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ORTHOMODULAR LOGIC. A PROPOSAL OF A LOGIC FOR QUANTUM PHYSICS

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Abstract

Classical physics are widely known to be closely related to classical propositional calculus, whereas there does not exist a strongly settled analogue for quantum physics. We will focus on orthomodular logic and, in particular, we will study two different sentential logics that have been purposed with this aim over othomodular lattices. Thus, we will introduce their semantics from the foundations of quantum mechanics and presenting them by means of two different approaches whose equivalence will be shown. Additionally, we will give an adequate syntax for each proposal, the first one due to M. L. dalla Chiara and R. Giuntini and the last one to G. Kalmbach. Finally, completeness theorems will be discussed as well as the results that have been reached.

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Contents

Introduction

Quantum logic is usually introduced by means of the great expectations that it initially generated after being introduced by J. von Neumann and G. Birkhoff in [1] in 1936, where they showed how quantum logic emerged from the lack of distributivity of empirical propositions that are referred to processes governed by quantum mechanics.

Hence, even if their original construction was later set aside in favour of new proposals based on weaker algebraic structures, researchers such as H. Putnam regarded the creation of a logic for quantum mechanics with great optimism. Actually, he particularly suggested in [18] that quantum logic would make us reconsider our classical logic, comparing it to how general relativity gave rise to Riemannian geometry, enriching the classical Euclidean notions.

In spite of the fact that this expectations have not been accomplished (at least until our days), from this pursuit of the correct logic for quantum mechanics have emerged many non-classical logics that are generally known as quantum logic, regarding their original aim.

We are going to present two similar sentential logics that have been widely regarded as very good representatives of quantum logic for a long period of time. In particular, the proposals that we will study are closely related to the relation established by H. Putnam between quantum and classical logic, which has been usually regarded as the confrontation between Boolean algebras and orthomodular lattices. More specifically, we will work on two sentential logics based on these orthomodular lattices.

Therefore, we will construct the semantics of our quantum logic from the foundations of quantum mechanics, that will be succinctly recalled, showing the close link between quantum logic and orthomodular lattices. Thus, we will present two different approaches to quantum logic, one based on algebraic methods and the other one on what is known as Kripke semantics. During this development we will understand how two different propositional calculi can be constructed, and, additionally, we will develop them by giving their related syntaxes, one due to G. Kalmbach (cf. [11]) and the other to M. L. dalla Chiara and R. Giuntini (cf. [5]). In the meantime, we will also focus on the construction of appropriate implication connectives for any proposal of quantum logic.

Finally, we will review the propositional calculi that we will have developed all along the previous sections, bearing in mind the more recent related discoveries.

Furthermore, this discoveries will shed some light to the initial ideas of H. Putnam, giving rise to some answers but many more questions and lines of research.

As a final observation, we must remark that we will refer to the logic that we will develop as quantum logic, although as we have already pointed, making this identification is not totally correct in the sense that, perhaps, we should refer to it as orthomodular logic.

Chapter 1

The need for quantum logic

In this section we will firstly show the connection between classical physics and classical logic, taking into account the algebraic structure of the propositions within this frame, which will turn out to be a Boolean algebra. After this, we will repeat the same schema with the quantum case, briefly introducing the formalism of quantum mechanics, in order to analyse the algebraic structure of the propositions within this other frame. Given these points, we will understand how quantum physics does not match with classical logic, making clear the need for a new one. All algebraic definitions and references to any kind of lattice can be found extensively developed by G. Birkohff in [2].

1.1 Classical physics and classical logic

1.1.1 Elementary concepts of classical mechanics

As is widely known, the predictions of classical physics are based on classical (or Newtonian) mechanics which are governed by the laws of motion developed by Isaac Newton. Although this construction works well, there have been later reformulations made by Joseph-Louis de Lagrange and William R. Hamilton, which happen to be far more tractable and do not have problems like the existence of fictitious forces.

In the first place, let us recall some basic definitions related to the formulation of classical mechanics made by Hamilton (cf. [10]).

Definition 1.1. Given a physical system of n particles in positions $\mathbf{r_1}, ..., \mathbf{r_n}$, a ***holonomic constraint*** is a condition that restricts their motion and can be expressed as an equation of the form: $f(\mathbf{r_1}, ..., \mathbf{r_n}, t) = 0$.

