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Master's Final Project

A theoretical study of a short rate model

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Contents

Abstract

The goal of this project is to do a theoretical study of a short interest rate model under the risk neutral probability, which is able to represent long range dependence. In order to do this, it will be explained the necessary literature to understand the model.

Furthermore, we will expose the consequences of adapting this model for evaluating bonds and derivatives. In order to do this, we will use ambit processes which in general are not semimartingales.

Our purpose is to see if these new models can capture the features of the bond market by extending popular models like the Vasicek model.

Keywords: Bond market, Gaussian processes, Non-semimartingales, Short rates, Volatility, Price, SDE, Hedging.

Mathematics Subject Classification: 91G30.

Chapter 1

Introduction

Long-range dependence, long memory or long-range persistence is a phenomenon based on the dependence or the relation between two points with increasing time interval between them.

One question that has remained a topic of active research is whether financial time series display long-range dependence [10]. Current observations have received confirmation about correlation between distant past with present, for example, observations about natural phenomena of meteorology, hydrology and geophysics. Motivated by cyclical patterns, early theories of business cycles argued that economic time series are long-range dependent. The presence of long memory dependence has serious implications for some paradigms used in financial economics. For example, Merton (1987) [19] and LeRoy (1989) [11] relied heavily on the absence of long-range dependence of the stock market. Long-range dependence in some cases is inconsistent with much of the literature on derivative pricing, which utilizes martingale methods and stochastic process. Rogers (1997) [20] shows that a long-range dependent process inevitably leads to arbitrage opportunities with derivative assets, but also shows how a modified process can retain long memory whereas eliminating arbitrage opportunities.

Mandelbrot (1967, 1971) [13] [14] was the first to identify long-range dependence in asset markets. The statistical vehicle for Mandelbrot's analysis was the *rescaled range* of R/S statistic. Rescaled range is a statistical measure of the variability of time series introduced in 1951 by Harnold Edwin Hurst ($1880-1978$). Its purpose is to provide an assessment of how apparent variability of series changes with the length of the time period being contained. In several seminal papers Mandelbrot and Wallis (1969) [16] and Mandelbrot and Taqqu (1979) [15] demonstrated the superiority of R/S analysis of determining long-range dependence. Greene and Fielitz (1977) [9] supported to Mandelbrot's findings using classical R/S analysis on 200 time series of security prices listed on the New York Stock Exchange, and they found that many of the series were characterized by long-term memory.

Even though it is well-established that R/S analysis can detect long-range dependence time series, Lo (1991) [12] developed a *modified R/S statistic* which is designed to be robust with respect to short-range dependence structures in a time

series.

Most recently, Willinger, Taqqu and Teverovsky (1999) [26] identified a number of problems associated with Lo's method and its use in practice. The have shown that Lo's *modified R/S* analysis has a strong preference for not rejecting the null hypothesis of no long-range dependence and they concluded that Lo's method is not adequate test for long-range dependence.

One way of capture long-range dependence is by using fractional processes as fractional Brownian motion, commonly denoted by *fBm*. As a difference of the classical Brownian motion, the increments of *fBm* need not be independent. Fractional Brownian motion, *fBm*, is a continuous-time gaussian process $B_H(t)$ on $[0, T]$, with zero expectation for all $t \in [0, T]$ and it has the following covariance function:

$$E[B_H(t)B_H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}),$$

where H is a real in $(0, 1)$ called the *Hurst index*.

The value of H determines what kind of process the *fBm* is:

- If $H < 1/2$, the increment of the process are negatively correlated.
- If $H = 1/2$, then *fBm* is a Brownian motion.
- If $H > 1/2$, the increments of the process are positively correlated.

For $H > 1/2$, the process exhibits long-range dependence.

Prediction problems arise in many financial and technical applications. One of these applications is the modeling of bond financial markets because of empirical evidence of long-range dependence in short rates.

One of the main reasons of modeling a bond market with fractional process like fractional Brownian motion, *fBm*, is because of their non-Markovianity. This non-Markovianity property of *fBm* allows catching the real market behavior because this type of processes takes into account the past. However, since all the past is considered, the models using *fBm* processes make the prediction more complicated as we can see in [6], [7] and [8].

In this project we are going to consider a short-rate model where r does not follow a *fBm* but a process close to it, in the sense that is not a semimartingale, nor Markovian and can capture the long-range dependence as well. The only thing we lose is the homogeneity of the increments. This process is a particular case of an ambit process and it allows us to give analytical formulas for bond prices and derivatives based on the bonds.

Chapter 2

A short rate model using ambit process article

2.1 Abstract of the chapter

This chapter consists on a detailed study of [2]. We are going to consider a bond market where short rates evolve as:

$$r_t = \mu_t + \int_{-\infty}^{t} g(t-s)\sigma_s \, dW_s,$$

and where we assume the following conditions:

- μ_t is deterministic.
- $g : (0, \infty) \to \mathbb{R}$ is deterministic.
- σ is positive and deterministic.
- W is the stochastic Wiener measure.

This type of processes are particular cases of ambit processes, which in general are not semimartingale. We are going to see if these new models can explain the behavior of the bond market by extending popular models such as the Vasicek model.

Affine models are popular models as short rate models, but this type of models implies a perfect correlation between bond prices and short rates (and this is not a real market situation), so we are not going to use them.

2.2 The model of short rates

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered and complete probability space with $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}}$. Consider in this probability space

$$r_t = \mu_t + \int_{-\infty}^{t} g(t-s)\sigma_s \, dW_s, \tag{2.2.1}$$

where:

- μ deterministic.
- $g \in L^2((0,\infty))$, deterministic and càdlàg.
- σ is positive, deterministic and càdlàg.
- W is the stochastic Wiener measure under a risk neutral probability, $P^* \sim P$. By $(\mathcal{F}_t)$-stochatic Wiener measure we understand an L^2-valued measure such that for any $A \in \mathcal{B}((-\infty,t])$ (Borelian set), if we consider $\int_{-\infty}^{t} \mathbb{1}_A(s)dW_s = W(A)$ with $E(W(A)^2) < \infty$, verifies that $W(A) \sim N(0, m(A))$ where m is a Lebesgue measure and if $A \subseteq [t,\infty)$, $W(A)$ is independent of $\mathcal{F}_t$.

We shall assume from now on that $\int_{-\infty}^{t} g^2(t-s)\sigma_s^2\, ds < \infty$ in order to define its stochastic integral. Also, this fact ensures that $r_t < \infty$ a.s.

Definition 2.2.1 *Consider the interval $[a,b] \subseteq \mathbb{R}$ and $\mathcal{P}$ the set of partitions of $[a,b]$. Let $f : [a,b] \to \mathbb{R}$ be a deterministic function. Define:*

$$V_a^b(f) := \sup\left\{\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| : P = \{a = t_1 < t_2 < \ldots < t_n = b\} \in \mathcal{P}\right\}.$$

We say that the function f is of bounded variation if $V_a^b(f) < +\infty$.

Definition 2.2.2 *Let X be a stochastic process. We say that X has finite variation if its trajectories are of bounded variation on every bounded time interval with probability* 1.

Definition 2.2.3 *Let $(X_t)_{t\in\mathbb{R}}$ be a stochastic process. We say that $(X_t)_{t\in\mathbb{R}}$ is a semimartingale if it can be decomposed as the sum of a local martingale and an adapted finite-variation process. That is, $(X_t)_{t\in\mathbb{R}}$ admits the following decomposition:*

$$X_t = X_0 + M_t + A_t, \quad t \in \mathbb{R},$$

where $M = (M_t)_{t\in\mathbb{R}}$ is a local martingale and $A = (A_t)_{t\in\mathbb{R}}$ is a process of finite variation.

Now, we define the quasimartingale concept [22]:

Definition 2.2.4 *An adapted process $(X_t)_{t\in\mathbb{R}}$ is a quasimartingale if*

$$\sup \sum_{i=0}^{n-1} E[|X_{t_i} - E[X_{t_{i+1}}|\mathcal{F}_{t_i}]|] < \infty.$$

The supremum is taken over all the finite partitions $a = t_0 < t_1 < \ldots < t_n < b$, where $-\infty < a < b < \infty$.

We have that $g' \notin L^2((0,\infty))$ *if and only if r is not semimartingale* by Knight's theorem stated and proved in detail in Appendix B.

2.3 Pricing and Hedging

Consider $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, P)$ be a filtered probability space, and assume that $(\mathcal{F}_t)_t$ is the filtration generated by the Brownian measure $(W_t)_t$. In this context, we introduce the riskless assets (or bank account):

$$S_t^0 = \exp\left(\int_0^t r_s ds\right),$$

where $(r_s)_s$ is an adapted process and $\int_0^T |r_s| ds < \infty$. In our market, we assume also the existence of risky assets (the bonds).

Definition 2.3.1 *A zero coupon bond with maturity T is a contract that guarantees one euro at time T.*

For each t, we define an adapted process $(P(t,T))_{t \leq T}$ satisfying that $P(t,t) = 1$. $P(t,T)$ denotes the price of a zero-coupon bond that starts at t and has maturity T. Assume by hypothesis that there exists P^* such that the discounted prices defined as:

$$\tilde{P}(t,T) := \frac{P(t,T)}{S_t^0} = \frac{P(t,T)}{\exp\left(\int_0^t r_s ds\right)} = e^{-\int_0^t r_s ds} P(t,T)$$

are P^*-martingale. Taking into account this hypothesis, we have the following fact:

$$P(t,T) = E_{P^*}\left(e^{-\int_t^T r_s ds}\middle|\mathcal{F}_t\right) \text{ for } 0 \leq t \leq T$$

Let us prove it:

$$\tilde{P}(t,T) \underbrace{=}_{\text{martingale}} E_{P^*}\left(\tilde{P}(T,T)|\mathcal{F}_t\right) \underbrace{=}_{\text{definition}} E_{P^*}\left(e^{-\int_0^T r_s ds} P(T,T)|\mathcal{F}_t\right) \underbrace{=}_{\text{P(T,T)=1}}$$
$$= E_{P^*}\left(e^{-\int_0^T r_s ds}|\mathcal{F}_t\right).$$

Then, since $e^{\int_0^t r_s ds}$ is $\mathcal{F}_t$-measurable,

$$P(t,T) = \tilde{P}(t,T) e^{\int_0^t r_s ds} = E_{P^*}\left(e^{-\int_0^T r_s ds} e^{\int_0^t r_s ds}|\mathcal{F}_t\right) =$$
$$= E_{P^*}\left(e^{-\int_t^T r_s ds}|\mathcal{F}_t\right).$$

To know the evolution of r under the risk neutral probability P^*, using the stochastic Fubini's theorem (whose proof is in Appendix B), we develop the following expression. According to (2.2.1):

$$\int_t^T r_s ds = \int_t^T \left(\int_{-\infty}^s g(s-u)\sigma_u \, dW_u\right) ds + \int_t^T \mu_s \, ds =$$
$$= \int_{-\infty}^t \sigma_u \left(\int_t^T g(s-u)\, ds\right) dW_u + \int_t^T \sigma_u \left(\int_u^T g(s-u)\, ds\right) dW_u + \int_t^T \mu_s \, ds.$$

For $u \leq t$, let us call:

$$c(u;t,T) = \int_t^T g(s-u)\,ds.$$

Then, we have that:

$$\int_t^T r_s ds = \int_{-\infty}^t \sigma_u c(u;t,T)\,dW_u + \int_t^T \sigma_u c(u;u,T) dW_u + \int_t^T \mu_s\,ds.$$

Hence:

$$P(t,T) = E_{P^*}\left[\exp\left(-\int_{-\infty}^t \sigma_u c(u;t,T)\,dW_u + \int_t^T \sigma_u c(u;u,T)\,dW_u + \int_t^T \mu_s\,ds\right)\Big|\mathcal{F}_t\right].$$

Now, applying logarithms at both sides we have that:

$$\log(P(t,T)) =$$
$$\log\left(E_{P^*}\left[\exp\left(-\int_{-\infty}^t \sigma_u c(u;t,T)\,dW_u + \int_t^T \sigma_u c(u;u,T)\,dW_u + \int_t^T \mu_s\,ds\right)\Big|\mathcal{F}_t\right]\right).$$

Let us call:

$$A(t,T) = \log\left(E_{P^*}\left[\exp\left(\int_t^T \sigma_u c(u;u,T)\,dW_u - \int_t^T \mu_s\,ds\right)\Big|\mathcal{F}_t\right]\right).$$

Since the exponential function is measurable, $\int_t^T \sigma_u c(u;u,T)\,dW_u$ is $\mathcal{F}_t$-measurable and $\int_t^T \mu_s\,ds$ is $\mathcal{F}_t$-measurable, we can omit the conditional expectation and by properties of the function logarithm:

$$A(t,T) = \log\left(E_{P^*}\left[\exp(\int_t^T \sigma_u c(u;u,T)\,dW_u)\right]\right) + \log\left(\exp\left(\int_t^T -\mu_s\,ds\right)\right)$$
$$= -\int_t^T \mu_s\,ds + \log\left(E_{P^*}\left[\exp\left(\int_t^T \sigma_u c(u;u,T)\right)\,dW_u\right]\right).$$

Now we focus on

$$E_{P^*}\left[\exp\left(\int_t^T \sigma_u c(u;u,T)\right)\,dW_u\right].$$

Call

$$Y := \int_t^T \sigma_u c(u;u,T)\,dW_u,$$

whose integrand is deterministic. Consider $\eta^2 := \int_t^T |\sigma_u c(u;u,T)|^2\,du$. By the construction of the Brownian motion, it holds that $Y \sim N(0,\eta^2)$, so by symmetry

$-Y \sim N(0, \eta^2)$. Taking this fact into account,

$$E_{P^*}(e^{-Y}) = \int \frac{1}{\sqrt{2\pi\eta^2}} e^{-y} e^{-\frac{1 \cdot y^2}{2 \cdot \eta^2}} \, dy = \int \frac{1}{\sqrt{2\pi\eta^2}} e^{-\frac{1}{2 \cdot \eta^2}(y^2 + 2\eta^2 y)} dy =$$
$$\int \frac{1}{\sqrt{2\pi\eta^2}} e^{-\frac{(y+\eta^2)^2}{2 \cdot \eta^2}} e^{\frac{\eta^4}{2\eta^2}} dy = 1 \cdot e^{\frac{\eta^2}{2}}.$$

Hence,

$$E_{P^*}(e^{Y}) = E_{P^*}\left(\exp\left(\int_{-t}^{T} \sigma_u dW_u\right)\right) = \exp\left(\frac{1}{2}\int_t^T |\sigma_u c(u; u, T)|^2 du\right) =$$
$$= \exp\left(\frac{1}{2}\int_t^T \sigma_u^2 c^2(u; u, T) du\right).$$

Thus, we have that:

$$A(t, T) = -\int_t^T \mu_s ds + \log\left(\exp\left(\frac{1}{2}\int_t^T \sigma_u^2 c^2(u; u, T)\, du\right)\right) =$$
$$= -\int_t^T \mu_s ds + \frac{1}{2}\int_t^T \sigma_u^2 c^2(u; u, T)\, du.$$

Then, we arrive at:

$$\log(P(t, T)) = -\int_t^T \mu_s ds + \frac{1}{2}\int_t^T \sigma_u^2 c^2(u; u, T)\, du$$
$$+ \log\left(E_{P^*}\left[\exp\left(-\int_{-\infty}^{t} \sigma_u c(u; t, T)\, dW_u\right) \middle| \mathcal{F}_t\right]\right).$$

Let $z = \exp(-\int_{-\infty}^{t} \sigma_u c(u; t, T) W(du))$. Since the exponential function is measurable and the function inside the exponential is $\mathcal{F}_t$-measurable, we have that z is $\mathcal{F}_t$-measurable. Then, by the conditional expectation properties we have that $E_{P^*}[z|\mathcal{F}_t] = z$. Therefore,

$$\log(P(t, T)) = -\int_t^T \mu_s ds + \frac{1}{2}\int_t^T \sigma_u^2 c^2(u; u, T) du + \log\left(\exp\left(-\int_{-\infty}^{t} \sigma_u c(u; t, T)\, dW_u\right)\right).$$

Hence,

$$\log(P(t, T)) = -\int_t^T \mu_s ds + \frac{1}{2}\int_t^T \sigma_u^2 c^2(u; u, T) du - \int_{-\infty}^{t} \sigma_u c(u; t, T)\, dW_u.$$

Thus, we have:

$$\mathrm{Var}\left(-\frac{1}{T-t}\log(P(t, T)\right) = \left(\frac{1}{T-t}\right)^2 Var\left(-\int_{-\infty}^{t} \sigma_u c(u; t, T)\, dW_u\right).$$

Notice that, since $\sigma_u c(u;t,T)$ is deterministic,

$$\int_{-\infty}^{t} \sigma_u c(u;t,T) dW_u \sim N\left(0, \int_{-\infty}^{t} |\sigma_u c(u;t,T)|^2 du\right)$$

Then, it holds that:

$$\begin{aligned}\mathrm{Var}\left(-\frac{1}{T-t}\log(P(t,T)\right) &= \left(\frac{1}{T-t}\right)^2 \int_{-\infty}^{t} |\sigma_u c(u;t,T)|^2 du \\ &= \left(\frac{1}{T-t}\right)^2 \int_{-\infty}^{t} \sigma_u^2 c^2(u;t,T) du.\end{aligned}$$

2.3.1 Interest rates

Consider that at time t we sell a bond with maturity S and with the money we receive, $P(t,S)$, we buy $P(t,S)/P(t,T)$ bonds with maturity $T > S$. By this operation we have a contract such that we pay 1 euro at time S and we receive $P(t,S)/P(t,T)$ at time T. This change of 1 euro at time S to $P(t,S)/P(t,T)$ at time T, can be quoted by simple or continuously compounded interest rates in the period $[S,T]$.

