Fractional exclusion statistics and shot noise in ballistic conductors

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We study the noise properties of ballistic conductors with carriers satisfying fractional exclusion statistics. To test directly the nature of exclusion statistics we found that systems under weakly degenerate conditions should be considered. Typical of these systems is that the chemical potential μ is in the thermal range $|\mu| < 3k_BT$. In these conditions the noise properties under current saturation are found to depend upon the statistical parameter g, displaying suppressed shot noise for $1/2 \leqslant g \leqslant 1$, and enhanced shot noise for $0 < g < 1/2$, according to the attractive or repulsive nature of the carrier exclusion statistics.

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I. INTRODUCTION

The concept of fractional exclusion statistics (FES) was introduced by Haldane[1] as a phenomenological description of excitations with mixed statistical properties (intermediate between fermions and bosons) in strongly correlated many-body systems. FES is based on the assumption that the change of the number of available one-particle states in a system, for a given volume and with fixed boundary conditions ΔD depends linearly on the change of the number of quasiparticles ΔN, i.e., $\Delta D = -g\Delta N$. Here, g is the statistical parameter with the properties of (i) being independent of the number of quasiparticles and (ii) being determined solely by the interaction strength between particles. In spite of its phenomenological ground, FES has been shown to be realized by, or directly related to, several physical models, as, for instance, (i) models for strongly correlated particles in one dimension[2–7] and two dimensions,[8,9] (ii) anyons,[10,11] and (iii) quasiparticle excitations in the fractional quantum Hall effect (FQHE) systems.[12–15]

The quantum statistical mechanics of a generalized gas of particles obeying FES was pioneered in Refs. 10,16,17. Since then, a series of works were devoted to the thermodynamic properties of this generalized system.[18–22] In addition, transport properties have been widely investigated. The linear response of one-dimensional systems obeying FES has been addressed by means of the Landauer approach[23–25] and the correlation function method.[26] The nonlinear response for edge excitations in the FQHE in terms of FES has been also analyzed.[15]

By contrast, the fluctuation properties of systems satisfying FES have received only a minor consideration. A significative attempt to this subject was presented in Ref. 25, where a generalization of the first quantized Landauer approach to conductance and noise was proposed for particles obeying FES. The application of the formalism to the case of strongly degenerate systems revealed that the effects of carrier statistics on the noise properties is only appreciated in the transition region between thermal and shot noise. Surprisingly enough, the shot noise suppression factor was found to be independent of the particle statistics. Thus, no evidence of the ''attractive'' or ''repulsive'' nature of the different statistics was expected.

The aim of the present paper is, precisely, to analyze under which conditions the noise properties of systems obeying FES are expected to depend on the statistics of the carriers, thus shedding new light on the attractive or repulsive nature of FES. To this purpose, we investigate the noise properties of a one-dimensional ballistic system obeying FES with current voltage (I-V) characteristics ranging from linear to current saturation regimes. As prototype of a one-dimensional ballistic system, we consider a perfectly transmitting channel where the random injection of carriers from the contacts is the only source of noise. Being related to the occupation probabilities, this noise source is only present at temperatures different from zero and it depends on the carrier statistics. Therefore, the above prototype represents the simplest system where to study the interrelations between FES and noise.

The content of the paper is organized as follows. In Sec. II we detail the system under study. In Sec. III we analyze the transport and noise properties for the case of strongly degenerate conditions. In Sec. IV the case of weakly degenerate conditions are considered. Finally, in Sec. V we sum up the main conclusions of the paper.

