 + Ex \quad (2.7)$$

has a well-known spectrum. We thus obtain the effective action by perturbing around the eigenbasis of  $h_0$ , with  $1/B$  as the small parameter. It should be remarked that the actual parameter is the ratio of any gauge invariant local configuration of the dimensions of  $[\text{mass}]^2$  constructed out of  $a^\mu$ , with  $B$ . In the following we have this in mind when we mention the order in  $1/B$  of any given term.

We set up a gauge-independent algebraic procedure for extracting the spectrum of  $h_0$ . We take

$$A^0 \equiv E\hat{x}\alpha,$$

$$A^x \equiv (1 - \alpha)E\hat{t} + \beta B\hat{y}, \quad (2.8)$$

$$A^y \equiv -(1 - \beta)B\hat{x},$$

where it is clear that these potentials lead to the correct background fields irrespective of the arbitrary parameters  $\alpha$  and  $\beta$ . We further define the single-particle operators

$$\hat{\pi}^x \equiv -i\partial_x - A^x, \quad (2.9)$$

$$\hat{\pi}^y \equiv -i\partial_y - A^y + m \frac{E}{B}. \quad (2.10)$$

The associated commutator

$$[\hat{\pi}^x, \hat{\pi}^y] = -iB \quad (2.11)$$

is seen to be independent of the arbitrary parameters  $\alpha$  and  $\beta$ . We further define the single-particle operators

$$\hat{X}^0 \equiv \hat{t}, \quad (2.12)$$

$$\hat{X} \equiv \hat{x} - \frac{1}{B} \hat{\pi}^y, \quad (2.13)$$

$$\hat{Y} \equiv \hat{y} + \frac{1}{B} \hat{\pi}^x, \quad (2.14)$$

with the associated commutators

$$[\hat{X}, \hat{Y}] = \frac{i}{B}, \quad (2.15)$$

$$[\hat{\Pi}^0, \hat{X}^0] = i. \quad (2.16)$$

These commutators are also seen to be independent of  $\alpha$  and  $\beta$ .  $\hat{X}$  and  $\hat{Y}$  are referred to in the literature as guiding center coordinates. We can also see quite easily that

$$[\hat{\pi}^x, \hat{X}] = [\hat{\pi}^y, \hat{X}] = [\hat{\pi}^x, \hat{Y}] = [\hat{\pi}^y, \hat{Y}] = 0. \quad (2.17)$$

Further,

$$[\hat{\Pi}^0, \hat{\pi}^x] = [\hat{\Pi}^0, \hat{\pi}^y] = [\hat{\Pi}^0, \hat{X}] = [\hat{\Pi}^0, \hat{Y}] = 0$$

and

$$[\hat{X}^0, \hat{\pi}^x] = [\hat{X}^0, \hat{\pi}^y] = [\hat{X}^0, \hat{X}] = [\hat{X}^0, \hat{Y}] = 0.$$

These relations are also insensitive to the choice of  $\alpha$  and  $\beta$ . They further indicate that the spectrum of the  $\hat{\Pi}^i$  (the cyclotron motion) is independent of that of  $\hat{X}$  and  $\hat{Y}$  (the guiding center motion).

We can now construct holomorphic and antiholomorphic combinations of the operators

$$\hat{\pi} \equiv \hat{\pi}^x - i \hat{\pi}^y, \quad (2.18)$$

$$\hat{\pi}^\dagger \equiv \hat{\pi}^x + i \hat{\pi}^y, \quad (2.19)$$

$$\hat{Z} \equiv \hat{X} + i \hat{Y}, \quad (2.20)$$

$$\hat{\bar{Z}} \equiv \hat{X} - i \hat{Y}, \quad (2.21)$$

and the corresponding commutation relations are

$$[\hat{\pi}, \hat{\pi}^\dagger] = 2B \quad (2.22)$$

and

$$[\hat{Z}, \hat{\bar{Z}}] = \frac{2}{B}. \quad (2.23)$$

With these operators,  $h_0$  can be reexpressed as

$$h_0 = \frac{1}{2m} \hat{\pi}^\dagger \hat{\pi} + E\hat{X} + \frac{\omega_c}{2} + \frac{m}{2} \left( \frac{E}{B} \right)^2, \quad (2.24)$$

$\omega_c \equiv B/m$  is the cyclotron frequency. It is easily seen that  $h_0$  is diagonalized by the basis  $\{|n, X\rangle\}$ , where

$$\hat{\pi}|n, X\rangle = \sqrt{2Bn}|n-1, X\rangle, \quad (2.25)$$

$$\hat{\pi}^\dagger \hat{\pi}|n, X\rangle = 2Bn|n, X\rangle, \quad (2.26)$$

$$\hat{X}|n, X\rangle = X|n, X\rangle, \quad (2.27)$$

$$h_0|n, X\rangle = E_n(X)|n, X\rangle, \quad (2.28)$$

with

$$E_n(X) \equiv n\omega_c + EX + \frac{\omega_c}{2} + \frac{m}{2} \left( \frac{E}{B} \right)^2. \quad (2.29)$$

The index  $n$  labels the Landau levels. In the event that  $E=0$ ,  $X$  measures the degeneracy of each Landau level (LL). The electric field lifts this degeneracy. It is important to note that the unperturbed single-particle spectrum is unbounded below. In the second-quantized many-body problem, however, the ground state will be specified as the one where all the single-particle states are occupied up to a given energy. This is done by an appropriate choice of the chemical potential. For definiteness, we shall take the chemical potential such that all the energy levels up to  $E_0(0)$  are filled up. This is equivalent to redefining the origin of energies by dropping the constant term  $(\omega_c/2) + (m/2)(E/B)^2$  in (2.29), which we shall do in the rest of the paper. Hence

$$E_n(X) \rightarrow E_n(X) = n\omega_c + EX.$$

For a given  $n$ , the shifted energy becomes negative when

$$X \leq -X_n \equiv -\frac{\omega_c}{E} n. \quad (2.29a)$$

Furthermore, for  $|E| \ll B$ , only  $X_0=0$  remains finite. All other  $X_n$  for  $n \neq 0$  go to minus infinity. Thus in this case the filling of all negative-energy single-particle states is tantamount to filling only the negative-energy states corresponding to  $n=0$ . Thus the system exists only upto  $X=0$ , which for  $n=0$  is the same as  $x=0$ . Hence, for  $n=0$ , the system exhibits an edge in real space at  $x=0$ .<sup>5</sup> For  $|E| \approx B$ , however, all LL's contribute to the ground state. In this case, an edge develops for each  $X_n$ .

An interesting observation that has to be made is that away from  $X=X_n$ , the size of  $\langle n, X|h|n, X\rangle$  is roughly  $E_n(X)$ , which is  $O(B)$ . However, in the neighborhood of  $X \sim X_n$ , since  $E_n(X) \sim 0$  the size of  $\langle n, X|h|n, X\rangle$  depends on the slowly varying electromagnetic potentials and turns out to be  $O(1)$ . Thus, near the edges given by  $X_n$ , the perturbation theory about  $h_0$  fails. To ensure its validity, we excise  $\epsilon$  neighborhoods of the  $X_n$  out from the integral over  $X$ , which arises when we calculate the trace (2.6) in the  $|n, X\rangle$  basis, and treat the fermion modes in these neighborhoods separately. For the remaining modes, the perturbation is obviously valid. These fermion modes that are treated separately lead to the edge states of the system.