- The simple forward interest rate (LIBOR), $L = L(t;S,T)$ is the solution of the equation:
$$\frac{P(t,S)}{P(t,T)} - \underbrace{1}_{\text{we pay 1 euro at time S}} = L(T-S).$$
- The continuously compounded interest rates, $R = R(t;S,T)$ is the solution of the equation:
$$e^{R(T-S)} = \frac{P(t,S)}{P(t,T)}.$$

Definition 2.3.2 *We define the continuously compounded forward rate contracted at time $t \in [S,T]$ as:*

$$R(t;S,T) = \frac{\log(P(t,S)) - \log(P(t,T))}{T-S}.$$

Definition 2.3.3 *We define the continuously compounded spot rate for $[t,T]$ as:*

$$R(t;t,T) = \frac{-\log(P(t,T))}{T-t}.$$

Definition 2.3.4 *We define the instantaneous forward rate with maturity T contracted at t as:*

$$\lim_{T\to S} R(t;S,T) = \frac{-\partial \log(P(t,S))}{\partial S} = f(t,S).$$

Notice that $\frac{-\partial \log(P(t,S))}{\partial S} = f(t,S)$ can be seen as a differential equation to be solved for $\log(P(t,T))$ under the initial condition $P(T,T) = 1$. Indeed, for $0 \le t \le T$:

$$\log(P(t,T)) = \log(P(t,T)) - \log(P(t,t)) = \int_t^T \frac{\partial \log(P(t,s))}{\partial s} ds = -\int_t^T f(t,s) ds.$$

This last expression is equivalent to $P(t,T) = \exp\left(-\int_t^T f(t,s)ds\right)$.

2.3.2 Absence of Arbitrage condition

We have that $f(t,T) = -\partial_T \log(P(t,T))$ is the instantaneous forward rate, and recall that $c(u;u,T) = \int_u^T g(s-u)ds$, so by the theorem of Exchanging derivative and stochastic integral proved in detail in Appendix B, we have the following expression:

$$f(t,T) = \partial_T \left(\int_t^T \mu_s ds \right) - \partial_T \left(\frac{1}{2} \int_t^T \sigma_u^2 c^2(u;u,T) du \right) + \partial_T \left(\int_\infty^t \sigma_u c(u;t,T) dW_u \right) =$$
$$= \mu_T - \partial_T \left(\frac{1}{2} \int_t^T \sigma_u^2 \int_u^T g(s-u) ds\, c(u;u,T) du \right) + \partial_T \left(\int_{-\infty}^t \sigma_u \int_t^T g(s-u) ds dW_u \right)$$
$$= \mu_T - \int_t^T \left(\sigma_u^2 g(T-u) c(u;u,T) du \right) + \int_{-\infty}^t \left(\sigma_u g(T-u) dW_u \right)$$

Notice that $\mathrm{Var}(f(t,T)) = \int_{-\infty}^t \sigma_u^2 g^2(T-u) du$. We have that:

$$d_t f(t,T) = \sigma_t^2 g(T-t) c(t;t,T) dt + \sigma_t g(T-t) dW_u.$$

Call $\alpha(t,T) := \sigma_t^2 g(T-t) c(t;t,T)$ and $\sigma(t,T) := \sigma_t g(T-t)$.

Next, we show that with these conditions there is no arbitrage opportunity using the following fact:

Fact: Under the condition $\alpha(t,T) = \sigma(t,T) \int_t^T \sigma(t,s) ds$, $t \in [0,T]$, which is known as the Heath-Jarrow-Morton (HJM) absence of arbitrage condition, the discounted prices defined as $\tilde{P}(t,T) = e^{-\int_0^t r_s ds} P(t,T)$ become martingale under P^*.

Proof. Consider the relation $r_t = f(t,t)$. We have that $P(t,T) = \exp\left(-\int_t^T f(t,s) ds \right)$ for $t \leq T$, so $f(t,s)$ represents the instantaneous rates at time s anticipated by the market t. Suppose that under the risk neutral probability, P^*:

$$df(t,T) = \alpha(t,T) dt + \sigma(t,T) dW_t, \text{ for } T \leq 0$$

with initial condition $f(0,T)$ and α and σ deterministic. Notice that $\tilde{P}(t,T) = e^{-\int_0^t r_s ds} P(t,T) = \exp\left(-\int_0^t r_s ds - \int_t^T f(t,s) ds \right)$. Let

$$Y_t = \exp\left(-\int_0^t r_s ds - \int_t^T f(t,s) ds \right).$$

We want to prove that $(Y_t)_{0 \leq t \leq T}$ is martingale under P^*. Let

$$Z_t = -\int_0^t r_s ds - \int_t^T f(t,s) ds.$$

By Itô's formula:

$$dY_t = Y_t\, dZ_t + \frac{1}{2} Y_t\, (dZ_t)^2.$$

We have that:

$$f(t,T) = f(0,T) + \int_0^t \alpha(u,T)\,du + \int_0^t \sigma(u,T)\,dW_u,\ \forall T \geq 0.$$

Denote

$$H_t := \int_t^T f(t,s)ds = \int_t^T f(0,s)ds + \int_t^T \int_0^t \alpha(u,s)du\,ds + \int_t^T \int_0^t \sigma(u,s)dW_u\,ds =$$
$$\underbrace{=}_{\text{Stochastic Fubini}} \int_0^T f(0,s)ds + \int_0^t \left(\int_t^T \alpha(u,s)ds\right) du + \int_0^t \left(\int_t^T \sigma(u,s)ds\right) dW_u.$$

On the other hand, using the fact that $f(s,s) = r_s$,

$$\int_t^T f(0,s)ds = -\int_t^T \left(\int_0^s \frac{\partial f(u,s)}{\partial u}du\right) ds + \int_t^T r_s ds \underbrace{=}_{\text{Fubini}}$$
$$= -\int_0^t \left(\int_t^T \frac{\partial f(u,s)}{\partial u}ds\right) du - \int_t^T \left(\int_u^T \frac{\partial f(u,s)}{\partial u}ds\right) du + \int_t^T r_s ds =$$
$$= \int_0^t \left(\int_u^T \frac{\partial f(u,s)}{\partial u}ds - \int_t^T \frac{\partial f(u,s)}{\partial u}ds\right) du - \int_0^T \left(\int_u^T \frac{\partial f(u,s)}{\partial u}ds\right) du + \int_t^T r_s ds =$$
$$= \int_0^t \left(\int_u^t \frac{\partial f(u,s)}{\partial u}ds\right) du - \int_0^T \left(\int_u^T \frac{\partial f(u,s)}{\partial u}ds\right) du + \int_t^T r_s ds.$$

Hence,

$$H_t = \int_0^t \left(\int_u^t \frac{\partial f(u,s)}{\partial u}ds\right) du - \int_0^T \left(\int_u^T \frac{\partial f(u,s)}{\partial u}ds\right) du + \int_t^T r_s ds +$$
$$+ \int_0^t \left(\int_t^T \alpha(u,s)ds\right) du + \int_0^t \left(\int_t^T \sigma(u,s)ds\right) dW_u.$$

Then,

$$dH_t = \left(-r_t + \int_t^T \alpha(t,s)ds\right) dt + \left(\int_t^T \sigma(t,s)ds\right) dW_t.$$

Therefore,

$$dZ_t = -r_t dt - dH_t = -\left(\int_t^T \alpha(t,s)ds\right) dt - \left(\int_t^T \sigma(t,s)ds\right) dW_t.$$

Then,

$$dY_t = Y_t \left(-\int_t^T \alpha(t,s)ds + \frac{1}{2}\left(\int_t^T \sigma(t,s)ds\right)^2\right) dt + Y_t \left(-\int_t^T \sigma(t,s)ds\right) dW_t.$$

To finish the proof we want

$$-\int_t^T \alpha(t,s)ds + \frac{1}{2}\left(\int_t^T \sigma(t,s)ds\right)^2 = 0.$$

Notice that, since we are assuming that $\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)\,ds$, $t \in [0,T]$:

$$\frac{d}{dT}\left(-\int_t^T \alpha(t,s)ds + \frac{1}{2}\left(\int_t^T \sigma(t,s)ds\right)^2\right) =$$
$$= -\alpha(t,T) + \sigma(t,T)\int_t^T \sigma(t,s)ds = 0.$$

This means that $-\int_t^T \alpha(t,s)ds + \frac{1}{2}\left(\int_t^T \sigma(t,s)ds\right)^2$ is constant. Evaluating at $T = t$ we obtain that

$$-\int_t^T \alpha(t,s)ds + \frac{1}{2}\left(\int_t^T \sigma(t,s)ds\right)^2 = 0,$$

as wanted.

□

Since,

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds = \sigma_t g(T-t)\int_t^T \sigma_t g(t-s)ds = \sigma_t^2 g(T-t)c(t;t,T),$$

the HJM condition of absence of arbitrage explained before is satisfied.

2.4 Completeness of the market

Recall that the price at time t of a zero-coupon bond with maturity T is $P(t,T)$, and the discounted price is given by:

$$\tilde{P}(t,T) = \exp\left(-\int_0^t r_s ds\right) P(t,T).$$

We had that:

$$A(0,T) = -\int_0^T \mu_s ds + \frac{1}{2}\int_t^T \sigma_u^2 c^2(u;u,T)du =$$
$$= -\int_0^t \mu_s ds - \int_t^T \mu_s ds + \frac{1}{2}\int_t^T \sigma_u^2 c^2(u;u,T)du + \frac{1}{2}\int_0^t \sigma_u^2 c^2(u;u,T)du =$$
$$= -\int_0^t \mu_s ds + A(t,T) + \frac{1}{2}\int_0^t \sigma_u^2 c^2(u;u,T)du.$$

Therefore,

$$A(t,T) = A(0,T) + \int_0^t \mu_s ds - \frac{1}{2}\int_0^t \sigma_u^2 c^2(u;u,T)du.$$

So

$$P(t,T) = \exp\left(-\int_t^T \mu_s ds + \frac{1}{2}\int_t^T \sigma_u^2 c^2(u;u,T)du - \int_{-\infty}^t \sigma_u c(u;t,T)dW_u\right) =$$

$$= \exp\left(A(t,T) - \int_{-\infty}^t \sigma_u c(u;t,T)dW_u\right) =$$

$$= \exp\left(A(0,T) - \int_{-\infty}^0 \sigma_u c(u;0,T)dW_u + \int_{-\infty}^0 \sigma_u c(u;0,T)dW_u + \int_0^t \mu_s ds -\right.$$

$$\left. - \frac{1}{2}\int_0^t \sigma_u^2 c^2(u;u,T)du - \int_{-\infty}^0 \sigma_u c(u;t,T)dW_u - \int_0^t \sigma_u c(u;t,T)dW_u\right)$$

$$= \exp\left(A(0,T) - \int_{-\infty}^0 \sigma_u c(u;0,T)dW_u\right)\exp\left(-\frac{1}{2}\int_0^t \sigma_u^2 c^2(u;u,T)du + \int_0^t \mu_s ds\right)\cdot$$

$$\cdot \exp\left(\int_{-\infty}^0 \sigma_u (c(u;0,T) - c(u;t,T))dW_u\right)\exp\left(-\int_0^t \sigma_u c(u;t,T)dW_u\right).$$

We have that

$$P(0,T) = \exp\left(A(0,T) - \int_{-\infty}^0 \sigma_u c(u;0,T)dW_u\right).$$

Thus,

$$P(t,T) = P(0,T)\exp\left(\frac{-1}{2}\int_0^t \sigma_u^2 c^2(u;u,T)du + \int_0^t \mu_s ds\right)\cdot$$

$$\cdot \exp\left(\int_{-\infty}^0 \sigma_u c(u;0,T)dW_u\right)\exp\left(-\int_{-\infty}^t \sigma_u c(u;t,T)dW_u\right).$$

On the other hand, by Fubini's stochastic theorem,

$$\exp\left(-\int_0^t r_s ds\right) = \exp\left(-\int_0^t \left(\int_{-\infty}^s \sigma_u g(s-u)dW_u\right)ds - \int_0^t \mu_s ds\right) =$$

$$\exp\left(-\int_{-\infty}^0 \sigma_u c(u;0,t)dW_u - \int_0^t \sigma_u c(u;u,t)dW_u - \int_0^t \mu_s ds\right).$$

Then,

$$\tilde{P}(t,T) = P(0,T)\exp\left(-\frac{1}{2}\int_0^t \sigma_u^2 c^2(u;u,T)du - \int_0^t \sigma_u c(u;t,T)dW_u -\right.$$

$$\left. - \int_0^t \sigma_u c(u;u,t)dW_u\right) = P(0,T)\exp\left(-\int_0^t \sigma_u c(u;u,T)dW_u - \frac{1}{2}\int_0^t \sigma_u^2 c^2(u;u,T)du\right).$$

Let

$$Y_t = -\int_0^t \sigma_u c(u;u,T)\,dW_u - \frac{1}{2}\int_0^t \sigma_u^2 c^2(u;u,T)\,du$$

and $X_t = \exp(Y_t)$. Let

$$\phi(X) = e^X \in \mathcal{C}^2.$$

Applying Itô's formula,

$$X_t = e^{Y_t} = \phi(Y_t) = \phi(Y_0 = 0) + \int_0^t \phi'(Y_s)dYs + \frac{1}{2}\int_0^t \phi''(Y_s)(dY_s)^2 =$$
$$= 1 + \int_0^t X_s dY_s + \frac{1}{2}\int_0^t X_s(dY_s)^2 = 1 + \int_0^t X_s\{-\sigma_s c(s;s,T)dW_s - \frac{1}{2}\sigma_s^2 c^2(s;s,T)ds\} +$$
$$+ \frac{1}{2}\int_0^t X_s\{-\sigma_s c(s;s,T)dW_s - \frac{1}{2}\sigma_s^2 c^2(s;s,T)ds\}^2 =$$
$$= 1 + \int_0^t X_s\{-\sigma_s c(s;s,T)dW_s - \frac{1}{2}\sigma_s^2 c^2(s;s,T)ds\} + \frac{1}{2}\int_0^t X_s\sigma_s^2 c^2(s;s,T)ds$$
$$= 1 - \int_0^t X_s\sigma_s c(s;s,T)dW_s.$$

Then,

$$\tilde{P}(t,T) = P(0,T)X_t = P(0,T)\left(1 - \int_0^t X_s\sigma_s c(s;s,T)dW_s\right) =$$
$$= P(0,T) - P(0,T)\int_0^t X_s\sigma_s c(s;s,T)\,dW_s.$$

Hence,

$$d\tilde{P}(t,T) = -P(0,T)X_t\sigma_t c(t,t,T)\,dW_t = -\tilde{P}(t,T)\sigma_t c(t;t,T)\,dW_t, \quad t \geq 0.$$

Let $\mathbf{X}$ be a positive, P^*-square integrable and $\mathcal{F}_T$-measurable payoff. Consider $M_t := E_{P^*}[\tilde{X}|\mathcal{F}_t]$, where

$$\tilde{X} = e^{-\int_0^T r_s ds}X.$$

Notice that M_t is a martingale with respect to $(\mathcal{F}_t)$. Indeed, if $s \leq t$, by tower property

$$E_{P^*}[M_t|\mathcal{F}_s] = E_{P^*}[E_{P^*}[e^{-\int_0^T r_s ds}X|\mathcal{F}_t]|\mathcal{F}_s] = E_{P^*}[e^{-\int_0^T r_s ds}X|\mathcal{F}_s] = M_s.$$

Moreover, M_t is square integrable under P^*:

$$E_{P^*}[M_t^2] = E_{P^*}[E_{P^*}[e^{-\int_0^T r_s ds}X|\mathcal{F}_t]^2] \underbrace{\lesssim}_{\substack{\text{Jensen's}\\ \text{inequality}}} E_{P^*}[E_{P^*}[e^{-2\int_0^T r_s ds}X^2|\mathcal{F}_t]] =$$
$$= e^{-2\int_0^T r_s ds}E_{P^*}[X^2] < \infty,$$

and this is finite, because X is P^*-square integrable. Then, by the extension of the representation theorem proved in detail in Appendix B, we have:

$$M_t = c + \int_{-\infty}^t H_s dW_s,$$

where c is a constant and H_s is square integrable and $\mathcal{F}_t$-measurable. Hence, $dM_t = H_t dW_t$. Let us define $\phi_t^1 := \frac{H_t}{-\tilde{P}(t,T)\sigma_t c(t;t,T)}$. Then,

$$M_t = c + \int_{-\infty}^{t} \phi_s^1[-\tilde{P}(s,T)\sigma_s c(s;s,T)]dW_s =$$
$$= c + \int_{-\infty}^{t} \phi_s^1 d\tilde{P}(s,T).$$

Now, define $\phi_t^0 := M_t - \phi_t^1 \tilde{P}(t,T)$. Consider the portfolio built by a bank account and a T-bond. Its value at time t is given by:

$$V_t(\phi) = \phi_t^0 S_t^0 + \phi_t^1 P(t,T) = \phi_t^0 e^{\int_0^t r_s ds} + \phi_t^1 P(t,T).$$

Its discounted value is given by:

$$\tilde{V}_t(\phi) = \frac{V_t(\phi)}{e^{\int_0^t r_s ds}} = \phi_t^0 + \phi_t^1 \frac{P(t,T)}{e^{\int_0^t r_s ds}} = \phi_t^0 + \phi_t^1 \tilde{P}(t,T).$$

So,

$$\tilde{V}_t(\phi) = \phi_t^0 + \phi_t^1 \tilde{P}(t,T) = M_t.$$

Hence,

$$V_t(\phi) = e^{\int_0^t r_s ds} \tilde{V}_t(\phi) = E\left[e^{-\int_t^T r_s ds} X \middle| \mathcal{F}_t \right].$$

Finally, since $M_t = c + \int_{-\infty}^{t} \phi_s^1 d\tilde{P}(s,T)$, the strategy $\phi = (\phi^0, \phi^1)$ is self-financing.