II. SYSTEM UNDER STUDY

We consider a standard two terminal experiment where two reservoirs are adiabatically inter-connected by a one-dimensional channel with the following assumptions. (i) The channel is ballistic and perfectly coupled with the reservoirs acting as ideal contacts so that no reflections take place at the interfaces. (ii) Each reservoir is in quasiequilibrium at a temperature T, and electrochemical potential ϕ. (iii) All the band bending occurs in the channel and the relative position of the conduction band and the electrochemical potential does not change in the contacts, i.e., the chemical potential $\mu = \phi - E_C$, with E_C being the bottom of the conduction energy band, is independent of the applied bias (for simplicity we assume the same value for the two contacts). Therefore, when the bias $qV = \phi_2 - \phi_1$, with $-q$ the carriers charge (taken as negative), is changed, the potential can vary exclusively inside the ballistic channel, and the contacts can be excluded from consideration.

The carriers in the system are assumed to satisfy FES and are injected from the reservoirs into the channel in accordance with the corresponding equilibrium distribution function. For a generalized gas satisfying FES, standard quantum statistics shows that its equilibrium distribution function

$f_g(\varepsilon-\mu)$ satisfies the implicit equation[10,16,17]

$$[1-gf_g(z)]^g[1+(1-g)f_g(z)]^{1-g}=f_g(z)e^{z/k_BT}, \quad (1)$$

where ε is the carrier energy, k_B the Boltzmann constant and T the temperature. The limiting values $g=1$ and $g=0$ correspond to the Fermi-Dirac and Bose-Einstein statistics, respectively. Valuable approximations of the above distributions are (i) under strongly degenerate conditions $f_g(\varepsilon-\mu)=(1/g)\theta(\mu-\varepsilon)$ with θ the Heaviside function, (ii) under nondegenerate conditions, $f_g(\varepsilon-\mu)=\exp[-(\varepsilon-\mu)/k_BT]$, which being the Maxwell-Boltzmann distribution is independent of g . Here, for the sake of convenience we will take $\mu>3k_BT$ and $\mu<-3k_BT$ as synonymous of strongly degenerate and nondegenerate conditions, respectively. These conditions are well verified for all FES. Finally, we note that the charge of carriers, q, is here taken to be independent of carrier statistics to allow us focusing only on purely statistical effects. To include the combined effect of fractional charge and fractional statistics one only needs to substitute q by ge in the final results.

III. STRONGLY DEGENERATE SYSTEMS

For the sake of simplicity we first consider the case of strongly degenerate conditions (i.e., $\mu>3k_BT$). Under these conditions, one can neglect long range Coulomb interaction effects on the fluctuations,[27] thus simplifying considerably the calculations.

A. Transport properties

Under the conditions that long range Coulomb interaction can be neglected, the average current flowing through a ballistic one dimensional channel can be written as[23,25]

$$I=(2s+1)\frac{q}{h}\int_0^{+\infty}[f_g(\varepsilon-\mu)-f_g(\varepsilon+qV-\mu)]d\varepsilon, \quad (2)$$

where s is the spin factor, h is the Planck constant, and use is made that $\phi_2-\phi_1=qV$. For FES, with the help of the relation

$$f_g'(z)=\frac{-1}{k_BT}f_g(z)[1-gf_g(z)][1+(1-g)f_g(z)], \quad (3)$$

which follows from Eq. (1), one can integrate the current equation obtaining the I-V characteristics in explicit form

$$I=I_0\left[\ln\left(\frac{1+(1-g)f_g(-\mu)}{1-gf_g(-\mu)}\right)-\ln\left(\frac{1+(1-g)f_g(qV-\mu)}{1-gf_g(qV-\mu)}\right)\right]. \quad (4)$$

Here, $I_0=(2s+1)(q/h)k_BT=G_0(k_BT/q)$, with $G_0=(2s+1)(q^2/h)$.

