The use of flexible quantile-based measures in risk assessment

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Abstract A new family of distortion risk measures -GlueVaR- is proposed in Belles-Sampera et al. (2014) to procure a risk assessment lying between those provided by common quantile-based risk measures. GlueVaR risk measures may be expressed as a combination of these standard risk measures. We show here that this relationship may be used to obtain approximations of GlueVaR measures for general skewed distribution functions using the Cornish-Fisher expansion. A subfamily of GlueVaR measures satisfies the tail-subadditivity property. An example of risk measurement based on real insurance claim data is presented, where implications of tail-subadditivity in the aggregation of risks are illustrated.
Keywords: quantiles, subadditivity, tails, risk management, Value-at-Risk.

1 Introduction

Management practitioners in insurance and financial institutions are used to deal with risk measures. Risk measures pursue to quantify the risk undertaken in a particular context, where the risk measure provides information related to an underlying random variable of interest that is frequently associated to losses[1]. Multiple applications are derived from the quantification of the risk by means of risk measures, including reserves and capital allocation. One of the main applications is the assessment of economic reserves. In the European Union, capital requirements of insurance and financial institutions are established by regulators to be assessed according to a particular risk measure. The risk measure value determines the minimum cushion of economic liquidity required to the institution such as banks and insurance companies. Another field in which risk measures are extensively employed is in the context of capital allocation applications. Capital allocation problems appear when managers have to distribute an amount across different risk units. All capital allocation principles are determined by a capital allocation criterion and a given risk measure. A general theoretical framework where the most common capital allocation principles may be accommodated is provided by Dhaene et al. (2012).

The most frequently used risk measures are the quantile-based risk measures Value-at-Risk (VaR) and Tail Value-at-Risk (TVaR). Both risk measures may be defined as two particular cases of the family of distortion risk measures. Distortion risk measures were introduced by Wang (Wang, 1995, 1996) and are closely related to the distortion expectation theory (see Yaari, 1987). In order

[1] See, for instance, Szëgo (2002) for an extensive introduction to risk measures.

to preserve the benefits of diversification when aggregating risks, an appealing property of a risk measure is subadditivity. The subadditivity property ensures that the risk measure value of the aggregated risk is lower than or equal to the sum of individual risk measure values. The subadditivity characteristic is guaranteed for the TVaR but not for the VaR risk measure.

Belles-Sampera et al. (2014) argued that, in practice, main concerns of managers are related to the performance of aggregated risks in the tail region. They proposed a new family of risk measures -GlueVaR- which belongs to the class of distortion risk measures, and showed that GlueVaR measures can be defined as a linear combination of common quantile-based risk measures. The authors investigated the properties of these new measures in tails, where theoretical foundations of the tail-subadditivity were established. They showed that a subfamily of GlueVaR risk measures satisfies this property.

In this article we extend the analysis of GlueVaR risk measures. We provide approximations to GlueVaR risk measures for general skewed distribution functions using a Cornish-Fisher expansion of their quantiles. In insurance applications managers often face to highly skewed random variables with right fat tails. In many of these situations, however, they do not know whether the underlying random variable of interest is distributed according to a known parametric distribution function. In those situations that the distribution is unknown, the value of the common quantile-based risk measures is routinely approximated by practitioners. In this study, we show that approximations of GlueVaR risk measures for general unknown skewed distribution functions can be straightforwardly obtained by means of the relationship of GlueVaR risk measures and the standard quantile-based risk measures.

Implications of tail-subadditivy are investigated. Subadditivity of GlueVaR risk measures in tails was investigated from a theoretical perspective in Belles-Sampera et al. (2014). However, implications of tail-subadditivity for insurance and financial institutions in comparison to subadditivity in the whole domain were not analyzed. In this article the tail-subadditivity is analyzed from a practical perspective. We empirically examine the subadditivity and tail-subadditivity properties of GlueVaR risk measures in the aggregation of risks, both illustrated with a numerical example based on real insurance claim data.

The article is structured as follows. Main concepts related to risk measures are briefly described in section 2 where GlueVaR risk measures are introduced. The approximation of the GlueVaR risk measure for general skewed distribution functions is provided and illustrated with an empirical example in section 3. Tail-subadditivity implications of GlueVaR risk measures are exemplified in section 4. Finally, concluding remarks are provided in section 5.

2 Risk assessment using GlueVaR measures

2.1 Quantile-based distortion risk measures used in risk assessment

Let assume a probability space $(\Omega, \mathcal{A}, \mathcal{P})$ with sample space Ω, a σ-algebra $\mathcal{A}$ and a probability $\mathcal{P}$ from $\mathcal{A}$ to $[0, 1]$, and the set of all random variables

defined on this space. Any risk measure ρ is a mapping from the set of random variables to the real line $\mathbb{R}$, $X \mapsto \rho(X) \in \mathbb{R}$. The Value-at-Risk (VaR) and Tail Value-at-Risk (TVaR) are the quantile-based risk measures commonly used in insurance and finance. Given a random variable X, VaR at level α is the α-quantile of the random variable X, i.e.

$$\mathrm{VaR}_\alpha(X) = \inf\{x \mid F_X(x) \geq \alpha\} = F_X^{-1}(\alpha), \tag{1}$$

where F_X is the cumulative distribution function (cdf) of X and α is the confidence level (also called tolerance level) $0 \leq \alpha \leq 1$. VaR is a standard tool to assess the risk in the financial industry, where the random variable X is usually related to losses. In fact, VaR measure has been widely adopted by regulators of the financial and insurance industry to calculate solvency capital requirements of banking and insurance institutions (Basel accords, Solvency II, etc.). However, the use of the VaR measure presents limitations in practice. Catastrophic losses are not properly captured. VaR risk measure can be understood as the maximum potential loss with a given confidence level. Then, this risk measure does not provide information related to the size of those losses falling above that quantile point. That is, VaR risk measure only considers the minimum loss in most adverse cases. However, the rest of (catastrophic) losses in those highly adverse scenarios are not taking into account. Additionally, the subadditivity of VaR cannot be generalized as indicated by Artzner et al. (1999) and Acerbi and Tasche (2002). Subadditivity is an appealing property of a risk measure whether risks are aggregated. A risk measure is subadditive when the associated value of the aggregated risk is always less than or equal to the sum of associated values of individual risks.