Moreover, given a physical system of n particles in positions $\mathbf{r_1}, ..., \mathbf{r_n}$; since it is in a 3-dimensional space, it will have $3n$ independent coordinates. If it also has k holonomic constraints, i.e., k equations having the form $f(\mathbf{r_1}, ..., \mathbf{r_n}, t) = 0$, then, we will be able to use them to eliminate k dependent coordinates and introduce a new set of $3n - k$ independent variables $q_1, ..., q_{3n-k}$, such that the position of the ith-particle can be expressed as $\mathbf{r_i}(q_1, ..., q_{3n-k}, t)$. We will call these new variables ***generalized coordinates***.

Definition 1.2. Given a physical system of n particles with k generalized coordinates $q_1, ..., q_k$; we will call the ***Lagrangian of the system*** the scalar function:

$$L(q_1, ..., q_k, \dot{q}_1, ..., \dot{q}_k, t) = T(q_1, ..., q_k, \dot{q}_1, ..., \dot{q}_k, t) - V(q_1, ..., q_k, \dot{q}_1, ..., \dot{q}_k, t),$$

where T and V are the kinetic and the potential energy, respectively, and $\dot{q}_i$ refers to the time derivative of q_i.

Definition 1.3. Given a physical system of n particles with k generalized coordinates $q_1, ..., q_k$; the ***generalized momentum*** associated to the ith-generalized coordinate is:

$$p_i = \frac{\partial L(q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n, t)}{\partial \dot{q}_i}.$$

Definition 1.4. Given a physical system S with k generalized coordinates $q_1, ...,$ q_k, and k generalized momenta, $p_1, ..., p_k$ (all of them independent), we will call the ***phase space*** of S the $2k$-dimensional space whose coordinates are those $2k$ independent variables. Moreover, we will denote it by Σ.

As can be easily noticed, given a classical physical system it is possible to identify its current state with a point in the phase space. Moreover, if we solve the equations of Hamilton, we will find the temporal evolution of S, which will actually be an orbit over the phase space containing the point related to the current state.

1.1.2 Algebraic structure of the propositions of classical physics

Bearing in mind all these definitions, we now have the tools needed to specify what are we going to mean by the logic related to classical dynamics: a propositional calculus whose propositions are the empirical possible states of a given system S within the classical approximation. Additionally, we will call this propositions ***classical events***. Therefore, we must define as good as possible the mathematical representatives of these propositions, so that the logical system arising does coherently correspond to the "classical world".

The widely accepted way to do this (see, for example, [1]) is by considering as the set of propositions, the set of all Lebesgue-measurable subsets of the phase space, Σ, of the system under consideration, $\mathcal{F}(\Sigma)$. The physical justification of this choice is quite obvious, since any classical event can intuitively be understood as a set of possible states of the given system, i.e., as a subset of Σ. From a mathematical point of view, it is basically linked to the preservation of the set-theoretic union, intersection and complementation, since they will be identified with the logical disjunction, conjunction and negation, respectively; and because of the fact of containing an upper and lower bound. On the other hand, the assumption of

Lebesgue-measurability will make classical events more tractable rather than simply considering propositions as any subset of the phase space.

Consequently, we can wonder what is the algebraic structure of the set of all propositions of the propositional calculus related to classical physics, i.e., what is the algebraic structure of $\mathcal{F}(\Sigma)$. As is well known, the set of all Lebesgue-measurable subsets of any set gives rise to a Boolean algebra, whose operations are the set-theoretic union ($\cup$), intersection ($\cap$) and complementation ($'$); which are related to the logical disjunction ($\vee$), conjunction ($\wedge$) and negation ($\neg$), respectively. Hence, we may denote the Boolean algebra of the set of propositions of classical physics as $\langle \mathcal{F}(\Sigma), \cap, \cup, ', \mathbf{1}, \mathbf{0} \rangle$, where $\mathbf{1}$ and $\mathbf{0}$ are the upper and lower bounds, respectively.

It is interesting to note that some authors do not make the assumption of Lebesgue-measurability when defining classical events, whereas Birkhoff and von Neumann do. In spite of this fact, they reach exactly the same results, inasmuch as their set of propositions is the power set of the phase space of the given system, $\mathcal{P}(\Sigma)$, which will obviously give rise to another Boolean algebra: $\langle \mathcal{P}(\Sigma), \cap, \cup, ', \mathbf{1}, \mathbf{0} \rangle$.

1.1.3 Classical logic arising from Boolean algebras

Once we have seen that the set of propositions (classic events) of a given system conforms a Boolean algebra, it can be easily shown that the propositional calculus arising from it matches with classical propositional calculus.