Hence, X can be replicated, because $V_T(\phi) = E_{P^*}[X|\mathcal{F}_t] = X$. Then, since any option with positive payoff X, square integrable with respect to P^* is replicable, the market is complete.

2.5 Option Prices

We want to find a general formula for a European option, so let us assume that we are given a financial market with short interest rate and strictly positive bond prices. We are going to see a procedure to calculate prices of options in a bond market related with forward measure. By definition of the neutral probability P^*, the discounted prices $(\tilde{P}(t,T))_{0 \le t \le T}$ are martingales for all values of T. Let us fix a maturity time T and consider the values of bonds with another maturity time $\bar{T} > T$ in terms of bonds with maturity T:

$$U(t,T,\bar{T}) := \frac{P(t,T)}{P(t,\bar{T})}.$$

In this case, notice that we are taking as a numeraire the price of a bond with maturity $\bar{T}$. Let $P^{\bar{T}}$ be a probability such that $(U(t,T,\bar{T}))$ is a martingale for all $\bar{T} > T$. This $P^{\bar{T}}$ is called the forward measure. In order to find the formula for the

price, we start by the following fact:

Fact: Let $(S_t)_{0\leq t\leq T}$ be the price of a strictly positive bond at time t. Then the price of a call option on the asset S, with maturity T and strike K, is given by:

$$\Pi(t;S) = S_t P^T(S_T \geq K|\mathcal{F}_t) - KP(t,T)P^T(S_T \geq K|\mathcal{F}_t).$$

Proof. Taking into account that: $S = (S_T - K)_+ = \begin{cases} S_T - K, \text{ if } S_T \geq K \\ 0, \text{ otherwise} \end{cases}$,

$$\begin{aligned}
&\Pi(t;S) = E_{P^*}\left(e^{-\int_t^T r_s ds}(S_T - K)_+|\mathcal{F}_t\right) = \\
&= E_{P^*}\left(e^{-\int_t^T r_s ds}(S_T - K)\cdot \mathbb{1}_{\{S_T\geq K\}}|\mathcal{F}_t\right) = \\
&= E_{P^*}\left(e^{-\int_t^T r_s ds}S_T\mathbb{1}_{\{S_T\geq K\}}|\mathcal{F}_t\right) - KE_{P^*}\left(e^{-\int_t^T r_s ds}\mathbb{1}_{\{S_T\geq K\}}|\mathcal{F}_t\right) = \\
&= S_t P^T\left(S_T \geq K|\mathcal{F}_t\right) - KP(t,T)P^T\left(S_T \geq K|\mathcal{F}_t\right).
\end{aligned}$$

□

Now suppose that S is another bond with maturity $\bar{T} > T$, then the option with maturity T on this bond has a price [1]:

$$\begin{aligned}
\Pi(t;S) &= P(t,\bar{T})P^{\bar{T}}(P(T,\bar{T}) \geq K|\mathcal{F}_t) - KP(t,T)P^T(P(T,\bar{T}) \geq K|\mathcal{F}_t) = \\
&= P(t,\bar{T})P^{\bar{T}}\left(\frac{P(T,T)}{P(T,\bar{T})} \leq \frac{1}{K}\middle|\mathcal{F}_t\right) - KP(t,T)P^{\bar{T}}\left(\frac{P(T,\bar{T})}{P(T,T)} \geq K\middle|\mathcal{F}_t\right),
\end{aligned}$$

where P^T and $P^{\bar{T}}$ are the T-forward measure and the $\bar{T}$-forward measure. We defined before $U(t,T,\bar{T}) := \frac{P(t,T)}{P(t,\bar{T})}$. Then, since

$$P(t,T) = \exp\left(A(t,T) - \int_{-\infty}^t \sigma_u c(u;t,T)\,dW_u\right)$$

and

$$P(t,\bar{T}) = \exp\left(A(t,\bar{T}) - \int_{-\infty}^t \sigma_u c(u;t,\bar{T})\,dW_u\right),$$

we have that:

$$\begin{aligned}
U(t,T,\bar{T}) &= \exp\left(A(t,T) - \int_{-\infty}^t \sigma_u c(u;t,T)dW_u - A(t,\bar{T}) - \int_{-\infty}^t \sigma_u c(u,t,\bar{T})dW_u\right) = \\
&= \exp\left(-A(t,\bar{T}) + A(t,T) - \int_{-\infty}^t \sigma_u(c(u;t,T) - c(u;t,\bar{T}))dW_u\right).
\end{aligned}$$

Now, taking the $\bar{T}$-forward measure, $P^{\bar{T}}$, by Girsanov's Theorem stated and proved in Appendix B we have that:

$$dW_u = dW_u^{\bar{T}} - a(u)du,$$

where $dW_u^{\bar{T}}$ is the random Wiener measure in $\mathbb{R}$ again and $a(u)$ is deterministic. As we said, $U(t,T,\bar{T})$ has to be martingale with respect to $P^{\bar{T}}$. Recall that:

$$A(t,\bar{T}) = \frac{1}{2}\int_t^{\bar{T}} \sigma_u^2 c^2(u;u,\bar{T})du - \int_t^{\bar{T}} \mu_s ds, \quad u \leq t,$$

$$A(t,T) = \frac{1}{2}\int_t^{T} \sigma_u^2 c^2(u;u,\bar{T})du - \int_t^{T} \mu_s ds.$$

Hence, we can write $U(t,T,\bar{T})$ as:

$$U(t,T,\bar{T}) = \exp\left(-\int_{-\infty}^t \sigma_u(c(u;t,T) - c(u;t,\bar{T}))dW_u^{\bar{T}} - \frac{1}{2}\int_{-\infty}^t \sigma_u^2(c(u;t,T) - c(u;t,\bar{T}))^2 du\right).$$

Then if we call $U(T) := U(T,T,\bar{T})$ and analogously $U^{-1}(T) = U(T,\bar{T},T)$ we have the following expressions:

$$U(T) = U(t,T,\bar{T})\exp\left(\int_t^T \sigma_u c(u;T,\bar{T})dW_u^{\bar{T}} - \frac{1}{2}\int_t^T \sigma_u^2 c(u;T,\bar{T})^2 du\right)$$

and

$$U^{-1}(T) = U^{-1}(t,T,\bar{T})\exp\left(-\int_t^T \sigma_u c(u;T,\bar{T})dW_u^{\bar{T}} - \frac{1}{2}\int_t^T \sigma_u^2 c(u;T,\bar{T})^2 du\right).$$

Hence, taking into account that if $a > 0$ and X is positive random variable $P(X \leq a) = P(\log(X) \leq \log(a))$,

$$\begin{aligned}
&\Pi(t;S) = P(t,\bar{T})P^{\bar{T}}\left(U(T) \leq \frac{1}{K}\middle|\mathcal{F}_t\right) - KP(t,T)P^T(U^{-1}(T) \geq K|\mathcal{F}_t) = \\
&= P(t,\bar{T})P^{\bar{T}}\left(\log(U(T)) \leq \log\left(\frac{1}{K}\right)\middle|\mathcal{F}_t\right) - KP(t,T)P^T(\log(U^{-1}(T)) \geq \log(K)|\mathcal{F}_t) = \\
&= P(t,\bar{T})P^{\bar{T}}(\log(U(T)) \leq -\log(K)|\mathcal{F}_t) - KP(t,T)P^T(\log(U^{-1}(T)) \geq \log(K)|\mathcal{F}_t) = \\
&= P(t,\bar{T})\phi(d_+) - KP(t,T)\phi(d_-),
\end{aligned}$$

where the function ϕ denotes the normal accumulate function and

$$d_\pm := \frac{\log(\frac{P(t,\bar{T})}{KP(t,T)}) \pm \frac{1}{2}\sum_{t,T,\bar{T}}^2}{\sum_{t,T,\bar{T}}},$$

where

$$\sum_{t,T,\bar{T}}^2 := \int_t^T \sigma_u^2 c(u;T,\bar{T})^2 du.$$

Example 1:

Let $g(t) = e^{-bt}$, $\sigma_u = \sigma$ and $\mu = a$. Then

$$\begin{aligned}
r_t &= \int_{-\infty}^{t} g(t-s)\sigma_s dW_s + \mu_t = \\
&= \int_{-\infty}^{0} g(t-s)\sigma_s dW_s + \int_{0}^{t} g(t-s)\sigma_s dW_s + \mu_t = \\
&= e^{-bt}\int_{-\infty}^{0} e^{bs}\sigma dW_s + \int_{0}^{t} e^{-b(t-s)}\sigma dW_s + a = \\
&= r_0 e^{-bt} - ae^{-bt} + \int_{0}^{t} e^{-b(t-s)}\sigma dW_s + a = \\
&= r_0 e^{-bt} + a(1-e^{-bt}) + e^{-bt}\int_{0}^{t} e^{bs}\sigma dW_s.
\end{aligned}$$

This is the Vasicek model. Taking into account that $g(T-u) \neq 0$ and positive:

$$\begin{aligned}
P(t,T) &= \exp\left(A(t,T) - \int_t^T \left(\int_{-\infty}^{T}\left(\int_{-\infty}^{t} \sigma g(s-u)dW_u\right)ds\right)\right) = \\
&= \exp\left(A(t,T) - \int_t^T \left(\int_{-\infty}^{t} \sigma\frac{g(s-u)}{g(t-u)}g(t-u)dW_u\right)ds\right) = \\
&= \exp\left(A(t,T) - \int_{-t}^{T}\int_{-\infty}^{t} \sigma e^{-b(s-u)+b(t-u)-b(t-u)}\sigma dW_u ds\right) = \\
&= \exp\left(A(t,T) - \int_{-t}^{T} e^{-b(s-t)}\int_{-\infty}^{t} e^{-b(t-u)}\sigma dW_u ds\right) = \\
&= \exp\left(A(t,T) - \left(\int_t^T e^{-b(s-t)}ds\right)(r_t - a)\right) = \\
&= \exp(A(t,T) + aB(t,T) - r_t B(t,T)),
\end{aligned}$$

where

$$B(t,T) = \int_t^T e^{-b(s-t)}ds.$$

We have that:

$$\begin{aligned}
A(t,T) &= \frac{1}{2}\int_t^T \sigma_u^2 c^2(u;u,T)du - \int_t^T \mu_s ds = \\
&= \frac{1}{2}\int_t^T \sigma^2 c^2(u;u,T)du - a(T-t),
\end{aligned}$$

where $c(u;u,T) = \int_u^T g(s-u)ds$. Then:

$$
\begin{aligned}
A(t,T) &= \frac{\sigma^2}{2}\int_t^T \left(\int_u^T g(s-u)ds\right)^2 du - a(T-t) = \\
&= \frac{\sigma^2}{2}\int_t^T \left(\int_u^T e^{-b(s-u)}ds\right)^2 du - a(T-t) = \\
&= \frac{\sigma^2}{2}\int_t^T \left(-\frac{1}{b}\left[1-e^{-b(T-u)}\right]\right)^2 du - a(T-t) = \\
&= \frac{\sigma^2}{2}\int_t^T B(u,T)^2 du - a(T-t).
\end{aligned}
$$

As we saw in section 2.3,

$$
\mathrm{Var}\left(-\frac{1}{T-t}\log(P(t,T)\right) = \frac{1}{(T-t)^2}\int_{-\infty}^t \sigma^2 c^2(u;t,T)du,
$$

where

$$
\begin{aligned}
c(u;t,T) &= \int_t^T g(s-u)ds = \int_t^T e^{-b(s-u)}ds = \\
&= \frac{-1}{b}\left[e^{-b(T-u)} - e^{-b(t-u)}\right] = \frac{1}{b}\left[e^{-b(t-u)} - e^{-b(T-u)}\right]
\end{aligned}
$$

for $u \le t \le T$. Thus,

$$
\begin{aligned}
&\mathrm{Var}\left(\frac{-1}{T-t}\log(P(t,T))\right) = \frac{1}{(T-t)^2}\int_{-\infty}^t \frac{\sigma^2}{b^2}\left(e^{2bu}(e^{-bt}-e^{-bT})^2\right)du = \\
&= \frac{1}{(T-t)^2}\frac{\sigma^2}{b^2}(e^{-bt}-e^{-bT})^2\int_{\infty}^T e^{2bu}du = \\
&= \frac{\sigma^2}{(T-t)^2}\frac{1}{2b^3}(1-e^{-b(T-t)})^2 \sim \frac{1}{T^2}.
\end{aligned}
$$

Then, when $T \to \infty$, as we saw in section 2.3, the forward rates are given by:

$$
\begin{aligned}
f(t,T) &= -\int_t^T \sigma_u^2 g(T-u)c(u;u,T)du + \int_{-\infty}^t \sigma_u g(T-t)dW_u + \mu_T = \\
&= -\int_t^T \left(\sigma_u e^{-b(T-u)}\int_u^T e^{-b(s-u)}ds\right)du + \int_{-\infty}^t \sigma_u g(T-t)dW_u + a = \\
&= -\int_t^T \sigma^2 e^{-b(T-u)}\left[\frac{1}{b}(1-e^{-b(T-u)})\right]du + \sigma(r_t-a)e^{-b(T-t)} + a = \\
&= -\frac{\sigma^2}{2b^2}(1-e^{-b(T-t)})^2 + \sigma e^{-b(T-t)}(r_t-a) + a
\end{aligned}
$$

and

$$\mathrm{Var}(f(t,T)) = \int_{-\infty}^{t} \sigma_u^2 g(T-u)du = \int_{-\infty}^{t} \sigma_u e^{-2b(T-u)}du =$$
$$= \sigma^2 \int_{-\infty}^{t} e^{-2b(T-u)}du = \frac{\sigma^2}{2b} e^{-2b(T-t)} \sim e^{-2bT},$$

when $T \to \infty$. Then

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t,$$

where

$$\alpha(t,T) = \sigma_t^2 g(T-t)c(t;t,T) = \sigma^2 e^{-b(T-t)} \int_t^T g(s-t)ds = \sigma^2 e^{-b(T-t)} \frac{1}{b}[1-e^{-b(T-t)}].$$

The volatility is given by

$$\sigma(t,T) = \sigma_t g(T-t) = \sigma e^{-b(T-t)}.$$

Next, let us see another example.

