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Multi-sided assignment games on m-partite graphs

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Abstract: We consider a multi-sided assignment game with the following characteristics: (a) the agents are organized in m sectors that are connected by a graph that induces a weighted m-partite graph on the set of agents, (b) a basic coalition is formed by agents from different connected sectors, and (c) the worth of a basic coalition is the addition of the weights of all its pairs that belong to connected sectors. We provide a sufficient condition on the weights to guarantee balancedness of the related multi-sided assignment game. Moreover, when the graph on the sectors is cycle-free, we prove the game is strongly balanced and the core is described by means of the cores of the underlying two-sided assignment games associated with the edges of this graph. Moreover, once selected a spanning tree of the cycle-free graph on the sectors, the equivalence between core and competitive equilibria is established.

JEL Codes: C71, C78.

Keywords: Cooperative games, multi-sided assignment games, core, competitive equilibria.

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1 Introduction

Two-sided assignment games (Shapley and Shubik, 1972) have been generalized to the multi-sided case. In this case, agents are distributed in m disjoint sectors. Usually it is assumed that these agents are linked by a hypergraph defined by the (basic) coalitions formed by exactly one agent from each sector (see for instance Kaneko and Wooders, 1982; Quint, 1991). A matching for a coalition S is a partition of the set of agents of S in basic coalitions and, since each basic coalition has a value attached, the worth of an arbitrary coalition of agents is obtained by maximizing, over all possible matchings, the addition of values of basic coalitions in a matching.

If we do not require that each basic coalition has exactly one agent of each side but allow for coalitions of smaller size, as long as they do not contain two agents from the same sector, we obtain a larger class of games, see Atay et al. (2016) for the three-sided case. But in both cases, the classical multi-sided assignment market and this enlarged model, the core of the corresponding coalitional game may be empty, and this is the main difference with the two-sided assignment game of Shapley and Shubik (1972), where the core is always non-empty.

A two-sided assignment game can also be looked at in another way. There is an underlying bi-partite (weighted) graph, where the set of nodes corresponds to the set of agents and the weight of an edge is the value of the basic coalition formed by its adjacent nodes. From this point of view, the generalization to a market with $m > 2$ sectors can be defined by a weighted m-partite graph G. In an m-partite graph the set of nodes N is partitioned in m sets $N_1, N_2, \ldots, N_m$ in such a way that two nodes in a same set of the partition are never connected by an edge. Each node in G corresponds to an agent of our market and each set N_i, for $i \in \{1, 2, \ldots, m\}$, to a different sector. We do not assume that the graph is complete but we do assume that the subgraph determined by any two sectors N_i and N_j, with $i \neq j$, is either empty or complete. Because of that, the graph G determines a quotient graph $\overline{G}$, the nodes of which are the sectors and two sectors are connected in $\overline{G}$ whenever their corresponding subgraph in G is non-empty.

For each pair of sectors N_r and N_s, $r \neq s$, that are connected in $\overline{G}$, we have a bilateral assignment market with valuation matrix $A^{\{r,s\}}$. For each $i \in N_r$ and $j \in N_s$, entry $a_{ij}^{\{r,s\}}$ is the weight in G of the edge $\{i, j\}$, and represents the value created by the cooperation of i and j.

Given the m-partite graph G, a coalition of agents in N is basic if it does not contain two agents from the same sector and its members are connected in G. Then, the worth of a basic coalition is the addition of the weights of the edges in G that are determined by nodes in the coalition. An optimal matching in this market is a partition of N in basic coalitions such that the sum of values is maximum among all possible such partitions.

We show that if there exists an optimal matching for the multi-sided m-partite market that induces an optimal matching in each bilateral market determined by the connected sectors, then the core of the multi-sided market is non-empty. Moreover, a core element can be obtained by the merging of one core element from each of the underlying bilateral markets associated to the connected sectors.

Secondly, if the quotient graph $\overline{G}$ is cycle-free, then the above sufficient condition for a non-empty core always holds and, moreover, the core of the multi-sided assignment game is fully described by the "merging" or "composition" of the cores of the underlying bilateral games. As a consequence, we prove several properties of the core of this multi-sided market. For instance, for each sector there exists a core allocation where all agents in the sector achieve their marginal contribution.

This model of multi-sided assignment market on an m-partite graph G where the quotient graph $\overline{G}$ is cycle-free can be related to the locally-additive multi-sided assignment games of Stuart (1997), where the sectors are organized on a chain and the worth of a basic coalition is also the addition of the worths of pairs of consecutive sectors. However, in Stuart's model all coalitions of size smaller than m have null worth. It can also be related with a model in Quint (1991) in which a value is attached to each pair of agents of different sectors and then the worth of an m-tuple is the addition of the values of its pairs. Again, the difference with our model is that in Quint (1991) the worth of smaller coalitions is zero. In particular, the worth of a two-player coalition is taken to be zero instead of the value of this pair. Notice that in these models the cooperation of one agent from each side is needed to generate some profit. Compared to that, in our model, any set of connected agents from different sectors yields some worth that can be shared.

For arbitrary coalitional games, cooperation restricted by communication graphs was introduced by Myerson (1977) and some examples of more recent studies are van Velzen et al. (2008), Khmelnitskaya and Talman (2014), and González-Arangüena et al. (2015). The difference with our work is that in the multi-sided assignment game on an m-partite graph there exist well-structured subgames, the two-sided markets between connected sectors, that provide valuable information about the multi-sided market.

Section 2 introduces the model. In Section 3, for an arbitrary m-partite graph, we provide a sufficient condition for the non-emptiness of the core. Section 4 focuses on the case in which the quotient graph is cycle-free. In that case, we completely characterize the non-empty core in terms of the cores of the two-sided markets between connected sectors. From that fact, additional consequences on some particular core elements are derived. In Section 5, once selected a spanning tree of the cycle-free graph $\overline{G}$, we characterize the core of the multi-sided assignment game in terms of competitive prices. Finally, Section 6 concludes with some remarks.

2 The model

Let N be the finite set of agents in a market situation. The set N is partitioned in m sets $N_1, N_2, \ldots, N_m$, each sector maybe representing a set of agents with a specific role in the market. There is a graph $\overline{G}$ with set of nodes $\{N_1, N_2, \ldots, N_m\}$, that we simply denote $\{1, 2, \ldots, m\}$ when no confusion arises, and we will identify the graph with its set of edges.[1] The graph $\overline{G}$ induces another graph on the set of agents N such that $\{i, j\} \in G$ if and only if there exist $r, s \in \{1, 2, \ldots, m\}$ such that $r \neq s$, $i \in N_r$, $j \in N_s$ and $\{r, s\} \in \overline{G}$. Notice that the graph G is an m-partite graph, that meaning that two agents on the same sector are not connected in G. We say that graph $\overline{G}$ is the quotient

[1] A graph consists of a (finite) set of nodes and a set of edges, where an edge is a subset formed by two different nodes. If $\{r, s\}$ is an edge of a given graph, we say that the nodes r and s belong to this edge or are adjacent to this edge.

graph of G.[2]

For any pair of connected sectors $\{r,s\} \in \overline{G}$, there is a non-negative valuation matrix $A^{\{r,s\}}$ and for all $i \in N_r$ and $j \in N_s$, $v(\{i,j\}) = a_{ij}^{\{r,s\}}$ represents the value obtained by the cooperation of agents i and j. Notice that these valuation matrices, $A = \{A^{\{r,s\}}\}_{\{r,s\}\in\overline{G}}$, determine a system of weights on the graph G, and for each pair of connected sectors $\{r,s\} \in \overline{G}$, $(N_r, N_s, A^{\{r,s\}})$ defines a bilateral assignment market. Sometimes, to simplify notation, we will write A^{rs}, with $r < s$, instead of $A^{\{r,s\}}$.

Then, $\gamma = (N_1, N_2, \ldots, N_m; G; \{A^{\{r,s\}}\}_{\{r,s\}\in\overline{G}})$ is a *multi-sided assignment market on an m-partite graph*. When necessary, we will write G^A to denote the weighted graph with the nodes and edges of G and the weights defined by the matrices $\{A^{\{r,s\}}\}_{\{r,s\}\in\overline{G}}$. Given any such market γ, a coalition $S \subseteq N$ defines a submarket $\gamma_{|S} = (S \cap N_1, \ldots, S \cap N_m; G_{|S}; A_{|S})$ where $G_{|S}$ is the subgraph of G defined by the nodes in S and $A_{|S}$ consists of the values of A that correspond to edges $\{i,j\}$ in the subgraph $G_{|S}$.

We now introduce a coalitional game related to the above market situation. To this end, we first define the worth of some coalitions that we name *basic coalitions* and then the worth of arbitrary coalitions will be obtained just imposing superadditivity. A basic coalition E is a subset of agents belonging to sectors that are connected in the quotient graph $\overline{G}$ and with no two agents of the same sector. That is, $E = \{i_1, i_2, \ldots, i_k\} \subseteq N$ is a basic coalition if $(i_1, i_2, \ldots, i_k) \in N_{l_1} \times N_{l_2} \times \cdots \times N_{l_k}$ and the sectors $\{l_1, l_2, \ldots, l_k\}$ are all different and connected in $\overline{G}$. Sometimes we will identify the basic coalition $E = \{i_1, i_2, \ldots, i_k\}$ with the k-tuple $(i_1, i_2, \ldots, i_k)$. For the sake of notation, we denote by $\mathcal{B}^N$ the set of basic coalitions of market γ, though we should write $\mathcal{B}^{N_1,\ldots,N_m}$, since which coalitions are basic heavily depends on the partition in sectors of the set of agents. Notice that all edges of G belong to $\mathcal{B}^N$. Moreover, if $S \subseteq N$, we denote by $\mathcal{B}^S$ the set of basic coalitions that have all their agents in S: $\mathcal{B}^S = \{E \in \mathcal{B}^N \mid E \subseteq S\}$.

The valuation function, until now defined on the edges of G, is extended to all basic coalitions by additivity: the value of a basic coalition $E \in \mathcal{B}^N$ is the addition of the weights of all edges in G with adjacent nodes in E. For all $E \in \mathcal{B}^N$,

$$v(E) = \sum_{\{i,j\}\in G_{|E}} v(\{i,j\}) = \sum_{\substack{i\in E\cap N_r, j\in E\cap N_s\\ \{r,s\}\in\overline{G}}} a_{ij}^{\{r,s\}}. \tag{1}$$

A *matching* μ for the market γ is a partition of $N = N_1 \cup N_2 \cup \cdots \cup N_m$ in basic coalitions in $\mathcal{B}^N$. We denote by $\mathcal{M}(N_1, N_2, \ldots, N_m)$ the set of all matchings. Similarly, a matching for a submarket $\gamma_{|S}$ with $S \subseteq N$ is a partition of S in basic coalitions in $\mathcal{B}^S$.

A matching $\mu \in \mathcal{M}(N_1, N_2, \ldots, N_m)$ is an *optimal matching* for the market $\gamma = (N_1, N_2, \ldots, N_m; G; \{A^{\{r,s\}}\}_{\{r,s\}\in\overline{G}})$ if it holds $\sum_{T\in\mu} v(T) \geq \sum_{T\in\mu'} v(T)$ for all other matching $\mu' \in \mathcal{M}(N_1, N_2, \ldots, N_m)$. We denote by $\mathcal{M}_\gamma(N_1, N_2, \ldots, N_m)$ the set of optimal matchings for market γ.