The way to see this is based on the fact that classical propositional calculus is precisely defined over the two-element Boolean algebra, in the sense that any interpretation of a given formula is either true or false. However, some related definitions must be given in order to properly show the connection between the logic related to classical physics and classical propositional calculus. Hence, let us come back to this point in the end of Section 2.1.

1.2 Quantum physics and quantum logic

1.2.1 Elementary concepts of quantum mechanics

When it comes to quantum physics, we must firstly point out that this theory is useful when we are dealing with processes such that their action is similar to the Planck constant or, putting it in a more intuitive way, such that the range of energies involved in this processes is close to the Planck constant. This is a consequence of two facts: on one hand, in such processes classical mechanics does not hold any more and its predictions do not match with the reality, and on the other hand, although quantum mechanics also holds when the action is much bigger than the Planck constant, i.e., in the macroscopic world; in such case it will be much easier to work in the classical framework, since it will simplify all the calculations very much and we will obtain almost the same results.

In order to settle the basis of quantum mechanics, let us recall its six postulates and some related definitions (cf. [6]).

Definition 1.5. A ***complex Hilbert space*** $\mathcal{H}$ is a $\mathbb{C}$-vector space with an inner product

$$\begin{array}{rccc} \langle\cdot|\cdot\rangle: & \mathcal{H}\times\mathcal{H} & \longrightarrow & \mathbb{C} \\ & (u,v) & \longmapsto & \langle u|v\rangle \end{array} \quad \text{such that} \begin{cases} \langle u|\lambda v+\beta w\rangle = \lambda\langle u|v\rangle + \beta\langle u|w\rangle \\ \langle u|v\rangle = \overline{\langle v|u\rangle} \\ \langle u|u\rangle \geq 0 \text{ and } \langle u|u\rangle = 0 \Rightarrow u = \vec{0} \end{cases}$$

that is complete for the norm $\|u\| = \langle u|u\rangle^{1/2}$.

Moreover, $\mathcal{H}$ will be ***separable*** if, and only if, it has a countable orthonormal basis.

Definition 1.6. Given a bounded linear operator $A : \mathcal{H} \longrightarrow \mathcal{H}$, the ***adjoint operator*** of A, $A^\dagger$, is the unique operator such that $\langle A^\dagger u|v\rangle = \langle u|Av\rangle \ \forall u, v \in \mathcal{H}$.

Definition 1.7. An operator A is called ***Hermitian*** or ***self-adjoint*** whenever $A = A^\dagger$.

A relevant property of Hermitian operators is that all their eigenvalues are real numbers. From now on, we will always deal with Hermitian operators even if we do not explicitly note it.

Definition 1.8. Given two linear operators A and B, we will define their ***commutator*** as $[A, B] \equiv AB - BA$.

Definition 1.9. Given a complex, separable Hilbert space, $\mathcal{H}$, the ***Hilbertian inner product*** is given by:

$$\langle u|v\rangle = \begin{pmatrix} \overline{u_1} & \dots & \overline{u_n} \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

Let us also introduce the ***notation of Dirac***: given $u, v \in \mathcal{H}$ and an operator A, we will write:

$$\langle u| \equiv \begin{pmatrix} \overline{u_1} & \dots & \overline{u_n} \end{pmatrix}, \quad |u\rangle \equiv \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \text{ and}$$

$$\langle u|A|v\rangle \equiv \begin{pmatrix} \overline{u_1} & \dots & \overline{u_n} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

We are now ready to state the postulates of quantum mechanics:

1st Postulate: The maximum information about a physical system in an instant of time, t, that one can have is its quantum state, represented by a unitary vector with an arbitrary phase factor, $|\psi(t)\rangle \in \mathcal{H}$, where $\mathcal{H}$ is a separable complex Hilbert space.

2nd Postulate: Any measurable magnitude (also called observable) of the system under consideration has an associated Hermitian operator defined over the vectorial space of the quantum states.

3rd Postulate: The result of a measure of an observable A over a state $|\psi(t)\rangle$ can just be one of its eigenvalues (which, considering A is Hermitian, will be real), and the probability that the state $|\psi(t)\rangle$ has a value of the measurable A equal to λ_i is:

$$P_{|\psi(t)\rangle}(A : \lambda_i) = |\langle e_i|\psi(t)\rangle|^2,$$

where $|e_i\rangle$ is the eigenvector corresponding to the eigenvalue λ_i.