TVaR at level α is defined as,

$$\mathrm{TVaR}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_\lambda(X)\, d\lambda. \tag{2}$$

Roughly speaking, the TVaR is understood as the mathematical expectation of losses given that those losses fall above the associated VaR value. Catastrophic losses are considered in the quantification of the TVaR since this risk measure computes average losses in most adverse cases. Moreover, the TVaR risk measure satisfies the subadditivity property. From a theoretical viewpoint, then, TVaR measure seems more adequate to assess risks faced by companies than VaR risk measure. Unlike VaR risk measure, TVaR takes into account losses in most adverse scenarios in the risk assessment and, as a subadditive risk measure, TVaR preserves benefits of diversification when aggregating risks. However, TVaR has not been widely accepted in the industry because practitioners interpret that reserve values associated to this risk measure are excessively high.

VaR and TVaR risk measures form part of a wider class referred to as distortion risk measures which were introduced by Wang (Wang, 1995, 1996). A distortion risk measure has associated a distortion function g, where $g : [0,1] \to [0,1]$ is a function such that $g(0) = 0$, $g(1) = 1$ and g is non-decreasing. A distortion risk measure is defined as follows:

Distortion risk measure. Consider a random variable X and its survival function $S_X(x) = P(X > x)$. Function ρ_g defined by

$$\rho_g(X) = \int_{-\infty}^{0} [g(S_X(x)) - 1]\, dx + \int_{0}^{+\infty} g(S_X(x))\, dx \tag{3}$$

is called a distortion risk measure where g is the associated distortion function.

Note that, given a random variable X, the distortion risk measure $\rho_g(X)$ can be understood as the Choquet Integral[2]of X with respect to the set function $\mu = g \circ \mathcal{P}$, where $\mathcal{P}$ is the probability function associated with the probability space in which X is defined. An straightforward interpretation of distortion risk measures can be then obtained. First, the survival function of the random variable is distorted ($g \circ S_X$) and, second, the mathematical expectation of the distorted random variable is computed. The mathematical expectation is then a particular case of distortion risk measure whose distortion function is the identity function, $\rho_{id}(X) = \mathbb{E}(X)$ (see, for instance, Denuit et al., 2005).

The associated distortion function of the VaR risk measure is as follows,

$$\psi_\alpha(u) = \begin{cases} 0 \text{ if } & 0 \le u < 1-\alpha \\ 1 \text{ if } & 1-\alpha \le u \le 1 \end{cases} \tag{4}$$

and for the TVaR,

$$\gamma_\alpha(u) = \begin{cases} \dfrac{u}{1-\alpha} & \text{if } \quad 0 \le u < 1-\alpha \\ 1 & \text{if } \quad 1-\alpha \le u \le 1 \end{cases} \tag{5}$$

Based on their distortion functions, it can be easily proved that $\mathrm{VaR}_\alpha(X) \le \mathrm{TVaR}_\alpha(X)$ for any random variable X. The proof is derived from the fact that $\psi_\alpha(u) \le \gamma_\alpha(u)$ for any u once α is fixed.

2.2 GlueVaR risk measures

A new class of distortion risk measures, named GlueVaR risk measures, were introduced by Belles-Sampera et al. (2014). They defined the GlueVaR risk measures by means of its distortion function as follows. Given a confidence level α, the distortion function associated to a GlueVaR risk measure is:

$$\kappa_{\beta,\alpha}^{h_1,h_2}(u) = \begin{cases} \dfrac{h_1}{1-\beta} u & \text{if } \quad 0 \le u < 1-\beta \\ h_1 + \dfrac{h_2 - h_1}{\beta - \alpha}[u - (1-\beta)] & \text{if } \quad 1-\beta \le u < 1-\alpha \\ 1 & \text{if } \quad 1-\alpha \le u \le 1 \end{cases} \tag{6}$$

[2] In honour of Gustave Choquet who introduced the concept of the integral for non-additive measures (Choquet, 1954). The asymmetric Choquet Integral with respect to a set function μ of a μ-measurable function $X : \Omega \to \overline{\mathbb{R}}$ is denoted as $\int X d\mu$ and is equal to $\int X d\mu = \int_{-\infty}^{0} [S_{\mu,X}(x) - \mu(\Omega)]\, dx + \int_{0}^{+\infty} S_{\mu,X}(x) dx, \quad \text{if } \mu(\Omega) < \infty,$ where $S_{\mu,X}(x) = \mu(\{X > x\})$ denotes the *survival function* of X with respect to μ and Ω denotes a set, which in financial and insurance applications is the sample space of a probability space. See Denneberg (1994) for more details.

where the parameter β is an additional confidence level introduced in the definition of GlueVaR risk measures so that $\alpha \leq \beta$ for $\alpha, \beta \in [0,1]$. The shape of the GlueVaR distortion function is determined by the distorted survival probabilities h_1 and h_2 at levels $1-\beta$ and $1-\alpha$, where $h_1 \in [0,1]$ and $h_2 \in [h_1, 1]$.