Then, the *multi-sided assignment game* associated with the market γ is the pair (N, w_γ), where the worth of an arbitrary coalition $S \subseteq N$ is the addition of the values

[2] Equivalently, we could introduce the model by first imposing a (weighted) m-partite garph on $N = N_1 \cup N_2 \cup \ldots \cup N_m$ with the condition that its restriction to $N_r \cup N_s$ for all $r, s \in \{1, \ldots, m\}$ and different, is either empty or a bi-partite complete graph. Then, the quotient graph $\overline{G}$ is easily defined.

of the basic coalitions in an optimal matching for this coalition S:

$$w_\gamma(S) = \max_{\mu \in \mathcal{M}(S\cap N_1,\dots,S\cap N_m)} \sum_{T\in\mu} v(T), \tag{2}$$

with $w_\gamma(\emptyset) = 0$. Notice that if $S \subseteq N$ is a basic coalition, $w_\gamma(S) = v(S)$, since no partition of S in smaller basic coalitions can yield a higher value, because of its definition (1) and the non-negativity of weights. Trivially, the game (N, w_γ) is superadditive as it is a special type of partitioning game introduced by Kaneko and Wooders (1982).

Multi-sided assignment games on m-partite graphs combine the idea of cooperation structures based on graphs (Myerson, 1977) and also the notion of (multi-sided) matching that only allows for at most one agent of each sector in a basic coalition. It is clear that for $m = 2$, multi-sided assignment games on bi-partite graphs coincide with the classical Shapley and Shubik (1972) assignment games. Notice also that for $m = 3$, multi-sided assignment games on 3-partite graphs are a particular case of the generalized three-sided assignment games in Atay et al. (2016), with the constraint that the value of a three-person coalition is the addition of the values of all its pairs.

As for the related quotient graphs, for $m = 2$ the quotient graph $\overline{G}$ consists of only one edge while, for $m = 3$, $\overline{G}$ can be either a complete graph[3] or a chain. Figure 1 illustrates both the graph G and its quotient graph $\overline{G}$ for the cases $m = 2$ and $m = 3$.

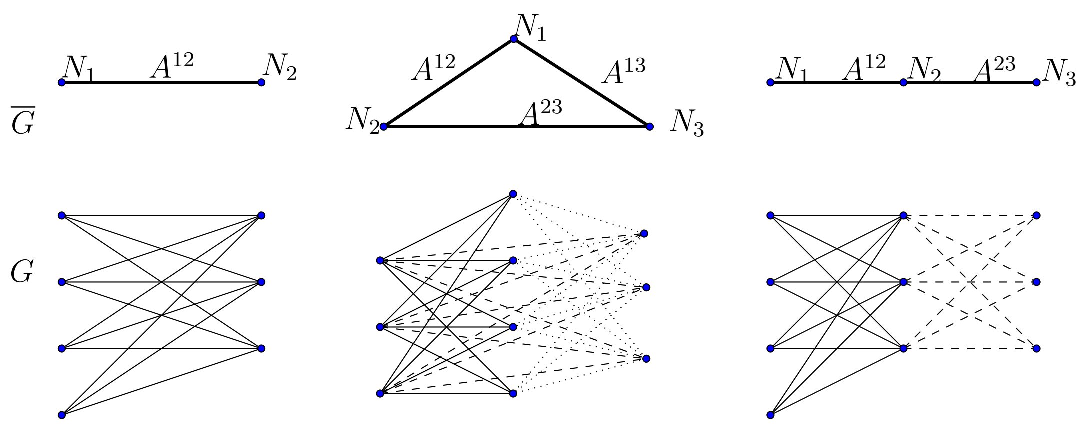

Figure 1: 2-partite and 3-partite graphs, and their quotient representation

As in any coalitional game, the aim is to allocate the worth of the grand coalition in such a way that preserves the cooperation among the agents. Given a multi-sided assignment market on an m-partite graph $\gamma = (N_1, N_2, \dots, N_m; G; \{A^{\{r,s\}}\}_{\{r,s\}\in\overline{G}})$, a vector $x \in \mathbb{R}^N$, where $N = N_1 \cup N_2 \cup \dots \cup N_m$, is a *payoff vector*. An *imputation* is a payoff vector $x \in \mathbb{R}^N$ that is *efficient*, $\sum_{i\in N} x_i = w_\gamma(N)$, and *individually rational*, $x_i \geq w_\gamma(\{i\}) = 0$ for all $i \in N$. Then, the *core* $C(w_\gamma)$ is the set of imputations that no coalition can object, that is $\sum_{i\in S} x_i \geq w_\gamma(S)$ for all $S \subseteq N$. Because of the definition of

[3] A graph is *complete* if any two of its nodes are connected by an edge. Hence, an m-partite graph with more than one node in some of the sectors is never complete in this sense. A *complete m-partite* graph is an m-partite graph such that any two nodes from different sectors are connected by an edge.

the characteristic function w_γ in (2), given any optimal matching $\mu \in \mathcal{M}_\gamma(N_1, \ldots, N_m)$, the core is described by

$$C(w_\gamma) = \left\{ x \in \mathbb{R}^N \,\middle|\, \sum_{i \in E} x_i = v(E) \text{ for all } E \in \mu, \sum_{i \in E} x_i \geq v(E), \text{ for all } E \in \mathcal{B}^N \right\}.$$

A multi-sided assignment game on an m-partite graph is *balanced* if it has a non-empty core. Moreover, and following Le Breton et al. (1992), we will say an m-partite graph $(N_1, N_2, \ldots, N_m; G)$ is *strongly balanced* if for any set of non-negative weights $\{A^{\{r,s\}}\}_{\{r,s\} \in \overline{G}}$ the resulting multi-sided assignment game is balanced. Recall from Shapley and Shubik (1972) that bi-partite graphs are strongly balanced. Our aim is to study whether this property extends to m-partite graphs or balancedness depends on properties of the weights or the structure of the graph.

3 Balancedness conditions

The first question above is easily answered. For $m \geq 3$, m-partite graphs are not strongly balanced. Take for instance a market with three agents on each sector. Sectors are connected by a complete graph: $N_1 = \{1, 2, 3\}$, $N_2 = \{1', 2', 3'\}$, $N_3 = \{1'', 2'', 3''\}$, and $\overline{G} = \{(N_1, N_2), (N_1, N_3), (N_2, N_3)\}$. From Le Breton et al. (1992) we know that a graph is strongly balanced if any balanced collection[4] formed by basic coalitions contains a partition. In our example, the collection

$$\mathcal{C} = \{\{1, 1'\}, \{1, 2''\}, \{2', 1''\}, \{2, 3'\}, \{3, 2''\}, \{3', 1''\}, \{3, 3''\}, \{2, 1'\}, \{2', 3''\}\}$$

is balanced (notice each agent belongs to exactly two coalitions in $\mathcal{C}$) but we cannot extract any partition. To better understand what causes the core to be empty we complete the above 3-partite graph with a system of weights and analyse some core constraints.

Example 1. Let us consider the following valuations on the complete 3-partite graph with three agents in each sector:

$$A^{12} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \!\! \begin{array}{c} \begin{array}{ccc} 1' & 2' & 3' \end{array} \\ \begin{pmatrix} 1 & 0 & \mathbf{0} \\ \mathbf{9} & 0 & 4 \\ 0 & \mathbf{0} & 0 \end{pmatrix} \end{array} \quad A^{13} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \!\! \begin{array}{c} \begin{array}{ccc} 1'' & 2'' & 3'' \end{array} \\ \begin{pmatrix} 0 & \mathbf{5} & 0 \\ \mathbf{0} & 0 & 0 \\ 0 & 2 & \mathbf{4} \end{pmatrix} \end{array} \quad A^{23} = \begin{array}{c} \\ 1' \\ 2' \\ 3' \end{array} \!\! \begin{array}{c} \begin{array}{ccc} 1'' & 2'' & 3'' \end{array} \\ \begin{pmatrix} 0 & \mathbf{0} & 0 \\ 4 & 0 & \mathbf{6} \\ \mathbf{2} & 0 & 0 \end{pmatrix} \end{array}.$$

In boldface we show the optimal matching for each two-sided assignment market. Now, applying (1), the reader can obtain the worth of all three-player basic coalitions and check that the optimal matching of the three-sided market is

$$\mu = \{(2, 1', 1''), (1, 3', 2''), (3, 2', 3'')\}.$$

[4]Given a coalitional game (N, v), a collection of coalitions $\mathcal{C} = \{S_1, S_2, \ldots, S_k\}$ with $S_l \subsetneq N$ for all $l \in \{1, 2, \ldots, k\}$, is balanced if there exist positive numbers $\delta_{S_l} > 0$ such that, for all $i \in N$, it holds $\sum_{i \in S_l \subseteq \mathcal{C}} \delta_{S_l} = 1$.

Notice that $v(\{2,1',1''\}) = 9+0+0 = 9$, $v(\{1,3',2''\}) = 0+5+0 = 5$ and $v(\{3,2',3''\}) = 0+4+6 = 10$.

Take $x=(u,v,w)\in\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}\times\mathbb{R}^{N_3}$. If $x=(u,v,w)\in C(w_\gamma)$, from core constraints $u_2+v_1+w_1=9$ and $u_2+v_1\geq 9$ we obtain $w_1=0$. Then, from $v_3+w_1\geq 2$ we deduce $v_3\geq 2$. Hence, $u_1+v_3+w_2=5$ implies $u_1+w_2\leq 3$, which contradicts the core constraint $u_1+w_2\geq 5$. Therefore, $C(w_\gamma)=\emptyset$.

We observe that the optimal matching μ in the above example induces a matching $\mu^{23}=\{(1',1''),(3',2''),(2',3'')\}$ for the market $(N_2,N_3,A^{\{2,3\}})$ which is not optimal. Let us relate more formally the matchings in a multi-sided assignment market on an m-partite graph with the matchings of the two-sided markets associated with the edges of the quotient graph.

Definition 2. Given $\gamma=(N_1,N_2,\ldots,N_m;G;\{A^{\{r,s\}}\}_{\{r,s\}\in\overline{G}})$, for each matching $\mu\in\mathcal{M}(N_1,\ldots,N_m)$ and each adjacent sectors $\{r,s\}\in\overline{G}$, we define a matching $\mu^{\{r,s\}}\in\mathcal{M}(N_r,N_s)$ by

$$\{i,j\}\in\mu^{\{r,s\}} \text{ if and only if there exists } E\in\mu \text{ such that } \{i,j\}\subseteq E. \tag{3}$$

We then say that μ is the composition of $\mu^{\{r,s\}}$ for $\{r,s\}\in\overline{G}$ and write

$$\mu=\bigoplus_{\{r,s\}\in\overline{G}}\mu^{\{r,s\}}.$$

Conversely, the composition of matchings of each underlying two-sided market not always results in a matching of the multi-sided assignment market. Take for instance matchings $\mu^{\{1,2\}}=\{(2,1'),(1,3'),(3,2')\}$, $\mu^{\{1,3\}}=\{(1,2''),(2,1''),(3,3'')\}$ and $\mu^{\{2,3\}}=\{(1',2''),(2',3''),(3',1'')\}$ in Example 1. Since $(1',2'')\in\mu^{\{2,3\}}$, $(2,1')\in\mu^{\{1,2\}}$ and $(1,2'')\in\mu^{\{1,3\}}$, both 1 and 2 should be in the same coalition when composing $\mu^{\{1,2\}}\oplus\mu^{\{1,3\}}\oplus\mu^{\{2,3\}}$, but then this coalition would not be basic since it contains two agents from N_1, and the composition would not be a matching of the three-sided market.