The I-V characteristics given by Eq. (4) is plotted in Fig. 1 for different statistics and strongly degenerate conditions given by $\mu/k_BT=8$. In all the cases we found that the I-V characteristics are linear for applied potentials lower than the chemical potential, while they tend to saturate for applied potentials higher than the chemical potential. The occurrence of current saturation is due to the fact that (i) all carriers injected from the contact at higher voltage are collected by the opposite contact and (ii) all carriers injected from the contact at lower voltage return back to the same contact. We anticipate, that the saturation condition plays a determinant role in evidencing the effects of the statistics on the noise properties of the system. As can be seen in Fig. 1, the main effect of the carrier statistics is to determine the slope of the linear regime (i.e., the conductance) and the value of the saturation current. The dependence of these quantities on the statistical parameter can be obtained straightforwardly. Indeed, from Eq. (2) the (differential) conductance is given by[23–25]

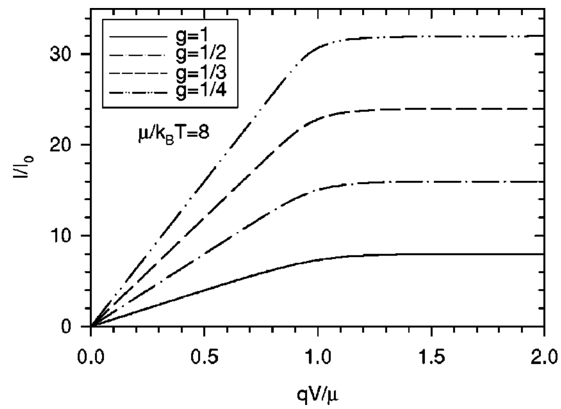

FIG. 1. I-V characteristics of the ballistic conductor for different statistics under strongly degenerate conditions. Here, $I_0=(2s+1)qk_BT/h$ and μ is the chemical potential.

$$G=\frac{dI}{dV}=G_0\int_0^{+\infty}[-f_g'(\varepsilon+qV-\mu)]d\varepsilon=G_0f_g(qV-\mu), \quad (5)$$

thus yielding for the (linear) conductance (when $qV\ll\mu$)

$$G^{eq}=G_0f_g(-\mu). \quad (6)$$

Under strongly degenerate conditions the conductance thus becomes

$$G^{eq,\deg}=\frac{G_0}{g}=(2s+1)\frac{q^2}{gh}. \quad (7)$$

Therefore, $G^{eq,\deg}$ is found to depend inversely upon the statistical parameter, in agreement with existing results.[23–26]

For the saturation current we obtain the expression

$$I_s=(2s+1)\frac{q}{h}\int_0^{+\infty}f(\varepsilon-\mu)d\varepsilon=I_0\ln\left(\frac{1+(1-g)f_g(-\mu)}{1-gf_g(-\mu)}\right), \quad (8)$$

which, under strongly degenerate conditions, leads to

$$I_s^{\deg}=G^{eq,\deg}\frac{\mu}{q}=(2s+1)\frac{q\mu}{gh}. \quad (9)$$

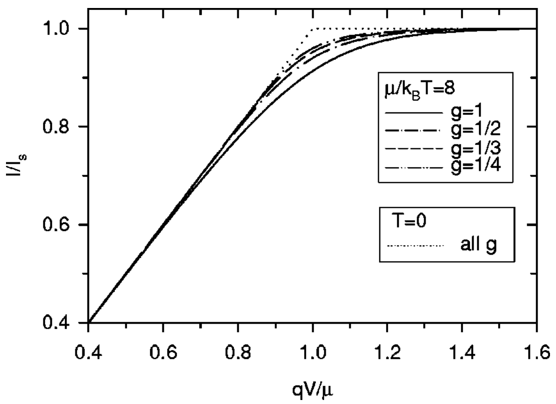

FIG. 2. Renormalized *I-V* characteristics of the ballistic conductor for different statistics under strongly degenerate conditions. I_s is the saturation current. For comparison the results for $T=0$ are also reported.

The saturation current is a function of the statistical parameter only through the value of $G^{eq,\deg}$, and thus it also depends inversely upon the statistical parameter. This dependence is due to the fact that for a given value of g the occupation number goes as $1/g$ while the velocity of the injected carries remains the same for all the statistics, once a value of the chemical potential is given. We remark, that if the unit charge of carriers depends on the statistical parameter as $q=ge$, then the value of the saturation current is independent of carrier statistics.