In addition, if the notation is modified as $\omega_1 = h_1 - \dfrac{(h_2 - h_1)(1-\beta)}{\beta - \alpha}$, $\omega_2 = \dfrac{h_2 - h_1}{\beta - \alpha}(1-\alpha)$ and $\omega_3 = 1 - \omega_1 - \omega_2 = 1 - h_2$, then the GlueVaR risk measure can be expressed as a linear combination of quantile-based risk measures as follows,

$$\mathrm{GlueVaR}_{\beta,\alpha}^{h_1,h_2}(X) = \omega_1 \mathrm{TVaR}_\beta(X) + \omega_2 \mathrm{TVaR}_\alpha(X) + \omega_3 \mathrm{VaR}_\alpha(X). \quad (7)$$

An interesting interpretation of GlueVaR risk measures is that the GlueVaR measure value summarizes information of three possible scenarios. Given the two levels of severity α and β with $\alpha < \beta$, then the risk can be measured in the highly conservative scenario with TVaR at level β, in the conservative scenario with TVaR at level α and in the less conservative scenario with VaR at level α. More details can be found in Belles-Sampera et al. (2014).

3 GlueVaR approximation for general skewed distribution functions

3.1 The Cornish-Fisher approximation of GlueVaR risk measures

Analytical closed-form expressions of the GlueVaR risk measure for the most frequently used distribution functions in the insurance and financial context are shown in Belles-Sampera et al. (2014), including the Normal, Lognormal, Student-t or Pareto distributions among others. In many situations, however, decision-makers do not know the distribution function of the random variable X. Approximations to the risk measure values are an interesting alternative when the true distribution function is unknown.

In actuarial and financial applications the random variable of interest is frequently highly skewed. The Cornish-Fisher expansion is widely used by practitioners to approximate the $\mathrm{VaR}_\alpha(X)$ and $\mathrm{TVaR}_\alpha(X)$ values when the random variable follows a skewed unknown distribution (see Cornish and Fisher, 1937, Fisher and Cornish, 1960, Johnson and Kotz, 1970, McCune and Gray, 1982). The VaR and TVaR measure values can be approximated as $\mathrm{VaR}_\alpha(X) \simeq \mu + q_{v,\alpha}\sigma$ and $\mathrm{TVaR}_\alpha(X) \simeq \mu + q_{tv,\alpha}\sigma$, where $\mu = \mathbb{E}[X]$, $\sigma^2 = \mathbb{V}[X]$ and both $q_{v,\alpha}$ and $q_{tv,\alpha}$ are modified quantiles of the standard normal distribution that take into account the skewness of the distribution function of X.

Following Sandström (2007), the modified quantiles $q_{v,\alpha}$ and $q_{tv,\alpha}$ are computed as follows. Let us consider $\gamma = \mathbb{E}\left[(X-\mu)^3\right]/\sigma^3$ as a measure of the

skewness of the distribution[3]. If $q_\alpha = \Phi^{-1}(\alpha)$ and ϕ are the α-quantile and the density function of the standard normal distribution, respectively, then $q_{v,\alpha}$ and $q_{tv,\alpha}$ can be written as,

$$\begin{aligned} q_{v,\alpha} &= \Phi^{-1}(\alpha) + \frac{\gamma}{6}\left[\left(\Phi^{-1}(\alpha)\right)^2 - 1\right] = q_\alpha + \frac{\gamma}{6}\left[q_\alpha^2 - 1\right], \\ q_{tv,\alpha} &= \frac{\phi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}\left[1 + \frac{\gamma}{6}\left(\Phi^{-1}(\alpha)\right)^3\right] = \frac{\phi(q_\alpha)}{1-\alpha}\left[1 + \frac{\gamma}{6}q_\alpha^3\right]. \end{aligned} \tag{8}$$

According to the interpretation of GlueVaR measure as a linear combination of risk measures shown in (7), the approximation for the GlueVaR of X random variable following the Cornish-Fisher expansion can be obtained as,

$$\mathrm{GlueVaR}_{\beta,\alpha}^{h_1,h_2}(X) \simeq \omega_1(\mu + q_{tv,\beta}\sigma) + \omega_2(\mu + q_{tv,\alpha}\sigma) + \omega_3(\mu + q_{v,\alpha}\sigma). \tag{9}$$

If modified quantiles defined in (8) are considered and notation is changed as $\omega_1 = h_1 - \dfrac{(h_2 - h_1)(1-\beta)}{\beta - \alpha}$, $\omega_2 = \dfrac{h_2 - h_1}{\beta - \alpha}(1-\alpha)$ and $\omega_3 = 1 - h_2$, then the Corner-Fisher approximation of the GlueVaR can be expressed as,

$$\begin{aligned} \mathrm{GlueVaR}_{\beta,\alpha}^{h_1,h_2}(X) \simeq \mu + \sigma\Bigg[&\left(\frac{h_1}{1-\beta} - \frac{h_2 - h_1}{\beta - \alpha}\right)\phi(q_\beta)\left(1 + \frac{\gamma}{6}q_\beta^3\right) \\ &+ \left(\frac{h_2 - h_1}{\beta - \alpha}\right)\phi(q_\alpha)\left(1 + \frac{\gamma}{6}q_\alpha^3\right) + (1 - h_2)\left(q_\alpha + \frac{\gamma}{6}\left(q_\alpha^2 - 1\right)\right)\Bigg]. \end{aligned} \tag{10}$$

The error of the approximation is upper bounded by the maximum error incurred when approximating $\mathrm{VaR}_\alpha(X)$, $\mathrm{TVaR}_\alpha(X)$ and $\mathrm{TVaR}_\beta(X)$ using the equivalent Cornish-Fisher expansion for skewed distributions. This result is straightforwardly derived from the linear relationship shown in (7) and (9). Note that weights ω_1, ω_2 and ω_3 are lower or equal than one and it holds that $\omega_1 + \omega_2 + \omega_3 = 1$.

3.2 Illustration of risk measurement using GlueVaR

Data for the cost of claims involving property damages and medical expenses from a major Spanish motor insurer are used to illustrate the application of GlueVaR measures in risk measurement. The sample consists of $n = 518$ observations of the cost of individual claims in thousands of euros. These data were previously analyzed in Bolancé et al. (2008) and Guillén et al. (2011).