Having discussed these finer points, let us now go on to organize the perturbative expansion. The perturbations on  $h_0$  are given by

$$V \equiv h - h_0 = a^0 + \frac{1}{2m} [\mathbf{a}^2 - \mathbf{a} \cdot (-i\nabla - \mathbf{A}) - (-i\nabla - \mathbf{A}) \cdot \mathbf{a}]. \quad (2.30)$$

We define the holomorphic and the antiholomorphic combinations of the components of the vector potential. That is,

$$A \equiv a^x + ia^y$$

and

$$\bar{A} \equiv a^x - ia^y.$$

Further,

$$z \equiv x + iy$$

and

$$\bar{z} \equiv x - iy.$$

We note that

$$[\hat{\pi}, A(\hat{z}, \bar{\hat{z}})] = -2i \partial_{\bar{z}} A \quad (2.31)$$

and

$$[\bar{A}(\hat{z}, \bar{\hat{z}}), \hat{\pi}^\dagger] = 2i \partial_{\bar{z}} \bar{A}. \quad (2.32)$$

Here we have used the relations

$$\begin{aligned} \hat{z} &= \hat{Z} - \frac{i}{B} \hat{\pi}^\dagger, \\ \bar{\hat{z}} &= \bar{\hat{Z}} + \frac{i}{B} \hat{\pi}, \end{aligned} \quad (2.33)$$

that are obtained from (2.13), (2.14), and (2.18)–(2.21). In terms of these quantities, the perturbation (2.30) is written as

$$V = a^0 + \frac{E}{B} a^y - \frac{1}{2m} b + \frac{1}{2m} \mathbf{a}^2 - \frac{1}{2m} (A \hat{\pi} + \hat{\pi}^\dagger \bar{A}).$$

Here  $b = \partial_x a^y - \partial_y a^x$ . Furthermore, any analytic function

$$f(\hat{z}, \bar{\hat{z}}) = \sum_{p,q} \frac{1}{p!q!} \left( \frac{-i}{B} \right)^p \left( \frac{i}{B} \right)^q \hat{\pi}^{\dagger p} \hat{\pi}^q \# \partial_Z^p \partial_{\bar{Z}}^q f(\hat{Z}, \bar{\hat{Z}}) \#.$$

Here the symbol  $\# \#$  simply indicates the antinormal ordering forced onto  $\hat{Z}, \bar{\hat{Z}}$  due to the normal ordering of  $\hat{\pi}, \hat{\pi}^\dagger$ . Using these, (2.30) may be rewritten and expanded in powers of  $1/B$  as

$$V = V^{(1/2)} + V^{(0)} + V^{(-1/2)} + V^{(-1)} + \dots, \quad (2.34)$$

where

$$V^{(1/2)} = -\frac{1}{2m} (A \hat{\pi} + \hat{\pi}^\dagger \bar{A}), \quad (2.35)$$

$$V^{(0)} = \Omega - \frac{i}{2mB} [\hat{\pi}^2 \partial_{\bar{z}} A - (\hat{\pi}^\dagger)^2 \partial_z \bar{A} - \hat{\pi}^\dagger \hat{\pi} (\partial_z A - \partial_{\bar{z}} \bar{A})], \quad (2.36)$$

$$\begin{aligned} V^{(-1/2)} &= \frac{i}{B} (\hat{\pi} \partial_{\bar{z}} \Omega - \hat{\pi}^\dagger \partial_z \Omega) \\ &\quad - \frac{1}{4mB^2} \hat{\pi}^\dagger \hat{\pi}^2 (2 \partial_z \partial_{\bar{z}} A - \partial_z^2 \bar{A}) \\ &\quad - \frac{1}{4mB^2} \hat{\pi}^{\dagger 2} \hat{\pi} (2 \partial_z \partial_{\bar{z}} \bar{A} - \partial_z^2 A), \end{aligned} \quad (2.37)$$

$$V^{(-1)} = \frac{1}{B^2} \hat{\pi}^\dagger \hat{\pi} \partial_z \partial_{\bar{z}} \Omega + \frac{i}{4mB^3} \partial_z \partial_{\bar{z}} (\partial_z A - \partial_{\bar{z}} \bar{A}) \hat{\pi}^{\dagger 2} \hat{\pi}^2. \quad (2.38)$$

Here

$$\Omega \equiv a^0 + \frac{E}{B} a^y - \frac{1}{2m} b + \frac{1}{2m} \mathbf{a}^2. \quad (2.39)$$

Here all the functions are antinormal ordered with respect to  $\hat{Z}, \bar{\hat{Z}}$ , even though the antinormalization symbol has not been displayed explicitly. In (2.37) and (2.38), we have omitted off-diagonal terms that do not contribute to the order we are working at. Reinserting all this back into the expression for the effective action given in (2.6), expanding the logarithm and retaining contributions upto  $O(1/B^2)$ , we obtain

$$\begin{aligned} S_{\text{eff}} &= i \text{Tr} \left[ -\ln D + \sum_{i=1}^6 L_i \right], \\ L_1 &= \frac{1}{D} (V^{(0)} + V^{(-1)}), \\ L_2 &= \frac{1}{2} \frac{1}{D} V^{(1/2)} \frac{1}{D} V^{(1/2)}, \\ L_3 &= \frac{1}{D} V^{(1/2)} \frac{1}{D} V^{(-1/2)}, \\ L_4 &= \frac{1}{2} \frac{1}{D} V^{(0)} \frac{1}{D} V^{(0)}, \\ L_5 &= \frac{1}{D} V^{(1/2)} \frac{1}{D} V^{(1/2)} \frac{1}{D} V^{(0)}, \\ L_6 &= \frac{1}{D} V^{(1/2)} \frac{1}{D} V^{(1/2)} \frac{1}{D} V^{(1/2)} \frac{1}{D} V^{(1/2)}, \end{aligned} \quad (2.40)$$

where  $D \equiv \hat{\Pi}_0 - h_0$ . The first term in the trace in (2.40),  $-\ln D$ , does not involve the perturbative electromagnetic potentials. It is just the vacuum energy, and we are not interested in it in the present paper. The trace is realized in terms of the tensor product of the basis of  $h_0$  given earlier with the basis of eigenfunctions of  $\hat{\Pi}^0$ . Indeed, since  $[\hat{\Pi}^0, h_0] = 0$ , the eigenvectors of  $D$  read

$$D|n, X, \omega\rangle = \Gamma_n(X, \omega)|n, X, \omega\rangle, \quad (2.41)$$

$$\Gamma_n(X, \omega) = \omega - E_n(X),$$

such that  $|n, X, \omega\rangle = |n, X\rangle|\omega\rangle$ , where

$$\hat{\Pi}^0|\omega\rangle = \omega|\omega\rangle. \quad (2.42)$$

The specification of the ground state of the system, namely, that all the negative-energy single-particle states are filled, means that the integral over the frequency  $\omega$  must be defined such that

$$\int \frac{d\omega}{2\pi} \frac{1}{\Gamma_n(X, \omega)} = i \theta[-E_n(X)]. \quad (2.43)$$