It is worth noting that, a part from the values of the linear conductance and the saturation current, the effects of the carrier statistics is also manifested in the shape of the region of transition between the linear and the saturation regimes. This effect is evidenced in Fig. 2, where we report the current normalized to its saturation value as a function of the applied voltage. For the sake of comparison, in this figure we also plot the *I-V* characteristics at $T=0$ which, with the used normalization, becomes a universal function, independent of statistics, given by

$$\left.\frac{I}{I_s}\right|_{T=0}=\begin{cases}\dfrac{qV}{\mu} & \text{for} \quad \dfrac{qV}{\mu}<1,\\[2ex] 1 & \text{for} \quad \dfrac{qV}{\mu}>1.\end{cases} \tag{10}$$

As can be seen in the figure, the difference between the different statistics amounts to a few percent of the value in the transition region.

B. Noise properties

As stated before, in the present structure current fluctuations originate solely from the randomness of carrier injection from the contacts. The carrier injection into the ballistic region must be, of course, consistent with the statistics of the quasiparticles. For the case of contacts at equilibrium, this consistency is satisfied by assuming that the low frequency current spectral density in the energy range $(\varepsilon,\varepsilon+d\varepsilon)$, is given by $s_I(\varepsilon-\mu)d\varepsilon$, with[25,26]

$$s_I(\varepsilon-\mu)=2G_0k_BT\frac{\partial f_g(\varepsilon-\mu)}{\partial\mu}. \tag{11}$$

This expression is reminiscent of that for the variance of the mean occupation number at equilibrium $\langle\delta f_g^2\rangle=k_BT[\partial f_g(\varepsilon-\mu)/\partial\mu]$, whose validity for FES was proved in Ref. 18. Note that, for the case of Fermi-Dirac statistics one has $s_I=2G_0f_{\mathrm{FD}}(1-f_{\mathrm{FD}})$, which shows the familiar term $f(1-f)$, while for FES in general it is $s_I=2G_0f_g(1-gf_g)[1+(1-g)f_g]$. In all the cases with $g>0$, in the zero temperature limit the low frequency noise vanishes since $f_g\to 1/g\,\theta(\mu-\varepsilon)$, and hence the transport becomes fully coherent.

The current fluctuations can be computed as[25]

$$S_I=\int_0^{+\infty}[s_I(\varepsilon-\mu)+s_I(\varepsilon+qV-\mu)]d\varepsilon, \tag{12}$$

where we have used that the channel is perfectly ballistic and transmitting and that, because of strongly degenerate conditions, long range Coulomb interaction can be neglected. After substituting Eq. (11) in Eq.(12), one obtains

$$S_I=2G_0k_BT[f_g(-\mu)+f_g(qV-\mu)]=2k_BT(G^{eq}+G). \tag{13}$$

We note that, independently of carrier statistics, at thermal equilibrium one always has

$$S_I^{eq}=4k_BTG_0f_g(-\mu)=4k_BTG^{eq}, \tag{14}$$

thus recovering Nyquist theorem, as it should be. In addition, under current saturation conditions one can neglect the contribution from one of the contacts and obtain

$$S_I^s=2G_0k_BTf(-\mu) \tag{15}$$

$$=2k_BTG^{eq}, \tag{16}$$

which corresponds to half of Nyquist thermal noise. Similarly to the case of the *I-V* characteristics, the effects of the statistics are noticeable in both the linear and saturation regimes through G^{eq}. In addition, the transition region between these two regimes also shows a slight but significant dependence on the statistics, as evidenced in Fig. 3. Here we plot the low frequency spectral density of current fluctuations as given in Eq. (13) normalized to its saturation value, as a function of the applied potential, for strongly degenerate conditions.