In Table 1 a set of quantile-based risk measures including three different GlueVaR are displayed. The table is divided into three blocks, each block representing the corresponding risk figures for the cost of claims for property

[3] Extensions of the Cornish-Fisher expansion that consider moments of higher order than γ have been provided in the literature (see, for instance, Giamouridis, 2006). More details can be found in Appendix B of Sandström (2011).

damage (X_1), the cost of claims of medical expenses (X_2) and the aggregate cost of claims ($X_1 + X_2$). Risk measure values using the empirical distribution (first row) are compared with outcomes when Normal, Lognormal, Student-t with 4 df and Generalized Pareto distributions are fitted to data. In the last two rows of each block outcome results are shown when risk measure values are approximated by the Cornish-Fisher expansion shown in expression (10). The sample mean ($\hat{\mu} = \bar{z}$), the sample deviation ($\hat{\sigma}^2 = \sum_i (Z_i - \bar{z})^2 / (n-1)$) and sample skewness ($\hat{\gamma} = \hat{\sigma}^{-3}(\sum_i (Z_i - \bar{z})^3 / n)$) are considered as estimators of μ, σ and γ when Z is one of the three random variables $X_1, X_2, X_1 + X_2$. Sample statistics were computed using observations that fall below the 99.5% quantile in order to exclude the effect of catastrophic losses on estimates (first Cornish-Fisher approximation). That means, a subsample of the first 516 increasingly ordered elements of the random variable were used to estimate parameters. Let remind that the sample size is 518 observations. Therefore, the two highest values were considered as catastrophic losses and were not included. Outcome values of risk measures were compared with the risk measure approximations when all the observations are included on sample estimates (second Cornish-Fisher approximation)[4].

The selection of the three GlueVaR risk measures included in Table 1 deserves further explanation. The two confidence levels considered are $\alpha = 95\%$ and $\beta = 99.5\%$. Three different scenarios of risk assessment are presented by the heights (h_1, h_2) equal to $(11/30, 2/3)$, $(0, 1)$ and $(1/20, 1/8)$ respectively. The heights (h_1, h_2) are selected in order to represent different risk attitudes of risk evaluation. In the scenario involving $h_1 = 11/30$ and $h_2 = 2/3$ the associated weights take the same value, i.e. $\omega_1 = \omega_2 = \omega_3 = 1/3$. So, the $\text{GlueVaR}^{11/30,2/3}_{99.5\%,95\%}$ corresponds to a balanced attitude because the three quantile-based risk measures $\text{TVaR}_{99.5\%}$, $\text{TVaR}_{95\%}$ and $\text{VaR}_{95\%}$ are equally important, $\omega_1 = \omega_2 = \omega_3$. A different attitude is symbolized in the second scenario by $\text{GlueVaR}^{0,1}_{99.5\%,95\%}$ with associated heights $h_1 = 0$ and $h_2 = 1$ or, equivalently, weights $\omega_1 = -1/9$, $\omega_2 = 10/9$ and $\omega_3 = 0$. It corresponds to a scenario in which the manager overweights $\text{TVaR}_{95\%}$ and allocates the lowest feasible weight to $\text{TVaR}_{99.5\%}$ given that a zero weight is allocated to $\text{VaR}_{95\%}$. Finally, $\text{GlueVaR}^{1/20,1/8}_{99.5\%,95\%}$ in the third scenario reflects a risk measurement attitude just a little bit more conservative than the one represented by using $\text{VaR}_{95\%}$, assigning low weights to $\text{TVaR}_{99.5\%}$ and $\text{TVaR}_{95\%}$, i.e. $h_1 = 1/20$ and $h_2 = 1/8$ or, analogously, $\omega_1 = 1/24$ and $\omega_2 = 1/12$.

As it is shown in Table 1, $\text{GlueVaR}^{11/30,2/3}_{99.5\%,95\%}$ is more conservative than the other two selected GlueVaR measures. This result can be generalized to all situations because the associated distortion function of $\text{GlueVaR}^{11/30,2/3}_{99.5\%,95\%}$ is greater than the other two distortion functions in the whole domain. Note that it is also observed in Table 1 that $\text{GlueVaR}^{1/20,1/8}_{99.5\%,95\%} \leq \text{GlueVaR}^{0,1}_{99.5\%,95\%}$. It is only valid to these data and an ordering between them can not be generalized. However, a relationship between these two GlueVaR risk measures and quantile-

[4] Calculations were made in R and MS Excel. R commands and spreadsheets are available from the authors upon request.

Table1. Examples of risk measurement of costs of insurance claims using quantile-based risk measures