Next proposition states that whenever the composition of optimal matchings of the underlying two-sided markets results in a matching of the multi-sided market on an m-partite graph, then that matching is optimal and the core of the multi-sided assignment market is non-empty. To show this second part we need to combine payoff vectors of each underlying two-sided market $(N_r,N_s,A^{\{r,s\}})$, with $\{r,s\}\in\overline{G}$, to produce a payoff vector $x\in\mathbb{R}^N$ for the multi-sided market γ. We write $C(w_{A^{\{r,s\}}})$ to denote the core of these two-sided assignment games.

Definition 3. Given $\gamma=(N_1,N_2,\ldots,N_m;G;\{A^{\{r,s\}}\}_{\{r,s\}\in\overline{G}})$, let $x^{\{r,s\}}\in\mathbb{R}^{N_r}\times\mathbb{R}^{N_s}$ for all $\{r,s\}\in\overline{G}$. Then,

$$x=\bigoplus_{\{r,s\}\in\overline{G}}x^{\{r,s\}}\in\mathbb{R}^N \quad \text{is defined by}$$

$$x_i=\sum_{\{r,s\}\in\overline{G}}x_i^{\{r,s\}},\ \text{for all } i\in N_r, r\in\{1,2,\ldots,m\}.$$

We then say that the payoff vector $x = \bigoplus_{\{r,s\}\in\overline{G}} x^{\{r,s\}} \in \mathbb{R}^N$ is the composition of the payoff vectors $x^{\{r,s\}} \in \mathbb{R}^{N_r} \times \mathbb{R}^{N_s}$. Similarly, we denote the set of payoff vectors in $\mathbb{R}^N$ that result from the composition of core elements of the underlying two-sided assignment markets by $\bigoplus_{\{r,s\}\in\overline{G}} C(w_{A^{\{r,s\}}})$.

Proposition 4. *Let $\gamma = (N_1, N_2, \ldots, N_m; G; \{A^{\{r,s\}}\}_{\{r,s\}\in\overline{G}})$ be a multi-sided assignment market on an m-partite graph. If there exists $\mu \in \mathcal{M}(N_1, \ldots, N_m)$ such that $\mu^{\{r,s\}}$ is an optimal matching of $(N_r, N_s, A^{\{r,s\}})$ for all $\{r,s\} \in \overline{G}$, then*

1. *μ is optimal for γ and*
2. *γ is balanced and moreover $\bigoplus_{\{r,s\}\in\overline{G}} C(w_{A^{\{r,s\}}}) \subseteq C(w_\gamma)$.*

Proof. To see that $\mu = \bigoplus_{\{r,s\}\in\overline{G}} \mu^{\{r,s\}}$ is optimal for γ, take any other matching $\tilde{\mu} \in \mathcal{M}(N_1, \ldots, N_m)$ and let $\tilde{\mu}^{\{r,s\}} \in \mathcal{M}(N_r, N_s)$, for $\{r,s\} \in \overline{G}$, be the matching $\tilde{\mu}$ induces in each underlying two-sided market. That is, $\tilde{\mu} = \bigoplus_{\{r,s\}\in\overline{G}} \tilde{\mu}^{\{r,s\}}$. Now, applying (1),

$$\begin{aligned}
\sum_{E\in\mu} v(E) &= \sum_{E\in\mu} \sum_{\substack{i\in N_r\cap E\\ j\in N_s\cap E\\ \{r,s\}\in\overline{G}}} v(\{i,j\}) = \sum_{\{r,s\}\in\overline{G}} \sum_{\{i,j\}\in\mu^{\{r,s\}}} v(\{i,j\}) \\
&\geq \sum_{\{r,s\}\in\overline{G}} \sum_{\{i,j\}\in\tilde{\mu}^{\{r,s\}}} v(\{i,j\}) = \sum_{E\in\tilde{\mu}} v(E),
\end{aligned}$$

where the inequality follows from the assumption on the optimality of $\mu^{\{r,s\}}$ in each market $(N_r, N_s, A^{\{r,s\}})$, for $\{r,s\} \in \overline{G}$. Hence, μ is optimal for the multi-sided market γ.

Take now, for each $\{r,s\} \in \overline{G}$, $x^{\{r,s\}} \in C(w_{A^{\{r,s\}}})$. Define the payoff vector $x \in \mathbb{R}^N$ as in Definition 3, $x_i = \sum_{\{r,s\}\in\overline{G}} x_i^{\{r,s\}}$, for all $i \in N_r$, $r \in \{1, 2, \ldots, m\}$. We will see that $x \in C(w_\gamma)$. Given any basic coalition $E \in \mathcal{B}^N$,

$$\begin{aligned}
\sum_{i\in E} x_i &= \sum_{r=1}^{m} \sum_{i\in E\cap N_r} x_i = \sum_{r=1}^{m} \sum_{i\in E\cap N_r} \sum_{\{r,s\}\in\overline{G}} x_i^{\{r,s\}} \\
&\geq \sum_{r=1}^{m} \sum_{i\in E\cap N_r} \sum_{\substack{\{r,s\}\in\overline{G}\\ E\cap N_s\neq\emptyset}} x_i^{\{r,s\}} = \sum_{\substack{\{r,s\}\in\overline{G}\\ E\cap N_r\neq\emptyset\\ E\cap N_s\neq\emptyset}} \sum_{\substack{i\in E\cap N_r\\ j\in E\cap N_s}} \left(x_i^{\{r,s\}} + x_j^{\{r,s\}}\right) \\
&\geq \sum_{\substack{\{r,s\}\in\overline{G}\\ i\in E\cap N_r\\ j\in E\cap N_s}} v(\{i,j\}) = v(E),
\end{aligned}$$

where both inequalities follow from $x^{\{r,s\}} \in C(w_{A^{\{r,s\}}})$ for all $\{r,s\} \in \overline{G}$. Notice also that if $E \in \mu$ the above inequalities cannot be strict and hence $\sum_{i\in E} x_i = v(E)$. Indeed, if $i \in E \cap N_r$, $\{r,s\} \in \overline{G}$ and $E \cap N_s = \emptyset$, then i is unmatched by $\mu^{\{r,s\}}$ and, because of the optimality of $\mu^{\{r,s\}}$, $x_i^{\{r,s\}} = 0$. Similarly, if $i \in E \cap N_r$ and $j \in E \cap N_s$, then $\{i,j\} \in \mu^{\{r,s\}}$ and hence $x_i^{\{r,s\}} + x_j^{\{r,s\}} = v(\{i,j\})$. □

The above proposition gives a sufficient condition for optimality of a matching and for balancedness of a multi-sided assignment game on an m-partite graph. However, this condition is not necessary. The matching μ in Example 1 is optimal while $\mu^{\{2,3\}}$ is not. The core of the market in Example 1 is empty, but one can find similar examples with non-empty core (see Example 15).

Finally, even under the assumption of the proposition, that is, when the composition of optimal matchings of the two-sided markets leads to a matching of the multi-sided market, the core may contain more elements than those produced by the composition of the cores of $(N_r, N_s, A^{\{r,s\}})$, for $\{r, s\} \in \overline{G}$ (see Atay et al. (2016) for an example in the three-sided case). The inclusion $\bigoplus_{\{r,s\}\in\overline{G}} C(w_{A^{\{r,s\}}}) \subseteq C(w_\gamma)$ will become an equality for some particular graphs.

4 When $\overline{G}$ is cycle-free: strong balancedness

In this section we assume that the quotient graph $\overline{G}$ of the m-partite graph G does not contain cycles. We will assume without loss of generality that it is connected, since the results in that case are easily extended to the case of a finite union of disjoint cycle-free graphs.

We select a node of $\overline{G}$ as a source, that is, we select a spanning tree of $\overline{G}$. Define the distance d of any other node as the number of edges in the unique path that connects this node to the source. Then, without loss of generality, we rename the nodes of $\overline{G}$ in such a way that the source has label 1 and, given two other nodes r and s, if $d(1, r) < d(1, s)$ then $r < s$. Notice that the labels of nodes at the same distance to the source are assigned arbitrarily.

A partial order is defined on the set of nodes of a tree in the following way: given two nodes r and s, we say that s *follows* r, and write $s \succeq r$, if given the unique path in the tree that connects s to the source, $\{s_1 = 1, s_2, \ldots, s_q = s\}$, it holds $r = s_p$ for some $p \in \{1, \ldots, q-1\}$. If $r = s_{q-1}$ we say that s is an *immediate follower* of r. We denote by $\mathcal{S}_r^{\overline{G}}$ the set of followers of $r \in \{1, 2, \ldots, m\}$, we write $\hat{\mathcal{S}}_r^{\overline{G}} = \{r\} \cup \mathcal{S}_r^{\overline{G}}$ when we need to include sector r, and we denote by $\mathcal{I}_r^{\overline{G}}$ the set of immediate followers of $r \in \{1, 2, \ldots, m\}$.

Our main result states that an m-partite graph G where the quotient graph $\overline{G}$ is a tree is strongly balanced.

Theorem 5. *Let $\gamma = (N_1, N_2, \ldots, N_m; G; \{A^{\{r,s\}}\}_{\{r,s\}\in\overline{G}})$ be a multi-sided assignment market on an m-partite graph. If $\overline{G}$ is cycle-free, then (N, w_γ) is balanced and*

$$C(w_\gamma) = \bigoplus_{\{r,s\}\in\overline{G}} C(w_{A^{\{r,s\}}}).$$

Proof. Notice first that when $\overline{G}$ is a tree, the composition of optimal matchings $\mu^{\{r,s\}}$ of each underlying two-sided market $(N_r, N_s, A^{\{r,s\}})$, for $\{r, s\} \in \overline{G}$, leads to a matching in $\mathcal{M}(N_1, N_2, \ldots, N_m)$. To see that, we define a binary relation on the set of agents $N = N_1 \cup N_2 \cup \cdots \cup N_m$. Two agents $i \in N_r$ and $j \in N_s$, with $r \leq s$, are related if either $i = j$ or there exist sectors $\{r = s_1, s_2, \ldots, s_t = s\} \subseteq \{1, 2, \ldots, m\}$ and agents

$i_k \in N_{s_k}$ for $k \in \{1, 2, \ldots, t\}$ such that $\{s_k, s_{k+1}\} \in \overline{G}$ and $\{i_k, i_{k+1}\} \in \mu^{\{s_k, s_{k+1}\}}$, for all $k \in \{1, 2, \ldots, t-1\}$. This is an equivalence relation and, because $\overline{G}$ is a tree, in each equivalence class there are no two agents of the same sector. Hence, the set μ of all equivalence classes is a matching and by its definition it is the composition of the matchings $\mu^{\{r,s\}}$ of the two-sided markets: $\mu = \bigoplus_{\{r,s\} \in \overline{G}} \mu^{\{r,s\}}$. Now, by Proposition 4, μ is an optimal matching for the multi-sided market γ and $\bigoplus_{\{r,s\} \in \overline{G}} C(w_{A^{\{r,s\}}}) \subseteq C(w_\gamma)$, which guarantees balancedness.

We will now prove that the converse inclusion also holds.

Let it be $u = (u^1, u^2, \ldots, u^m) \in C(w_\gamma)$. We will define, for each $\{r, s\} \in \overline{G}$, a payoff vector $(x^{\{r,s\}}, y^{\{r,s\}}) \in \mathbb{R}^{N_r} \times \mathbb{R}^{N_s}$. Take the optimal matching $\mu = \bigoplus_{\{r,s\} \in \overline{G}} \mu^{\{r,s\}}$ and $E \in \mu$. Let us denote by $\overline{E} = \overline{G}_{|E}$ the subtree in $\overline{G}$ determined by the sectors containing agents in E and take as the source of $\overline{E}$ its sector s_1 with the lowest label. Take any leaf[5] s_r of $\overline{E}$ and let $\{s_1, s_2 \ldots, s_q, s_{q+1}, \ldots, s_{r-1}, s_r\}$ be the unique path in $\overline{E}$ connecting s_r to the source s_1. Let s_q be the sector in this path with the highest label among those that have more than one immediate follower in $\overline{E}$ (let us assume for simplicity that s_q has two immediate followers, s_{q+1} and $s_{q'+1}$). Figure 2 depicts such a subtree $\overline{E}$.