Example 2:

Now let $\sigma_t = \sigma \mathbb{1}_{t\geq 0}$ and $g(t-u) = e^{-b(t-u)} \int_0^{t-u} e^{bs}\beta s^{\beta-1}ds$, where $\beta \in (1/2, 1)$. We have that

$$c(u;t,T) = \int_t^T g(s-u)ds.$$

Hence, since

$$c(0;0,x) = \int_0^x g(s-0)ds = \int_0^x e^{-bs} \int_0^s e^{bu}\beta u^{\beta-1}\, du\, ds,$$

it holds that:

$$c(0;0,T-u) = \int_0^{T-u} g(s)ds = e^{-b(T-u)} \int_0^{T-u} e^{bs}s^{\beta}ds$$

and

$$c(0;0,t-u) = \int_0^{t-u} g(s)ds = e^{-b(t-u)} \int_0^{t-u} e^{bs}s^{\beta}ds.$$

Then, doing a change of variable we have:

$$c(u;t,T) = \int_t^T g(s-u)ds = \int_{t-u}^{T-u} g(s)ds = \int_0^{T-u} g(s)ds - \int_0^{t-u} g(s)ds =$$
$$= c(0;0,T-u) - c(0,0,t-u).$$

Thus,

$$\mathrm{Var}\left(\frac{-1}{T-t}\log(P(t,T))\right) = \frac{1}{(T-t)^2}\int_{-\infty}^{t}\sigma_u c^2(u;t,T)du =$$
$$= \frac{1}{(T-t)^2}\int_0^t \sigma^2\left[c(0;0,T-u)-c(0,0,t-u)\right]^2 du.$$

Now we show that, when $T \to +\infty$,

$$\mathrm{Var}\left(\frac{-1}{T-t}\log(P(t,T))\right) \to \frac{1}{T^2}\int_0^T c(0,0,T-u)^2 du \sim \frac{T^{2\beta}}{T^2},$$

$$c(0;0,s) = e^{-bs}\int_0^x e^{bs}s^\beta ds = x^\beta \int_0^x e^{-bs}\left(1-\frac{s}{x}\right)^\beta ds.$$

By the M.C.T,

$$\lim_{x\to+\infty}\int_0^x e^{bs}\left(1-\frac{s}{x}\right)^\beta ds = \int_0^{+\infty} e^{-bs}ds = \frac{1}{b}.$$

Furthermore, it holds that:

$$\mathrm{Var}\left(\frac{-1}{T-t}\log(P(t,T))\right) \sim \frac{T^{2\beta}}{T^2},$$

as wanted, because since for $x \geq 0$

$$g(x) = e^{-bx}\int_0^x e^{bs}\beta s^{\beta-1}ds = \beta x^{\beta-1}\int_0^x e^{-bs}\left(1-\frac{s}{x}\right)ds =$$
$$= \beta x^{\beta-1}\left(\int_0^{x/2} e^{-bs}\left(1-\frac{s}{x}\right)^{\beta-1}ds + \int_{x/2}^x e^{-bs}\left(1-\frac{s}{x}\right)\right),$$

and since

$$\lim_{x\to\infty}\int_0^{x/2} e^{-bs}\left(1-\frac{s}{x}\right)^{\beta-1}ds = \int_0^{+\infty} e^{-bs}ds = \frac{1}{b},$$

we have that:

$$\int_{x/2}^x e^{-bs}\left(1-\frac{s}{x}\right)^{\beta-1}ds \leq e^{-bx/2}\int_{x/2}^x\left(1-\frac{s}{x}\right)^{\beta-1}ds =$$
$$= xe^{-bx/2}\int_0^{1/2} v^{\beta-1}dv = \frac{xe^{-bx/2}}{\beta 2^\beta} \xrightarrow{x\to\infty} 0.$$

Notice that, for $\beta \in (-1/2, 0)$, if we consider

$$g(x) = e^{-bx}x^\beta + \beta\int_0^x (e^{-b(x-u)} - e^{-bx})u^{\beta-1}du \sim x^{\beta-1}$$

when $x \to \infty$, we obtain analogous results as those when $\beta \in (0, 1/2)$.

2.6 A SDE approach

Let us consider:

$$r_t = \int_0^t g(t-s)\sigma_s dW_s + \mu_t.$$

We want to show that $(r_t)_{t\in\mathbb{R}}$ can be seen as the solution of a SDE. Firstly, assume that r_t is such that:

$$dr_t = b(a - r_t)dt + \sigma dW_t,$$

where a, b and σ are deterministic. This last expression is the same as

$$dr_t + br_t\,dt = ba\,dt + \sigma\,dW_t.$$

Multiplying at both sides by e^{bt}, we get:

$$e^{bt}(dr_t + br_t\,dt) = bae^{bt} + \sigma e^{bt}\,dW_t.$$

Then,

$$d(r_t e^{bt}) = bae^{bt} + \sigma e^{bt} dW_t.$$

Integrating this last expression in $[0, t]$, we obtain:

$$r_t e^{bt} - r_0 = a(e^{bt} - 1) + \int_0^t \sigma e^{bs} dW_s.$$

Finally, if we multiply at both sides by e^{-bt}, we arrive at

$$r_t = r_0 e^{-bt} + a(1 - e^{-bt}) + \int_0^t \sigma e^{-b(t-s)} dW_s.$$

Then,

$$r_t = r_0 e^{-bt} + a(1 - e^{-bt}) + e^{-bt} \int_0^t e^{bs}\sigma dW_s,$$

and the solution is unique. If we take $r_0 = \int_{-\infty}^0 e^{bs}\sigma dW_s + a$, we get that:

$$r_t = a + \int_{-\infty}^t e^{-b(t-s)}\sigma dW_s.$$

And this corresponds to $\mu_t = a$, $g(t) = e^{-bt}$ and $\sigma_t = \sigma$.

2.6.1 Ambit process as noises of SDE

Let us consider the process W^g given by:

$$W_t^g = \int_{-\infty}^t g(s,t)dW_s,$$

where $g : \mathbb{R}^2 \to \mathbb{R}$ deterministic and continuously differentiable with respect $g(s, \cdot)$.

Let us assume that $g(s,t) = 0$ for $s > t$ and $\int_{-\infty}^{t} g^2(s,t)ds < \infty$, then W_t^g is well-defined. First, formally we have that:

$$dW_t^g = g(t,t)dW_t + \left(\int_{-\infty}^{t} \partial_t g(s,t)dW_s\right) dt.$$

Then, if we consider $f(\cdot,\cdot)$ deterministic, we can define:

$$\begin{aligned}
&\int_{-\infty}^{t} f(u,t)dW_u^g = \int_{-\infty}^{t} f(u,t)((g(u,u)dW_u + \int_{-\infty}^{u} \partial_u g(s,u)dW_s)du) = \\
&= \int_{-\infty}^{t}\int_{-\infty}^{u} f(u,t)\partial_u g(s,u)dW_s + \int_{-\infty}^{t} f(u,t)g(u,u)dW_u + \\
&+ \int_{-\infty}^{t}\int_{s}^{t} f(s,t)\partial_u g(s,u)dudW_s - \int_{-\infty}^{t}\int_{s}^{t} f(s,t)\partial_u g(s,u)dudW_s = \\
&= \int_{-\infty}^{t}\int_{-\infty}^{u} f(u,t)\partial_u g(s,u)dW_s + \int_{-\infty}^{t} f(u,t)g(u,u)dW_u + \\
&+ \int_{-\infty}^{t}\int_{s}^{t} f(s,t)\partial_u g(s,u)dudW_s - \int_{-\infty}^{t}\int_{-\infty}^{u} f(s,t)\partial_u g(s,u)dW_s du.
\end{aligned}$$

Thus, we have that, for $-\infty \leq u \leq s \leq t$,

$$\begin{aligned}
&\int_{-\infty}^{t} f(u,t)dW_u^g = \int_{-\infty}^{t} \left(\int_{-\infty}^{u} (f(u,t) - f(s,t)\right) \partial_u g(s,u)dW_s)du + \\
&\int_{-\infty}^{t} \left(\int_{s}^{t} f(s,t)\partial_u g(s,u)du\right) dW_s + \int_{-\infty}^{t} f(u,t)g(u,u)dW_u.
\end{aligned}$$

Then,

$$\begin{aligned}
&\int_{-\infty}^{t} f(u,t)dW_u^g = \int_{-\infty}^{t} f(s,t)g(s,t)dW_s + \\
&+ \int_{-\infty}^{t} \left(\int_{s}^{t} (f(u,t) - f(s,t))\partial_u g(s,u)du\right) dW_s = \\
&= \int_{-\infty}^{t}\int_{s}^{t} ((f(u,t) - f(s,t))\partial_u g(s,u)du + f(s,t)g(s,t))dW_s.
\end{aligned}$$

If we assume that:

$$\int_{-\infty}^{t}\int_{s}^{t} ((f(u,t) - f(s,t))\partial_u g(s,u)du + f(s,t)g(s,t))^2 ds < +\infty,$$

it holds that last integral,

$$\int_{-\infty}^{t}\int_{s}^{t} ((f(u,t) - f(s,t))\partial_u g(s,u)du + f(s,t)g(s,t))dW_s$$

is well-defined, because it is square integrable. Now, let us define the following operator:

$$K_t^g(f)(s,t) := \int_s^t (f(u,t) - f(s,t))\partial_u g(s,t)du + f(s,t)g(s,t).$$

Hence, we can define:

$$\int_{-\infty}^t f(s,t)dW_s^g := \int_{-\infty}^t K_t^g(f)(s,t)dW_s,$$

provided that $f(\cdot,t) \in (K_t^g)^{-1}(L^2(-\infty,t])$. If $g(s,s) = 0$, since $f(s,t)$ does not depend on u:

$$K_t^g(s,t) = \int_s^t f(u,t)\partial_u g(s,u)du - f(s,t)\int_s^t \partial_u g(s,u)du + f(s,t)g(s,t)$$

Then, we can write

$$K_t^g(f)(s,t) := \int_s^t f(u,t)\partial_u g(s,u)du = -\int_s^t f(u,t)g(s,u)du + [f(u,t)g(s,u)]_s^t.$$

applying integration by parts. Since $g(s,s) = 0$,

$$K_t^g(f)(s,t) = -\int_s^t f(u,t)g(s,u)du + [f(u,t)g(s,u)]_s^t =$$
$$= -\int_s^t f(u,t)g(s,u)du + f(t,t)g(s,t).$$

In the particular case:

$$\frac{\partial}{\partial t}f(u,t) + \frac{\partial}{\partial u}f(u,t) = 0,$$

since

$$\partial_t \int_s^t f(u,t)g(s,u)du = f(t,t)g(s,t) + \int_s^t \partial_t f(u,t)g(s,u)du,$$

we will have that

$$K_t^g(f)(s,t) = \partial_t \int_s^t f(u,t)g(s,u)du.$$

Notice that this can be seen as a convolution. Then, in this case:

$$K_t^g(f)(s,t) = \partial_t(f * g)(s,t).$$

So, since we defined

$$\int_{-\infty}^t f(s,t)dW_s^g := \int_{-\infty}^t K_t^g(f)(s,t)dW_s,$$

we have that:

$$\int_{-\infty}^{t} f(s,t)dW_s^g = \int_{\infty}^{t} \left(\partial_t \int_s^t f(u,t)g(s,u)du \right) dW_s =$$
$$= \frac{\partial}{\partial t} \int_{-\infty}^{t} \int_s^t f(u,t)g(s,u)dudW_s.$$

Since

$$\int_s^t (f(u,t)g(s,u))^2 du \leq \int_{-\infty}^{t} (f(u,t)g(s,t))^2 du < +\infty,$$

we can apply stochastic Fubini and we get:

$$\int_{-\infty}^{t} f(s,t)dW_s^g = \frac{\partial}{\partial t} \int_{-\infty}^{t} f(u,t)(\int_{-\infty}^{u} g(s,u)dW_u)du =$$
$$\frac{\partial}{\partial t} \int_{-\infty}^{t} f(u,t)dW_u^g,$$

where the last equality is by definition of W_u^g. Now, let us consider:

$$r_t = b \int_0^t (a - r_s)ds + \sigma \int_0^t (t-s)^\beta dW_s$$

with $\beta \in (-1/2, 0) \cup (0, 1/2)$. Then, if we define

$$W_t^\beta := \int_0^t (t-s)^\beta dW_s,$$

we can write:

$$r_t = b \int_0^t (a - r_s)ds + \sigma W_t^\beta.$$

Differentiating this last expression we get

$$dr_t = b(a - r_t)dt + \sigma dW_t^\beta,$$

and this is the same as

$$dr_t + br_t\, dt = ba\, dt + \sigma\, dW_t^\beta.$$

If we multiply at both sides by e^{bt}, we obtain:

$$e^{bt}(dr_t + br_t\, dt) = bae^{bt} + \sigma e^{bt}\, dW_t^\beta.$$

Then,

$$d(r_t e^{bt}) = bae^{bt} + \sigma e^{bt} dW_t^\beta.$$

Integrating this last expression in $[0, t]$, we get

$$r_t e^{bt} - r_0 = a(e^{bt} - 1) + \int_0^t \sigma e^{bs} dW_s^\beta.$$

Therefore,

$$r_t = r_0 e^{-bt} + a(1 - e^{-bt}) + e^{-bt} \int_0^t e^{bs} \sigma dW_s^\beta$$

and

$$r_t = r_0 e^{-bt} + a(1 - e^{-bt}) + \int_0^t \sigma g(t-s) dW_u.$$

Let $\beta \in (0, 1/2)$. Since $K_t^g(f)(s,t) = \int_s^t f(u,t)\partial_u g(s,u)du$,

$$\begin{aligned}
&\int_0^t e^{-b(t-s)} dW_s^\beta = \\
&= \int_0^t K_t^\beta(f)(s,t) dW_s = \int_0^t \int_u^t e^{-b(t-s)} \partial_s (t-s)^\beta ds dW_u = \\
&= \int_0^t \int_u^t e^{-b(t-s)} \beta (s-u)^{\beta-1} ds dW_u = \int_0^t \int_0^{t-u} e^{-b(t-u-s)} \beta s^{\beta-1} ds dW_u = \\
&= \int_0^t e^{-b(t-u)} \int_0^{t-u} e^{bs} \beta s^{\beta-1} ds dW_u,
\end{aligned}$$

in such a way that

$$g(t-s) = e^{-b(t-s)} \int_0^{t-s} e^{bu} \beta u^{\beta-1}.$$

And analogously, if $\beta \in (-1/2, 0)$, we get:

$$g(t-s) = e^{-b(t-s)}(t-s)^\beta + \beta e^{-b(t-s)} \int_0^{t-s} (e^{bu} - 1) v^{\beta-1} du.$$

2.7 A defaultable zero coupon bond

In this section we are going to price a zero coupon bond with possibility of default. A zero coupon bond with default is a contract with maturity time T and payoff $X = \mathbb{1}_{\{\tau > T\}}$ where τ is the random default time. Then, the arbitrage price at time $t < \tau$ is given by:

$$D(t,T) = \mathbb{1}_{\{\tau > t\}} E_{P*}(\mathbb{1}_{\{\tau > T\}} e^{-\int_t^T r_s ds} | \mathcal{G}_t),$$

where $0 \leq t \leq T$ and $(\mathcal{G}_t)_{t>0}$ is a filtration. Let $\vee$ and $\wedge$ be the maximum and the minimum respectively:

$$\begin{cases} u \vee v = \max\{u, v\} \\ u \wedge v = \min\{u, v\} \end{cases}$$

We are considering two filtrations:

- $(\mathcal{F}_t)_{t \geq 0}$, which incorporates the history of the short rates.

- $(\mathcal{G}_t)_{t\geq 0}$, where $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\tau \wedge t)$ in such a way that τ is a $(\mathcal{G}_t)$ stopping time and it represents the available information of the market up to time t.