Furthermore, as we mentioned earlier, we have to excise  $\epsilon$

neighborhoods of  $X = -X_n$  to ensure the validity of the perturbation theory. This has the following direct consequence. Consider the derivative of (2.43) with respect to  $E_n(X)$ ,

$$\int \frac{d\omega}{2\pi} \frac{1}{\Gamma_n^2(X, \omega)} = -i\delta(E_n(X)). \quad (2.44)$$

Since, for a given  $n$ ,  $E_n(X) = 0$  only when  $X = -X_n$ , the  $\delta$  function never contributes as  $X = -X_n$  is not permitted according to the criteria adopted above to define a valid perturbation theory. Hence coincident denominators can be dropped owing to the exclusion of the  $\epsilon$  neighborhoods. We shall do so consistently in the sequel.

### III. SAMPLE CALCULATION

In this section, we provide a nontrivial instance where the perturbative calculation outlined in Sec. II is actually performed. We choose to work out the term  $L_3$  in Eq. (2.40) in detail. Realizing the trace operation as mentioned earlier, this term is

$$L_3 = \sum_{n=0}^{\infty} \int dX \int \frac{d\omega}{2\pi} \langle n, X, \omega | \frac{1}{D} V^{(1/2)} \frac{1}{D} V^{(-1/2)} | n, X, \omega \rangle. \quad (3.1)$$

We now use the action of  $\hat{\pi}$  and  $\hat{\pi}^\dagger$  on the basis states from (2.25) and its complex conjugate, and that of  $D$  from (2.41) to obtain

$$\begin{aligned} L_3 = & \frac{1}{2m} \sum_{n=0}^{\infty} \int dX \int \frac{d\omega}{2\pi} \left[ \frac{1}{\Gamma_n \Gamma_{n+1}} \langle X, n, \omega | [2i(n+1)] \# A \# \# \partial_z \Omega \# + \left( \frac{2n(n+1)}{m} \right) \# A \# \# (\partial_z \partial_{\bar{z}} \bar{A} - \frac{1}{2} \partial_z^2 A) \# | X, n, \omega \rangle \right. \\ & \left. + \frac{1}{\Gamma_n \Gamma_{n-1}} \langle X, n, \omega | (-2in) \# \bar{A} \# \# \partial_{\bar{z}} \Omega \# + \left( \frac{2n(n-1)}{m} \right) \# \bar{A} \# \# (\partial_z \partial_{\bar{z}} A - \frac{1}{2} \partial_{\bar{z}}^2 \bar{A}) \# | X, n, \omega \rangle \right]. \end{aligned} \quad (3.2)$$

We have written  $\Gamma_n$  as a shorthand for  $\Gamma_n(X, \omega)$ . We have also assumed that the commutator of  $1/D$  with any function  $f(\hat{Z}, \hat{Z}, t)$  is suppressed by  $1/B$  [see (A9)], and hence it does not contribute to  $L_3$  at the order we are considering. However, there are contributions to  $L_2$  emanating from such a commutator. We can now use properties (2.25) and (2.26) to bring the Landau-level index  $n$  in the states to zero with no other alteration in (3.2). We next introduce the identity  $\int dt |t\rangle \langle t|$  in the  $|\omega\rangle$  subspace. Since  $\langle t | \omega \rangle = e^{-i\omega t}$  all the  $\omega$  dependence in the states disappears, and it only remains in  $\Gamma_n$ . We have

$$\begin{aligned} & \frac{1}{2m} \sum_{n=0}^{\infty} \int dX \int \frac{d\omega}{2\pi} \int dt \left[ \frac{1}{\Gamma_n \Gamma_{n+1}} \langle X, 0 | [2i(n+1)] \# A \# \# \partial_z \Omega \# + \left( \frac{2n(n+1)}{m} \right) \# A \# \# (\partial_z \partial_{\bar{z}} \bar{A} - \frac{1}{2} \partial_z^2 A) \# | X, 0 \rangle \right. \\ & \left. + \frac{1}{\Gamma_n \Gamma_{n-1}} \langle X, 0 | (-2in) \# \bar{A} \# \# \partial_{\bar{z}} \Omega \# + \left( \frac{2n(n-1)}{m} \right) \# \bar{A} \# \# (\partial_z \partial_{\bar{z}} A - \frac{1}{2} \partial_{\bar{z}}^2 \bar{A}) \# | X, 0 \rangle \right]. \end{aligned} \quad (3.3)$$

We use  $|X\rangle$  for  $|X, 0\rangle$  in the following.

At this point we recall that all the functions of  $\hat{Z}$  and  $\hat{\bar{Z}}$  above are separately antinormal ordered. However, as displayed in the Appendix in (A6), the product of two individually antinormal ordered quantities can itself be reexpressed as an infinite series of antinormal ordered quantities, more and more subleading in  $(1/B)$ . In the order we are working, only the first term in the series contributes to  $L_3$ . However, the next to leading term in the series contributes to  $L_2$ . Hence we can just substitute the separate antinormal orderings in (3.3) by a global antinormal ordering which we will not explicitly write down [e.g., for the first term in (3.3) we have  $\# A \# \# \partial_z \Omega \# \sim \# A \partial_z \Omega \# \rightarrow A \partial_z \Omega$ ]. We then use Eqs. (A10) and (A11) from the Appendix to disentangle the multiple denominators in (3.2). This yields

$$\begin{aligned} L_3 = & \frac{1}{2m} \sum_{n=0}^{\infty} \int dX \int dt \int \frac{d\omega}{2\pi} \left[ \frac{2im}{B} (n+1) \left( \frac{1}{\Gamma_{n+1}} - \frac{1}{\Gamma_n} \right) \right. \\ & \times \langle X | A \partial_z \Omega | X \rangle + \frac{2}{B} n(n+1) \left( \frac{1}{\Gamma_{n+1}} - \frac{1}{\Gamma_n} \right) \langle X | A (\partial_z \partial_{\bar{z}} \bar{A} \\ & - \frac{1}{2} \partial_z^2 A) | X \rangle + \frac{(-2im)}{B} n \left( \frac{1}{\Gamma_n} - \frac{1}{\Gamma_{n-1}} \right) \langle X | \bar{A} \partial_{\bar{z}} \Omega | X \rangle \\ & \left. + \frac{2}{B} n(n-1) \left( \frac{1}{\Gamma_n} - \frac{1}{\Gamma_{n-1}} \right) \langle X | \bar{A} (\partial_z \partial_{\bar{z}} A - \frac{1}{2} \partial_{\bar{z}}^2 \bar{A}) | X \rangle \right], \end{aligned} \quad (3.4)$$

which can be further simplified using (A12)–(A15). Thus we reduce all the denominators to the same form  $\Gamma_n$ . This en-

ables us to use (2.43) to integrate over  $\omega$  and obtain the  $\theta$  function displayed therein. This means that from (3.4) we obtain

$$L_3 = \frac{1}{2m} \sum_{n=0}^{\infty} \int dX \int dt \left[ \frac{2m}{B} \theta[-E_n(X)] \langle X | (A \partial_z \Omega - \bar{A} \partial_{\bar{z}} \Omega | X) - \frac{4in}{B} \theta[-E_n(X)] \langle X | A (\partial_z \partial_{\bar{z}} \bar{A} - \frac{1}{2} \partial_z^2 A) + \bar{A} (\partial_z \partial_{\bar{z}} A - \frac{1}{2} \partial_{\bar{z}}^2 \bar{A}) | X \rangle \right]. \quad (3.5)$$