It is interesting to mention that, given a state $|\psi(t)\rangle$, the expected value of the observable A will be:

$$\langle A\rangle_{|\psi(t)\rangle} = \sum_{i=1}^{n} \lambda_i P_{|\psi(t)\rangle}(A : \lambda_i) = \langle\psi(t)|A|\psi(t)\rangle \equiv \tilde{\lambda}.$$

And the dispersion of the measure of A will be:

$$\sigma_A^2\left(|\psi(t)\rangle\right) = \sum_{i=1}^{n} \left(\lambda_i - \tilde{\lambda}\right)^2 P_{|\psi(t)\rangle}(A : \lambda_i) = \langle\psi(t)|A^2|\psi(t)\rangle - \langle\psi(t)|A|\psi(t)\rangle^2 .$$

We can now enunciate the uncertainty principle of Heisenberg, which states that:

$$\sigma_A^2\left(|\psi\rangle\right)\sigma_B^2\left(|\psi\rangle\right) \geq \frac{1}{2}|\,\langle\psi|[A,B]|\psi\rangle\,|$$

4th Postulate: If we measure any observable A over any state $|\psi(t)\rangle$, getting as a result of the measure the eigenvalue of A: λ_i, then, immediately after the measure, the quantum state $|\psi(t)\rangle$ will have collapsed into the normalised eigenvector whose eigenvalue is λ_i.

5th Postulate: Between measures, the system is going to evolve governed by the equation of Schrödinger:

$$i\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle = \mathscr{H}\left|\psi(t)\right\rangle, \quad \text{where } \mathscr{H} \text{ is the Hamiltonian of the system.}$$

6th Postulate: The commutators related to the operators of the position and the momentum in Cartesian coordinates, X and P, respectively; satisfy that $\forall i, j \in \{x, y, z\}$:

- $[X_i, X_j] = X_iX_j - X_jX_i = 0.$
- $[P_i, P_j] = P_iP_j - P_jP_i = 0.$

- $[X_i, P_j] = X_i P_j - P_j X_i = \delta_{ij}$.

After this brief introduction to quantum mechanics, some qualitative discussion may be very handy. The first observation that must be made is that quantum mechanics is a probabilistic theory where every system is characterized by its quantum state, also called wave function, that is going to evolve over time governed by the equation of Schrödinger. To be totally correct, we should add here that if our system was going at a very high speed (comparable to that of light), then we should take into account the relativistic effects by using the equation of Dirac instead of the one of Schrödinger, but it is not within the scope of this text.

In either case, once we have the wave function of our system, we are able to figure out the probability that any measurable magnitude (observable) equals a certain value. Here comes another big difference with the classical case, now the values an observable can take are not arbitrary any longer. This is a consequence of the second and third postulates, one ensures that every observable will have an associated Hermitian operator in the Hilbert space, while the other one tells us that the result of the measure can just be one of the eigenvalues of the Hermitian operator (which will necessarily be a real number).

On the other hand, it also tells us how to compute the probability that the result of the measure is a certain eigenvalue of the operator. To do this we just have to project our state (which is a vector of the Hilbert space) onto the subspace generated by the eigenvectors corresponding to that certain eigenvalue, and the result can be either 0 (if the state is orthogonal to the subspace), 1 (if it belongs to the subspace), or an intermediate value (if it is neither orthogonal nor parallel). Moreover, the fourth postulate tells us that if, for example, we measure the spin on the x-direction of an electron whose state is $|\psi\rangle$, finding out that it is $+\hbar/2$, then, immediately after having got that result, the state of the electron will have turned from $|\psi\rangle$ into $|S_x, +\rangle$, the eigenvector corresponding to the eigenvalue $+\hbar/2$.

1.2.2 Algebraic structure of the propositions of quantum physics

Similarly to what we have already seen for the classic case in Section 1.1.2, any empirical proposition referred to a quantum system will be identified with the set of possible quantum states of that given system that satisfy that particular proposition. We will call this sets ***quantum events***. The following procedure can be found in many references, such as [1, 3, 5].