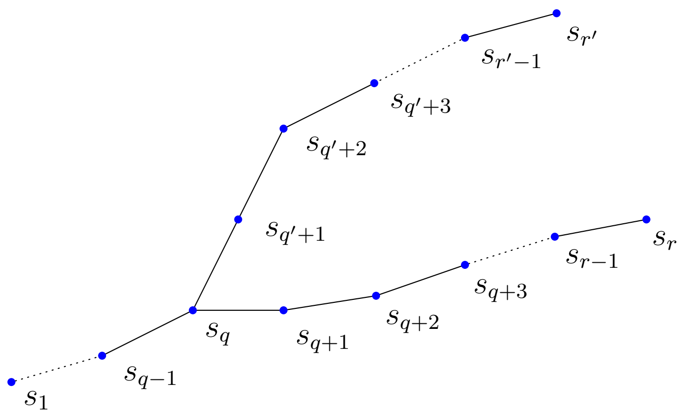

Figure 2: A subtree $\overline{E}$ for $E \in \mu$

For each sector s_t with $t \in \{1, 2, \ldots, r\}$ we denote by i_t the unique agent in E that belongs to this sector. Then, we define

$$y^{\{s_{r-1},s_r\}}_{i_r} = u^{s_r}_{i_r}, \tag{4}$$

$$x^{\{s_{r-1},s_r\}}_{i_{r-1}} = a^{\{s_{r-1},s_r\}}_{i_{r-1}i_r} - y^{\{s_{r-1}s_r\}}_{i_r}, \text{ and} \tag{5}$$

$$y^{\{s_{r-2},s_{r-1}\}}_{i_{r-1}} = u^{s_{r-1}}_{i_{r-1}} - x^{\{s_{r-1},s_r\}}_{i_{r-1}}. \tag{6}$$

Iteratively, for all $t \in \{q+1, \ldots, r-2\}$, we define

$$x^{\{s_t,s_{t+1}\}}_{i_t} = a^{\{s_t,s_{t+1}\}}_{i_t i_{t+1}} - y^{\{s_t,s_{t+1}\}}_{i_{t+1}}, \text{ and} \tag{7}$$

$$y^{\{s_{t-1},s_t\}}_{i_t} = u^{s_t}_{i_t} - x^{\{s_t,s_{t+1}\}}_{i_t}, \tag{8}$$

while for sector s_q we define $x^{\{s_q,s_{q+1}\}}_{i_q} = a^{\{s_q,s_{q+1}\}}_{i_q i_{q+1}} - y^{\{s_q,s_{q+1}\}}_{i_{q+1}}$, and, assuming $x^{\{s_q,s_{q'+1}\}}_{i_q}$ has been defined analogously from the branch $\{s_{q'+1}, s_{q'+2}, \ldots, s_{r'-1}, s_{r'}\}$, we also define

[5] Given a tree, a leaf is a node with no followers.

$y_{i_q}^{\{s_{q-1},s_q\}} = u_{i_q}^{s_q} - \left(x_{i_q}^{\{s_q,s_{q+1}\}} + x_{i_q}^{\{s_q,s_{q'+1}\}}\right)$. More generally, if s_q has several immediate followers in $\overline{E}$, then

$$y_{i_q}^{\{s_{q-1},s_q\}} = u_{i_q}^{s_q} - \sum_{\substack{\{s_q,s_l\}\in\overline{E}\\ s_q<s_l}} x_{i_q}^{\{s_q,s_l\}}. \tag{9}$$

We proceed backwards until we reach $x_{i_1}^{\{s_1,s_l\}}$ for all $\{s_1,s_l\} \in \overline{E}$ with $s_1 < s_l$.

In addition, if $i \in N_r$ and for some $\{r,s\} \in \overline{G}$, $r < s$, i is unmatched by $\mu^{\{r,s\}}$, define $x_i^{\{r,s\}} = 0$. Similarly, if $i \in N_r$ and for all $\{s,r\} \in \overline{G}$, $s < r$, i is unmatched by $\mu^{\{s,r\}}$, define $y_i^{\{s,r\}} = 0$.

We will first check that the payoff vectors $(x^{\{r,s\}}, y^{\{r,s\}})$ we have defined are non-negative for all $\{r,s\} \in \overline{G}$. From (4) to (9) above, it follows that, for all maximal path in $\overline{E}$ starting at s_1, $\{s_1,s_2,\ldots,s_r\}$, and all $t \in \{1,2,\ldots,r-1\}$, we can express $x_{i_t}^{\{s_t,s_{t+1}\}}$ in terms of the payoffs in u to agents in following sectors in $\overline{E}$:

$$\begin{aligned} x_{i_t}^{\{s_t,s_{t+1}\}} &= a_{i_t i_{t+1}}^{\{s_t,s_{t+1}\}} - y_{i_{t+1}}^{\{s_t,s_{t+1}\}} = a_{i_t i_{t+1}}^{\{s_t,s_{t+1}\}} - (u_{i_{t+1}}^{s_{t+1}} - \sum_{\substack{\{s_{t+1},l\}\in\overline{E}\\ l>s_{t+1}}} x_{i_{t+1}}^{\{s_{t+1},l\}}) \\ &= \cdots = a_{i_t i_{t+1}}^{\{s_t,s_{t+1}\}} + \sum_{\substack{i\in N_r\cap E\\ j\in N_s\cap E\\ \{r,s\}\in\overline{E}, r,s\in\hat{\mathcal{S}}_{s_{t+1}}^{\overline{E}}}} a_{ij}^{\{r,s\}} - \sum_{\substack{k\in N_r\cap E\\ r\in\hat{\mathcal{S}}_{s_{t+1}}^{\overline{E}}}} u_k^r. \end{aligned} \tag{10}$$

Hence, if $T = \{i_t\} \cup \{i \in E \mid i \in N_r,\ r \in \hat{\mathcal{S}}_{s_{t+1}}^{\overline{E}}\}$, we have

$$x_{i_t}^{\{s_t,s_{t+1}\}} = v(T) - u(T \setminus \{i_t\}). \tag{11}$$

Notice that for $t = 1$, because of efficiency of $u \in C(w_\gamma)$, we obtain

$$\sum_{\{s_1,l\}\in\overline{E}} x_{i_1}^{\{s_1,l\}} = v(E) - \sum_{\substack{k\in E\cap N_r\\ k\neq i_1}} u_k^r = u_{i_1}^{s_1}. \tag{12}$$

Equation (10), together with (9) gives, for all $t \in \{2,\ldots,r\}$,

$$\begin{aligned} y_{i_t}^{\{s_{t-1},s_t\}} &= u_{i_t}^{s_t} - \sum_{\substack{\{s_t,s_l\}\in\overline{E}\\ s_t<s_l}} x_{i_t}^{\{s_t,s_l\}} \\ &= u_{i_t}^{s_t} - \sum_{\substack{\{s_t,s_l\}\in\overline{E}\\ s_t<s_l}} \left(a_{i_t i_l}^{\{s_t,s_l\}} + \sum_{\substack{i\in N_r\cap E\\ j\in N_s\cap E\\ \{r,s\}\in\overline{E}, r,s\in\hat{\mathcal{S}}_{s_l}^{\overline{E}}}} a_{ij}^{\{r,s\}} - \sum_{\substack{k\in N_r\cap E\\ r\in\hat{\mathcal{S}}_{s_l}^{\overline{E}}}} u_k^r \right) \geq 0, \end{aligned}$$

where the inequality follows from the core constraint satisfied by $u \in C(w_\gamma)$ for coalition $T = \{i_t\} \cup \{i \in E \mid i \in N_r, r \in \mathcal{S}_{s_t}^{\overline{E}}\}$, that is, $y_{i_t}^{\{s_{t-1},s_t\}} = u(T) - v(T) \geq 0$.

Now, again making use of (4) to (12), we express $x_{i_t}^{\{s_t,s_{t+1}\}}$ in terms of the payoffs in u to agents in sectors that do not follow s_t in $\overline{E}$:

$$\begin{aligned} x_{i_t}^{\{s_t,s_{t+1}\}} &= u_{i_t}^{s_t} - y_{i_t}^{\{s_{t-1},s_t\}} - \sum_{\substack{\{s_t,l\}\in\overline{E}\\ l>s_t, l\neq s_{t+1}}} x_{i_t}^{\{s_t,l\}} \\ &= u_{i_t}^{s_t} - a_{i_{t-1}i_t}^{\{s_{t-1},s_t\}} + x_{i_{t-1}}^{\{s_{t-1},s_t\}} - \sum_{\substack{\{s_t,l\}\in\overline{E}\\ l>s_t, l\neq s_{t+1}}} x_{i_t}^{\{s_t,l\}} = \cdots \\ &= u_{i_t}^{s_t} - a_{i_{t-1}i_t}^{\{s_{t-1},s_t\}} + x_{i_{t-1}}^{\{s_{t-1},s_t\}} - \sum_{\substack{\{s_t,l\}\in\overline{E}\\ s_t<l\neq s_{t+1}}} (v(T_l) - u(T_l \setminus \{i_t\})), \end{aligned}$$

where $T_l = \{i_t\} \cup \{i \in E \mid i \in N_r,\ r \in \hat{\mathcal{S}}_l^{\overline{E}}\}$. Recursively applying the same argument (in first place to $x_{i_{t-1}}^{\{s_{t-1},s_t\}}$), we eventually obtain

$$x_{i_t}^{\{s_t,s_{t+1}\}} = u((T' \setminus T) \cup \{i_t\}) - v((T' \setminus T) \cup \{i_t\}) \geq 0,$$

with $T' = \{i_1\} \cup \{i \in E \mid i \in N_r, r \in \mathcal{S}_{s_1}^{\overline{E}}\}$, T as defined above, and where the inequality also follows from $u \in C(w_\gamma)$.

Once proved that for all $\{r,s\} \in \overline{G}$, $(x^{\{r,s\}}, y^{\{r,s\}})$ is a non-negative payoff vector, let us check it is in $C(w_{A^{\{r,s\}}})$. If $(i,j) \in \mu^{\{r,s\}}$ for some $\{r,s\} \in \overline{G}$, then i and j belong to the same basic coalition E of μ and $x_i^{\{r,s\}} + y_j^{\{r,s\}} = a_{ij}^{\{r,s\}}$ follows by definition from equations (5) and (7).

It only remains to prove that if $i \in N_r$, $j \in N_s$, with $\{r,s\} \in \overline{G}$, $r < s$, and $(i,j) \notin \mu^{\{r,s\}}$, then $x_i^{\{r,s\}} + y_j^{\{r,s\}} \geq a_{ij}^{\{r,s\}}$. Since i and j are not matched in $(N_r, N_s, A^{\{r,s\}})$, they belong to different basic coalitions in μ. Let E and E' be the basic coalitions containing i and j respectively. Let us consider a maximal path $\{s_1, s_2, \ldots, s_t, \ldots, s_p\}$ in $\overline{E}$ with origin in the node in $\overline{E}$ with the lowest label (that we will name the source of the subtree $\overline{E}$) and such that there exists $t \in \{1, \ldots, q\}$ with $r = s_t$. We write $i_1 \in E \cap N_{s_1}$. Similarly, let $\{s'_1, s'_2, \ldots, s'_l, \ldots, s'_p\}$ be the maximal path in $\overline{E'}$ with origin in the node in $\overline{E'}$ with the lowest label (the source) and such that there exists $l \in \{1, \ldots, p\}$ with $s = s'_l$.