Obviously, $D(t,T)$ will depend on the model for τ. There are different approaches. One of them is the intensity approach. In this approach the total information available for the investors is given by the filtration $(\mathcal{G}_t)_{t\geq 0}$ defined before. The default time, τ is not necessary a $(\mathcal{F}_t)$, but is a $(\mathcal{G}_t)$ stopping time. It is assumed that there exists a non negative adapted process $(\mathcal{F}_t)_t$ such that:

$$P^*(\tau > t|\mathcal{F}_t) = e^{-\int_0^t \lambda_s ds}$$

Now we show that:

$$D(t,T) = \mathbb{1}_{\{\tau>t\}} E(\mathbb{1}_{\tau>T} e^{-\int_t^T r_s ds}|\mathcal{G}_t) = \mathbb{1}_{\{\tau>T\}} E(e^{-\int_t^T (r_s+\lambda_s)ds}|\mathcal{F}_t).$$

To prove this, a priory we will show the following fact:

$$\mathbb{1}_{\{t<\tau\}} E_{P^*}(X|\mathcal{G}_t) = \frac{E_{P^*}(X\mathbb{1}_{\{t<\tau\}}|\mathcal{F}_t)}{E_{P^*}(\mathbb{1}_{\{t<\tau\}}|\mathcal{F}_t)}.$$

By definition of conditional expectation this is equivalent to prove that:

$$E_{P^*}(\mathbb{1}_{\{t<\tau\}} X \mathbb{1}_A) = E_{P^*}\left(\mathbb{1}_{\{t<\tau\}} \frac{E_{P^*}(X\mathbb{1}_{\{t<\tau\}}|\mathcal{F}_t)}{E_{P^*}(\mathbb{1}_{\{t<\tau\}}|\mathcal{F}_t)} \mathbb{1}_A\right) \tag{2.7.1}$$

for all $A \in \mathcal{G}_t$. Hence, it is enough to consider sets of the form $A = B \cap \{\tau \leq s\}$, where $0 \leq s \leq t$, $B \in \mathcal{F}_t$ or $A \in \mathcal{F}_t$. If $A = B \cap \{\tau \leq s\}$ where $B \in \mathcal{F}_t$, $\mathbb{1}_{\{t<\tau\}}\mathbb{1}_A = 0$, so (2.7.1) is zero at both sides. If $A \in \mathcal{F}_t$,

$$\begin{aligned} &E_{P^*}\left(\mathbb{1}_{\{t<\tau\}} \frac{E_{P^*}(X\mathbb{1}_{\{t<\tau\}}|\mathcal{F}_t)}{E_{P^*}(\mathbb{1}_{\{t<\tau\}}|\mathcal{F}_t)} \mathbb{1}_A\right) = E_{P^*}\left(E_{P^*}\left(\mathbb{1}_{\{t<\tau\}} \frac{E_{P^*}(X\mathbb{1}_{\{t<\tau\}}|\mathcal{F}_t)}{E_{P^*}(\mathbb{1}_{\{t<\tau\}}|\mathcal{F}_t)} \mathbb{1}_A\right)\right) = \\ &= E_{P^*}(E_{P^*}(X\mathbb{1}_{\{t<\tau\}}\mathbb{1}_A|\mathcal{F}_t)) = E_{P^*}(X\mathbb{1}_{\{t<\tau\}}\mathbb{1}_A), \end{aligned}$$

as wanted. Taking into account this fact:

$$\begin{aligned} D(t,T) &= E_{P^*}\left(\mathbb{1}_{\{T<\tau\}} e^{\int_t^T r_s ds}|\mathcal{G}_t\right) = E_{P^*}\left(\mathbb{1}_{\{t<\tau\}}\mathbb{1}_{\{T<\tau\}} e^{-\int_t^T r_s ds}|\mathcal{G}_t\right) = \\ &= \mathbb{1}_{\{t<\tau\}} E_{P^*}\left(\mathbb{1}_{\{T<\tau\}} e^{-\int_t^T r_s ds}|\mathcal{G}_t\right) = \mathbb{1}_{\{t<\tau\}} \frac{E_{P^*}\left(\mathbb{1}_{\{t<\tau\}}\mathbb{1}_{\{T<\tau\}} e^{-\int_t^T r_s ds}|\mathcal{G}_t\right)}{E_{P^*}\left(\mathbb{1}_{\{t<\tau\}}|\mathcal{G}_t\right)} = \\ &= \mathbb{1}_{\{t<\tau\}} \frac{E_{P^*}\left(e^{-\int_0^T \lambda_s ds} e^{-\int_t^T r_s ds}|\mathcal{F}_t\right)}{e^{-\int_0^t \lambda_s ds}}. \end{aligned}$$

We arrive at

$$D(t,T) = \mathbb{1}_{\{\tau>t\}} E(\mathbb{1}_{\tau>T} e^{-\int_t^T r_s ds}|\mathcal{G}_t) = \mathbb{1}_{\{\tau>T\}} E(e^{-\int_t^T (r_s+\lambda_s)ds}|\mathcal{F}_t).$$

Then, we need to model $(\lambda_t)_{t\geq 0}$ and $(r_t)_{t\geq 0}$. One alternative to model these processes is considering the Vasicek model (seen in the first example):

$$dr_t = b(a - r_t)dt + \sigma dW_t \text{ and}$$

$$d\lambda_t = b(a - \lambda_t)dt + \sigma d\bar{W}_t$$

where $\bar{W}$ and W are the correlated Brownian motions with respect to $\mathcal{F}_t = \sigma(W_s, \bar{W}_s, 0 \leq s \leq t)$. Now taking into account the section before, we extend the model in such a way that:

$$r_t = \int_{\infty}^{t} \sigma_s g(t-s) dW_s + \mu_t,$$

$$\lambda_t = \int_{\infty}^{t} \bar{\sigma}_s \bar{g}(t-s) d\bar{W}_s + \bar{\mu}_t.$$

Therefore, the free arbitrage price of a zero coupon bond at time t will be given by the following formula:

$$D(t,T) = \mathbb{1}_{\{\tau > t\}} \exp(A(t,T) - \int_{\infty}^{t} (\sigma_u c(u;t,T) dW_u + \bar{\sigma}_u \bar{c}(u;t,T)) d\bar{W}_u),$$

where for $u < t$

$$c(u;u,T) = \int_t^T g(s-u)\, ds$$

and

$$A(t,T) = \frac{1}{2} \int_t^T (\sigma_u^2 c^2(u;t,T) + \bar{\sigma}_u^2 \bar{c}^2(u;t,T) + 2\rho\sigma_u\bar{\sigma}_u c(u;t,T)\bar{c}(u;t,T)) du$$
$$- \int_t^T (\mu_u + \bar{\mu}_u) du,$$

where ρ is the correlation coefficient between W and $\bar{W}$. An interesting case is when $\sigma_u = \sigma \mathbb{1}_{\{u\geq 0\}}$, $\bar{\sigma}_u = \bar{\sigma} \mathbb{1}_{\{u\geq 0\}}$, $\mu_u = \mu$, $\bar{\mu}_u = \bar{\mu}$ and

$$g(t-s) = e^{-b(t-s)} \int_0^{t-s} e^{bu} \beta u^{\beta-1} du,$$

$$\bar{g}(t-s) = e^{-\bar{b}(t-s)} \int_0^{t-s} e^{\bar{b}u} \bar{\beta} u^{\bar{\beta}-1} du,$$

for $\beta, \bar{\beta} \in (-1/2, 0) \cup (0, 1/2)$. In this case, according with what we said before,

$$\text{Var}\left(-\frac{1}{T-t} \log(D(t,T))\right) \sim T^{2(\beta \vee \bar{\beta})-2}.$$

2.8 The analogous of a CIR model

One of the drawbacks of the previous model is that it allows negative short rates. One obvious way to treat this problem is by taking squares in such a way that

$$r_t = \sum_{i=1}^{d} \left(\int_0^t g(t-s)\sigma_s dW_i(s) \right)^2 + r_0, \tag{2.8.1}$$

where $r_0 > 0$, $t \geq 0$ and $(W_i)_{1 \leq i \leq d}$ is a Brownian motion in $\mathbb{R}^d$.

2.8.1 Bond prices

Consider (2.8.1) with $r_0 = 0$ by simplicity:

$$r_t = \sum_{i=1}^{d} \int_0^t \int_0^t g(t-u)g(t-v)\sigma_s\sigma_u dW_i(u)dW_i(v).$$

Hence,

$$\sum_{i=1}^{d} \int_t^T \left(\int_0^s \int_0^s g(s-u)g(u-v)\sigma_u\sigma_v dW_i(u)dW_i(v) \right) ds =$$
$$= \sum_{i=1}^{d} \int_0^t \int_0^t \sigma_u\sigma_v \left(\int_t^T g(s-u)g(s-v)ds \right) dW_i(u)dW_i(v).$$

Now, we prove this last equality. To prove this we show that the regions of integration are exactly equal.

Let $0 \leq u, v \leq s$ and $t \leq s \leq T$. Let us call:

$$\begin{aligned}
A &= \{(u,v,s) : 0 \leq u, v \leq s, t \leq s \leq T\} \\
B_1 &= \{(u,v,s) : t \leq s \leq T, 0 \leq u, v \leq t\} \\
B_2 &= \{(u,v,s) : u \leq s \leq T, t \leq u \leq T, 0 \leq v \leq t\} \\
B_3 &= \{(u,v,s) : v \leq s \leq T, t \leq v \leq T, 0 \leq u \leq t\} \\
B_4 &= \{(u,v,s) : u \vee v \leq s \leq T, t \leq u, v \leq T\}
\end{aligned}$$

Notice that it is enough to show that $A = B_1 \cup B_2 \cup B_3 \cup B_4$. We prove the double implication:

($[A \subseteq \cup_{i=1}^4 B_i]$):
If $(u,v,s) \in A$, it holds that $t \leq s \leq T$. Then,

- If $0 \leq u \leq t$, $t \leq v \leq s \to (u,v,s) \in B_3$.
- If $0 \leq v \leq t$, $t \leq u \leq s \to (u,v,s) \in B_2$.
- If $0 \leq u, v \leq t \to (u,v,s) \in B_1$.

- If $t \leq u, v \leq s \to (u, v, s) \in B_4$.

([$A \supseteq \cup_{i=1}^4 B_i$]):

- If $(u, v, s) \in B_1 \to 0 \leq u, v \leq s, t \leq s \leq T \to (u, v, s) \in A$
- If $(u, v, s) \in B_2 \to 0 \leq v \leq t \leq s, 0 \leq u \leq s \to (u, v, s) \in A$
- If $(u, v, s) \in B_3 \to 0 \leq u \leq s, v \leq s\ , t \leq s \leq T \to (u, v, s) \in A$
- If $(u, v, s) \in B_4 \to 0 \leq u, v \leq s, t \leq s \leq T \to (u, v, s) \in A$

Now since the B_i's are disjoint it holds that $\int_A = \int_{B_1} + \int_{B_2} + \int_{B_3} + \int_{B_4}$. Furthermore, according to the domain of the sets, it holds (we omit the integrands):

$$\int_A = \int_t^T \int_0^s \int_0^s dudvds$$

$$\int_{B_1} = \int_0^t \int_0^t \int_t^T dudvds$$

$$\int_{B_2} = \int_0^t \int_t^T \int_u^T dsdudv$$

$$\int_{B_3} = \int_0^t \int_t^T \int_v^T dsdvdu = \int_{B_2}$$

(since the integrand is symmetric in u and v, we can change u by v and conversely)

$$\int_{B_4} = \int_t^T \int_t^T \int_{u \vee v}^T dsdudv.$$

Then, the equality is proved. Now, consider $c_2(u, v; t, T) := \int_t^T g(s-u)g(s-v)ds$. By simplicity we take $t = 0$:

$$P(0, T) = E\left(e^{-\int_0^T r_s ds}\right) =$$

$$= E\left(\exp\left(-\sum_{i=1}^d \int_0^T \int_0^T \sigma_u \sigma_v c_2(u, v; u \vee v, T) dW_i(u) dW_i(v)\right)\right) =$$

$$= \prod_{i=1}^d E\left(\exp\left(-T \int_0^1 \int_0^1 \sigma_{Tu} \sigma_{Tv} c_2(T_u, T_v; T(u \vee v), T) dW_i(u) dW_i(v)\right)\right),$$

where the second equality is by independence. Using the idea of the formula of Fourier transform of a Gaussian:

$$= \prod_{i=1}^d E\left(\exp\left(-T \int_0^1 \int_0^1 \sigma_{Tu} \sigma_{Tv} c_2(T_u, T_v; T(u \vee v), T) dW_i(u) dW_i(v)\right)\right) =$$

$$= \left(1 + \sum_0^\infty \frac{2^n T^n}{n!} \int_0^1 \cdots \int_0^1 \det \begin{pmatrix} R(s_1, s_1) & \dots & R(s_1, s_n) \\ \vdots & \ddots & \vdots \\ R(s_n, s_1) & \dots & R(s_n, s_n) \end{pmatrix} ds_1 \cdots ds_n (\sigma^{2n} \cdot T^n)\right)^{-d/2}.$$

Example 3:

Assume that $g(t) = \mathbb{1}_{t\geq 0}$ and $\sigma_t = \sigma$. Then for $u, v \geq t$ we have that:

$$c_2(u,v;t,T) = \int_t^T g(s-u)g(s-v)ds = \int_t^T \mathbb{1}_{u\leq s}\mathbb{1}_{v\leq s}ds =$$
$$= \int_t^T \mathbb{1}_{u,v\leq s}ds = \int_t^T \mathbb{1}_{u\vee v\leq s}ds = T - (u\vee v).$$

where $\vee$ denotes the maximum between both values. Recall that:

$$R(u,v) = \sigma^2 \cdot T - ((Tu)\vee(Tv)) = \sigma^2 \cdot T \cdot (1 - u\vee v).$$

We had that:

$$P(0,T) = \left(1 + \sum_0^\infty \frac{2^n T^n}{n!}\int_0^1 \cdots \int_0^1 \det\begin{pmatrix} 1 - s_1 V s_1 & \dots & 1 - s_1 \vee s_n \\ \vdots & \ddots & \vdots \\ 1 - s_n \vee s_n & \dots & 1 - s_n \vee s_n \end{pmatrix} ds_1 \cdots ds_n (\sigma^{2n}\cdot T^n)\right)^{-d/2},$$

where

$$D(\lambda) = \left(1 + \sum_0^\infty \frac{\lambda^n}{n!}\int_0^1 \cdots \int_0^1 \det\begin{pmatrix} R(s_1,s_1) & \dots & R(s_1,s_n) \\ \vdots & \ddots & \vdots \\ R(s_n,s_1) & \dots & R(s_n,s_n) \end{pmatrix} ds_1 \cdots ds_n (\sigma^{2n}\cdot T^n)\right)$$

is the Fredholm determinant. The solution of a Gaussian expression is often expressed in terms of Fredholm determinant. A curious procedure to compute this Fredholm determinant is the following, which is given by [24]: Consider $K(u,v)$ separable kernel of a continuous operator, which has the form:

$$K(u,v) = M(u\vee v)N(u\wedge v),$$

where

$$\begin{cases} u\vee v = \max\{u,v\} \\ u\wedge v = \min\{u,v\} \end{cases},$$

that is, we can express the kernel as a function of the maximum an the minimum. The Fredholm determinant is:

$$D(\lambda) = \prod(1 + \lambda^{-1}\lambda_i),$$

where λ is real and λ_i are the eigenvalues of $K(u,v)$. According to [24] we can write:

$$\ln D(\lambda) = \ln\det(B_\lambda(T)),$$

where $B_\lambda(T)$ is defined by the linear differential equation system:

$$\begin{pmatrix} A'_\lambda(t) \\ B'_\lambda(t) \end{pmatrix} = \lambda \begin{pmatrix} -N(t)M(t) & N^2(t) \\ -M^2(t) & N(t)M(t) \end{pmatrix} \begin{pmatrix} A_\lambda(t) \\ B_\lambda(t) \end{pmatrix}$$

and

$$\begin{pmatrix} A_\lambda(t) \\ B_\lambda(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In our case, we got that $R(u,v) = \sigma^2 \cdot T \cdot (1 - u \vee v) = K(u,v)$, so we can write $R(u,v) = M(u \vee v)N(u \wedge v)$, where $M(u \vee v) = \sigma^2 \cdot T \cdot (1 - u \vee v)$ and $N(u \wedge v) = 1$. We have that

$$D(\lambda) = B_\lambda(1)$$

and this implies that

$$P(0,T) = (B_{2T}(1))^{-d/2},$$

where $B_\lambda(t)$ is defined as:

$$\begin{pmatrix} A'_\lambda(t) \\ B'_\lambda(t) \end{pmatrix} = \lambda \begin{pmatrix} -\sigma^2 T(1-t) & 1 \\ -\sigma^4 T^2 (1-t)^2 & \sigma^2 T(1-t) \end{pmatrix} \begin{pmatrix} A_\lambda(t) \\ B_\lambda(t) \end{pmatrix}$$

and

$$\begin{pmatrix} A_\lambda(t) \\ B_\lambda(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

If we solve this equation system in Mathematica using the function *Dsolve*, we get that

$$B_\lambda(t) = \sigma^2 T^2 \left((1-t) \frac{e^{\sigma\sqrt{\lambda T}t} - e^{-\sigma\sqrt{\lambda T}t}}{\sigma\sqrt{\lambda T}} + \frac{e^{\sigma\sqrt{\lambda T}t} + e^{-\sigma\sqrt{\lambda T}t}}{(\sigma\sqrt{\lambda T})^2} \right).$$

Then substituting t by 1 and λ by $2T$ in last formula we have a formula for $P(0,T)$:

$$P(0,T) = \left(\sigma^2 T^2 \left(\frac{e^{\sigma\sqrt{2T^2}} + e^{-\sigma\sqrt{2T^2}}}{(\sigma\sqrt{2T^2})^2} \right) \right)^{-d/2}.$$

Now, let us see another example more complicated which is the classical *CIR* model. In order to solve it, we are going to use the same method used in the third example, which is based on [24].