At this point, we already have enough tools to wonder what is going to be the set of possible events in the quantum case, i.e., when the classical approximation does not work well any longer. One could start trying to use the same mathematical representatives for quantum events as for the classical ones, i.e., all Lebesgue-measurable subsets of the “quantum phase space”, which will actually be the Hilbert space, since all possible quantum states live there as vectors. The problem is, though, that we are not taking into account an important point: let p be any quantum proposition and $\{|\phi_i\rangle\}_{i\in I}$ a family of states that satisfy it. Moreover, let us consider the quantum state $|\psi\rangle$ defined as a linear combination of those states:

$$|\psi\rangle = \sum_{i\in I} \lambda_i|\phi_i\rangle, \quad \lambda_i \in \mathbb{C} \text{ and } |\psi\rangle, |\phi_i\rangle \in \mathcal{H},$$

where $\sum_{i\in I} |\lambda_i|^2 = 1$ in order that $|\psi\rangle$ is a proper quantum state. Then, it follows straightforwardly from the postulates introduced above that, provided that the probability that p holds equals one for any $|\phi_i\rangle$, then it will also be one for $|\psi\rangle$, i.e., $|\psi\rangle$ will also satisfy p and, consequently, $|\psi\rangle$ will belong to the quantum event related to p, too.

This is the reason why we cannot simply take the subsets of the Hilbert space as representatives of quantum propositions, we also need them to be closed under finite and infinite linear combinations. The mathematical objects that accomplish this are, as G. Birkhoff and J. von Neumann already said in [1], *closed linear subspaces* of the Hilbert space (subspaces closed under linear combinations and Cauchy sequences). However, it should be noted that for some reasons that are going to be discussed in the conclusions they did only consider the finite-dimensional case. Accordingly to the related literature, we will denote the set of all closed subspaces of the Hilbert space $\mathcal{H}$ as $\mathcal{C}(\mathcal{H})$ and we will identify quantum propositions with the elements of $\mathcal{C}(\mathcal{H})$.

Additionally, given a proposition $p \in \mathcal{C}(\mathcal{H})$ we will denote by $P_{|\psi\rangle}(p)$ the probability that the state $|\psi\rangle \in \mathcal{H}$ satisfies the proposition p. Trivially, it will be 1 if $|\psi\rangle \in p$, 0 whenever $|\psi\rangle$ is orthogonal to all states of p, and any other value within $(0, 1)$, otherwise.

When it comes to its algebraic structure, we must start wondering what are going to be, in the quantum framework, the mathematical representatives for negation, conjunction and disjunction (see Figure 1 for graphic examples of each operation).

Negation ($\neg$): By negation of a proposition $p \in \mathcal{C}(\mathcal{H})$, we mean another proposition, that we will denote as $\neg p$, such that $P_{|\psi\rangle}(p) = 1$ if and only if $P_{|\psi\rangle}(\neg p) = 0$, or, in other words, such that $\neg p$ consists of all states that are orthogonal to all states of p. Birkhoff and von Neumann called $\neg p$ the orthogonal complement of p.

Conjunction ($\wedge$): For the conjunction of two propositions p and q, denoted as $p \wedge q$, we just have to consider the intersection of both closed linear subspaces (that will obviously be another closed linear subspace), and the result, similarly to the classic case, is that $P_{|\psi\rangle}(p \wedge q) = 1$ if and only if $P_{|\psi\rangle}(p) = 1$ and $P_{|\psi\rangle}(q) = 1$.

Disjunction ($\vee$): As for the disjunction of p and q, $p \vee q$, we cannot just consider the mere union of them, since nor would it even be a closed linear subspace, i.e., a proposition. This is why, as we generally do for the sum of vector subspaces, we take as $p \vee q$ the smallest closed linear subspace containing p and q.

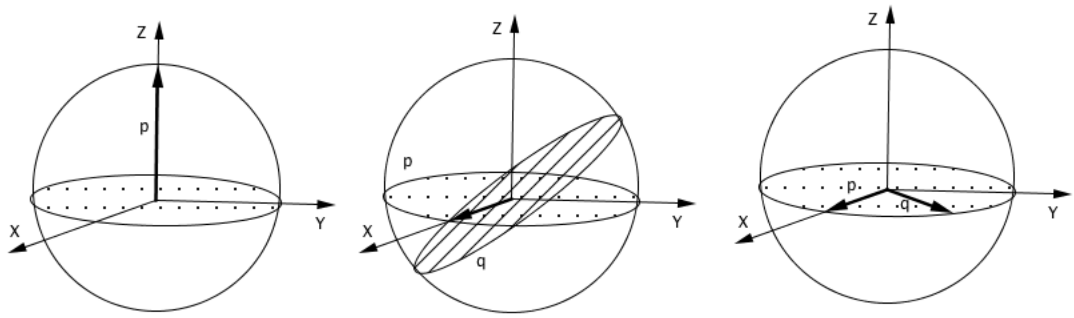

Figure 1: From left to right: the first picture shows the subspace related to $\neg p$ (the stippled area), the second one represents the conjunction operation of quantum events p and q, which gives rise to the closed linear subspace generated by the thick vector of its intersection. Finally, the last picture shows the subspace generated by the disjunction of two propositions p an q, from which arises the stippled area.