	Model	$\mathbf{VaR}_{95\%}$	$\mathbf{TVaR}_{95\%}$	$\mathbf{TVaR}_{99.5\%}$	$\mathbf{GlueVaR}^{h_1,h_2}_{99.5\%,95\%}$ $\left(\frac{\mathbf{11}}{\mathbf{30}},\frac{\mathbf{2}}{\mathbf{3}}\right)$	$(\mathbf{0},\mathbf{1})$	$\left(\frac{\mathbf{1}}{\mathbf{20}},\frac{\mathbf{1}}{\mathbf{8}}\right)$
	Empirical	38.8	112.5	440.0	197.1	76.1	61.7
	Normal	78.9	96.1	130.4	101.8	92.3	82.5
X_1	Lognormal	42.5	106.3	364.0	170.9	77.7	61.2
	Student-t (4 d.f.)	99.0	143.2	272.1	171.4	128.9	109.9
	Pareto	38.3	82.4	264.5	128.4	62.2	51.4
	Cornish-Fisher$^{(1a)}$	61.3	169.2	724.3	318.3	107.5	98.0
	Cornish-Fisher$^{(1b)}$	262.1	1,081.9	5,437.9	2,260.6	597.9	546.1
	Empirical	6.4	18.4	54.2	26.3	14.4	9.4
	Normal	10.2	12.4	16.7	13.1	11.9	10.7
X_2	Lognormal	6.5	14.5	44.6	21.9	11.2	8.8
	Student-t (4 d.f.)	12.8	18.3	34.5	21.9	16.5	14.2
	Pareto	5.9	12.4	38.5	18.9	9.5	7.8
	Cornish-Fisher$^{(2a)}$	14.3	45.4	207.3	89.0	27.4	24.9
	Cornish-Fisher$^{(2b)}$	22.1	76.1	359.4	152.5	44.6	40.6
	Empirical	47.6	125.5	479.0	217.4	86.2	72.1
	Normal	87.0	105.9	143.4	112.1	101.7	90.9
X_1+X_2	Lognormal	48.9	119.1	397.8	188.6	88.1	69.3
	Student-t (4 d.f.)	109.0	157.5	298.6	188.4	141.8	120.9
	Pareto	44.2	94.6	301.4	146.7	71.6	59.1
	Cornish-Fisher$^{(3a)}$	71.3	198.0	850.7	373.3	125.4	114.4
	Cornish-Fisher$^{(3b)}$	283.6	1,164.0	5,840.3	2,429.3	644.4	588.5

$^{(1a)}$ $\hat{\mu}=9.0$, $\hat{\sigma}=17.9$ and $\hat{\gamma}=4.5$. Subsample without catastrophic losses. The two largest values of X_1 are excluded.
$^{(1b)}$ $\hat{\mu}=11.0$, $\hat{\sigma}=41.3$ and $\hat{\gamma}=15.6$. Full sample.
$^{(2a)}$ $\hat{\mu}=1.5$, $\hat{\sigma}=3.7$ and $\hat{\gamma}=6.4$. Subsample without catastrophic losses. The two largest values of X_2 are excluded.
$^{(2b)}$ $\hat{\mu}=1.7$, $\hat{\sigma}=5.2$ and $\hat{\gamma}=8.0$. Full sample.
$^{(3a)}$ $\hat{\mu}=10.5$, $\hat{\sigma}=20.6$ and $\hat{\gamma}=4.6$. Subsample without catastrophic losses. The two largest values of X_1+X_2 are excluded.
$^{(3b)}$ $\hat{\mu}=12.7$, $\hat{\sigma}=45.2$ and $\hat{\gamma}=15.3$. Full sample.

based risk measures can be established. Belles-Sampera et al. (2014) showed that $\mathrm{VaR}_\alpha \leq \mathrm{GlueVaR}^{h_1,h_2}_{\beta,\alpha} \leq \mathrm{TVaR}_\alpha$ if $h_1 \leq (1-\beta)/(1-\alpha)$. That means, $\mathrm{VaR}_{95\%} \leq \mathrm{GlueVaR}^{0,1}_{99.5\%,95\%} \leq \mathrm{TVaR}_{95\%}$, because $0 \leq 0.1$, and $\mathrm{VaR}_{95\%} \leq \mathrm{GlueVaR}^{1/20,1/8}_{99.5\%,95\%} \leq \mathrm{TVaR}_{95\%}$, because $0.05 \leq 0.1$. Although results in Table 1 invite to deduce that $\mathrm{TVaR}_{95\%} \leq \mathrm{GlueVaR}^{11/30,2/3}_{99.5\%,95\%} \leq \mathrm{TVaR}_{99.5\%}$, it can not be generalized because conditions on the parameters of the GlueVaR risk measure to satisfy $\mathrm{TVaR}_\alpha \leq \mathrm{GlueVaR}^{h_1,h_2}_{\beta,\alpha} \leq \mathrm{TVaR}_\beta$ are $h_1 \geq (1-\beta)/(1-\alpha)$ and $h_2 = 1$. In our case it holds $0.37 \geq 0.1$ but $h_2 \neq 1$.

Some comments related to outcome values for the Cornish-Fisher approximation of the quantile-based risk measures should be made. According to our results, it seems that this kind of risk measurement corresponds to a conservative attitude for the two types of approximations shown in Table 1. Relevant differences are observed depending on the approximation finally used. If the first Cornish-Fisher approximation is considered, i.e. when sample statistics were estimated excluding catastrophic losses, we observe that the outcome values for this approximation are in most of the cases larger than those values associated with the empirical or the parametric distributions. It happens in thirteen cases among the sixteen examples. Although conservative values are obtained with this approximation, results are in general comparable with those computed with the empirical and parametric distributions. Unlike values of this first Cornish-Fisher approximation, outcome values related to the second Cornish-Fisher approximation are drastically larger than the rest in all the examples. These outcome values would be associated to a excessively conservative (unrealistic) attitude. Let remind that only the two largest losses are not included in the sample estimates involving the first approximation. In other words, the Cornish-Fisher approximation shows a poor performance when the data are severely right skewed distributed, as in our case. However, the performance of this approximation seems to be improved when catastrophic losses are excluded for the sample estimates of parameters.

An important issue that arises from results is the model risk. Even when the same risk measure is used, huge differences are observed depending on the hypothesis about the underlying distribution of the claim cost random variables. Let us assume that the regulator is focused on the $\mathrm{VaR}_{95\%}$ for the aggregate cost $X_1 + X_2$ as a measure of pure underwriting risk (without taking into account the premium paid by the policyholders). If it is supposed that the random variable is Pareto distributed, then the institution will need 44.2 thousands of euros for regulatory solvency purposes. The company should set aside almost 2.5 times this economic amount whether the underlying distribution is Student-t with 4 degrees of freedom. This topic is out of the scope of this paper. The interested reader is addressed, for instance, to the study of Alexander and Sarabia (2012) which deals with VaR model risk.