Recall that, $y_j^{\{r,s\}} = u(R) - v(R)$, where $R = \{j\} \cup \{b \in E' \mid b \in N_k, k \in \mathcal{S}_{s'_l}^{\overline{E'}}\}$, and $x_i^{\{r,s\}} = u((T' \setminus T) \cup \{i\}) - v((T' \setminus T) \cup \{i\})$, where $T = \{i\} \cup \{b \in E \mid b \in N_k, k \in \mathcal{S}_{s_t}^{\overline{E}}\}$ and $T' = \{i_1\} \cup \{b \in E \mid b \in N_k, k \in \mathcal{S}_{s_1}^{\overline{E}}\}$. Since $E \cap E' = \emptyset$, $(T' \setminus T) \cup \{i\}$ and R are also disjoint. Then,

$$x_i^{\{r,s\}} + y_j^{\{r,s\}} = u((T' \setminus T) \cup \{i\}) + u(R) - v((T' \setminus T) \cup \{i\}) - v(R) \geq a_{ij}^{\{r,s\}}$$

since $v((T' \setminus T) \cup \{i\} \cup R) = v((T' \setminus T) \cup \{i\}) + v(R) + a_{ij}^{\{r,s\}}$ and $u \in C(w_\gamma)$. This completes the proof of $C(w_\gamma) = \bigoplus_{\{r,s\}\in\overline{G}} C(w_{A^{\{r,s\}}})$. □

The fact that the core of the multi-sided assignment game on an m-partite graph is completely described by the cores of all underlying two-sided markets allows us to deduce some properties of $C(w_\gamma)$ from the known properties of $C(w_{A^{\{r,s\}}})$, with $\{r,s\} \in \overline{G}$.

One of these consequences is that, for each sector $r \in \{1, 2, \dots, m\}$, there is a core element $u \in C(w_\gamma)$ where all agents in sector r simultaneously receive their maximum core payoff, which is their marginal contribution to the grand coalition. This is one property of two-sided assignment markets that does not extend to arbitrary multi-sided markets but it is preserved when sectors are connected by a tree and the value of basic coalitions is defined additively as in (1).

Proposition 6. *Let $\gamma = (N_1, N_2, \dots, N_m; G; \{A^{\{r,s\}}\}_{\{r,s\}\in\overline{G}})$ be a multi-sided assignment market on an m-partite graph. If $\overline{G}$ is a tree, then for each sector $k \in \{1, 2, \dots, m\}$ there exists $u \in C(w_\gamma)$ such that $u_i = w_\gamma(N) - w_\gamma(N \setminus \{i\})$ for all $i \in N_k$.*

Proof. Take any $k \in \{1, 2, \dots, m\}$. For all $s \in \{1, 2, \dots, m\}$ with $\{k, s\} \in \overline{G}$, take $(x^{\{k,s\}}, y^{\{k,s\}}) = (\overline{x}^{\{k,s\}}, \underline{y}^{\{k,s\}})$ the element of $C(w_{A^{\{k,s\}}})$ that is optimal for all agents in N_k. Similarly, for all $r \in \{1, 2, \dots, m\}$ such that $\{r, k\} \in \overline{G}$, take the element $(x^{\{r,k\}}, y^{\{r,k\}}) = (\underline{x}^{\{r,k\}}, \overline{y}^{\{r,k\}})$ of $C(w_{A^{\{r,k\}}})$ that is optimal for the agents in N_k. These optimal core elements exist in any bilateral assignment market (see Shapley and Shubik, 1972). Moreover, by Demange (1982) and Leonard (1983), it is known that for all $i \in N_k$, $\overline{x}_i^{\{k,s\}} = w_{A^{\{k,s\}}}(N_k \cup N_s) - w_{A^{\{k,s\}}}(N_k \cup N_s \setminus \{i\})$ and $\overline{y}_i^{\{r,k\}} = w_{A^{\{r,k\}}}(N_r \cup N_k) - w_{A^{\{r,k\}}}(N_r \cup N_k \setminus \{i\})$. Finally, for all $\{r, s\} \in \overline{G}$ with $r \neq k$ and $s \neq k$, take an arbitrary core element $(x^{\{r,s\}}, y^{\{r,s\}}) \in C(w_{A^{\{r,s\}}})$.

Now, if we consider the composition of the core elements defined above, we get, given $k \in \{1, 2, \dots, m\}$, $\overline{u}^k = \bigoplus_{\{r,s\}\in\overline{G}} (x^{\{r,s\}}, y^{\{r,s\}})$.

Then, for all $i \in N_k$, if $\{r, k\} \in \overline{G}$ with $r < k$,

$$\overline{u}_i^k = \overline{y}_i^{\{r,k\}} + \sum_{\substack{\{k,s\}\in\overline{G} \\ k<s}} \overline{x}_i^{\{k,s\}} \geq u_i$$

for all other $u \in C(w_\gamma)$, as a consequence of Theorem 5.

Moreover, if $k \in \{1, 2, \dots, m\}$ is such that there exists $r \in \{1, 2, \dots, m\}$ with $\{r, k\} \in \overline{G}$ and $r < k$, and there exists $s \in \{1, 2, \dots, m\}$ with $\{k, s\} \in \overline{G}$ and $k < s$, then

$$\begin{aligned} w_\gamma(N) - w_\gamma(N \setminus \{i\}) &= [w_{A^{\{r,k\}}}(N_r \cup N_k) - w_{A^{\{r,k\}}}(N_r \cup N_k \setminus \{i\})] \\ &+ \sum_{\substack{\{k,s\}\in\overline{G} \\ k<s}} [w_{A^{\{k,s\}}}(N_k \cup N_s) - w_{A^{\{k,s\}}}(N_k \cup N_s \setminus \{i\})] \\ &= \overline{u}_i^k, \end{aligned}$$

for all $i \in N_k$.

Similarly, if k is a leaf of $\overline{G}$, then

$$w_\gamma(N) - w_\gamma(N \setminus \{i\}) = w_{A^{\{r,k\}}}(N_r \cup N_k) - w_{A^{\{r,k\}}}(N_r \cup N_k \setminus \{i\}) = \overline{u}_i^k$$

for the only $r \in \{1, 2, \dots, m\}$ such that $\{r, k\} \in \overline{G}$ and for all $i \in N_k$. Also, if k is the source of the tree $\overline{G}$, then

$$w_\gamma(N) - w_\gamma(N \setminus \{i\}) = \sum_{\substack{\{k,s\}\in\overline{G} \\ k<s}} [w_{A^{\{k,s\}}}(N_k \cup N_s) - w_{A^{\{k,s\}}}(N_k \cup N_s \setminus \{i\})] = \overline{u}_i^k,$$

for all $i \in N_k$.

Then, for all $k \in \{1, 2, \dots, m\}$ we have $\overline{u}_i^k = w_\gamma(N) - w_\gamma(N \setminus i)$ for all $i \in N_k$. □

Once proved in Theorem 5 that for an assignment market on an m-partite graph with a cycle-free quotient graph $\overline{G}$ the core can be completely described from the cores of the two-sided markets between connected sectors, the question arises whether some other single-valued cooperative solutions of the market can be obtained in the same way.

As a first consequence we obtain that all extreme core allocations of the multi-sided assignment game are obtained as the composition of extreme core allocations of the underlying two-sided markets.

Proposition 7. *Let $\gamma = (N_1, N_2, \ldots, N_m; G; \{A^{\{r,s\}}\}_{\{r,s\}\in\overline{G}})$ be a multi-sided assignment market on an m-partite graph. If $\overline{G}$ is cycle-free, then any extreme core allocation $x \in Ext(C(w_\gamma))$ is the composition of extreme core allocations of the underlying two-sided markets, $x = \bigoplus_{\{r,s\}\in\overline{G}} x^{\{r,s\}}$, where $x^{\{r,s\}} \in Ext(C(w_{A^{\{r,s\}}}))$ for all $\{r,s\} \in \overline{G}$.*

Proof. From Theorem 5, it is straightforward to see that $x \in Ext(C(w_\gamma))$ satisfies $x = \bigoplus_{\{r,s\}\in\overline{G}} x^{\{r,s\}}$ with $x^{\{r,s\}} \in C(w_{A^{\{r,s\}}})$. Assume now that $x^{\{r',s'\}} \notin Ext(C(w_{A^{\{r',s'\}}}))$ for some $\{r',s'\} \in \overline{G}$. Then, there exist two different elements, $y^{\{r',s'\}}$ and $z^{\{r',s'\}}$, in $C(w_{A^{\{r',s'\}}})$ such that $x^{\{r',s'\}} = \frac{1}{2}y^{\{r',s'\}} + \frac{1}{2}z^{\{r',s'\}}$.

We now consider two different elements in $C(w_\gamma)$ by composing $\bigoplus_{\substack{\{r,s\}\in\overline{G}\\ \{r,s\}\neq\{r',s'\}}} x^{\{r,s\}}$ either with $y^{\{r',s'\}}$ or $z^{\{r',s'\}}$,

$$x^y = \left(\bigoplus_{\substack{\{r,s\}\in\overline{G}\\ \{r,s\}\neq\{r',s'\}}} x^{\{r,s\}} \right) \oplus y^{\{r',s'\}} \text{ and } x^z = \left(\bigoplus_{\substack{\{r,s\}\in\overline{G}\\ \{r,s\}\neq\{r',s'\}}} x^{\{r,s\}} \right) \oplus z^{\{r',s'\}}.$$

It is then straightforward to check that $x = \frac{1}{2}x^y + \frac{1}{2}x^z$, which contradicts the assumption $x \in Ext(C(w_\gamma))$. □

However, the converse implication does not hold, that is, the composition of extreme core allocations of the underlying two-sided markets provides an element in $C(w_\gamma)$ which may not be an extreme point (see Example 14 in the Appendix).

We now consider single-valued core selections that are not extreme points but usually interior core points. As a consequence of Theorem 5, the composition $\eta^{\oplus}(w_\gamma) = \oplus_{\{r,s\}\in\overline{G}}\eta(w_{A^{\{r,s\}}})$ of the nucleolus[6] of the two-sided markets between connected sectors belongs to $C(w_\gamma)$. Moreover, well-known algorithms to compute the nucleolus of a two-sided assignment game (Solymosi and Raghavan, 1994; Martínez de Albéniz et al., 2014) can be used to obtain $\eta^{\oplus}(w_\gamma)$. However, this composition does not coincide with the nucleolus of the initial m-sided market $\gamma = (N_1, N_2, \ldots, N_m; G; \{A^{\{r,s\}}\}_{\{r,s\}\in\overline{G}})$, as Example 13 in the Appendix shows.

If we select the τ-value or fair-division point[7] as the cooperative solution concept to distribute the profits in each bilateral market, we can propose the composition

[6]The nucleolus of a coalitional game (N, v) is the payoff vector $\eta(v) \in \mathbb{R}^N$ that lexicographically minimizes the vector of decreasingly ordered excesses of coalitions among all possible imputations (Schmeidler, 1969). An imputation for the game (N, v) is a payoff vector $x \in \mathbb{R}^N$ that satisfies $\sum_{i\in N} x_i = v(N)$ and $x_i \geq v(\{i\})$ for all $i \in N$. The excess of coalition $S \subseteq N$ at $x \in \mathbb{R}^N$ is $v(S) - \sum_{i\in S} x_i$.