If we consider conjunction and disjunction, both of them binary operations, we can easily see that $\langle \mathcal{C}(\mathcal{H}), \wedge, \vee \rangle$ has the algebraic structure of a ***lattice***, since it satisfies the following properties $\forall p, q, r \in \mathcal{C}(\mathcal{H})$:

- **Idempotency:** $\begin{cases} p \wedge p = p \\ p \vee p = p \end{cases}$
- **Associativity:** $\begin{cases} p \wedge (q \wedge r) = (p \wedge q) \wedge r \\ p \vee (q \vee r) = (p \vee q) \vee r \end{cases}$
- **Commutativity:** $\begin{cases} p \wedge q = q \wedge p \\ p \vee q = q \vee p \end{cases}$
- **Absorption:** $\begin{cases} p \wedge (p \vee q) = p \\ p \vee (p \wedge q) = p \end{cases}$

Furthermore, they define a partial order relation (reflexive, antisymmetric and transitive) within $\mathcal{C}(\mathcal{H})$, $\leq$ such that:

$$p \leq q \text{ iff } p \wedge q = p \text{ iff } p \vee q = q.$$

In particular, the following holds for all $p \in \mathcal{C}(\mathcal{H})$:

$$p \geq \varnothing \text{ and } p \leq \mathcal{H},$$

or, equivalently,

$$p \wedge \mathcal{H} = p \text{ and } p \vee \varnothing = p.$$

Hence, $\mathcal{C}(\mathcal{H})$ contains an upper ($\mathcal{H}$) and a lower ($\varnothing$) bound with respect to this partial order relation that are commonly known as the ***top*** and the ***bottom*** of the lattice. Then, we have seen that $\langle \mathcal{C}(\mathcal{H}), \wedge, \vee, \mathcal{H}, \varnothing \rangle$ is a ***bounded lattice***.

If now we take a look at the unary operation $\neg$, one could easily prove that it is an ***involution***, given that for all $p, q \in \mathcal{C}(\mathcal{H})$,

$$\neg(\neg p) = p \text{ and if } p \leq q, \text{ then, } \neg q \leq \neg p.$$

Furthermore, it is a ***orthocomplementation***, since for all $p \in \mathcal{H}$,

$$p \vee \neg p = \mathcal{H} \text{ and } p \wedge \neg p = \varnothing.$$

Therefore, we have seen that $\langle \mathcal{C}(\mathcal{H}), \wedge, \vee, \neg, \mathcal{H}, \varnothing \rangle$ is a ***orthocomplemented lattice*** (or ***ortholattice***). Similarly, the laws of de Morgan also hold:

$$p \wedge q = \neg(\neg p \vee \neg q) \text{ and } p \vee q = \neg(\neg p \wedge \neg q).$$

We have reached a crucial point, if we saw now that this orthocomplemented lattice satisfies the distributive property, then, we would have proved that it is a Boolean algebra, too. Thus, quantum propositions would exactly work as classic ones do, and quantum world would also be governed by classical propositional calculus. Instead of this, quantum events satisfy a weaker form of distributivity, but before going through this, let us illustrate this fact with a very easy example:

Example. *Let an electron be in the quantum state $|S_x, +\rangle$, which represents the eigenvector of the operator related to the spin in the x-direction with eigenvalue equal to $+\hbar/2$. Moreover, let us consider the following propositions:*

p: The spin in the x-direction equals $+\hbar/2$.

q: The spin in the y-direction equals $+\hbar/2$.

r: The spin in the y-direction equals $-\hbar/2$.

Thus, distributivity may be expressed in this particular case as the following equality:

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r) \tag{$*$}$$

If we saw that it does not hold, then we would have found a counterexample that would make clear the fact that distributivity does not hold for quantum events.

On the one hand, given that the spin in the y-direction can only equal $+\hbar/2$ or $-\hbar/2$, then, $q \vee r$ covers all the range of possible values of S_y. Therefore, $q \vee r$ would be true for any quantum state. Moreover, as the electron is in the quantum state $|S_x, +\rangle$, then, p would also hold for it. Consequently, $p \wedge (q \vee r)$ would hold.