4 Tail-subadditivity of GlueVaR risk measures

4.1 Subadditivity in tails

Risk measures are normally claimed to satisfy a set of properties. Frequently, risk measures are required to be coherent in Artzner sense. Artzner et al. (1999) established that the set of axioms that a risk measure should satisfy was positive homogeneity, translation invariance, monotonicity and subadditivity. They referred to such risk measures as coherent risk measures. Distortion risk measures satisfy the first three properties[5]. However, the subadditivity property

[5] Additional properties for distortion risk measures can be found in Jiang (2008) and Balbás et al. (2009).

of the distortion risk measure is only ensured whether the associated distortion function is concave (Denneberg, 1994, Wang and Dhaene, 1998, Wirch and Hardy, 2002). As a consequence, TVaR is an Artzner-sense coherent risk measure but VaR is not. Since GlueVaR risk measures may be interpreted as a linear combination of VaR and TVaR risk measures, subadditivity property will be inherited by GlueVaR if $\omega_1 \geq 0$ and $\omega_3 = 0$, that is, if there is none negative weight and the weight associated to VaR is zero.

Although subadditivity in the whole domain is in general not satisfied by GlueVaR risk measures, Belles-Sampera et al. (2014) showed that a subfamily of GlueVaR measures satisfy the subadditivity property in the tail region. The behavior of aggregate risks in the tail region has received huge attention by researchers in last years (Cheung, 2009, Song and Yan, 2009, Hua and Joe, 2012). Belles-Sampera et al. (2014) defined tail-subadditivity for a pair of risks as follows.

Let the random variable Z be defined on the probability space with sample space Ω and the α-quantile be defined as $s_\alpha(Z) = \inf\{z \mid S_Z(z) \leq 1-\alpha\}$. Given a confidence level α, the tail region of the random variable Z is defined as $\mathcal{Q}_{\alpha,Z} := \{\omega \mid Z(\omega) > s_\alpha(Z)\} \subseteq \Omega$. Let X, Y be two risks defined on the same probability space. The common tail for both risks is defined as $\mathcal{Q}_{\alpha,X,Y} := \mathcal{Q}_{\alpha,X} \cap \mathcal{Q}_{\alpha,Y} \cap \mathcal{Q}_{\alpha,X+Y}$.

Tail-subadditivity. Given a confidence level $\alpha \in [0,1]$, a distortion risk measure ρ_g is subadditive in the tail for the pair X, Y if $\mathcal{Q}_{\alpha,X,Y} \neq \emptyset$ and

$$\int_{\mathcal{Q}_{\alpha,X,Y}} (X+Y)\, d(g \circ \mathcal{P}) \leq \int_{\mathcal{Q}_{\alpha,X,Y}} X d(g \circ \mathcal{P}) + \int_{\mathcal{Q}_{\alpha,X,Y}} Y d(g \circ \mathcal{P}), \quad (11)$$

where the integral symbol stands for Choquet Integrals with respect to the set function $g \circ \mathcal{P}$.

Given a confidence level α, a GlueVaR risk measure is tail-subadditive if its associated distortion function is concave in $[0, 1-\alpha)$. In next section we show that tail-subadditivity is a convenient property to preserve the benefits of diversification in extremely adverse cases.

4.2 Illustration of tail-subadditivity in risk aggregation

In this section we follow the example described in section 3.2 to investigate the tail-subadditivity property of GlueVaR risk measures. A comment on the subadditivity of risk measures when the Cornish-Fisher approximation is used should be made before. Unlike the VaR risk measure, we previously discussed that the TVaR risk measure satisfies the subadditivity property. In our example, however, the second Cornish-Fisher approximation of the TVaR risk measure value fails subadditivity. Note that it is deduced from Table 1 that $\mathrm{TVaR}_\beta(X_1) + \mathrm{TVaR}_\beta(X_2) < \mathrm{TVaR}_\beta(X_1 + X_2)$ $(5,797.3 < 5,840.3)$ and $\mathrm{TVaR}_\alpha(X_1) + \mathrm{TVaR}_\alpha(X_2) < \mathrm{TVaR}_\alpha(X_1 + X_2)$ $(1,158.0 < 1,164.0)$. Therefore, the subadditivity property of the TVaR measure is not ensured when

the risk measure value is approximated by the second Cornish-Fisher approximation. This result supports the statement that the second Cornish-Fisher approximation is not adequate to estimate quantile-based risk measure values for highly right skewed data.

In relation to tail-subadditivity of GlueVaR risk measures, in this example both $\mathrm{GlueVaR}^{11/30,2/3}_{99.5\%,95\%}$ and $\mathrm{GlueVaR}^{1/20,1/8}_{99.5\%,95\%}$ are candidates to satisfy subadditivity in tails for a pair of risks at confidence level $\alpha = 95\%$. Note that it holds in both cases that $h_2 \leq h_1 (1-\alpha) / (1-\beta)$ ($2/3 \leq 11/3$ and $1/8 \leq 1/2$, respectively). However, this inequality is not fulfilled by $\mathrm{GlueVaR}^{0,1}_{99.5\%,95\%}$ and, then, this GlueVaR risk measure does not satisfy the tail-subadditivity property. In fact, Table 1 seems to reflect subadditivity of $\mathrm{GlueVaR}^{11/30,2/3}_{99.5\%,95\%}$ and $\mathrm{GlueVaR}^{0,1}_{99.5\%,95\%}$ risk measures. Indeed, the risk measure outcomes for the aggregate risk are lower than the sum of individual risk values in all of the models with the exception of the outcomes associated to the second Cornish-Fisher approximation. We must emphasize that this result is strongly related to these data but subadditivity property can not be generalized to all the circumstances.