Example 4:

Assume that $g(t) = e^{-b(s-u)} e^{-b(s-v)}$, where $u, v < s$ and $\sigma_t = \sigma$. In this case,

$$R(u,v) = \sigma^2 \int_{T(u \vee v)}^{T} e^{-b(s-u)} e^{-b(s-v)} ds = \frac{\sigma^2}{2b} e^{-b(T((u \wedge v)-1)} \left(e^{-b(T((u \vee v)-1)} - e^{b(T((u \vee v)-1)} \right).$$

Then, $R(u,v)$ can be seen also as a function of the maximum ($\vee$) and the minimum ($\wedge$) . That is, we can write:

$$R(u,v) = \frac{\sigma}{\sqrt{2b}} \left(e^{-bT(t-1)} - e^{bT(t-1)} \right) \frac{\sigma}{\sqrt{2b}} e^{bT(t-1)} = M(u \vee v) N(u \wedge v),$$

where

$$\begin{cases} M(u \vee v) = \frac{\sigma}{\sqrt{2b}} \left(e^{-bT(t-1)} - e^{bT(t-1)} \right) \\ N(u \wedge v) = \frac{\sigma}{\sqrt{2b}} e^{bT(t-1)} \end{cases}$$

Recall the linear system defined before,

$$\begin{pmatrix} A'_\lambda(t) \\ B'_\lambda(t) \end{pmatrix} = \lambda \begin{pmatrix} -N(t)M(t) & N^2(t) \\ -M^2(t) & N(t)M(t) \end{pmatrix} \begin{pmatrix} A_\lambda(t) \\ B_\lambda(t) \end{pmatrix}$$

and

$$\begin{pmatrix} A_\lambda(t) \\ B_\lambda(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

So, in this case, multiplying and dividing by $e^{bT(t-1)}$ in $M(u \vee v)$, we have that:

$$\begin{pmatrix} A'_\lambda(t) \\ B'_\lambda(t) \end{pmatrix} = \frac{\lambda\sigma^2}{2b} \begin{pmatrix} e^{2bT(t-1)}(1-e^{-2bT(t-1)}) & e^{2bT(t-1)} \\ -(e^{2bT(t-1)}-1)(1-e^{-2bT(t-1)}) & -(e^{2bT(t-1)}-1) \end{pmatrix} \begin{pmatrix} A_\lambda(t) \\ B_\lambda(t) \end{pmatrix}$$

and

$$\begin{pmatrix} A_\lambda(t) \\ B_\lambda(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

As in the third example:

$$D(\lambda) = B_\lambda(1)$$

and therefore,

$$P(0,T) = (B_{2T}(1))^{-d/2},$$

where $B_\lambda(1)$ was defined by the system before. Hence, if we solve in *Mathematica* and we substitute λ by $2T$ in this last system we arrive at:

$$B_{2T} = \frac{-e^{-2bT}\sigma^2}{2Tb\sqrt{b^2+2\sigma^2}}(e^{-2bT(t-1)}-1)(e^{Tt(b+\sqrt{b^2+2\sigma^2})} - e^{Tt(b-\sqrt{b^2+2\sigma^2})})$$
$$\frac{2bT\sigma^2}{2Tb\sqrt{b^2+2\sigma^2}}\left(\frac{e^{Tt(-b+\sqrt{b^2+2\sigma^2})}}{-b+\sqrt{b^2+2\sigma^2}} - \frac{e^{Tt(-b-\sqrt{b^2+2\sigma^2})}}{-b-\sqrt{b^2+2\sigma^2}}\right).$$

Then, substituting t by 1 in this last formula we have a formula for $P(0,T)$:

$$(P(0,T))^{d/2} = \frac{2bT\sigma^2}{2Tb\sqrt{b^2+2\sigma^2}}\left(\frac{e^{T(-b+\sqrt{b^2+2\sigma^2})}}{-b+\sqrt{b^2+2\sigma^2}} - \frac{e^{T(-b-\sqrt{b^2+2\sigma^2})}}{-b-\sqrt{b^2+2\sigma^2}}\right).$$

Simplifying this last expression, we arrive at:

$$(P(0,T))^{d/2} = \frac{1}{2\sqrt{b^2+2\sigma^2}}\left(e^{T(-b+\sqrt{b^2+2\sigma^2})}(b+\sqrt{b^2+2\sigma^2}) + e^{-T(b+\sqrt{b^2+2\sigma^2})}(-b+\sqrt{b^2+2\sigma^2})\right).$$

Appendix A: Itô integral

In order to understand the notation related with the Itô's integral used throughout the project, in this section we will sketch out the Itô's integral construction.
Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a probability space with $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the filtration generated by the Brownian motion.
We say that $u \in L^2_{a,T}$, where $L^2_{a,T}$ consists of stochastic processes such that:

- $u : [0,T] \times \Omega \to \mathbb{R}$ is measurable with respect to the product of σ-fields $\mathcal{B}([0,T]) \times \mathcal{F}$.
- u_t is $\mathcal{F}_t$-measurable $\forall t \in [0,T]$ (adapted).
- $E\left[\int_0^T u_s^2 ds\right] < \infty$.

Then,

$$E\left[\int_0^T u_s^2 ds\right] = \int_\Omega dP \left(\int_0^T u_s^2 ds\right) = \int_{\Omega\times[0,T]} u_s^2 dP ds,$$

where the last inequality is because of the classic Fubini's theorem applied to positive functions. Hence, if $u \in L^2_{a,T}$, it holds that $u \in L^2(\Omega \times [0,T])$.

Now, consider $0 = t_0 < t_1 < \ldots < t_n = T$. Assume that u is a step process, that is $u \in \epsilon$ is of the form $u_t = \sum_{j=1}^n u_j \mathbb{1}_{(t_{j-1},t_j]}(t)$, where u_j are random variables such that u_j is $\mathcal{F}_{t_{j-1}}$-measurable and $E[u_j^2] < \infty$.

Remark: $\epsilon \subseteq L^2_{a,T}$.

Take $u \in \epsilon$: we define $\int_0^T u_s dW_s = \sum_{j=1}^n u_j(W_{t_j} - W_{t_{j-1}})$, where $0 = t_0 < t_1 < \ldots < t_n = T$. Call $I_T(u) = \int_0^T u_s dW_s$, $u \in \epsilon$. This integral has the following three properties:

- $I_T(u)$ is a random variable with $E[I_T(u)] = 0$.
- Isometry property: $E[I_T(u)^2] = E[\int_0^T u_t^2 dt] < \infty$.
- Linearity: Let a and b in $\mathbb{R}$ and u, v in ϵ. Then $I_T(au + bv) = aI_T(u) + bI_T(v)$.

Given $u \in L^2_{a,T}$, there exists a sequence $\{u_n\}_{n\geq 1} \subseteq \epsilon$, with $\lim_n E\left[\int_0^T (u_n(t) - u(t))^2\right] =$

0. We define $I_T(u) = L^2(\Omega) - \lim_n I_T(u_n)$, that is $E[(I_T(u_n) - I_T(u))^2] \to 0$ when $n \to \infty$.

Definition 2.8.1 *Given $u \in L^2_{a,T}$, there exists $\{u_n\}_{n\geq 1} \subseteq \epsilon$, with $\lim_n \|u_n - u\|_2$. We define $I_T(u) = \|\cdot\|_2 - \lim_n I_T(u_n)$. This $I_T(u)$ is well-defined and it does not depend on the particular $\{u_n\}_{n\geq 1} \subseteq \epsilon$.*

Remark: If a stochastic process u lies in $L^2_{a,T}$, the three properties exposed before hold.

2.8.2 Indefinite Stochastic Integral

Definition 2.8.2 *An indefinite stochastic integral is $\{\int_0^T u_s dW_s : t \in [0,T]\}$. It is defined $\int_0^t u_s dW_s = \int_0^T u_s \mathbb{1}_{[0,t]}(s) dW_s$.*

According to this last definition, since u is jointly measurable and $\mathbb{1}_{[0,t]}(s)$ is too, $u\mathbb{1}_{[0,t]}(\cdot)$ is jointly measurable. Since constants are adapted to any σ-field, $u_s\mathbb{1}_{[0,t]}(s)$ is $\mathcal{F}_s$-measurable. Moreover,

$$E\left[\int_0^T u_s^2 \mathbb{1}_{[0,t]}(s) ds\right] = E\left[\int_0^t u_s^2 ds\right] \leq E\left[\int_0^T u_s^2 ds\right] < \infty.$$

Then, $\{\int_0^T u_s dW_s : t \in [0,T]\}$ belongs to $L^2_{a,T}$.

Remark: $\{I_t(u) : t \in [0,T]\}$ is martingale with respect to the filtration generated by the Brownian motion $\{\mathcal{F}_t\}_{t\geq 0}$ (the integral inherits the martingale property of the Brownian motion).

2.8.3 Extension if the Itô Integral

We will extend the integral to the class $\Delta^2_{a,T}$. We will say that a stochastic process u is in $\Delta^2_{a,T}$ if:

- u is jointly-measurable in $[0,T] \times \Omega$
- u is $\mathcal{F}_t$-measurable (adapted).
- $\int_0^T u_s^2 \, ds < \infty$ a.s

We know that if $E[|X|] < \infty$, then X is finite a.s. Hence, $L^2_{a,T} \subseteq \Delta^2_{a,T}$.

Remark: In this extension of the Itô's integral, only the property of linearity among the three properties explained before holds.

Appendix B

In this section we will state and prove in detail all the theorems used during the project.

Firstly, we state and prove the knight theorem.

2.9 Knight theorem

Before proving the theorem, we state some remarks and a theorem that we will use to prove the Knight theorem. Also, to understand its proof we shall keep in mind the concepts of finite variation, semimartingale and quasimartingale that we defined at the beginning of Chapter 2.

Remark: By a theorem of [22], $(X_t)_{t\in\mathbb{R}}$ is quasimartingale if and only if

$$\int_0^{+\infty} E[|E[X_{t+h} - X_t|\mathcal{F}_t]|]dt = \mathcal{O}(h).$$

Remark: If a function is absolutely continuous then it is of finite variation.

Remark: If a process is a martingale, then in particular it is local martingale.

(Banach-Alaoglu's Theorem): Let X be a separable normed space. Denote by X^* the dual space. Let $B = \{\varphi \in X^* : \|\varphi\| \leq 1\}$. Then, for all sequence $(\varphi_n) \subseteq B$, there exist a subsequence (φ_{n_k}) and $\varphi \in X^*$ such that $\lim_k \varphi_{n_k}(x) = \varphi(x)\ \forall x \in X$.

Theorem 2.9.1 (Knight's Theorem) *Let $X_t := \int_{-\infty}^t g(t-s)\, dW_s$, where g is in $L^2((0,\infty))$ and deterministic. Then, $(X_t)_{t\in\mathbb{R}}$ is semimartingale with respect to the natural filtration generated by the Wiener measure if and only if g is absolutely continuous and $g' \in L^2(0,\infty)$.*

Proof. [$\leftarrow$]

Assume that $g(0) < +\infty$. Since g is absolutely continuous, there exists g' such that $g(t) = g(s) + \int_s^t g'(u)\, du$. Then

$$g(t-s) = g(0) + \int_s^t g'(z-s)dz$$

and

$$g(t+s) = g(s) + \int_0^t g'(z+s)dz$$

Let us assume that $\int_0^\infty (g'(u))^2\, du < +\infty$, so we will be able to use Fubini's stochastic theorem which will be proved in detail below this result. Notice that we can write $X_t = \int_{-\infty}^0 g(t-s)\, dW_s + \int_0^t g(t-s)\, dW_s$. Now we develop this expression:

$$\begin{aligned}
&\int_{-\infty}^0 g(t-s)\, dW_s + \int_0^t g(t-s)\, dW_s = \\
&= \int_0^{+\infty} g(t+s)\, d\bar{W}_s + \int_0^t g(t-s)\, dW_s = \\
&= \int_0^{+\infty} \left(g(s) + \int_0^t g'(z+s)dz \right) d\bar{W}_s + \int_0^t \left(g(0) + \int_s^t g'(z-s)dz \right) dW_s = \\
&\underbrace{=}_{u=z+s} \int_0^{+\infty} g(s)d\bar{W}_s + \int_0^{+\infty} \int_s^{t+s} g'(u)dud\bar{W}_s + \int_0^t g(0)dW_s + \int_0^t \int_s^t g'(z-s)dzdW_s = \\
&\underbrace{=}_{z=u-s} \int_0^{+\infty} g(s)d\bar{W}_s + \int_0^{+\infty} \int_0^t g'(z+s)dzd\bar{W}_s + \int_0^t g(0)dW_s + \int_0^t \int_s^t g'(z-s)dzdW_s = \\
&\underbrace{=}_{\substack{\text{Fubini}\\ \text{Stochastic}}} \int_0^{+\infty} g(s)d\bar{W}_s + \int_0^{+\infty} \int_0^t g'(z+s)dzd\bar{W}_s + g(0)\Delta W_t + \int_0^t \int_0^z g'(z-s)dW_s dz.
\end{aligned}$$

Since by hypothesis g is deterministic and $\int_0^\infty g(u)^2\, du < +\infty$, $\int_0^{+\infty} g(s)\, W(ds)$ is well-defined and it is a martingale. On the other hand, since $g(0) < \infty$, $g(0)\Delta W_t$ is clearly a martingale with respect the filtration generated by the stochastic Wiener measure, because the increment of Wiener measure is martingale with respect to the filtration generated by the stochastic Wiener measure. Finally, we have that $\int_0^{+\infty} \int_0^t g'(z+s)dz\bar{W}_s$ and $\int_0^t \int_0^z g'(z-s)dW_s dz$ are absolutely continuous.

Therefore, since X_t can be written as a sum of an absolutely continuous term (finite variation) and a martingale term (in particular locally martingale), it is a semimartingale.

Now, we proceed to prove the other implication.

$[\rightarrow]$

The proof of this implication is based on [23]. Given $0 \leq s \leq t$,

$$\begin{aligned}
E[X_t - X_s|\mathcal{F}_s] &= E\left[\int_{-\infty}^t g(t-u)dW_u - \int_{-\infty}^s g(s-u)dW_u \middle| \mathcal{F}_s \right] = \\
&= \underbrace{E\left[\int_S^t g(t-u)dW_u \middle| \mathcal{F}_s \right]}_{=0} + E\left[\int_{-\infty}^s (g(t-u) - g(s-u))dW_u \middle| \mathcal{F}_s \right] = \\
&= \int_{-\infty}^s (g(t-u) - g(s-u))dW_u.
\end{aligned}$$

Therefore,

$$E[|E[X_t - X_s|\mathcal{F}_s]|] = \left(\frac{2}{\pi}\int_{-\infty}^{s}(g(t-u)-g(s-u))^2 du\right)^{1/2}.$$

By the first remark of the chapter, $(X_t)_{t\in\mathbb{R}}$ is a quasimartingale if and only if

$$\int_0^\infty (g(t+u)-g(u))^2 du = \mathcal{O}(t^2), \text{ as } t \to 0.$$

And this is equivalent to saying that

$$\limsup_{t\to 0^+}\int_0^\infty \left(\frac{g(t+u)-g(u)}{t}\right)^2 du < \infty.$$

Then, the sequence $\{g_n\}$, where $g_n(t) := n(g(t+1/n)-g(t))$, is bounded in $L^2(\mathbb{R}^+)$. As each function in L^2 can be viewed as a bounded operator from L^2 to $\mathbb{C}$, we can see $\{g_n\}$ as a bounded sequence of bounded operators, and by Banach-Alaoglu's theorem stated before, there exist a subsequence $\{g_{n_l}\}_{l=1}^\infty$ and $\psi \in L^2(\mathbb{R}^+)$ such that

$$\lim_l \int_0^\infty g_{n_l}(t)h(t)\,dt = \int_0^\infty \psi(t)h(t)\,dt$$

for all $h \in L^2(\mathbb{R}^+)$.

As $g \in L^2(\mathbb{R}^+)$, g is locally integrable, so by the Lebesgue's differentiation theorem there exists a null set N such that, for all $t \in \mathbb{R}\backslash N$,

$$\lim_n n\int_t^{t+\frac{1}{n}} g(u)\,du = g(t).$$

Thus, for almost every $s \le t$,

$$\int_s^t \psi(u)\,du \underbrace{=}_{h=\mathbb{1}_{[s,t]}} \lim_l \int_s^t g_{n_l}(u)\,du = \lim_l\left(n_l\int_s^t g\left(u+\frac{1}{n_l}\right)du - n_l\int_s^t g(u)\,du\right) =$$

$$= \lim_l\left(n_l\int_{s+\frac{1}{n_l}}^{t+\frac{1}{n_l}} g(u)\,du - n_l\int_s^t g(u)\,du\right) = \lim_l\left(n_l\int_t^{t+\frac{1}{n_l}} g(u)\,du - n_l\int_s^{s+\frac{1}{n_l}} g(u)\,du\right) =$$

$$= g(t) - g(s).$$

Then g is absolutely continuous on any closed interval contained in $[0,\infty)$ and $g' = \psi$ almost surely, so $g' \in L^2(\mathbb{R}^+)$, as wanted.

□

Next, we prove the stochastic Fubini's Theorem, whose proof is proposed in Exercise 40, page 171, of [4].