On the other hand, as a consequence of the uncertainty principle, considering the propositions $(p \wedge q)$ and $(p \wedge r)$ would not make much sense, given that the operators S_x and S_y do not commutate and so they cannot be determined in the same time with an arbitrary accuracy. Actually, as the system is known to be in the state $|S_x, +\rangle$, which can be expressed in terms of the basis of eigenvectors of S_y as:

$$|S_x, +\rangle = \frac{1+i}{2}|S_y, +\rangle + \frac{1-i}{2}|S_y, -\rangle,$$

then, there would be a probability equal to $\left|\dfrac{1+i}{2}\right|^2 = 1/2$ *that the spin in the y-direction is positive, and equal to* $\left|\dfrac{1-i}{2}\right|^2 = 1/2$ *that it is negative. Hence, the spin in the y-direction would be totally undetermined and, as a result, neither* $(p \wedge q)$ *nor* $(p \wedge r)$ *may be true and, consequently, the equality* $(*)$ *does not hold.*

As can be seen, the lack of the distributive property for quantum events is a direct consequence of the uncertainty principle. Additionally, another archetypal counterexample that proves this lack is based on the non-commutativity between the operators of position and momentum. Furthermore, as the uncertainty principle will be present whenever two operators do not commute, the lack of distributivity will be quite a general anomaly. Nevertheless, it is important to realize that when we apply quantum logic to the macroscopic world, i.e., to processes that can be studied within the classical approximation, quantum logic is reduced to classical propositional calculus, as one could expect, given that the uncertainty principle becomes negligible and distributivity holds.

After all, we can conclude that, as we had already said, $\langle \mathcal{C}(\mathcal{H}), \wedge, \vee, \neg, \mathcal{H}, \varnothing \rangle$ is not a Boolean algebra but a weaker algebraic structure that generally satisfies a weaker form of distributivity. Two natural weakenings of distributivity have been thought to be satisfied by quantum events, namely modular and orthomodular laws, which can be sequentially obtained as follows:

Distributive property: $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$,

and restricting it to the case where $p \leq q$ we obtain:

Modular property: If $p \leq q$, then $\forall r \in \mathcal{C}(\mathcal{H})$ $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$,

which can be additionally weakened if we delimit it to the case where $r = \neg p$:

Orthomodular property: If $p \leq q$, then $p \vee (\neg p \wedge q) = q$.

The orthocomplemented lattice $\langle \mathcal{C}(\mathcal{H}), \wedge, \vee, \neg, \mathcal{H}, \varnothing \rangle$ satisfies modularity whenever the Hilbert space on which it is based has a finite dimension. Hence, given the way that G. Birkhoff and J. von Neumann understood the probabilities related to quantum mechanics, they thought that modularity had to be accomplished. This is why they restricted their construction in [1] to the finite-dimensional case.

As M. Rédei analyses throughout [19], J. von Neumann unsuccessfully attempted to overcome this limitation. However, it was later discovered that orthomodularity held for any finite and infinite-dimensional Hilbert spaces, and a great agreement was reached around the fact that it was the weaker form of distributivity that defined the algebraic structure of quantum events and, consequently, $\langle \mathcal{C}(\mathcal{H}), \wedge, \vee, \neg, \mathcal{H}, \varnothing \rangle$, is a ***orthomodular lattice***.

Given that this structure plays a fundamental role in the quantum logic that we are going to develop, let us define it more generally:

Definition 1.10. Any set L equipped with one unary and two binary operations ($\neg$, $\wedge$ and $\vee$, respectively) is a ***orthomodular lattice*** if the following conditions hold:

(i) Idempotency, associativity, commutativity and absorption are satisfied.

(ii) L is bounded with respect to the partial order relation, $\leq$, that emerge naturally from $\wedge$ and $\neg$ ($x \leq y$ iff $x \wedge y = x$ or, equivalently, $x \vee y = y$), i.e., if there exist two elements $\mathbf{0}, \mathbf{1} \in L$ such that $\forall x \in L$, $\mathbf{0} \leq x$ and $x \leq \mathbf{1}$.

(iii) The unary operation $\neg$ is an involution operation and, in particular, an orthocomplementation, i.e., $\forall x, y \in L$: $\neg(\neg x) = x$, $x \vee \neg x = \mathbf{1}$, $x \wedge \neg x = \mathbf{0}$ and $x \leq y$ implies that $\neg y \leq \neg x$.