Let us focus on the outcomes for the $\mathrm{GlueVaR}^{1/20,1/8}_{99.5\%,95\%}$ when the empirical distribution is considered. Table 1 shows that the $\mathrm{GlueVaR}^{1/20,1/8}_{99.5\%,95\%}$ fails to be subadditive for X_1 and X_2, since $61.7+9.4 \leq 72.1$. We investigate now the tail-subadditivity property of this GlueVaR risk measure. In order to analyze the tail-subadditivity property for the GlueVaR risk measure, the common right tail of the empirical distribution has to be firstly isolated. The common 5%-right tail for the empirical distribution is separated as follows. A subsample is selected which satisfies the criterion that each individual risk values are above its respective 95%-quantile given that the values of the aggregate random variable fall above its 95%-quantile and the values of the other individual random risk fall above its respective 95%-quantile as well. Risk measure values are then computed for this subsample, where the survival probabilities associated to the observations of this subsample have not been changed. Table 2 displays the values of their common 5%-right tail for the individual random variables and the aggregate random variable.

An illustration of tail-subadditivity of the $\mathrm{GlueVaR}^{1/20,1/8}_{99.5\%,95\%}$ risk measure is provided in Table 3, where results for the $\mathrm{GlueVaR}^{1/20,1/8}_{99.5\%,95\%}$ when aggregating risks in the whole domain are compared with those $\mathrm{GlueVaR}^{1/20,1/8}_{99.5\%,95\%}$ outcomes in the common 5%-right tail. In the second block, risk measure values are computed for the three random variables in the common 5%-right tail, i.e. using data shown in Table 2. Outcome results of the $\mathrm{GlueVaR}^{1/20,1/8}_{99.5\%,95\%}$ are in bold type to highlight differences between subadditivity in the whole range and subadditivity in tails. The last row of the second block illustrates numerically the 95% tail-subadditivity property of $\mathrm{GlueVaR}^{1/20,1/8}_{99.5\%,95\%}$ for the pair of risks X_1 and X_2, where diversification benefit is computed as the difference between the sum of GlueVaR outcome values for individual risks and the outcome value of the aggregate risk. On the common 5%-right tail, a benefit of diversification of 2.2 thousands of euros is observed for the $\mathrm{GlueVaR}^{1/20,1/8}_{99.5\%,95\%}$ risk measure.

Table2. Common 5%-right tail for X_1, X_2 and $X_1 + X_2$

$\mathbf{i}$	$\mathbf{x_{1,i}}$	$\mathbf{x_{2,i}}$	$\mathbf{s_i = x_{1,i} + x_{2,i}}$
1	829.0	71.3	900.3
12	108.2	23.7	131.9
32	55.0	44.3	99.3
185	121.6	32.5	154.1
189	74.2	13.2	87.4
198	88.8	30.1	118.9
213	57.5	10.0	67.5
214	148.7	10.2	158.9
289	145.4	42.2	187.6
294	44.8	7.5	52.3
297	221.5	8.3	229.8

A discrete finite probability space $\Omega = \{\varpi_1, \varpi_2, ..., \varpi_{518}\}$ is considered.
Each ith-observation $(x_{1,i}, x_{2,i}, s_i)$ corresponds to a realization of random event ϖ_i.
Note that all values in last three columns are greater or equal than their empirical quantiles at 95% level, where $\mathrm{VaR}_{95\%}(X_1)$=38.8; $\mathrm{VaR}_{95\%}(X_2) = 6.4$;$\mathrm{VaR}_{95\%}(X_1 + X_2)$=47.6.

In other words, the aggregate risk $X_1 + X_2$ is preferable than these risks individually taken in simultaneously adverse events for X_1 and X_2, according to the results of the $\mathrm{GlueVaR}^{1/20,1/8}_{99.5\%,95\%}$. However, it does not hold whether the whole domain of the random variables is considered. Last row in the first block shows a negative value for the diversification benefit associated to the $\mathrm{GlueVaR}^{1/20,1/8}_{99.5\%,95\%}$ on the whole domain. Therefore, these two risks should not be aggregated by managers when the whole domain is considered.

The underlying idea in the assessment of the incurred risk in this context is that the benefit of diversification in simultaneously adverse events is balanced by the cost of diversification in the rest of cases. Therefore, divergence decisions would be taken by managers depending on where the attention is paid, all the scenarios or highly adverse scenarios. As it is shown in Table 3, this phenomenon is due to the lack of subadditivity of $\mathrm{VaR}_{95\%}$ on the whole domain. By considering the common 5%-right tail, the effect of $\mathrm{VaR}_{95\%}$ on the whole domain is blurred on the tail.

5 Conclusions

Managers of insurance and financial institutions often focus on the performance of aggregated risks in adverse scenarios. The emphasis is put by decision-makers in the behavior of the common tail region and losses falling below this aggregate quantile point are not so relevant. GlueVaR risk measures introduced by Belles-Sampera et al. (2014) play an important role in this context. These measures can be expressed as linear combinations of standard risk measures. Concavity of the distortion function on the subrange $[0, 1 - \alpha)$ is necessary

Table3. Subadditivity and tail-subadditivity

		$\mathbf{X_1}$ (a)	$\mathbf{X_2}$ (b)	$\mathbf{X_1 + X_2}$ (c)	**Difference**$^{(*)}$ (a+b-c)
Whole range	$\mathrm{VaR}_{95\%}$	38.8	6.4	47.6	-2.4
	$\mathrm{TVaR}_{95\%}$	112.5	18.4	125.5	5.4
	$\mathrm{TVaR}_{99.5\%}$	440.0	54.2	479.0	15.2
	$\mathbf{GlueVaR}^{1/20,1/8}_{99.5\%,95\%}$	**61.7**	**9.4**	**72.1**	**-1.0**
Common 5%-right tail	$\mathrm{VaR}_{95\%}$	0.0	0.0	0.0	0.0
	$\mathrm{TVaR}_{95\%}$	75.3	12.5	76.8	11.0
	$\mathrm{TVaR}_{99.5\%}$	411.3	46.7	426.7	31.3
	$\mathbf{GlueVaR}^{1/20,1/8}_{99.5\%,95\%}$	**23.4**	**3.0**	**24.2**	**2.2**

$^{(*)}$ Benefit of diversification.

for tail-subadditivity of GlueVaR risk measures. We show that this milder condition (than full subadditivity) can be a sufficient requisite for risk measures when attention is paid on the right-tail. We provide an illustration in which benefits of diversification are not satisfied in the whole domain but are preserved in adverse scenarios.