2.10 Stochastic Fubini's Theorem

Theorem 2.10.1 (Stochastic Fubini's Theorem) *Let $(\Omega, \mathbb{F}, \mathcal{F}, P)$ be a filtered and complete probability space, where $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$, and let $W = \{W_t | 0 \le t \le T\}$ be the standard Brownian motion with respect to $(\mathcal{F}_t)_{0\le t\le T}$. Let us consider a process $(H(t,s))_{0\le t,s\le T}$ with two indexes satisfying the following properties:*

- *$\forall w$ the map $(t,s) \to H(t,s)(w)$ is continuous.*
- *The process $(H(t,s))_{0\le t\le T}$ is adapted for any $s \in [0,T]$.*

Also assume that

$$E\left(\int_0^T \left(\int_0^T H^2(t,s)dt\right) ds\right) < \infty. \tag{2.10.1}$$

Taking into account these conditions, we have that:

$$\int_0^T \left(\int_0^T H(t,s)\, dW_t\right) ds = \int_0^T \left(\int_0^T H(t,s)\, ds\right) dW_t$$

Proof.

We see first $\int_0^T \left(\int_0^T H(t,s)dW_t\right) ds < \infty$ almost surely:

$$E\left[\left|\int_0^T \left(\int_0^T H(t,s)dW_t\right) ds\right|\right] \le E\left[\int_0^T \left|\int_0^T H(t,s)dW_t\right| ds\right] =$$

$$= \int_0^T E\left[\left|\int_0^T H(t,s)dW_t\right|\right] ds = \int_0^T \left(E\left[\left|\int_0^T H(t,s)dW_t\right|\right]^2\right)^{1/2} ds \lesssim$$

$$\underbrace{\lesssim}_{\text{Jensen}} \int_0^T E\left[\left(\int_0^T H(t,s)dW_t\right)^2\right]^{1/2} ds \underbrace{=}_{\text{Itô isometry}} \int_0^T E\left[\int_0^T H(t,s)^2 dt\right]^{1/2} ds \lesssim$$

$$\underbrace{\lesssim}_{\text{Jensen}} \left(\int_0^T E\left[\int_0^T H(t,s)^2 dt\right] ds\right)^{1/2} < \infty.$$

Let

$$H^{(k,N)}(t,s) = \left(\sum_{i=0}^{N-1} H(t_i,s)\mathbb{1}_{(t_i,t_{i+1}]}(t)\right)\cdot \mathbb{1}_{\{H(t,s)\le k\}}(t,s) = \sum_{i=0}^{N-1} H(t_i,s)\cdot \mathbb{1}_{\{H(t_i,s)\le k\}}\mathbb{1}_{(t_i,t_{i+1}]}(t).$$

We have that:

$$L^2 - \lim_k \lim_n \int_0^T \left(\int_0^T H^{(k,N)}(t,s)ds\right) dW_t = \int_0^T \int_0^T H(t,s)ds\, dW_t.$$

Now we will prove that the left-hand side of this last formula also tends to $\int_0^T (\int_0^T H(t,s)dW_t)ds$. For that purpose, let

$$H^k(t,s) := H(t,s)\mathbb{1}_{H(t,s)\le k}.$$

Notice that $H^k(t,s)$ is bounded. We have that:

$$E\left[\left(\int_0^T\left(\int_0^T H(t,s)dW_t\right)ds-\sum_{i=0}^{N-1}\left(\int_0^T H^{(k,N)}(t_i,s)ds\right)(W_{t_{i+1}}-W_{t_i})\right)^2\right]=$$

$$=E\left[\int_0^T\left(\int_0^T\left(H(t,s)-H^{(k,N)}(t,s)\right)dW_t\right)ds\right]^2\leq T\int_0^T E\left[\int_0^T\left(H(t,s)-H^{(k,N)}\right)dW_t\right]^2 ds\leq$$

$$\leq 2T\left(\int_0^T\left(E\left(H(t,s)-H^k(t,s)\right)^2dt\right)ds+\int_0^T\left(E\left(\int_0^T H^k(t,s)-H^{(k,N)}(t,s)\right)^2dt\right)ds\right).$$

Since $H^k(t,s)-H^{(k,N)}(t,s)$ is bounded, applying the dominated convergence theorem, this last expression tends to zero as $k,N\to\infty$, as wanted.

This result can be also proved assuming instead of (2.10.1),

$$\int_0^T\left[E\left(\int_0^T H^2(t,s)\,dt\right)\right]^{1/2}ds<\infty.$$

As in the first proof $\int_0^T\left(\int_0^T H(t,s)dW_t\right)ds<\infty$ almost surely.

In this case we have:

$$\int_0^T\left(\int_0^T H(t,s)\,ds\right)dW_t=P-\lim_k\lim_N\sum_{i=0}^{N-1}\left(\int_0^T H^{(N,k)}(t_i,s)\,ds\right)\cdot(W_{t_{i+1}}-W_{t_i}).$$

It suffices to prove that

$$L^1-\lim_k\lim_N\sum_{i=0}^{N-1}\left(\int_0^T H^{(N,k)}(t_i,s)\,ds\right)\cdot(W_{t_{i+1}}-W_{t_i})=\int_0^T\left(\int_0^T H(t,s)\,dW_t\right)ds.$$

We have:

$$E\left[\int_0^T\left|\int_0^T(H(t,s)-H^{(k,N)}(t,s))\,dW_t\right|ds\right]=\int_0^T E\left[\int_0^T(H(t,s)-H^{(k,N)}(t,s))^2\,dt\right]^{1/2}ds\lesssim$$

$$\lesssim\int_0^T E\left[\int_0^T\underbrace{(H(t,s)-H^k(t,s))^2}_{H(t,s)^2\mathbb{1}_{\{H(t,s)\geq k\}}}\,dt\right]^{1/2}ds+\int_0^T E\left[\int_0^T(H^k(t,s)-H^{(k,N)}(t,s))^2\,dt\right]^{1/2}ds.$$

Both terms go to zero as $k\to\infty$ and $N\to\infty$ by the dominated convergence theorem.

□

Taking into account the proof of changing the derivate by an integral in the real case (whose proof is based of the classical Fubini Theorem), we will show this same result but with stochastic integrals, and for the proof we will use the stochastic Fubini's theorem stated and proved above.

2.11 Exchanging derivative and stochastic integral

Theorem 2.11.1 (Exchanging derivative and stochastic integral) *Suppose that, $\forall s \in [0,T]$, $g(s,\cdot) \in L^2_{a,T}$. Assume that for almost every $t \in [0,T]$ and for almost every $w \in \Omega$ $g(\cdot,t) \in L^1([0,T])$. Suppose that for almost every $w \in \Omega$, $\forall s \in [0,T]$ and for almost every $t \in [0,T]$, $\exists \frac{\partial g}{\partial s}(s,t)$ and*

$$\frac{\partial g}{\partial s} \in L^2([0,T] \times [0,T] \times \Omega). \tag{2.11.1}$$

Finally, suppose that $\{\frac{\partial g}{\partial s}(s,t)\}_{0 \le s \le T}$ is adapted for all $t \in [0,T]$ and that the real map $(s,t) \mapsto \frac{\partial g}{\partial s}(s,t)$ is continuous for almost every $w \in \Omega$.

Then, for almost every $w \in \Omega$,

$$\frac{\partial}{\partial s}\int_0^T g(s,t)dW_t = \int_0^T \frac{\partial}{\partial s} g(s,t)dW_t, \text{ for almost every } s \in [0,T].$$

Proof. We are going to consider this result from Analysis [25], Chapter 7, Theorem 7.21:

If we have a function $f : [a,b] \to \mathbb{C}$ such that $f \in L^1([a,b])$, there exists $f'(x)$ $\forall x \in [a,b]$ and $f' \in L^1([a,b])$, then $f \in AC([a,b])$.

Now, taking into account the hypotheses and the result we prove the theorem:

For almost every $t \in [0,T]$ and for almost every $w \in \Omega$, $g(\cdot,t)(w) \in L^1([0,T])$. On the other hand, $\frac{\partial}{\partial s} g(\cdot,t)(w) \in L^1([0,T])$ by Fubini applied in (2.11.1). Then by the stated result from Analysis, we have that $g(\cdot,t)(w) \in \mathrm{AC}([0,T])$. Hence, for almost every $t \in [0,T]$, for almost every $w \in \Omega$ and $\forall s \in [0,T]$, we have:

$$g(s,t)(w) = g(0,t)(w) + \int_0^s \partial_u g(u,t)(w)du.$$

Notice that, given $s \in [0,T]$,

$$E\left[\int_0^T \left(\int_0^s \partial_u g(u,t)du\right)^2 dt\right] \underbrace{\lesssim}_{\text{Jensen}} E\left[\int_0^T \int_0^s \partial_u g(u,t)^2 dudt\right] < \infty,$$

which is finite by (2.11.1). So, there exists $\int_0^T \int_0^s \partial_u g(u,t)du\, dW_t$ for all $s \in [0,T]$. Then,

$$\int_0^T g(s,t)dW_t = \int_0^T g(0,t)dW_t + \int_0^T \int_0^s \partial_u g(u,t)du\, dW_t, \text{ for all } s \in [0,T].$$

By Stochastic Fubini's Theorem it holds:

$$\int_0^T g(s,t)dW_t = \int_0^T g(0,t)dW_t + \int_0^s \int_0^T \partial_u g(u,t)dW_t\, du, \text{ for all } s \in [0,T].$$

Notice that $u \mapsto \int_0^T \partial_u g(u,t)dW_t$ is in $L^1([0,T])$ for almost every $w \in \Omega$. Indeed,

$$E\left[\int_0^T \left|\int_0^T \partial_u g(u,t)dW_t\right|^2 du\right] \underbrace{=}_{\substack{\text{Integrand positive}\\ \text{Fubini}}} \int_0^T E\left[\left(\int_0^T \partial_u g(u,t)dW_t\right)^2\right] du =$$

$$\underbrace{=}_{\text{Itô isometry}} \int_0^T \int_0^T E[(\partial_u g(u,t))^2]\, dt\, du < \infty.$$

The last expression is finite by (2.11.1). Thus, for almost every $w \in \Omega$,

$$\int_0^T \left|\int_0^T \partial_u g(u,t)dW_t\right|^2 du < \infty.$$

Hence, for almost every $w \in \Omega$, $u \mapsto \int_0^T \partial_u g(u,t)dW_t$ is in $L^2([0,T])$. Since the interval $[0,T]$ is bounded, we have that for almost every $w \in \Omega$, $u \mapsto \int_0^T \partial_u g(u,t)dW_t$ is in $L^1([0,T])$, as claimed.

So, for almost every $w \in \Omega$, we can apply the Lebesgue differentiation theorem and conclude that:

$$\exists \frac{\partial}{\partial s}\int_0^T g(s,t)(w)dW_t(w) = \int_0^T \partial_s g(s,t)(w)dW_t(w), \text{ for almost every } s \in [0,T].$$

□

2.12 Extension of the stochastic representation theorem for improper Itô's integrals

Since we have worked in this project with improper stochastic integral, we cannot use the classic representation theorem. Hence, we need to prove another version of the representation theorem, which will be an extension of it.

Let $\mathcal{S}$ be the set of step functions with compact support in $(-\infty, T]$, that is if f is in $\mathcal{S}$, f is of the form:

$$f = \sum_{j=1}^{n} \lambda_j \mathbb{1}_{]t_{j-1},t_j]}$$

Consider a wider set of functions:

$$\epsilon_T^f = \{\exp\left(\int_{-\infty}^T f_s dW_s - \frac{1}{2}\int_{-\infty}^T f_s^2 ds\right) \in L^2(\Omega)\}.$$

Lemma 2.12.1 *$\{\epsilon_T^f : f \in \mathcal{S}\}$ is total $L^2(\Omega)$.*

Proof. The idea of proof of this lemma is based on [5]. To prove that this set is total in $L^2(\Omega)$ is equivalent to show that the vector space formed by the finite linear

combinations of the elements of the set is dense in $L^2(\Omega)$. Since L^2 is a Hilbert space, we have that ϵ_T^f is total if and only if for all random variable Y orthogonal to ϵ_T^f, Y is exactly zero with probability one. That is, we want to show that:

$$E_{P^*}[\epsilon_T^f \cdot Y] = 0 \text{ then } P(Y, 0) = 1.$$

It is enough to prove it for $Y : \Omega \to \mathcal{R}$ such that $Y^{-1}(B) \subseteq \sigma(B_{t_1}, \cdots, B_{t_n})$ for all Borel set B of $\mathbb{R}$ [5]. Let:

$$\phi(z_1, \ldots, z_n) = e^{-\frac{1}{2}\int_{-\infty}^T f_s^2 ds} E_{p^*}\left[\exp\left(\sum_{j=1}^n z_j(B_{t_j} - B_{t_{j-1}}\right) \cdot Y\right].$$

We put outside the expectation the term $e^{-\frac{1}{2}\int_{-\infty}^T f_s^2 ds}$, because it is deterministic.

Notice that this function is analytic on $\mathbb{C}^n$. Let $\lambda_i \in \mathbb{R}$. Let $f \in \mathcal{S}$. Then,

$$\phi(\lambda_1, \ldots, \lambda_n) = e^{-\frac{1}{2}\int_{-\infty}^T f_s^2 ds} E_{P^*}[\epsilon_T^f \cdot Y] = 0.$$

Therefore we $\phi|_{\mathbb{R}^n} = 0$. By the analytic prolongation theorem, $\phi = 0$ in $\mathbb{C}^n$. Then,

$$e^{-\frac{1}{2}\int_{-\infty}^T f_s^2 ds} E_{P^*}\left[\exp\left(i\sum_{j=1}^n \lambda_j(B_{t_j} - B_{t_{j-1}})\right) \cdot Y\right] = \phi(i\lambda_1, \ldots, i\lambda_n) = 0.$$

Since,

$$E_{P^*}\left[\exp\left(i\sum_{j=1}^n \lambda_j(B_{t_j} - B_{t_{j-1}})\right) \cdot Y\right] = \int_\Omega e^{\sum_{j=1}^n \lambda_j(B_{t_j}(w) - B_{t_{j-1}}(w))} Y(w) dP(w).$$

If we call $X = Y(B_{t_1}(w), \ldots, B_{t_j}(w) - B_{t_{j-1}}(w), \ldots)$, we arrive at:

$$\int_\Omega \exp(i\sum_{j=1}^n \sum_{j=1}^n \lambda_j(B_{t_j}(w) - B_{t_{j-1}}(w))X(w))dP(w) = 0.$$

Then, $E_{P^*}[Y|\mathcal{F}_t] = 0$. Hence, when $n \to \infty$, $E_{P^*}[Y|\mathcal{F}] = 0$. So, since Y is measurable, we arrive at $Y = 0$. Hence, the image of $Y \cdot P$ by $w \to (B_{t_1}(w), \ldots, B_{t_j}(w) - B_{t_{j-1}}(w), \ldots)$ is zero because its Fourier transform is zero.

□

Taking this result into account we prove the extension representation theorem.

Theorem 2.12.2 *Let $Z \in L^2(\Omega)$, $\mathcal{F}_t$-measurable. Then, there exists $h \in L^2_{a,T}$, unique, such that:*

$$Z = E[Z] + \int_{-\infty}^T h_s dW_s. \tag{2.12.1}$$

Proof. This proof is similar to the proof in [21] for indefinite stochastic integrals. First we prove the uniqueness. Assume that there exist $h_1, h_2 \in L^2_{a,T}$ such that the extension of the representation theorem holds for both processes:

$$Z = E[Z] + \int_{-\infty}^{T} h_s^i dW_s, \text{ for } i \in \{1, 2\}.$$

Then,

$$\int_{-\infty}^{T} (h_s^1 - h_s^2) dW_s = 0.$$

Since the Itô integral is in $L^2(\Omega)$,

$$E\left[\left(\int_{-\infty}^{T} (h_s^1 - h_s^2) dW_s\right)^2\right] = 0.$$

By the isometry property,

$$E\left[\int_{-\infty}^{T} (h_s^1 - h_s^2)^2 ds\right] = 0.$$

Hence, $h^1 = h^2$ almost everywhere in $\Omega \times [-\infty, T]$. The lemma above tells us that, given $Z \in L^2(\Omega)$, there exists $\{Z_n\}_{n=1}^{+\infty}$ with $Z_n = \sum_{j=1}^{r_s} a_j \epsilon_T^f$, $f_j \in L^2(]\infty, T])$, $Z_n \to Z$ in $L^2(\Omega)$.

Let $\mathcal{H} = \{Z \in L^2(\Omega)$ such that (2.12.1) is true$\}$, which is closed in $L^2(\Omega)$ (i.e., if $(Z_n) \in \mathcal{H}$, then $L^2 - \lim_n Z_n$ in $\mathcal{H}$). Now, we prove this fact: suppose that exists $h^n \in L^2_{a,T}$, with $Z_n = E[Z_n] + \int_{-\infty}^{T} h_s^n dW_s$ and $Z_n \to Z$ in $L^2(\Omega)$. We want $h \in L^2_{a,T}$, with $Z_n = E[Z_n] + \int_{-\infty}^{T} h_s^n dW_s$. Pass to the L^2-limit: since $Z_n \to Z$ in $L^2(\Omega)$ and the limit when $n \to \infty$ of $E[Z_n] \to E[Z]$, there exists $\|.\|_2 - \lim_n \int_{-\infty}^{T} h_s^n dW_s$. Therefore, $\{\int_{-\infty}^{T} h_s^n dW_s\}_{\in \mathbb{N}}$ is Cauchy in $L^2(\Omega)$.