It is worth noting that, by using Eqs. (7), (9), and (16), the saturation value of the current spectral density under strongly degenerate conditions can be written as

$$(S_I^s)^{\deg}=\frac{2G_0k_BT}{g}=2q\left(\frac{k_BT}{\mu}\right)I_s, \tag{17}$$

which can be interpreted as shot noise suppressed by the degeneracy factor k_BT/μ.

A convenient figure of merit of shot noise is the Fano factor γ defined as

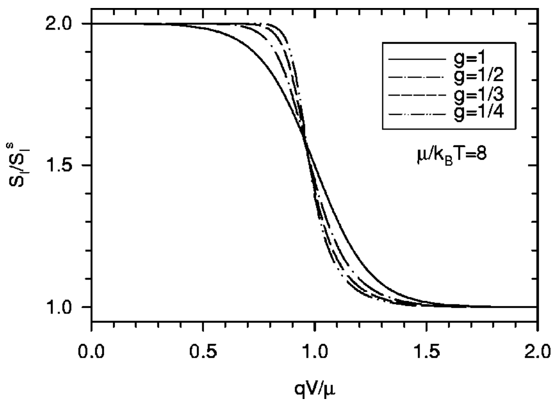

FIG. 3. Current spectral density at low frequency of the ballistic conductor as function of the applied bias for different statistics under strongly degenerate conditions. The current spectral density is normalized to its value under current saturation conditions.

$$\gamma = \frac{S_I}{2qI}. \tag{18}$$

In Fig. 4 we plot the Fano factor for different statistics as a function of the applied potential under strongly degenerate conditions. Here, at the lowest and highest potentials the Fano factor is independent of statistics. More precisely, in the former limit it decreases inversely with the applied voltage according to Nyquist relation as

$$\gamma = 2\frac{k_BT}{qV} \quad \text{for} \quad qV/\mu \ll 1. \tag{19}$$

By contrast, in the latter limit the Fano factor is found to saturate taking the value

$$\gamma = \frac{k_BT}{\mu} \quad \text{for} \quad qV/\mu \gg 1. \tag{20}$$

Between these limits, the Fano factor displays a transition region which depends slightly on the carrier statistics. We note that the value reached by γ at the highest voltages is

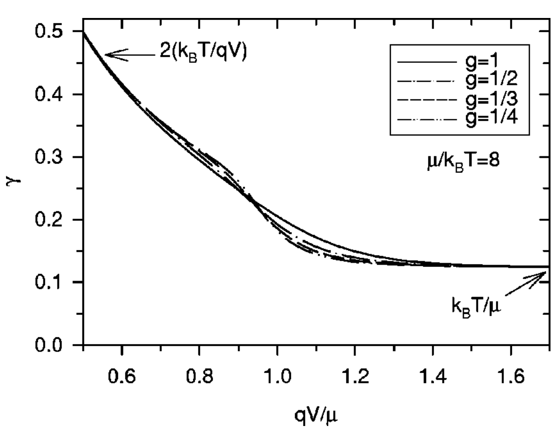

FIG. 4. Fano factor of the ballistic conductor as a function of the applied bias under strongly degenerate conditions for different statistics.

always less than 1, thus corresponding to suppressed shot noise, independently of carrier statistics.

In a certain sense, the picture emerging from the previous analysis is close to that presented in Ref. 25. Indeed, in both cases the system displays thermal noise at low bias and suppressed shot noise at higher bias, both limits being independent of carrier statistics, and with the effect of carrier statistics becoming only noticeable in the transition region between low and high bias voltages. The difference between the present results and those of Ref. 25 is that in our case, being the channel ballistic, the shot noise is reached under current saturation conditions when $qV > \mu$ and displays a suppression factor equal to k_BT/μ, while in Ref. 25 because the channel is nonballistic, the shot noise is reached for $qV < \mu$ and displays a suppression factor $1-t$, where t is the transmission of the channel.

