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Comment on: Remarkable polyhedra related to set functions, games and capacities

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This paper highlights the connections between different theories that are built on the notion of set-functions. For instance, the theory of cooperative games with transferable utility represents a game by means of a characteristic function (a set-function vanishing on the empty set) and decision theory under risk represents some generalizations of probability measures by means of capacities (set-functions vanishing on the empty set and monotone w.r.t. inclusion).

Due to the proficiency of the author in several of these related fields, one of the contributions of this survey is to make the reader aware of the fact that identical notions or results have been reached independently in the different research areas. The multilinear extension of a game is the pseudo-Boolean function related to its characteristic function; the Möbius transform of a game coincides with the set of Harsanyi dividends; the theorem that states that the marginal worth vectors are the extreme points of the core of a convex game (or of a capacity with non-negative Möbius transform) has been proved several times.

But the main goal of the paper is the study of the different polyhedra related to set-functions: the polytope of capacities, the polytope of p-additive capacities, the cone of supermodular games and the core of games. In all these instances, the question is the computation of the extreme points of the polytope or the extreme rays of the cone. Many questions are still open: the extreme points of the polytope of capacities are the 0-1 capacities (or simple games) but there is no closed formula for the computation of the number of extreme points; the extreme points of the polytope of p-additive capacities are only known for $p \leq 2$.

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My last comments will be referred to the core of cooperative games. The following questions would be also interesting, and probably more difficult, for games on set systems (or games with restricted cooperation, where the domain of the game is restricted to some family $\mathcal{F}$ of coalitions), but we will remain within the classical notion of game.

– It is mentioned in the paper that when the lower envelope of the core of a game v gives v back, then the game is said to be exact, but this is not always the case. As a consequence, we may have different set functions with the same core, and the one with the highest values is the "exact" one. For a class of games known as assignment games (Shapley and Shubik, 1972), taking the exact representative of the set-function allows for an easy computation of the set of extreme core points (Izquierdo et al., 2007). Could exactness simplify the computation of extreme points of the core for some other classes of games?

– Also from Section 7 in the paper, the core of a game coincides with the base polyhedron of its conjugate (or dual) game. It may happen that the representation of the core by means of the usual characteristic function is not exact while its representation by means of the conjugate set-function is exact (that meaning that each coalition attains its worth in the conjugate function at some core element). In fact, again for assignment games, the representation of the core by means of the conjugate function is always exact and this allows for another procedure to obtain all extreme core points in Núñez and Solymosi (2014).

– Certainly, not much is known of the set of extreme points of the core for games that are not supermodular (that is, convex). The author cites our works where the set of *reduced marginal worth vectors* are proved to be the set of extreme core points for two classes of games: almost convex games and assignment games. There is one such vector for each order on the set of players and, given an order, the first agent in the order receives his/her marginal contribution and the second one receives his/her marginal contribution in the reduced game. We proceed iteratively in the same way following the order and reducing the game at each step. For a given class of games, the set of reduced marginal worth vectors is the set of extreme core points if three conditions hold: a) every player must attain his/her marginal contribution in the core; b) the reduced game at a marginal contribution must be a game in the same class; c) in each extreme core point some agent must attain his/her marginal contribution. It would be interesting to know which other known classes of games satisfy these conditions.
  Recently, Trudeau and Vidal-Puga (2015) have proved that also for minimum cost spanning tree games, the set of extreme core allocations coincides with the set of reduced marginal worth vectors.
  Finally, and this is still work in progress with S. Miquel and T. Solymosi, the reduced marginal worth vectors turn out to be extreme core points in

the class of cyclic permutation games, although the converse inclusion does not hold.

References

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