Since:

$$E\left[\left(\int_{-\infty}^{T} (h_s^n - h_s^m) dW_s\right)^2\right] = E\left[\int_{-\infty}^{T} (h_s^n - h_s^m)^2 ds\right],$$

$\{h^n\}_{n=1}^{+\infty}$ is Cauchy in $L^2_{a,T}$ and therefore there is $h \in L^2_{a,T}$ with $\lim_n h^n = h$ in $L^2(\Omega \times (-\infty, T])$. This means that

$$\lim_n E\left[\int_{-\infty}^{T} (h_s^n - h_s^m)^2 ds\right] = 0,$$

so by Itô is isometry,

$$\lim_n E\left[\left(\int_{-\infty}^{T} (h_s^n - h_s^m) dW_s\right)^2\right] = 0,$$

then,

$$L^2(\Omega) - \lim_n \int_{-\infty}^{T} h_s^n dW_s = \int_{-\infty}^{T} h_s dW_s.$$

Notice that $\epsilon_T^f \in L^2(\Omega)$ for all $f \in L^2(]-\infty, T])$, because

$$\int_{-\infty}^T f_s dW_s \sim N\left(0, \int_{-\infty}^T f_s^2 ds\right),$$

so it has second order moments and the exponential of it too. Let

$$\text{Total} = \{ \text{ functions in } L^2(\Omega) \text{ and } \mathcal{F}_t - \text{measurable}\},$$

$$D = \{\epsilon_T^f, f \in \mathcal{S}\}.$$

If we prove

$$D \subseteq \mathcal{H} \subseteq \text{Total},$$

taking closures,

$$\text{Total} = \bar{D} \subseteq \mathcal{H},$$

so Total $= \mathcal{H}$. Hence, our goal is to show that $\epsilon_T^f \in \mathcal{H}$. Let:

$$X_t = \int_{-\infty}^t f_s dW_s - \frac{1}{2}\int_{-\infty}^t f_s^2 ds, \quad t \in]\infty, T].$$

We have that $f \in L^2(]-\infty, T]) \subseteq L^2_{a,T}$ and $\frac{-1}{2} f_s^2 \in L^1(]-\infty, T])$. Let $\varphi(x) = e^x$, so $\varphi^{'}(x) = e^x$ and $\varphi^{''}(x) = e^x$. Then, by Itô formula:

$$\begin{aligned}
&\epsilon_T^f = \epsilon_{-\infty}^f + \int_{-\infty}^T \exp\left(\int_{-\infty}^s f_r dW_r - \frac{1}{2}\int_{-\infty}^s f_r^2 dr\right) dX_r + \\
&+ \frac{1}{2}\int_{-\infty}^T \exp\left(\int_{-\infty}^s f_r dW_r - \frac{1}{2}\int_{-\infty}^s f_r^2 dr\right)(dX_r)^2 = 1 + \int_{-\infty}^T \epsilon_s^f \left[f_s dW_s - \frac{1}{2} f_s^2 ds\right] + \\
&+ \frac{1}{2}\int_{-\infty}^T \epsilon_s^f f_s^2 ds = 1 + \int_{-\infty}^T \epsilon_s^f f_s dW_s.
\end{aligned}$$

Since the expectation of the Itô integral is zero, we have $E[\epsilon_T^f] = 1$ and $\epsilon_T^f = E[\epsilon_T^f] + \int_{-\infty}^T \epsilon_s^f f_s dW_s$.

□

Corollary 2.12.3 *Let $M = \{M_t; t \in (-\infty, T]\}$ be a martingale with respect to the Brownian measure in $L^2(\Omega)$. Then, there exists $h \in L^2_{a,T}$ such that for all $t \in (-\infty, T]$, $M_t = E[M_0] + \int_{-\infty}^t h_s dW_s$.*

Proof. Notice that

$$E[M_t - M_s] = E[E[M_t - M_s|\mathcal{F}_s]] = E[0] = 0,$$

so $E[M_t] = E[M_0]\ \forall t$ (here the filtration we are considering is the filtration generated by the Brownian measure). We have that $M_t = E[M_T|\mathcal{F}_t]$. By the theorem before, there is a unique $h \in L^2_{a,T}$ with $M_T = E[M_0] + \int_{-\infty}^T h_s dW_s$. Then,

$$\begin{aligned}
&M_t = E[M_0] + E\left[\int_{-\infty}^T h_s dW_s \middle| \mathcal{F}_t\right] = E[M_0] + E\left[\int_{-\infty}^t h_s dW_s \middle| \mathcal{F}_t\right] + \\
&+ E\left[\int_t^T h_s dW_s \middle| \mathcal{F}_t\right].
\end{aligned}$$

Considering the notation seen in the Appendix A, let $\{u^n\}_{n=1}^{+\infty} \in \epsilon$ such that $\lim_n \|u^n - h\|_{L^2(\Omega\times[t,T])} = 0$.

$$E\left[E\left[\int_t^T h_s dW_s - \int_t^T u_s^n dW_s \middle| \mathcal{F}_t\right]^2\right] \underbrace{\lesssim}_{\text{Jensen}} E\left[E\left[\left(\int_t^T h_s dW_s - \int_t^T u_s^n dW_s\right)^2 \middle| \mathcal{F}_t\right]\right] =$$

$$= E\left[\left(\int_t^T h_s dW_s - \int_t^T u_s^n dW_s\right)^2\right] \underbrace{=}_{\text{Itô isometry}} E\left[\int_t^T (h_s - u_s^n)^2 ds\right],$$

and when $n \to 0$,

$$E\left[\int_t^T (h_s - u_s^n)^2 ds\right] \to 0.$$

Then,

$$M_t = E[M_0] + \int_{-\infty}^t h_s dW_s.$$

□

2.13 Girsanov's Theorem

Before stating and proving Girsanov's theorem, we need some results:

Definition 2.13.1 *We say that P and Q are equivalent probability measures on $(\Omega, \mathcal{F})$ if for all $A \in \mathcal{F}$, $P(A) = 0$ if and only if $Q(A) = 0$.*

Definition 2.13.2 *We say that P is absolutely continuous with respect Q, and we denote it by $P << Q$, if and only if for all $A \in \mathcal{F}$ $Q(A) = 0$ implies $P(A) = 0$. Also, $Q << P$ if and only if for all $A \in \mathcal{F}$, $P(A) = 0$ implies $Q(A) = 0$. Thus, P and Q are equivalent if and only if $P << Q$ and $Q << P$. We denote the equivalence between probability measures as $P \sim Q$.*

Lemma 2.13.3 *Let $L : \Omega \to \mathbb{R}$ be a random variable with $E[L] = 1$, $L > 0$. Set $Q(A) = E[\mathbb{1}_A L]$, $A \in \mathcal{F}$. Then, Q is a probability on $(\Omega, \mathcal{F})$.*

Proof. Firstly we prove that Q is a probability:

- $Q(\Omega) = E[\mathbb{1}_\Omega L] = E[L] = 1$
- $Q(A) \geq 0$,for all $A \in \mathcal{F}$

- Q is σ-additive: if $A_n \in \mathcal{F}$, $n \geq 1$, disjoint, then $Q(\cup_{n\geq 1} A_n) = \sum_{n=1}^{\infty} Q(A_n)$. Indeed,

$$Q(\cup_{n\geq 1} A_n) = E[\mathbb{1}_{\cup_{n=1}^{\infty} A_n} L] = E\left[\left(\sum_{n=1}^{\infty} \mathbb{1}_{A_n}\right) L\right] =$$
$$= \sum_{n=1}^{\infty} E[\mathbb{1}_{A_n} L] = \sum_{n=1}^{\infty} Q(A_n).$$

Now we discuss the equivalence:

- $P << Q$: if $A \in \mathcal{F}$ and $Q(A) = 0$, then $0 = E[\mathbb{1}_A L] = \int_A L\, dP$, $L > 0$. Hence, $P(A) = 0$.
- $Q << P$: if $A \in \mathcal{F}$ and $P(A) = 0$, then $0 = \int_A L\, dP = E[\mathbb{1}_A L] = Q(A)$.

□

Now a question that we ask ourselves is how to compute $E_Q[X]$. If $L = 1$, then $Q(A) = E_P[\mathbb{1}_A] = P(A)$, so the problem is the L. Then, it seems $E_Q[X] = E_P[LX]$. We are going to prove it for X with $X, LX \in L^1(\Omega)$. If $X = \mathbb{1}_A$, then $E_Q[\mathbb{1}_A] = Q(A) = E_P[\mathbb{1}_A L]$. If $X = \sum_{i=1}^{r} a_i \mathbb{1}_{A_i}$, by linearity $E_Q[X] = E_P[XL]$. In general, $X = \lim_n X_n$, X_n simple and $E_Q[X] = E_P[LX]$ holds passing to the limit. Let us show it: if $X \in L^1(\Omega)$, $X \geq 0$, take $\{X_n\}_{n=1}^{\infty}$ simple functions with $0 \leq X_n$ and $X_n \to X$ increasingly a.s. By the Monotone Convergence Theorem, $\lim_n E_Q[X_n] = E_Q[X]$. We have that $0 \leq X_n L$ and $X_n L \to XL$ increasingly a.s, so by the Monotone Convergence Theorem $\lim_n E_P[X_n L] = E_P[XL]$. Since $E_Q[X_n] = E_P[XL]$, we arrive at $E_Q[X] = E_P[XL]$. In general, we write $X = X^+ + X^-$. Take $\{X_n^1\}_{n=1}^{\infty}$ and $\{X_n^2\}_{n=1}^{\infty}$ simple functions with $0 \leq X_n^1$ and $X_n^1 \to X^+$ increasingly a.s and $0 \leq X_n^2$ and $X_n^2 \to X^-$ increasingly a.s. Let $X_n = X_n^1 - X_n^2$. Then, $X_n \to X$ a.e and $|X_n| = X_n^1 + X_n^2 \leq X^+ + X^- = |X|$, so by the Dominated Convergence Theorem, $\lim_n E_Q[X_n] = E_Q[X]$. Also, $|X_n L| \leq |LX| \in L^1$, so $\lim_n E_P[X_n L] = E_P[XL]$.

Example: Let $L_t = e^{-\lambda W_t - \frac{\lambda^2}{2} t}$, $t \in [0, T]$, $\lambda \in \mathbb{R}$. Then, $L_t = \epsilon_t^f$, $f = -\lambda \in L^2([0, T])$. By Itô's formula, $L_t = 1 - \lambda \int_0^t L_s dW_s$. We have that $\{L_t; t \in [0, T]\}$ is martingale , $L_t > 0$ and $E[L_t] = 1$. Then, $L_T = 1 - \lambda \int_0^T L_s dW_s$. If $Q(A) = E[\mathbb{1}_A L_T]$ on $\mathcal{F}_T$, we know that $P \sim Q$. We ask ourselves if we have another expression for $Q(A)$ when $A \in \mathcal{F}_t \subseteq \mathcal{F}_T$. We have:

$$Q(A) = E[\mathbb{1}_A L_T] = E[E[\mathbb{1}_A L_T | \mathcal{F}_t]] = E[\mathbb{1}_A E[L_T | \mathcal{F}_t]] \underbrace{=}_{\text{martingale}} E[\mathbb{1}_A L_t],$$

Lemma 2.13.4 *Let X be a random variable on $(\Omega, \mathcal{F}, P)$ and $\mathcal{G} \subseteq \mathcal{F}$ such that $E[e^{iuX} | \mathcal{G}] = e^{-\frac{-u^2\sigma^2}{2}}$ for some $\sigma > 0$. Then, X is independent of $\mathcal{G}$ and $X \sim N(0, \sigma^2)$.*

Proof. If $X \sim N(0,\sigma^2)$, then $\phi_X(u) = e^{\frac{-\sigma^2 u^2}{2}}$ (characteristic function). The fact that $E[e^{iuX}|\mathcal{G}] = e^{\frac{-\sigma^2 u^2}{2}}$ implies that $\forall G \in \mathcal{G}$, $E[\mathbb{1}_G e^{iuX}] = E[\mathbb{1}_G e^{\frac{-\sigma^2 u^2}{2}}]$. Taking $G = \Omega$, $E[e^{iuX}] = e^{\frac{-\sigma^2 u^2}{2}}$. Since the characteristic function determines the law, $X \sim N(0,\sigma^2)$. Let us see that X is independent of G:

$$E_A[e^{iuX}] = E[e^{iuX}|A] \underbrace{=}_{\text{definition}} \frac{E[\mathbb{1}_A e^{iuX}]}{P(A)} \underbrace{=}_{\text{definition}} \frac{E[\mathbb{1}_A e^{\frac{-\sigma^2 u^2}{2}}]}{P(A)} =$$
$$= \frac{E[\mathbb{1}_A] e^{\frac{-\sigma^2 u^2}{2}}}{P(A)} = e^{\frac{-\sigma^2 u^2}{2}}.$$

The characteristic function conditioned to A is the same as without conditioning, so the density of X does not change when conditioning to A.

□

Now we are in conditions to rise above the Girsanov's Theorem.

Theorem 2.13.5 (Girsanov's Theorem) *Let $\lambda \in \mathbb{R}$ and define $B_t = W_t + \lambda t$. Then $\{B_t; t \in [0,T]\}$ is a Brownian motion in a new probability space $(\Omega, \mathcal{F}_T, Q)$, where $\mathcal{F}_T = \sigma(W_s; 0 \le s \le T)$, $Q(A) = E_p[\mathbb{1}_A L_T]$ and $\{L_t = e^{-\lambda W_t - \frac{\lambda^2}{2}t}; t \in [0,T]\}$ and $Q \sim P$.*

Proof.

We have to see that $B_t - B_s \sim N(0, t-s)$ on $(\Omega, \mathcal{F}, Q)$ and $B_t - B_s$ is independent to $\mathcal{F}_s$, $(\sigma(W_s; 0 \le s \le t) = \sigma(B_s; 0 \le s \le t))$. If we prove that $E_Q[e^{iu(B_t - B_s)}|\mathcal{F}_s] = e^{\frac{-u^2(t-s)}{2}}$, by the lemma above we will have both properties. The term $e^{\frac{-u^2(t-s)}{2}}$ is a constant as a random variable, so it is $\mathcal{F}_s$-measurable (because a constant is $\mathcal{F}_0$-measurable). Hence, we want to prove that $\forall A \in \mathcal{F}_s$, it holds that:

$$E_Q[\mathbb{1}_A e^{iu(B_t - B_s)}] = E_Q\left[\mathbb{1}_A e^{\frac{-u^2(t-s)}{2}}\right] = e^{\frac{-u^2(t-s)}{2}} Q(A).$$

Since $A \in \mathcal{F}_s$, we have that $Q(A) = E[\mathbb{1}_A L_s]$ by the example above. Also, $E_Q[X] = E_P[L_s X]$. Since $\mathbb{1}_A e^{iu(B_t - B_s)}$ is $\mathcal{F}_t$-measurable:

$$E_Q[\mathbb{1}_A e^{iu(B_t - B_s)}] = E_P[\mathbb{1}_A e^{iu(B_t - B_s)} L_t] =$$
$$= E_P[\mathbb{1}_A e^{iu(W_t - W_s) + iu\lambda(t-s)} e^{-\lambda(W_t - W_s) - \frac{\lambda^2}{2}(t-s) - \lambda W_s - \frac{\lambda^2}{2}s}] =$$
$$= E_P[\mathbb{1}_A L_s e^{iu(W_t - W_s) + iu\lambda(t-s) - \lambda(W_t - W_s) - \frac{\lambda^2}{2}(t-s)}] = E_P[\mathbb{1}_A L_s e^{(iu-\lambda)(W_t - W_s)} e^{iu\lambda(t-s) - \frac{\lambda^2}{2}(t-s)}] =$$
$$= e^{iu\lambda(t-s) - \frac{\lambda^2}{2}(t-s)} E_P[\mathbb{1}_A L_s] E_P[e^{(iu-\lambda)(W_t - W_s)}] = e^{iu\lambda(t-s) - \frac{\lambda^2}{2}(t-s)} \cdot Q(A) \cdot \phi_{N(0,t-s)}(u + i\lambda).$$

We defined the characteristic function for reals, but it can be extended to the complex. We have

$$E_Q\left[\mathbb{1}_A e^{iu(B_t-B_s)}\right] = e^{iu\lambda(t-s)-\frac{\lambda^2}{2}(t-s)}Q(A)e^{-\frac{(u+i\lambda)^2(t-s)}{2}} =$$
$$= Q(A)e^{\frac{-u^2(t-s)}{2}}.$$

$\square$

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