In its turn, the  $\theta$  function is used to yield an  $n$ -dependent upper bound on the  $X$  integral. That is, the  $X$  integral goes from  $-\infty$  to  $-X_n$ . Furthermore, as shown in (A5), the integral over  $X$  of a function that is antinormal ordered and evaluated in the  $X$  basis can be easily reexpressed as the integral over real space of a  $c$ -number-valued function that can be readily obtained from the original antinormal-ordered  $q$ -number valued function. In this integral over real space, the  $x$  integral ranges from  $-\infty$  to  $-X_n$ , a consequence of the restricted range of integration of  $X$ . The  $y$  integral, on the other hand, is over the entire  $y$  axis. Thus, going through this last step, for  $L_3$  we have

$$L_3 = -\frac{i}{2\pi} \sum_{n=0}^{\infty} \int_{-\infty}^{-X_n} dx \int dy \int dt \left[ b \left( a^0 + \frac{E}{B} a^y - \frac{1}{2m} b \right) + \frac{1}{2m} \mathbf{a}^2 \right] + \frac{2n}{m} (A \partial_z \partial_{\bar{z}} \bar{A} + \bar{A} \partial_z \partial_{\bar{z}} A - \frac{1}{2} A \partial_z^2 A - \frac{1}{2} \bar{A} \partial_{\bar{z}}^2 \bar{A}) + \frac{i}{2\pi} \sum_{n=0}^{\infty} \int dy \int dt \left[ a^0 a^y - \frac{1}{2m} a^y (b - \mathbf{a}^2) + \frac{E}{B} (a^y)^2 \right] \Bigg|_{x=-X_n}. \quad (3.6)$$

We note here that apart from terms that obviously live in the ‘‘bulk,’’ there are terms that live at the edges of the bulk. These terms have arisen as a consequence of reexpressing some terms in the bulk in a convenient form through judicious integrations by parts. Notice also that we have taken the  $\epsilon$  neighborhoods to zero at the end of the calculation.

#### IV. RESULTS FOR THE BULK

Now that the perturbation theory in the bulk has been scrupulously defined, and a significant example of a calculation has been shown in Sec. III, we proceed to display the results for the various contributions in (2.40). Notice that the first term in (2.40) is just an irrelevant vacuum contribution that we drop. Let us organize the remaining contributions as follows:

$$S_{\text{eff}}^{\text{bulk}} = i \sum_{n=0}^{\infty} \int dx \int dy \int dt \int \frac{d\omega}{2\pi} \int dX |\langle X, 0, \omega | x, y, t \rangle|^2 \times \frac{1}{\Gamma_n(X, \omega)} \sum_{i=1}^6 L_i. \quad (4.1)$$

We obtain

$$L_1 = \Omega - \frac{n}{m} b + \frac{2n}{B} \partial_{\bar{z}} \partial_z \Omega - \frac{n(n-1)}{mB} \partial_z \partial_{\bar{z}} b, \quad (4.2)$$

$$L_2 = -\frac{1}{2m} A \bar{A} + \frac{1}{2mB} (\partial_{\bar{z}} A \partial_z \bar{A} + \partial_z A \partial_{\bar{z}} \bar{A}) - \frac{1}{4B} \left[ -i \bar{A} \partial_t A + i A \partial_t \bar{A} - \frac{E}{B} [\bar{A} (\partial_z A - \partial_{\bar{z}} \bar{A}) - A (\partial_z \bar{A} - \partial_{\bar{z}} \bar{A})] \right], \quad (4.3)$$

$$L_3 = \frac{i}{B} [\bar{A} \partial_{\bar{z}} \Omega - A \partial_z \Omega] - \frac{2n}{mB} [\bar{A} (-\frac{1}{2} \partial_z \partial_{\bar{z}} \bar{A} + \partial_{\bar{z}} \partial_z A) + A (-\frac{1}{2} \partial_z \partial_{\bar{z}} \bar{A} + \partial_{\bar{z}} \partial_z A)], \quad (4.4)$$

$$L_4 = -\frac{2n+1}{mB} \partial_z \bar{A} \partial_{\bar{z}} A, \quad (4.5)$$

$$L_5 = \frac{i}{2mB} (-\bar{A} \bar{A} \partial_{\bar{z}} A + A A \partial_z \bar{A} + \bar{A} A \partial_z A - \bar{A} A \partial_{\bar{z}} \bar{A}), \quad (4.6)$$

$$L_6 = 0. \quad (4.7)$$

A few comments are in order. The second and third terms in the first line of (4.3) arise from the renormal ordering of the first term [see (A6)]. The second line in (4.3) arises from the fact that  $D$  and  $V^{(1/2)}$  do not commute [see (A9)].

A few cancellations can be readily seen from (4.2)–(4.7). The term  $(1/2m)AA$  from (4.3) cancels against the same term in  $\Omega$  from (4.2). Equation (4.6) cancels against the terms with  $(1/2m)AA$  in  $\Omega$  from (4.4). As we indicated in (2.43), the integral over  $\omega$  gives rise to a step function. This step function, in turn, provides a finite upper limit to the  $X$  integral. We show in (A1)–(A5) how this translates into a finite upper bound for the  $x$  integral. Therefore partial integration with respect to  $x$  gives rise to boundary terms which must be kept. Let us divide the final result into bulk and boundary terms. In each contribution there are both mass-independent and mass-dependent terms. We also rewrite  $A$  and  $\bar{A}$  in terms of  $a^x$  and  $a^y$ , and  $\partial_z$  and  $\partial_{\bar{z}}$  in terms of  $\partial_x$  and  $\partial_y$ . Consider first the bulk effective action. We obtain

$$S^{\text{bulk}} = \frac{1}{2\pi} \sum_{n=0}^{\infty} \int_{-\infty}^{-X_n} dx \int dy \int dt (L^{\text{top}} + L^m), \quad (4.8)$$

$$L^{\text{top}} = -\frac{1}{2} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho - B \left( a_0 + \frac{E}{B} a^y \right), \quad (4.9)$$

$$L^m = -\frac{2n+1}{2m} b^2 + \frac{2n+1}{2m} bB. \quad (4.10)$$

The boundary effective action reads

$$S_n^{\text{bound}} = \frac{1}{2\pi} \int dy \int dt (L^{\text{top}} + L^m), \quad (4.11)$$

$$L^{\text{top}} = -\frac{n}{2} \partial_x \left( a^0 + \frac{E}{B} a^y \right) - \frac{1}{2} \left( a^y a^0 + \frac{E}{B} (a^y)^2 \right), \quad (4.12)$$

$$L^m = -\frac{n^2}{4mB} \partial_x b + \frac{2n+1}{4m} \partial_x (a^y)^2. \quad (4.13)$$

Notice that the bulk action is gauge invariant but the boundary action is not. This is so because on the one hand there are explicit gauge-noninvariant terms in (4.13) and (4.14), and on the other hand the gauge variation of (4.9) gives rise to boundary contributions due to the Chern-Simons term in (4.10). We shall see in Sec. V how the edge states fix up the lack of gauge invariance at the boundary.