The Cornish-Fisher approximation of the GlueVaR risk measures for general skewed distribution functions is given and a numerical illustration is provided to compare it with the most frequently used parametric distribution functions. Our results seem to indicate that it is a conservative risk assessment approximation. In particular, managers should be cautious with the use of the Cornish-Fisher approximation to estimate quantile-based risk measure values for highly right skewed data, where subadditivity property for approximated TVaR values is not ensured. An improvement of the performance of this approximation is observed when catastrophic losses are excluded for the parameters' estimation.

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Bibliography

Acerbi, C. and Tasche, D. (2002). On the coherence of expected shortfall. *Journal of Banking & Finance*, 26(7):1487–1503.

Alexander, C. and Sarabia, J. M. (2012). Quantile uncertainty and value-at-risk model risk. *Risk Analysis*, 32(8):1293–1308.

Artzner, P., Delbaen, F., Eber, J.-M., and Heath, D. (1999). Coherent measures of risk. *Mathematical Finance*, 9(3):203–228.

Balbás, A., Garrido, J., and Mayoral, S. (2009). Properties of distortion risk measures. *Methodology and Computing in Applied Probability*, 11(3, SI):385–399.

Belles-Sampera, J., Guillén, M., and Santolino, M. (2014). Beyond Value-at-Risk: GlueVaR distortion risk measures. *Risk Analysis*, 34(1):121–134.

Bolancé, C., Guillén, M., Pelican, E., and Vernic, R. (2008). Skewed bivariate models and nonparametric estimation for the CTE risk measure. *Insurance: Mathematics and Economics*, 43(3):386–393.

Cheung, K. C. (2009). Upper comonotonicity. *Insurance: Mathematics and Economics*, 45(1):35 – 40.

Choquet, G. (1954). Theory of Capacities. *Annales de l'Institute Fourier*, 5:131–295.

Cornish, E. A. and Fisher, R. A. (1937). Moments and cumulants in the specification of distributions. *Revue de l'Institut International de Statistique*, 4:307–320.

Denneberg, D. (1994). *Non-Additive Measure and Integral*. Kluwer Academic Publishers, Dordrecht.

Denuit, M., Dhaene, J., Goovaerts, M., and Kaas, R. (2005). *Actuarial Theory for Dependent Risks. Measures, Orders and Models.* John Wiley & Sons Ltd, Chichester.

Dhaene, J., Tsanakas, A., Valdez, E. A., and Vanduffel, S. (2012). Optimal Capital Allocation Principles. *Journal of Risk and Insurance*, 79(1):1–28.

Fisher, R. A. and Cornish, E. A. (1960). The percentile points of distributions having known cumulants. *Technometrics*, 2(2):209–225.

Giamouridis, D. (2006). Estimation risk in financial risk management: A correction. *Journal of Risk*, 8(4):121–125.

Guillén, M., Prieto, F., and Sarabia, J. M. (2011). Modelling losses and locating the tail with the Pareto Positive Stable distribution. *Insurance: Mathematics and Economics*, 49(3):454–461.

Hua, L. and Joe, H. (2012). Tail comonotonicity: Properties, constructions, and asymptotic additivity of risk measures. *Insurance: Mathematics and Economics*, 51(2):492–503.

Jiang, L. (2008). Convexity, translation invariance and subadditivity for g-expectations and related risk measures. *Annals of Applied Probability*, 18(1):245–258.

Johnson, N. L. and Kotz, S. (1970). *Distributions in Statistics: Continuous Univariate Distributions - 1.* John Wiley & Sons, New York.

McCune, E. and Gray, H. (1982). Cornish-Fisher and Edgeworth expansions. In Samuel Kotz and Norman L. Johnson, editor, *Encyclopedia of Statistical Sciences*, volume 2. John Wiley & Sons, New York.

Sandström, A. (2007). Solvency II: Calibration for skewness. *Scandinavian Actuarial Journal*, (2):126–134.

Sandström, A. (2011). *Handbook of Solvency for Actuaries and Risk Managers. Theory and Practice*. CRC Finance Series. Chapman & Hall / CRC , Boca Raton.

Song, Y. and Yan, J.-A. (2009). Risk measures with comonotonic subadditivity or convexity and respecting stochastic orders. *Insurance: Mathematics and Economics*, 45(3):459–465.

Szëgo, G. (2002). Measures of risk. *Journal of Banking and Finance*, 26(7):1253–1272.

Wang, S. S. (1995). Insurance pricing and increased limits ratemaking by proportional hazard transforms. *Insurance: Mathematics and Economics*, 17(1):43–54.

Wang, S. S. (1996). Premium calculation by transforming the layer premium density. *ASTIN Bullletin*, 26(1):71–92.

Wang, S. S. and Dhaene, J. (1998). Comonotonicity, correlation order and premium principles. *Insurance: Mathematics and Economics*, 22(3):235–242.

Wirch, J. L. and Hardy, M. R. (2002). Distortion risk measures: Coherence and stochastic dominance. IME Conference, Lisbon.

Yaari, M. E. (1987). The dual theory of choice under risk. *Econometrica*, 55(1):95–115.