At first sight the previous results are somewhat surprising. Because of the sensitivity of shot noise to carrier correlations, one would have expected a strong dependence of the Fano factor on carrier statistics. In particular, one would have expected some evidence of the attractive or repulsive nature of different statistics in the results obtained for strongly degenerate conditions. However, the weak dependence of the noise properties on the statistical parameter found above is a direct consequence of the strongly degenerate conditions assumed. This conclusion will be better clarified in the next section where the case of weakly degenerate conditions is investigated.

IV. WEAKLY DEGENERATE SYSTEMS

Under weakly degenerate conditions (i.e., $\mu < 3k_BT$), to evidence the effects of carrier statistics on the noise properties it suffices to consider the current saturation regime. This limit offers the advantage that the effects of long-range Coulomb interaction on fluctuations can be neglected, thus allowing us to resume results obtained in the previous section.

The general expression for the saturation current is given in Eq. (8). We note that under nondegenerate conditions (i.e., $\mu < -3k_BT$) one has $I_s^{n\text{-deg}} = I_0 f_g(-\mu) = I_0 \exp(\mu/k_BT)$, which is independent of g as it should be. For intermediate values of the chemical potential, the values of I_s follow Eq. (8), thus interpolating between the non-degenerate and the strongly degenerate values, given in Eq. (9).

Furthermore, under saturation conditions the current spectral density is given in general by Eq. (15), and the Fano factor under saturation by

$$\gamma_S = \frac{f_g(-\mu)}{\ln\left(\frac{1+(1-g)f_g(-\mu)}{1-gf_g(-\mu)}\right)}, \tag{21}$$

where we have used Eqs. (8), (15), and (18). The previous results can be written explicitly in terms of the saturation current, by combining Eqs. (8), (15), and (21). One then obtains the following expressions:

$$S_I^s = \frac{2G_0k_BT}{g + [\exp(I_s/I_0)-1]^{-1}}, \tag{22}$$

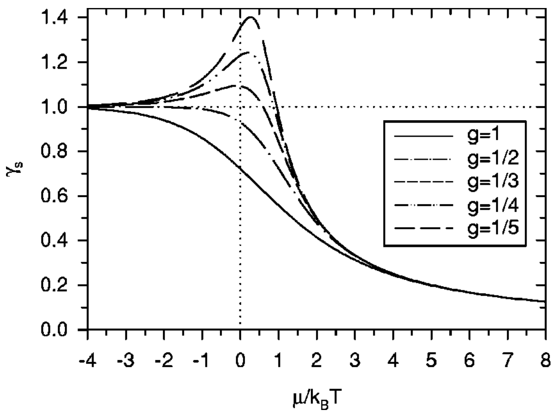

FIG. 5. Fano factor of the ballistic conductor under current saturation conditions as a function of the chemical potential for different statistics. k_BT is the thermal energy.

$$\gamma_s=\frac{I_0/I_s}{g+[\exp(I_s/I_0)-1]^{-1}}. \tag{23}$$

The above expressions give S_I^s and γ_s as a function of I_s, q (through G_0 and I_0), and g, and can be used to obtain direct information on particle statistics. When the injection of carriers occurs under strongly degenerate conditions (i.e., $\mu/k_BT>3$, $I_s/I_0\gg1$) we recover the results obtained in the previous section, that is

$$(S_I^s)^{\mathrm{deg}}=\frac{2G_0k_BT}{g}=2q\left(\frac{k_BT}{\mu}\right)I_s, \tag{24}$$

$$(\gamma_s)^{\mathrm{deg}}=\frac{k_BT}{\mu}, \tag{25}$$

thus corresponding to suppressed shot noise, with a Fano factor independent of carrier statistics given by k_BT/μ. Furthermore, under nondegenerate conditions (i.e., $\mu/k_BT<-3$, $I_s/I_0\ll1$) we can approximate Eqs. (22) and (23) by

$$(S_I^s)^{\mathrm{nondeg}}=2qI_s, \tag{26}$$

$$(\gamma_s)^{\mathrm{nondeg}}=1, \tag{27}$$

thus recovering full shot noise. This result is a direct consequence of the fact that the distribution function for nondegenerate conditions is the Maxwell-Boltzmann one. Obviously, in this limit the results are independent of carrier statistics.