## V. EDGE STATES

In Sec. II, we excluded small neighborhoods around  $X_n$  from the calculation, which correspond to zero modes of the Hamiltonian  $h_0$ . In these neighborhoods  $h_0$  is  $O(B^0)$  instead of  $O(B^1)$ , and hence the perturbation theory must be reorganized. Notice first that  $V^{-1/2}$  and  $V^{-1}$  are still perturbations. We shall ignore them for the moment, and see later on that they actually give rise to higher-order terms. Let us then split  $D$  into its diagonal and off-diagonal parts,

$$D = O_d + \Delta O,$$

$$O_d = \hat{\Pi}_0 - h_0 - a^0 - \frac{E}{B} a^y - \frac{1}{2m} A \bar{A} + \frac{1}{2m} \left( \frac{\pi^\dagger \pi}{B} + 1 \right) b, \quad (5.1)$$

$$\Delta O = \Delta O^{1/2} + \Delta O^0,$$

$$\Delta O^{1/2} = \frac{1}{2m} (A \pi + \pi^\dagger \bar{A}),$$

$$\Delta O^0 = \frac{i}{2mB} \partial_{\bar{z}} A \pi^2 - \frac{i}{2mB} \partial_z \bar{A} \pi^{\dagger 2}.$$

The effective action for all these neighborhoods reads

$$S^{\text{edges}} = -i \sum_{p=0}^{\infty} \int_{-X_p-\epsilon}^{-X_p+\epsilon} dX \sum_{n=0}^{\infty} \int \frac{d\omega}{2\pi} \langle n, X, \omega | \ln(O_d + \Delta O) | n, X, \omega \rangle, \quad (5.2)$$

where  $X_p$  is defined in (2.29a). Let us next focus on a fixed  $p$ . Notice that when  $\epsilon \rightarrow 0$ , only singular terms make a finite contribution to  $S^{\text{edge}}$ . Those may occur only when  $h_0$  is close to zero. Otherwise, a local expansion about  $h_0$ , which gives rise to piecewise continuous functions, as we have seen in Sec. IV, is legitimate.  $h_0$  gets close to zero only if  $n = p, p \pm 1$  at the order we are interested in. Consider then the formal expansion

$$S^{\text{edge } p} = -i \int_{-X_p-\epsilon}^{-X_p+\epsilon} dX \sum_{n=0}^{\infty} \int \frac{d\omega}{2\pi} \langle n, X, \omega | \left[ \ln O_d + \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r} \left( \frac{1}{O_d} \Delta O \right)^r \right] | n, X, \omega \rangle \quad (5.3)$$

Let us first focus on the first term in the expansion. For this term it is clear that only  $n = p$  gives rise to a small  $h_0$ . The remaining terms in (5.3) also make contributions with small  $h_0$  for  $n = p$  and for  $n = p \pm 1$ , which will be calculated further on. For  $n = p$ ,

$$\begin{aligned} S_0^{\text{edge } p} &= -i \int_{-X_p-\epsilon}^{-X_p+\epsilon} dX \int \frac{d\omega}{2\pi} \langle p, X, \omega | \ln \left[ \hat{\Pi}_0 - h_0 - a^0 \right. \\ &\quad \left. - \frac{E}{B} a^y - \frac{1}{2m} A \bar{A} + \frac{1}{2m} \left( \frac{\pi^\dagger \pi}{B} + 1 \right) b \right] | p, X, \omega \rangle \\ &= -i \int_{-X_p-\epsilon}^{-X_p+\epsilon} dX \int \frac{d\omega}{2\pi} \langle 0, X, \omega | \ln \Gamma_p | 0, X, \omega \rangle \\ \Gamma_p &:= \Gamma_p(\hat{\Pi}_0, \hat{X}, \hat{Y}) := \hat{\Pi}_0 - \frac{B}{m} p - E \hat{X} + f_p(\hat{X}, \hat{Y}), \end{aligned} \quad (5.4)$$

$$f_p(\hat{X}, \hat{Y}) := -a^0 - \frac{E}{B} a^y - \frac{1}{2m} A \bar{A} + \frac{1}{2m} (2p+1)b.$$

Let us next insert the identity into the subspace  $\langle \hat{Y}, t \rangle$ , and use the representation

$$\hat{X} = \frac{i}{B} \partial_Y, \quad \hat{\Pi}_0 = i \partial_t. \quad (5.5)$$

We have

$$\begin{aligned} S_0^{\text{edge } p} &= -i \int_{-X_p-\epsilon}^{-X_p+\epsilon} dX \int \frac{d\omega}{2\pi} \int dY \int dt \langle 0, X, \omega | 0, Y, t \rangle \\ &\quad \times \ln \Gamma_p \left( i \partial_t, \frac{i}{B} \partial_Y, Y \right) \langle 0, Y, t | 0, X, \omega \rangle. \end{aligned} \quad (5.6)$$

Using the explicit expression for the wave function,

$$\langle 0, Y, t | 0, X, \omega \rangle = \left( \frac{B}{2\pi} \right)^{1/2} e^{-i\omega t} e^{iBXY}, \quad (5.7)$$

and making the shift

$$X \rightarrow -X_p + \frac{k}{B}, \quad (5.8)$$

we have

$$\begin{aligned} S_0^{\text{edge } p} &= -i \int_{-\epsilon B}^{\epsilon B} \frac{dk}{2\pi} \int \frac{d\omega}{2\pi} \int dt \int dY \int dt \ln \left( \omega - \frac{E}{B} k \right. \\ &\quad \left. + i \partial_t - i \frac{E}{B} \partial_Y + f_p(X_p, Y) \right) + O(\epsilon). \end{aligned} \quad (5.9)$$

For  $\epsilon B \rightarrow \infty$ , the above is nothing but the effective action for a 1+1-dimensional chiral fermion, the Lagrangian for which reads

$$\begin{aligned} L_p^{\text{chf}} &= \int dt \int dY \psi^\dagger \left( i \partial_t - i \frac{E}{B} \partial_Y - a^0 - \frac{E}{B} a^y \right. \\ &\quad \left. + \frac{1}{2m} (2p+1)b - \frac{1}{2m} A \bar{A} \right) \psi. \end{aligned} \quad (5.10)$$

Notice that (5.9) is already  $O(B^0)$ , and hence any contribution suppressed by  $1/B$  should be consistently neglected. In particular, that includes contributions from  $V^{(-1/2)}$  and  $V^{(-1)}$  which we have already disregarded. Formula (5.10) is not yet the right Lagrangian for the edge states, but it con-

tains its essential features, which we would like to comment upon. First of all there is only one term, namely,  $-(1/2m)AA$ , which is not gauge invariant. This term cancels out with other terms which also contribute to the edge action, as we will see below, so we shall disregard it in the following discussion. The remaining terms can be split into mass-independent and mass-dependent terms. The mass-independent terms, which are identical for any edge ( $p$  independent), and hence identical to the ones obtained for the only edge that exist when the electric field is weak,<sup>5</sup> correspond to the usual Lagrangian for a 1+1 chiral fermion (chiral Schwinger model). The coefficient of the mass-dependent term depends on the particular edge ( $p$  dependent), and to our knowledge has not been found before.