[7]The fair-division point of a two-sided assignment market is the midpoint of the buyers-optimal and the sellers-optimal core allocations Thompson (1981), and it coincides with the τ-value of the corresponding assignment game (Núñez and Rafels, 2002).

of the τ-values of all connected two-sided markets, $\tau^{\oplus}(w_\gamma) = \oplus_{\{r,s\}\in\overline{G}}\tau(w_{A^{\{r,s\}}})$ as an allocation of the profit of the multi-sided assignment market with a tree quotient graph. Because of Theorem 5, this composition belongs to $C(w_\gamma)$ and can be considered as a fair division solution for the m-sided market. However, different to the two-sided case, it may not coincide with the τ-value of the initial m-sided market $\gamma = (N_1, N_2, \ldots, N_m; G; \{A^{\{r,s\}}\}_{\{r,s\}\in\overline{G}})$. In fact, the τ-value of a multi-sided assignment market on an m-partite graph may lie outside the core (see Example 12 in the Appendix), even when the quotient graph $\overline{G}$ is cycle-free.

5 Core and competitive prices in a market network

The aim of this section is to extend to multi-sided assignment games on an m-partite graph the equivalence between core and competitive equilibria that Shapley and Shubik (1972) prove for two-sided markets. To introduce prices and payments, we need to assign some roles of buyers and/or sellers to the agents in the network.

Consider now m sectors $N_1, N_2, \ldots, N_m$ connected by a tree $\overline{G}$ and assume that the source is at N_1. Let us denote by $L(\overline{G})$ the set of leaves of this tree, and by N_L the agents in these leaves. Each agent i in a sector $r \neq 1$ offers an object on sale and has a reservation value $c_i \geq 0$ for this object, meaning that he/she will not sell below that value. We denote by c the vector of sellers' reservation values. At the same time, each agent $i \in N_r$, with $r \notin L(\overline{G})$, is willing to buy one object from each sector $s > r$ such that $\{s, r\} \in \overline{G}$. Assume this agent $i \in N_r$ places a value of $w^i_j \geq 0$ on the object of agent $j \in N_s$ with $s > r$ and $\{r, s\} \in \overline{G}$, and we denote w^i the vectors of buyer i's valuations and by w the vector of all buyers' valuations. Notice that each agent can sell at most one object and buy several objects but at most one from the same sector.

This situation may represent a market network in which each agent at an intermediate sector acts independently both as a buyer in the downstream (higher labels) direction and as a seller in the upstream direction, and pulls together the payoffs obtained in both transactions. We assume all these transactions are independent, that is, an agent can sell an item even if he/she is unmatched in the markets where he/she acts as a buyer. That is, the basic coalitions are, as before, those coalitions connected by G and with no two agents belonging to the same sector. Recall we do not require all sectors to be present in each basic coalition.

These valuations (w, c) give rise to a multi-sided assignment market $\gamma = (N_1, N_2, ..., N_m; G; \{A^{r,s}\}_{\{r,s\}\in\overline{G}})$ on an m-partite graph G with a tree quotient graph $\overline{G}$, where for all $r, s \in \{1, 2, \ldots, m\}$, with $\{r, s\} \in \overline{G}$, $a^{\{r,s\}}_{ij} = \max\{0, w^i_j - c_j\}$ for all $i \in N_r$ and $j \in N_s$. We will then simply denote the market by $\gamma = (N_1, N_2, ..., N_m; G; w, c)$. Then, the value of a basic coalition E is

$$v^{w,c}(E) = \sum_{\substack{i\in E\cap N_r, j\in E\cap N_s \\ r<s, \{r,s\}\in\overline{G}}} \max\{0, w^i_j - c_j\},$$

and from this valuation function the coalitional game (N, w_γ) is defined as in (2).

Let us denote by $\mathcal{B}_k$ those basic coalitions containing buyer $k \in N_r$, for some $r \in \{1, 2, \ldots, m\}$, and only sellers in sectors that immediatly follow r. We refer to these coalitions as *k-basic coalitions*. That is,

$$\mathcal{B}_k = \{E \in \mathcal{B}^N \mid k \in E \cap N_r \text{ and } (E \setminus \{k\}) \cap N_t = \emptyset, \text{ for all } t \in \{1, 2, \dots, m\} \setminus \mathcal{I}_r^{\overline{G}}\}.$$

Recall that $\mathcal{I}_r^{\overline{G}}$ is the set of the immediate followers of r

$$\mathcal{I}_r^{\overline{G}} = \{s \in \{1, 2, \dots, m\} \mid s > r, \{r, s\} \in \overline{G}\}.$$

We want to show that each core allocation is obtained as the result of trading at competitive prices. Therefore, we need to introduce some previous definitions in order to define the notion of competitive price vector.

Given a multi-sided assignment market $\gamma = (N_1, N_2, ..., N_m; G; w, c)$ where $\overline{G}$ is a tree with source at N_1, a *feasible price vector* is $p \in \mathbb{R}_+^{N_2 \cup N_3 \cup \ldots \cup N_m}$ such that $p_j \geq c_j$ for all $j \in \bigcup_{l=2}^{m} N_l$. Next, for each feasible price vector $p \in \mathbb{R}_+^{N_2 \cup N_3 \cup \ldots \cup N_m}$ we introduce the *demand set* of each buyer $k \in N_r$, with $r \in \{1, 2, \dots, m\} \setminus L(\overline{G})$.

Definition 8. Let $\gamma = (N_1, N_2, ..., N_m; G; w, c)$ be a multi-sided assignment market where $\overline{G}$ is a tree with source at N_1. The *demand set* of a buyer $k \in N_r$, $r \in \{1, 2, \dots, m\} \setminus L(\overline{G})$, at a feasible price vector $p \in \mathbb{R}_+^{N_2 \cup N_3 \cup \ldots \cup N_m}$ is

$$\begin{aligned} D_k(p) = \{E \in \mathcal{B}_k \mid\, & w^k(E \setminus \{k\}) - p(E \setminus \{k\}) \\ & \geq w^k(E' \setminus \{k\}) - p(E' \setminus \{k\}), \forall E' \in \mathcal{B}_k\}, \end{aligned} \tag{13}$$

where for all coalition T of sellers, $w^k(T) = \sum_{j \in T} w_j^k$ and $p(T) = \sum_{j \in T} p_j$.

Note that $D_k(p)$ describes the set of k-basic coalitions that maximize the net valuation of buyer k at prices p. Notice also that the demand set of a buyer $k \in N_r$, for some $r \in \{1, 2, \dots, m\} \setminus L(\overline{G})$, is always non-empty since k can always demand $E = \{k\}$ with a net profit of 0. If $\mu \in \mathcal{M}(N_1, N_2, ..., N_m)$, for all $k \in N \setminus N_L$ we write $\mu(k)$ to denote the k-basic coalition E such that $k \in E \subseteq E' \in \mu$, that is, $\mu(k) = \{E \in \mathcal{B}_k \mid \text{ there exists } E' \in \mu \text{ such that } E \subseteq E'\}$. Notice that $\mu(k)$ may consist of only agent k, that meaning that k is not matched to any of his/her immediately follower sellers.

Given a matching $\mu \in \mathcal{M}(N_1, N_2, ..., N_m)$, we say a seller $j \in \bigcup_{l=2}^{m} N_l$ is *unassigned* (by μ) if there is no $k \in N_r$ for some $r \in \{1, 2, \dots, m\} \setminus L(\overline{G})$ such that $j \in \mu(k)$.

Now, we can introduce the notion of *competitive equilibrium* for the market γ on an m-partite graph where $\overline{G}$ is a tree.

Definition 9. Let $\gamma = (N_1, N_2, ..., N_m; G; w, c)$ be a multi-sided assignment market on an m-partite graph, where $\overline{G}$ is a tree with source at N_1. A pair (p, μ), where $p \in \mathbb{R}_+^{N_2 \cup N_3 \cup \ldots \cup N_m}$ is a feasible price vector and $\mu \in \mathcal{M}(N_1, N_2, ..., N_m)$, is a *competitive equilibrium* if

(i) for all buyer $k \in N_r$, $r \in \{1, 2, \dots, m\} \setminus L(\overline{G})$, it holds $\mu(k) \in D_k(p)$,

(ii) for all seller $j \in \bigcup_{l=2}^{m} N_l$, if j is unassigned by μ, then $p_j = c_j$.

If a pair (p,μ) is a competitive equilibrium, then we say that the price vector p is a *competitive equilibrium price vector*. The corresponding payoff vector for a given pair (p,μ) is called *competitive equilibrium payoff vector*. This payoff vector is $(u^1(p,\mu), u^2(p,\mu), ..., u^m(p,\mu)) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times ... \times \mathbb{R}^{N_m}$, defined by

$$
\begin{aligned}
u_k^1(p,\mu) &= w^k(\mu(k)\setminus\{k\}) - p(\mu(k)\setminus\{k\}) \quad \text{for all } k \in N_1,\\
u_k^r(p,\mu) &= w^k(\mu(k)\setminus\{k\}) - p(\mu(k)\setminus\{k\}) + p_k - c_k\\
&\quad \text{for all } k \in N_r,\ r \in \{2,\dots,m\}\setminus L(\overline{G}),\\
u_k^r(p,\mu) &= p_k - c_k \quad \text{for all } k \in N_r, r \in L(\overline{G}),
\end{aligned}
$$

where for all coalition T of sellers, $w^k(T) = \sum_{j\in T} w_j^k$ and $p(T) = \sum_{j\in T} p_j$. We denote the set of competitive equilibrium payoff vectors of market γ by $\mathcal{CE}(\gamma)$.

We now study the relationship between the core of a multi-sided assignment market on an m-partite graph $\gamma = (N_1, N_2, ..., N_m; G; w, c)$ where $\overline{G}$ is a tree, and the set of competitive equilibrium payoff vectors. First, we show that if a matching μ constitutes a competitive equilibrium with a feasible price vector p, then μ is an optimal matching.