In the transition region between weakly and strongly degenerate conditions (i.e., $|\mu|<3k_BT$) one must use the full expressions given in Eqs. (15) and (21), or alternatively in Eqs. (22) and (23). Figure 5 reports the Fano factor under current saturation conditions versus the chemical potential, as obtained from Eq. (21), with the values of the distribution function being computed from Eq. (1). Here, the Fano factor is found to depend significantly on particle statistics. In particular, in the transition region $|\mu|<3k_BT$ the ''attractive'' or ''repulsive'' nature of the exclusion statistics becomes manifest by the fact that the Fano factor takes values larger or smaller than one, respectively. By analyzing Eq. (21) it can be shown that for $0<g<1/2$ one has

$$\gamma_s>1 \quad \text{for} \quad \mu<\mu_c, \tag{28}$$

$$\gamma_s<1 \quad \text{for} \quad \mu>\mu_c, \tag{29}$$

where μ_c is the value of the chemical potential for which $\gamma_s=1$, and that can be calculated for each statistics from Eq. (21). On the other hand, for $1/2\leqslant g\leqslant1$ we obtain

$$\gamma_s<1 \quad \text{for all } \mu. \tag{30}$$

Therefore, for carriers following FES with $0<g<1/2$ we have proved their possibility to display enhanced shot noise ($\gamma_s>1$), thus evidencing the positive correlation (bunching) induced by the exclusion statistics. Remarkably, these positive correlations are only manifested under weakly degenerate conditions and when $\mu<\mu_c$. Under strongly degeneracy conditions, the shot noise is always suppressed since the system tends to a coherent transport, which implies noiseless conditions. By contrast, for carriers following FES with $1/2\leqslant g\leqslant1$ we have obtained always suppressed shot noise ($\gamma_s<1$), thus evidencing the negative correlation (antibunching) induced by the exclusion statistics.

V. CONCLUSIONS

We have studied the noise properties of perfect ballistic, one-dimensional channels with carriers satisfying FES. We have found that the attractive or repulsive nature of the carrier statistics can be really appreciated only in the case of systems which are under weakly degenerate conditions, that is with a chemical potential μ in the contacts satisfying the condition $|\mu|<3k_BT$. Then, under current saturation regime the Fano factor is found to display enhanced shot noise ($\gamma_s>1$), or suppressed shot noise ($\gamma_s<1$) depending on whether the statistical parameter is in the range $0<g<1/2$, or in the range, $1/2\leqslant g\leqslant1$, respectively. These results show that for particles with $0<g<1/2$ the statistics tends to bunch the carriers, while for particles with $1/2\leqslant g\leqslant1$ it tends to antibunch them.

In the remaining range of values of the chemical potential ($|\mu|>3k_BT$) we have found that the effects of the statistics on the noise properties are less pronounced. For the case of strongly degenerate systems ($\mu>3k_BT$) the information on the particle statistics is mostly contained in the linear region of the I-V characteristics, and to a less extent in the transition region between linear and current saturation conditions. In particular, independently of carrier statistics the system always displays suppressed shot noise under current saturation conditions, with a Fano factor $\gamma_s=k_BT/\mu<1$. On the other hand, for the case of nondegenerate systems ($\mu<-3k_BT$), the distribution function is well approximated by the Maxwell-Boltzman distribution, and hence the results are independent of the carrier statistics. In this case, under current saturation conditions the system displays full shot noise $\gamma_s=1$.

The present results suggest that to evidence the nature of the FES one should consider situations in which the system is under weakly degenerate conditions. This conclusion is expected to remain basically valid also for one dimensional channels with transmission less than one.

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