It is well known that the chiral Schwinger model is anomalous.<sup>6,7</sup> This means that not all the symmetries of the classical Lagrangian can be implemented at the quantum level. This is due to UV singularities, and implies that the quantum theory is defined up to local counterterms. The local counterterms encode the inherent ambiguity of an anomalous gauge theory.<sup>8</sup> They can also be regarded as a reminder of the possible different results one can obtain at the quantum level by using different regularizations. In the relativistic chiral Schwinger model, one usually requires Lorentz symmetry at the quantum level. This forces gauge invariance to be spoiled, and reduces the number of possible local counterterms. In our case the physical situation is quite different. In condensed-matter physics Lorentz symmetry is not supposed to play an important role. Although in the  $m \rightarrow \infty$  limit the edge action enjoys a 1+1 Lorentz symmetry (taking  $c = E/B$ ), it is not clear that it must be considered a fundamental symmetry. In fact, the only fundamental symmetry that we have in our system is the  $U(1)$  electromagnetic gauge invariance. Then the most reasonable criterion to define the edge effective action quantum mechanically is to require gauge invariance in the whole system. Since the bulk action contains a Chern-Simons term which gives rise to the gauge anomaly at the boundary, we must define the edge action in such a way that cancels out this gauge anomaly, together with the rest of non-gauge-invariant local counterterms. Recall that the cancellation of the gauge anomaly coming from the Chern-Simons term in the bulk cannot be carried out by means of local counterterms at the edge. It actually requires the existence of extra degrees of freedom at the edge, in this case the chiral fermion. Therefore once we have chosen a regularization for the edge effective action which cancels the

gauge anomaly, the non-gauge-invariant local counterterms are automatically fixed. The only freedom we have are gauge-invariant local counterterms (up to total derivatives in  $t$  and  $y$ ) at the edge. Let us list them below:

$$\begin{aligned} d=1: & a^0, \quad a^y, \\ d=2: & \partial_x a^0, \quad \partial_x a^y, \\ d=3: & \partial_x^2 a^0, \quad \partial_x^2 a^y. \end{aligned} \tag{5.11}$$

The coefficients of these terms contribute to the observables, and must be fitted experimentally or calculated numerically from the fundamental theory.

One may wonder at this point as to how it comes about that even though we started with a well-defined problem we end up with a few free parameters that we cannot calculate. The answer is that for the bulk and edge calculations, assumptions about  $\epsilon$ , the cutoff which separates the edge and bulk, are different. For the perturbation theory at the bulk to be correct one needs  $E\epsilon \gg (\sqrt{B}a^i/m)$ ,  $(a^i a^j/m)$ , and  $a^0$ , whereas for the derivation of the edge states  $E\epsilon \ll a^0$  is understood. Therefore there is a region, presumably small, in which none of the expansions hold. What is remarkable, however, is that our ignorance about this region can be summarized into only a few gauge-invariant local counterterms at the edges.

Let us next analyze the remaining terms in expansion (5.3). Recall first that  $1/O_d$  is  $O(B^0)$  at the edge but  $O(B^1)$  elsewhere, and that  $\Delta O$  contains a piece  $O(B^{1/2})$  and a piece  $O(B^0)$ . We notice that only a very particular class of terms makes contributions of the same order as (5.4). Indeed, if  $1/O_d$  is at the edge,  $\Delta O$  brings the following  $1/O_d$  out of the edge. The next  $\Delta O$  may either keep the next  $1/O_d$  out of the edge or bring it back to the edge. In the first case we obtain a term which is suppressed at least by a power of  $1/B$  with respect to (5.4). In the second case we obtain a contribution of the same order as (5.4) due to  $\Delta O^{1/2}$ , whereas  $\Delta O^0$  gives rise to contributions suppressed by  $1/B$ . Therefore we must add up all the terms corresponding to the second case in which only  $\Delta O^{1/2}$  is involved. From this discussion it is clear that only  $n = p, p \pm 1$  will contribute. For  $n = p$ , we have

$$\begin{aligned} S_{0'}^{\text{edge } p} &= -i \int_{-X_p - \epsilon}^{-X_p + \epsilon} dX \int \frac{d\omega}{2\pi} \sum_{j=1}^{\infty} -\frac{1}{2j} \left\langle p, X, \omega \left| \left( \frac{1}{O_d} \Delta O \frac{1}{O_d} \Delta O \right)^j \right| p, X, \omega \right\rangle \\ &= -i \int_{-X_p - \epsilon}^{-X_p + \epsilon} dX \int \frac{d\omega}{2\pi} \sum_{j=1}^{\infty} -\frac{1}{2j} \left\langle p, X, \omega \left| \left( \frac{1}{\Gamma_p} \left[ \frac{1}{\Gamma_{p-1}} \frac{p B A \bar{A}}{2m^2} + \frac{1}{\Gamma_{p-1}} \frac{(p+1) B A \bar{A}}{2m^2} \right] \right)^j \right| p, X, \omega \right\rangle \\ &\sim -i \int_{-X_p - \epsilon}^{-X_p + \epsilon} dX \int \frac{d\omega}{2\pi} \sum_{j=1}^{\infty} -\frac{1}{2j} \left\langle 0, X, \omega \left| \left( -\frac{1}{\Gamma_p} \frac{A \bar{A}}{2m} \right)^j \right| 0, X, \omega \right\rangle. \end{aligned} \tag{5.12}$$

For  $n = p \pm 1$ , we have



$$\begin{aligned}
S_{\pm}^{\text{edge } p} &= -i \int_{-X_p - \epsilon}^{-X_p + \epsilon} dX \int \frac{d\omega}{2\pi} \sum_{j=0}^{\infty} -\frac{1}{2(j+1)} \left\langle p \pm 1, X, \omega \left| \frac{1}{O_d} \Delta O \left( \frac{1}{O_d} \Delta O \frac{1}{O_d} \Delta O \right)^j \frac{1}{O_d} \Delta O \right| p \pm 1, X, \omega \right\rangle \\
&\sim -i \int_{-X_p - \epsilon}^{-X_p + \epsilon} dX \int \frac{d\omega}{2\pi} \sum_{j=0}^{\infty} -\frac{1}{2(j+1)} \langle 0, X, \omega | \left( \frac{(-1)^{1 \pm 1/2} \left( p + \frac{1 \pm 1}{2} \right) A \bar{A}}{\Gamma_p} \right)^j \left( -\frac{1}{\Gamma_p} \frac{A \bar{A}}{2m} \right) | 0, X, \omega \rangle. \quad (5.13)
\end{aligned}$$