Lemma 10. *Let $\gamma = (N_1, N_2, ..., N_m; G; w, c)$ be a multi-sided assignment market on an m-partite graph. If $\overline{G}$ is a tree and the pair (p,μ) is a competitive equilibrium, then μ is an optimal matching.*

Proof. Let us assume that $\overline{G}$ has a source at N_1, and hence competitive equilibria are defined as in Definition 9. In order to prove the statement, we need to show that if (p,μ) is a competitive equilibrium, then the matching μ is a partition of maximal value. We can assume without loss of generality that for all $E \in \mu$, if $i \in E \cap N_r$, $j \in E \cap N_s$ with $r < s$ and $\{r,s\} \in \overline{G}$, then it holds $w_j^i - c_j \geq 0$ and hence $v^{w,c}(E) = \sum_{\substack{i\in E\cap N_r, j\in E\cap N_s\\ r<s,\{r,s\}\in\overline{G}}} w_j^i - c_j$, since otherwise E could be partitioned in basic coalitions satisfying the above condition to obtain another matching that gives rise to the same value. Consider now another matching $\mu' \in \mathcal{M}(N_1, N_2, ..., N_m)$. Then,

$$
\begin{aligned}
\sum_{E\in\mu} v^{w,c}(E) &= \sum_{k\in N\setminus N_L} w^k(\mu(k)\setminus\{k\}) - c(\mu(k)\setminus\{k\})\\
&\geq \sum_{k\in N\setminus N_L} w^k(\mu'(k)\setminus\{k\}) - c(\mu(k)\setminus\{k\}) - p(\mu'(k)\setminus\{k\}) + p(\mu(k)\setminus\{k\})\\
&= \sum_{k\in N\setminus N_L} w^k(\mu'(k)\setminus\{k\}) - c(\mu(k)\setminus\{k\})\\
&- p\left(\bigcup_{k\in N\setminus N_L} \mu'(k)\setminus N_1\right)\\
&+ p\left(\bigcup_{k\in N\setminus N_L} \mu(k)\setminus N_1\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k\in N\setminus N_L} w^k(\mu'(k)\setminus\{k\}) - c\left(\bigcup_{k\in N\setminus N_L}\mu(k)\setminus N_1\right) \\
&- p\left(\left(\bigcup_{k\in N\setminus N_L}\mu'(k)\setminus\bigcup_{k\in N\setminus N_L}\mu(k)\right)\setminus N_1\right) \\
&+ p\left(\left(\bigcup_{k\in N\setminus N_L}\mu(k)\setminus\bigcup_{k\in N\setminus N_L}\mu'(k)\right)\setminus N_1\right) \\
&= \sum_{k\in N\setminus N_L} w^k(\mu'(k)\setminus\{k\}) - c\left(\bigcup_{k\in N\setminus N_L}\mu(k)\setminus N_1\right) \\
&- c\left(\left(\bigcup_{k\in N\setminus N_L}\mu'(k)\setminus\bigcup_{k\in N\setminus N_L}\mu(k)\right)\setminus N_1\right) \\
&+ p\left(\left(\bigcup_{k\in N\setminus N_L}\mu(k)\setminus\bigcup_{k\in N\setminus N_L}\mu'(k)\right)\setminus N_1\right) \\
&= \sum_{k\in N\setminus N_L} w^k(\mu'(k)\setminus\{k\}) - c\left(\bigcup_{k\in N\setminus N_L}\mu'(k)\setminus N_1\right) \\
&- c\left(\left(\bigcup_{k\in N\setminus N_L}\mu(k)\setminus\bigcup_{k\in N\setminus N_L}\mu'(k)\right)\setminus N_1\right) \\
&+ p\left(\left(\bigcup_{k\in N\setminus N_L}\mu(k)\setminus\bigcup_{k\in N\setminus N_L}\mu'(k)\right)\setminus N_1\right) \\
&\geq \sum_{k\in\setminus N_L} w^k(\mu'(k)\setminus\{k\}) - c(\mu'(k)\setminus\{k\}) = \sum_{E\in\mu'} v^{w,c}(E),
\end{aligned}
$$

where the first inequality follows from the definition of the demand set and the fact that (p,μ) is a competitive equilibrium:

$$w^k(\mu(k)\setminus\{k\}) \geq w^k(\mu'(k)\setminus\{k\}) - p(\mu'(k)\setminus\{k\}) + p(\mu(k)\setminus\{k\}).$$

The fourth equality follows from the fact that for all

$$j \in \left(\bigcup_{k\in N\setminus N_L}\mu'(k)\setminus\bigcup_{k\in N\setminus N_L}\mu(k)\right)\setminus N_1, \qquad p_j = c_j,$$

and the last inequality follows from the feasibility of the price vector p. $\square$

Now, we can give the main result in this section.

Theorem 11. *Let $\gamma = (N_1, N_2, ..., N_m; G; w, c)$ be a multi-sided assignment market on an m-partite graph, where $\overline{G}$ is a tree. The core of the market, $C(w_\gamma)$, coincides with the set of competitive equilibrium payoff vectors, $\mathcal{CE}(\gamma)$.*

Proof. Assume that $\overline{G}$ has a source at N_1, and hence competitive equilibria are defined as in Definition 9. First, we show the implication that states if (p, μ) is a competitive equilibrium, then its corresponding competitive equilibrium payoff vector $x = (u^1(p, \mu), u^2(p, \mu), ..., u^m(p, \mu)) \in \mathcal{CE}(\gamma)$ is a core element. As in the proof of Lemma 10, we can assume without loss of generality that $v^{w,c}(E) = \sum_{\substack{i \in E \cap N_r, j \in E \cap N_s \\ r<s, \{r,s\} \in \overline{G}}} (w^i_j - c_j)$.

Recall that by definition $u^r_k(p, \mu) = w^k(\mu(k) \setminus \{k\}) - p(\mu(k) \setminus \{k\}) + p_k - c_k$ for all $k \in N_r$ for $r \in \{1, 2, \ldots, m\} \setminus L(\overline{G})$. Let us check that for all basic coalitions $E \in \mathcal{B}^N$ it holds $x(E) \geq v^{w,c}(E)$. Notice that if E only contains one agent, then $v^{w,c}(E) = 0$ and hence the core inequality holds. Otherwise, take $E \in \mathcal{B}^N$ such that $k \in E$ for some $k \in N_r$ with $r \in \{1, 2, \ldots, m\} \setminus L(\overline{G})$. For each $k \in E \cap N_r$ with $r \in \{1, 2, \ldots, m\} \setminus L(\overline{G})$, denote by E_k the union of $\{k\}$ with the set of $j \in E \cap N_s$ for some $r < s$ with $\{r, s\} \in \overline{G}$. Notice that E_k is formed by agent k and those of his immediate followers that belong to E. Then,

$$\begin{aligned}
x(E) &= p(E \setminus N_1) - c(E \setminus N_1) + \sum_{r=1}^{m} \sum_{\substack{k \in E \cap N_r \\ r \notin L(\overline{G})}} w^k(\mu(k) \setminus \{k\}) - p(\mu(k) \setminus \{k\}) \\
&\geq p(E \setminus N_1) - c(E \setminus N_1) + \sum_{r=1}^{m} \sum_{\substack{k \in E \cap N_r \\ r \notin L(\overline{G})}} w^k(E_k \setminus \{k\}) - p(E_k \setminus \{k\}) \\
&= \sum_{\substack{k \in E \cap N_r, j \in E \cap N_s \\ r<s, \{r,s\} \in \overline{G}}} w^k_j - c_j = v^{w,c}(E),
\end{aligned}$$

where the inequality follows from the fact that (p, μ) is a competitive equilibrium. It remains to check that x is efficient. Since μ is a partition of $N = N_1 \cup N_2 \cup ... \cup N_m$, we get

$$\begin{aligned}
x(N) &= \sum_{r=1}^{m} \sum_{\substack{k \in N_r \\ r \notin L(\overline{G})}} \left[w^k(\mu(k) \setminus \{k\}) - p(\mu(k) \setminus \{k\}) \right] + p(N \setminus N_1) - c(N \setminus N_1) \\
&= \sum_{r=1}^{m} \sum_{\substack{k \in N_r \\ r \notin L(\overline{G})}} \left[w^k(\mu(k) \setminus \{k\}) - p(\mu(k) \setminus \{k\}) + p(\mu(k) \setminus \{k\}) - c(\mu(k) \setminus \{k\}) \right] \\
&+ \sum_{s=2}^{m} \sum_{\substack{q \in N_s, q \notin \mu(k) \\ \forall k \in N \setminus N_L}} (p_q - c_q) \\
&= \sum_{r=1}^{m} \sum_{\substack{k \in N_r \\ r \notin L(\overline{G})}} \left[w^k(\mu(k) \setminus \{k\}) - c(\mu(k) \setminus \{k\}) \right]
\end{aligned}$$

$$= \sum_{r=1}^{m} \sum_{\substack{k\in N_r \\ r\notin L(\overline{G})}} v^{w,c}(\mu(k)\setminus\{k\}) = \sum_{E\in\mu} v^{w,c}(E),$$

where the third equality holds since $p_q = c_q$ for unassigned sellers q.

We have shown that if (p,μ) is a competitive equilibrium, then its competitive equilibrium payoff vector x is a core allocation.

Next, we show that the reverse implication holds. That is, if $x \in \mathbb{R}^N$ is a core allocation, then it is the payoff vector related to a competitive equilibrium (p,μ), where μ is any optimal matching and p is a competitive equilibrium price vector. Recall from Theorem 5 that $x = \bigoplus_{\{r,s\}\in\overline{G}} x^{\{r,s\}}$, where $x^{\{r,s\}} \in C(w_{A^{\{r,s\}}})$.

For all $s \in \{2,\dots,m\}$ and all $j \in N_s$, define $p_j = x_j^{\{r,s\}} + c_j$, where r is the unique sector in $\{1,2,\dots,m\}$ such that $\{r,s\} \in \overline{G}$. Notice first that since $x \in C(w_{A^{\{r,s\}}})$, if an object $j \in N_s$ is not assigned by the matching μ to any $k \in N_r$, then $p_j = x_j + c_j = c_j$. Moreover, $x^{\{r,s\}}(\mu(k)) = v^{w,c}(\mu(k))$ for all $k \in N\setminus N_L$ and $x^{\{r,s\}}(E') \geq v^{w,c}(E')$ for all $E' \in \mathcal{B}_k$ where $k \in N \setminus N_L$.

Then, for all $k \in N\setminus N_L$ and all $E' \in \mathcal{B}_k$,

$$\begin{aligned}
w^k(\mu(k)\setminus\{k\}) - p(\mu(k)\setminus\{k\}) &= v^{w,c}(\mu(k)) + c(\mu(k)\setminus\{k\}) - p(\mu(k)\setminus\{k\}) \\
&= x^{\{r,s\}}(\mu(k)) + c(\mu(k)\setminus\{k\}) - p(\mu(k)\setminus\{k\}) \\
&= x_k^{\{r,s\}} \\
&\geq v^{w,c}(E') - x^{\{r,s\}}(E'\setminus\{k\}) \\
&= v^{w,c}(E') - \left[p(E'\setminus\{k\}) - c(E'\setminus\{k\})\right] \\
&= w^k(E'\setminus\{k\}) - p(E'\setminus\{k\})
\end{aligned}$$

where the inequality follows from the fact that $x \in C(w_{A^{\{r,s\}}})$. This shows that $\mu(k) \in D_k(p)$ which concludes the proof. □

Once shown that the set of competitive equilibrium payoff vectors of a multi-sided assignment market on a cycle-free quotient graph $\overline{G}$, $\mathcal{CE}(\gamma)$, coincides with the core of the market, $C(w_\gamma)$, we have that a competitive equilibrium always exists for this model, since we already know that the core is non-empty.

6 Concluding remarks

We have considered multi-sided markets where agents are on an m-partite graph that induces a cycle-free network among the sectors. Basic coalitions do not need to have agents from all sectors, it is enough not to have two agents from the same sector. Moreover, the worth of a basic coalition is the addition of the worths of all its pairs that are an edge of the m-partite graph.

A similar situation is considered in Stuart (1997), although restricted to the case in which the network that connects the sectors is a chain. There, the worth of a basic coalition is also defined additively, but, as in the classical multi-sided assignment games in Kaneko and Wooders (1982) and Quint (1991), the set of basic coalitions is smaller since it is required that a basic coalition contains exactly one agent of each side. Although the core of Stuart's multi-sided game is also non-empty, it does not contain the composition of all core elements of the underlying two sided markets.

Indeed, take $N_1 = \{1, 2, 3\}$, $N_2 = \{1', 2', 3'\}$ and $N_3 = \{1'', 2''\}$, and consider the chain $\overline{G} = \{\{N_1, N_2\}, \{N_2, N_3\}\}$. Assume also that $a_{ij}^{\{r,s\}} = 1$ for all $(i, j) \in N_r \times N_s$ such that $\{N_r, N_s\} \in \overline{G}$, but, unlike the model we present in this chapter, only triplets may have a positive value. It is easy to see that $(0.5, 0.5, 0.5; 0.5, 0.5, 0.5) \in C(w_{A^{\{1,2\}}})$ and $(0, 0, 0; 1, 1) \in C(w_{A^{\{2,3\}}})$. However,

$$z = x \oplus y = (0.5, 0.5, 0.5; 0.5, 0.5, 0.5; 1, 1) \notin C(w_\gamma),$$

since an optimal matching consists of two triplets and hence the unassigned agents in sectors N_1 and N_2 can only receive zero payoff in the core. Hence, our generalized multi-sided markets, together with the cycle-free network structure, where the set of basic coalitions has been enlarged, better inherits some properties of the core of the well-known two-sided markets.

A final remark on the computation of an optimal matching for multi-sided assignment markets is appropriate. Although the solution of the two-sided assignment problem is solvable in polynomial time, the solution of its multi-sided extension is NP-hard (see Garey and Johson, 1979). However, for a multi-sided assignment market on an m-partite graph where the quotient graph that connects the sectors is cycle-free, an optimal matching is computed in polynomial time. Indeed, from our Theorem 5 it follows that the composition of optimal matchings of each underlying two-sided market yields an optimal matching of the multi-sided assignment market. Since in a market with m sectors any tree connecting the sectors has $m - 1$ edges, we have $m - 1$ underlying two-sided markets and we only need to solve $m - 1$ linear programs to build an optimal matching for the multi-sided market.

A Appendix

We consign to this appendix two examples that show that for a multi-sided assignment game on a cycle-free quotient graph, the composition of the τ-values (or the nucleolus) of each underlying two-sided market may not coincide with the τ-value or the nucleolus of the initial multi-sided market. Similarly, the third example shows that by composition of arbitrary extreme core allocations of each two-sided market we may not obtain an extreme core allocation of the muti-sided market.

Example 12. Let us consider an assignment market γ on a 3-partite graph such that the quotient graph is $\overline{G} = \{\{1, 2\}, \{2, 3\}\}$ which is cycle-free. The sectors are $N_1 = \{1, 2\}$, $N_2 = \{1', 2'\}$, and $N_3 = \{1'', 2''\}$. The valuation matrices of the two underlying two-sided markets are

$$\begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1' & 2' \end{array} \\ \begin{pmatrix} 2 & 0 \\ 5 & 4 \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ 1' \\ 2' \end{array} \begin{array}{c} \begin{array}{cc} 1'' & 2'' \end{array} \\ \begin{pmatrix} 3 & 4 \\ 0 & 3 \end{pmatrix} \end{array},$$

and the value of triplets is given by the following three-dimensional matrix

$$\begin{array}{c} \\ 1 \\ 2 \\ \\ \end{array} \begin{array}{c} 1' \quad 2' \\ \begin{pmatrix} 5 & 0 \\ 8 & 4 \end{pmatrix} \\ 1'' \end{array} \qquad \begin{array}{c} \\ 1 \\ 2 \\ \\ \end{array} \begin{array}{c} 1' \quad 2' \\ \begin{pmatrix} 6 & 3 \\ 9 & 7 \end{pmatrix} \\ 2'' \end{array}.$$

The τ-value of this multi-sided market game is $\tau(\gamma) = (\frac{5}{9}, \frac{24}{9}; \frac{29}{9}, \frac{15}{9}; \frac{15}{9}, \frac{20}{9})$ which is not in the core. Hence, $\tau(\gamma)$ cannot coincide with $\tau(w_{A^{\{1,2\}}}) \oplus \tau(w_{A^{\{2,3\}}})$.

Example 13. Let us consider an assignment market γ on the following 4-partite graph related to the the quotient graph $\overline{G} = \{\{1,2\},\{2,3\},\{2,4\}\}$ which is cycle free. The sectors are $N_1 = \{1,2\}$, $N_2 = \{1',2'\}$, $N_3 = \{1'',2''\}$, $N_4 = \{1''',2'''\}$, and the valuation matrices of the two-sided markets are

$$A^{\{1,2\}} = \begin{pmatrix} \mathbf{2} & 3 \\ 0.5 & \mathbf{2} \end{pmatrix}, \quad A^{\{2,3\}} = \begin{pmatrix} \mathbf{3} & 0.8 \\ 4 & \mathbf{2} \end{pmatrix} \text{ and } A^{\{2,4\}} = \begin{pmatrix} \mathbf{2} & 0.6 \\ 2.4 & \mathbf{2} \end{pmatrix}.$$

The nucleolus of the three underlying two-sided markets are

$$\eta^{\{1,2\}} = (1.625, 0.375; 0.375, 1.625), \quad \eta^{\{2,3\}} = (0.45, 1.55; 2.55, 0.45)$$
$$\text{and } \eta^{\{2,4\}} = (0.55, 1.45; 1.45, 0.55)$$

and their composition is

$$\eta^{\oplus} = (1.625, 0.375; 1.375, 4.625; 2.55, 0.45; 1.45, 0.55),$$

while the nucleolus of the six-player game (N, w_γ) can be computed and is

$$\eta = (1.65, 0.4; 1.6, 4.75; 2.55, 0.45; 1.2, 0.4).$$

Example 14. Let us consider an assignment market γ on a 4-partite graph related to the quotient graph $\overline{G} = \{\{1,2\},\{2,3\},\{2,4\}\}$ which is cycle-free. The sectors are $N_1 = \{1,2\}$, $N_2 = \{1',2'\}$, $N_3 = \{1'',2''\}$, and $N_4 = \{1''',2'''\}$. The valuation matrices of the three underlying two-sided markets are

$$A^{\{1,2\}} = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} 1' \quad 2' \\ \begin{pmatrix} \mathbf{2} & 0 \\ 1 & \mathbf{2} \end{pmatrix} \end{array} \qquad A^{\{2,3\}} = \begin{array}{c} \\ 1' \\ 2' \end{array} \begin{array}{c} 1'' \quad 2'' \\ \begin{pmatrix} \mathbf{2} & 1 \\ 0 & \mathbf{2} \end{pmatrix} \end{array} \qquad A^{\{2,4\}} = \begin{array}{c} \\ 1' \\ 2' \end{array} \begin{array}{c} 1''' \quad 2''' \\ \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} \end{array}.$$

Take respective extreme core allocations of the three underlying two-sided markets $A^{\{1,2\}}$, $A^{\{2,3\}}$, and $A^{\{2,4\}}$: $(2,1;0,1)$, $(2,0;0,2)$, and $(1,0;0,1)$. Then, by composition we get a core allocation for the multi-sided assignment market, $x^{\oplus} = (2,1;3,1;0,2;0,1) \in C(w_\gamma)$. But, there exist two core elements

$$y = (1.8, 0.8; 3.2, 1.2; 0, 2; 0, 1) \in C(w_\gamma)$$

and

$$z = (2.2, 1.2; 2.8, 0.8; 0, 2; 0, 1) \in C(w_\gamma)$$

such that $x^{\oplus} = \frac{1}{2}y + \frac{1}{2}z$. Hence, $x^{\oplus} \notin Ext(C(w_\gamma))$.

This last example shows that assumptions of Proposition 4 are not necessary for the non-emptiness of the core. In this example, the core of the multi-sided assignment game is non-empty but the matching induced on one two-sided market is not optimal.

Example 15. Consider an assignment market γ on the complete 3-partite graph related to the quotient graph $\overline{G} = \{\{1,2\},\{2,3\},\{1,3\}\}$. The sectors are $N_1 = \{1,2\}$, $N_2 = \{1',2'\}$, $N_3 = \{1'',2''\}$, and the valuation matrices of the three underlying two-sided markets are

$$A^{\{1,2\}} = \begin{matrix} & 1' & 2' \\ 1 & \mathbf{2} & 0 \\ 2 & 0 & \mathbf{1} \end{matrix} \qquad A^{\{1,3\}} = \begin{matrix} & 1'' & 2'' \\ 1 & 0 & \mathbf{1} \\ 2 & \mathbf{1} & 0 \end{matrix} \qquad A^{\{2,3\}} = \begin{matrix} & 1'' & 2'' \\ 1' & \mathbf{1} & 0 \\ 2' & 0 & \mathbf{2} \end{matrix},$$

and the value of triplets is given by the following three-dimensional matrix

$$\underset{1''}{\begin{matrix} & 1' & 2' \\ 1 & \mathbf{3} & 0 \\ 2 & 2 & 2 \end{matrix}} \qquad \underset{2''}{\begin{matrix} & 1' & 2' \\ 1 & 3 & 3 \\ 2 & 0 & \mathbf{3} \end{matrix}}.$$

Note that there is only one optimal matching for the market γ. That is, $\mu = \{(1,1',1''),(2,2',2'')\}$. Notice also that $x = (1,1;1,1;1,1,1)$ satisfies core constraints. Hence, the core of the market γ, $C(\gamma)$, is non-empty. Now, if we decompose the optimal matching μ for the market γ, we observe that for the underlying two-sided market $(N_1, N_3, A^{\{1,3\}})$ it induces a matching $\mu^{13} = \{(1,1''),(2,2'')\}$ which is not optimal.

References

Atay, A., F. Llerena, and M. Núñez (2016), "Generalized three-sided assignment markets: core consistency and competitive prices." *TOP*, 24, 572–593.

Demange, G. (1982), "Strategyproofness in the assignment market game." *Laboratorie d'Économétrie de l'École Polytechnique, Mimeo, Paris.*

Garey, M. R. and D. S Johson (1979), *Computers and Intractability: A Guide to the Theory of NP-completeness.* W. H. Freeman, New York.

González-Arangüena, E., C. M. Manuel, and M. del Pozo (2015), "Values of games with weighted graphs." *European Journal of Operational Research*, 243, 248–257.

Kaneko, M. and M. Wooders (1982), "The core of partitioning games." *Mathematical Social Sciences*, 3, 313–327.

Khmelnitskaya, A. and R. Talman (2014), "Tree, web and average web values for cycle-free directed graph games." *European Journal of Operational Research*, 235, 233–246.

Le Breton, M., G. Owen, and S. Weber (1992), "Strongly balanced cooperative games." *International Journal of Game Theory*, 20, 419–427.

Leonard, H. B. (1983), "Elicitation of honest preferences for the assignment of individuals to positions." *Journal of Political Economy*, 91, 461–479.

Martínez de Albéniz, F. J., C. Rafels, and N. Ybern (2014), "A procedure to compute the nucleolus of the assignment game." *Operations Research Letters*, 41, 675–678.

Myerson, R. (1977), "Graphs and cooperation in games." *Mathematics of Operations Research*, 2, 225–229.

Núñez, M. and C. Rafels (2002), "The assignment game: the τ-value." *International Journal of Game Theory*, 31, 411–422.

Quint, T. (1991), "The core of an m-sided assignment game." *Games and Economic Behavior*, 3, 487–503.

Schmeidler, D. (1969), "The nucleolus of a characteristic function game." *SIAM Journal of Applied Mathematics*, 17, 1163–1170.

Shapley, L. S. and M. Shubik (1972), "The assignment game I: the core." *International Journal of Game Theory*, 1, 111–130.

Solymosi, T. and T. E. S. Raghavan (1994), "An algorithm for finding the nucleolus of assignment games." *International Journal of Game Theory*, 23, 119–143.

Stuart, H. (1997), "The supplier-firm-buyer game and its m-sided generalization." *Mathematical Social Sciences*, 34, 21–27.

Thompson, G. L. (1981), "Auctions and market games." In *Essays in Game Theory and Mathematical Economics in Honor of Oskar Morgenstern* (R. Aumann, ed.).

van Velzen, B., H. Hamers, and T. Solymosi (2008), "Core stability in chain component additive games." *Games and Economic Behaviour*, 62, 116–139.