A few technical comments are in order.  $\Gamma_p$  above must be understood as the  $(\hat{X}, \hat{Y})$ -valued operator defined in (5.4).  $A$  and  $\bar{A}$  are also functions of  $(\hat{X}, \hat{Y})$ . Although  $\hat{X}$  and  $\hat{Y}$  do not commute, its commutator is  $O(1/B)$ . Hence we have commuted  $A$  and  $\bar{A}$  at will. The commutator of  $1/\Gamma_{p \pm 1}$  with  $A$  and  $\bar{A}$  is also  $O(1/B)$ , and has also been neglected. However, the commutator of  $1/\Gamma_p$  with  $A$  and  $\bar{A}$  is  $O(B^0)$  and must be kept. Instead, the cyclic property of the trace has been used in (5.13) in order to bring the first  $A$  and  $\bar{A}$  together.

If we add (5.12) and (5.13) to (5.4), we see that the only effect on the latter is the cancellation of the gauge-noninvariant term,  $-(1/2m)A\bar{A}$ , as claimed above. We finally have

$$\begin{aligned}
L_p^{\text{chf}} &= \int dt \int dY \psi_p^\dagger \left( i \partial_t - i \frac{E}{B} \partial_Y - a^0 - \frac{E}{B} a^y \right. \\
&\quad \left. + \frac{1}{2m} (2p+1)b \right) \psi_p. \quad (5.14)
\end{aligned}$$

The last term in (5.4) is not instrumental in the maintenance of gauge invariance, which may be the reason why it has not been obtained before. This term depends on the electron mass and, more importantly, on the particular edge. If  $b$  were a constant field, we could reformulate the problem in terms of  $B+b$ , which would have resulted in a small shift in the location of the edges. In that event, this term would not appear in (5.14). A simple way to implement this in the edge action (5.14) is by a field redefinition  $\psi_p \rightarrow e^{i(1/2m)(p+1)bt} \psi_p$ . However, we have envisioned a perturbation that depends on space and time, for instance an electromagnetic wave, where this term cannot be removed.

In what follows, we argue that this term may be detected experimentally. For definiteness, we consider an experimental situation similar to the setup in Ref. 15. We take  $B = 15T$ , and  $E/c = 3 \times 10^{-5}T$ . From (2.29a) we obtain that the distance between successive edges is  $\Delta X \approx 10^5 \lambda_c \approx 10^2 \text{ \AA}$ , where  $\lambda_c$  is the Compton wavelength. Given the dimensions of the sample in Ref. 15, a relatively large number of edges ( $\sim 1000$ ) fit into it. If we imagine that the edges are probed by a monochromatic electromagnetic wave, we immediately see that the wavelength must be less than the interedge distance  $\Delta X$  in order to resolve between them. To ensure that we detect a different response from each edge, we must verify that the last term in (5.14) is at least of the order of the other terms. This implies, for a probe with wavelength  $\lambda$ ,

$$\frac{E}{cB} \sim \frac{\lambda_c}{\lambda}.$$

This in turn means that

$$\lambda \leq 5 \times 10^5 \lambda_c = 5 \Delta X.$$

This constraint will be met automatically if we choose  $\lambda$  less than  $\Delta X$ , as required by resolvability. From this discussion it follows that the pertinent term in (5.14) should be detectable in an actual experiment.

## VI. DISCUSSION AND CONCLUSIONS

In this paper, we have focused on the rather interesting phenomenon that, in an arbitrarily strong electric field along the plane, a system of planar electrons exhibits a multiplicity of edges. The calculation here exhibits a number of interesting features. First, we have shown that we do not have to make a specific gauge choice for the background electric and magnetic fields. In fact, we have set up a gauge-independent algebra for the set of operators that diagonalizes the unperturbed Hamiltonian  $h_0$ . This is to be contrasted with most calculations in the literature, where a gauge choice (either Landau or symmetric) is made at the onset for the background fields. Moreover, at no point in the calculation are we called upon to use the explicit wave functions of the higher Landau levels. It is at the very end of the calculation that we have to use the explicit wave functions of the lowest Landau level, in order to make contact with the real spatial coordinates. Most of the manipulations are at the operator level, and integrals have to be performed only at the ultimate stages. This alleviates the tedium of a derivative expansion considerably. Furthermore, the only expansion has been with respect to the large background magnetic field. The electron mass has been retained throughout as an arbitrary parameter, and hence finite-mass effects have already been incorporated. We have further seen that to set up a viable perturbation theory, an  $\epsilon$  neighborhood of the edge due to each Landau level has to be excised, and the fermion modes in these intervals treated separately. These modes cannot be integrated out and reexpressed through local terms involving the perturbative gauge potentials. We have dealt with these separately to obtain the chiral edge fermionic actions. The remaining fermionic modes can be integrated out and reexpressed in terms of a local effective action involving the gauge potentials. This is the ‘‘bulk’’ action. Thus a clear separation of the bulk and the edge has been effected in this calculation. The bulk action is expectedly not gauge invariant by itself. The edge fermionic systems also possess the well-known  $U(1)$  gauge anomaly. The basic anomaly is, however, seen to be insufficient to compensate for the non-invariance of the bulk. We also need to include local counterterms constructed out the operators in the edge fermionic action to render the total effective description gauge invariant. While this procedure is quite familiar to aficionados of

the theory of anomalies, it is instructive to encounter it in a more everyday condensed-matter context. Apart from the terms in the fermionic edge action which lead to the anomaly, we have also found an interesting gauge-invariant term that depends both on the electron mass and on the particular edge. We have argued that the existence of this term can be verified experimentally.

Turning from enumerating the virtues of our calculation, we outline future as well as ongoing projects in the general direction. We are currently involved in generalizing to the case of relativistic fermions, in which case the electronic spin is introduced in a natural manner. One of the objectives is to see whether the spin could become an important degree of freedom even in a strong magnetic field.<sup>9,10</sup> Another issue that will bear closer scrutiny is the effect of the Coulomb interaction between the electrons and the effect of it on the edge states of the system.<sup>11-13</sup> The effect of a strong electric field in the case of the fractional Hall effect can also be discussed by looking at the integral Hall effect for Jain's "super" fermions.<sup>14</sup> Work in this direction is underway.

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### APPENDIX

In this appendix, we shall provide some of the computational details omitted in the main body of the paper. As we noted in Sec. III, we have to translate results obtained in the  $\{|n, X, \omega\rangle\}$  basis to the space-time basis. For this, as we shall see below, we need  $|\mathbf{x}|0, X\rangle^2$ . This may be worked out by going to a specific gauge. Here we choose to work in the

$$\mathbf{A} \equiv (0, -Bx) \quad (\text{A1})$$

gauge. In this gauge, we have

$$\langle \mathbf{x}|0, X\rangle = \left(\frac{B}{\pi}\right)^{1/4} \left(\frac{B}{2\pi}\right)^{1/2} e^{-iB(X+mE/B^2)y} e^{-(B/2)(x-X)^2}. \quad (\text{A2})$$

In fact, in Sec. III, we were required to evaluate  $\int_{-\infty}^{-X_n} dX |\langle \mathbf{x}|0, X\rangle|^2$ . With the wave function from (A2), we obtain

$$\int_{-\infty}^{-X_n} dX |\langle \mathbf{x}|0, X\rangle|^2 = \frac{B}{2\pi} \text{erf}[-\sqrt{B}(x+X_n)], \quad (\text{A3})$$

where  $\text{erf}(x) \equiv (1/\sqrt{\pi}) \int_{-\infty}^x dy e^{-y^2}$ . In the limit of a strong magnetic field, from the value of the error function as its argument goes to either plus or minus infinity, we obtain

$$\int_{-\infty}^{-X_n} dX |\langle \mathbf{x}|0, X\rangle|^2 \rightarrow \frac{B}{2\pi} \theta[-(x+X_n)]. \quad (\text{A4})$$

This means that, on performing the  $X$  integral, the upper limit on the integral, due to the specification of the ground state, devolves on to the integral over the spatial coordinate  $x$ .

### 1. From the $|0, X\rangle$ basis to space-time functions

Next we indicate the simplification of the following expression which occurs frequently in the text:

$$\int_{-\infty}^{-X_n} dX \langle 0, X | \# f(\hat{Z}, \hat{\bar{Z}}) \# | 0, X \rangle.$$

This can be rewritten as

$$\int_{-\infty}^{-X_n} dX \langle 0, X | f(\hat{z}, \hat{\bar{z}}) | 0, X \rangle$$

in view of the fact that normal-ordered products of  $\hat{\pi}$  and  $\hat{\pi}^\dagger$  give zero matrix elements in the lowest Landau level [recall (2.33)]. Inserting the identity operator in terms of the spatial basis, namely  $\int d\mathbf{x} |\mathbf{x}\rangle \langle \mathbf{x}| = I$ , into the above expression, we obtain

$$\int d\mathbf{x} f(x, y) \int_{-\infty}^{-X_n} dX |\langle \mathbf{x}|0, X\rangle|^2.$$

Using (A4), this yields

$$\int_{-\infty}^{-X_n} dX \langle 0, X | \# f(\hat{Z}, \hat{\bar{Z}}) \# | 0, X \rangle = \frac{B}{2\pi} \int_{-\infty}^{-X_n} dx \int_{-\infty}^{\infty} dy f(x, y). \quad (\text{A5})$$

### 2. Reantinormal ordering

Again, in the text, we come across expressions of the form  $\# f \# \# g \#$ , where each of the functions  $f$  and  $g$  are separately antinormal ordered in  $\hat{Z}$  and  $\hat{\bar{Z}}$ . However, this can be written again in a suitably antinormal-ordered form by commuting  $\hat{Z}$  across  $\hat{\bar{Z}}$  wherever they do not occur in antinormal order in the product  $\# f \# \# g \#$ . As  $\hat{Z}$  and  $\hat{\bar{Z}}$  are canonically conjugate (up to factors), this leads to an infinite series of antinormal functions of increasingly higher order in  $1/B$ . That is,

$$\# f \# \# g \# = \# f g \# - \frac{1}{B} \# \partial_{\bar{z}} f \partial_z g \# + \dots, \quad (\text{A6})$$

where the  $\dots$  indicate terms of higher order in  $(1/B)$  that have been dropped. Hence, using (A5) and (A6), we have

$$\begin{aligned} \int_{-\infty}^{-X_n} dX \langle X | \# f \# \# g \# | X \rangle &= \frac{B}{2\pi} \int_{-\infty}^{-X_n} dx \int_{-\infty}^{\infty} dy \left[ fg \right. \\ &\quad \left. - \frac{1}{B} \partial_{\bar{z}} f \partial_z g + \dots \right]. \quad (\text{A7}) \end{aligned}$$

### 3. Commuting $1/D$ with $f(\hat{X}, \hat{Y}, \hat{t})$

Furthermore, we also encounter expressions like

$$\left\langle n, X, \omega \left| \#f\# \frac{1}{D} \#g\# \right| n, X, \omega \right\rangle.$$

$D$  does not commute with  $f$  or  $g$  as it contains  $\hat{X}$  and  $\hat{\Pi}_0$ . However, using

$$[D, f] = -i \frac{E}{B} \partial_y f + [\hat{\Pi}_0, f], \quad (\text{A8})$$

where  $[\hat{\Pi}_0, f] = i \partial_t f$  in the space-time basis, we have

$$\begin{aligned} & \left\langle n, X, \omega \left| \#f\# \frac{1}{D} \#g\# \right| n, X, \omega \right\rangle \\ &= \left\langle n, X, \omega \left| \frac{1}{D} \#f\# \#g\# \right| n, X, \omega \right\rangle \\ &+ \left\langle n, X, \omega \left| \frac{1}{D} [D, \#f\#] \frac{1}{D} \#g\# \right| n, X, \omega \right\rangle. \end{aligned} \quad (\text{A9})$$

The second term on the right-hand side  $O(1/B)$  suppressed with respect to the first term. The formula above can be further iterated until one obtains all  $1/D$ 's acting on the states, plus higher-order terms.

### 4. Multiple denominators

We also have to perform a bit of algebra with  $\Gamma_n(X)$  defined in Eq. (2.42). The principal results that we note here are

$$\frac{1}{\Gamma_n \Gamma_{n+1}} = \frac{1}{\omega_c} \left( \frac{1}{\Gamma_{n+1}} - \frac{1}{\Gamma_n} \right), \quad (\text{A10})$$

$$\frac{1}{\Gamma_n \Gamma_{n-1}} = \frac{1}{\omega_c} \left( \frac{1}{\Gamma_n} - \frac{1}{\Gamma_{n-1}} \right). \quad (\text{A11})$$

Furthermore, we have

$$\sum_{n=0}^{\infty} \frac{(n+1)}{\Gamma_{n+1}} = \sum_{n=0}^{\infty} \frac{n}{\Gamma_n}, \quad (\text{A12})$$

$$\sum_{n=0}^{\infty} \frac{n}{\Gamma_{n-1}} = \sum_{n=0}^{\infty} \frac{n+1}{\Gamma_n}, \quad (\text{A13})$$

$$\sum_{n=0}^{\infty} \frac{n(n+1)}{\Gamma_{n+1}} = \sum_{n=0}^{\infty} \frac{n(n-1)}{\Gamma_n}, \quad (\text{A14})$$

and

$$\sum_{n=0}^{\infty} \frac{n(n-1)}{\Gamma_{n-1}} = \sum_{n=0}^{\infty} \frac{n(n+1)}{\Gamma_n}. \quad (\text{A15})$$

Other results may be obtained as required by repeated applications of (A10) and (A11).

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