(iv) Orthomodular property holds, i.e., for all $x, y \in L$ such that $x \leq y$, $y = x \vee (y \wedge \neg x)$.

Under these circumstances, we will denote this orthomodular lattice as:

$$\mathcal{L} = \langle L, \wedge, \vee, \neg, \mathbf{0}, \mathbf{1} \rangle.$$

1.2.3 Quantum logic arising from orthomodular lattices

As we have seen until here, it has become quite obvious that, as a consequence of all deep differences between classical and quantum physics (its probabilistic character, the uncertainty principle, having a Hilbert space as a phase space...), some deep logical differences have arisen, mainly, the lack of distributivity, which may be a source of apparent paradoxes when we deal with quantum processes while we reason in a classic way. Because of all these, G. Birkhoff and J. von Neumann where the firsts to attempt to create an adequate quantum propositional calculus that includes all these anomalies (with respect to classic events).

According to what we have already pointed in the introduction, in the following chapters we are going to develop two similar propositional calculi based on orthomodular lattices with the objective of faithfully constructing the logical system that governs quantum events.

Chapter 2

Semantics for quantum logic

Given a formal language that will be defined recursively by means of connectives and variables, the semantics of the logic will give an interpretation to any formula of this formal language; giving, in some way, a meaning to them. Additionally, the semantic notion of truth will arise from the definitions of logical consequence and logical truth. The construction of the semantics of any logic can be made in several ways, giving rise to different results. If two different semantics preserve the logical consequence independently of how each semantic defines it, then, they will be equivalent and, in particular, have exactly the same logical truths.

On the other hand, the syntax of the logic is also based on the formal language but with the big difference that it does not give any importance to the possible meaning that the sentences may have. Even if it can also be given in many ways, in general terms, the syntax of a logic is based on settling different rules, axioms... and giving a method to determine whether a formula of the formal language is true or, at least, can be inferred from some other formulas. Here comes the syntactical analogue of logical truths (now called theorems) and logical consequence (deduction).

The aim of this section is to start the construction of quantum propositional calculus by giving its formal semantics. Although, as we have already said, it could be done in many ways, we will basically work on two approaches, the first one algebraic and the second one based on Kripke realizations. Moreover, we will prove that they are equivalent.

Before starting with this construction, though, we need to give the formal language on which quantum logic will be based. It will consist of:

Sentential letters or variables: They will all be contained in a denumerable set X and, according to the previous section, we will denote them by letters $p_1, p_2, p_3, ... \in X$.

Primitive connectives: We will just consider negation $\neg$ and conjunction $\wedge$, and we will define any other connective from these two.

Formulas: We will denote them using Greek letters α, β, γ...

Furthermore, we will define recursively the set of formulas that can be constructed by means of the variables of X and the primitive connectives, $Form(X)$, as follows:

(i) All variables of X are formulas.

(ii) If α is a formula, then $\neg\alpha$ is a formula.

(iii) If α and β are formulas, then $\alpha \wedge \beta$ is a formula.

Additionally, we will define disjunction using the laws of de Morgan and in the same way as for the classical propositional calculus, i.e., $\alpha \vee \beta := \neg(\neg\alpha \wedge \neg\beta)$. Furthermore, it is interesting to note that $Form(X)$ is the absolutely free algebra with the elements of X as free generators over the signature $\{\wedge, \neg\}$.

When it comes to the implication connective, it is not going to be as easy as using its classical analogue $\alpha \overset{CPC}{\rightarrow} \beta = \neg\alpha \vee \beta$, given that it does not behave as it should. Nevertheless, as the correct choice of a proper connective for the implication is not clear at all, we will work on it all along Chapter 3.

2.1 Algebraic semantics for quantum logic

Before starting studying algebraic semantics and in order to make the notation clearer, from now on we will work with general orthomodular lattices $\mathcal{L} = \langle L, \wedge, \vee, \neg, \mathbf{1}, \mathbf{0}\rangle$, assuming them to be as those based on the set of quantum events of a certain system, $\langle \mathcal{C}(\mathcal{H}), \wedge, \vee, \neg, \mathcal{H}, \varnothing\rangle$. Obviously, $\mathbf{1}$ is the top and $\mathbf{0}$ the bottom of the lattice, as they are commonly denoted. Additionally, the general construction of algebraic semantics can be found in many references, such as [5, 8, 11]. When it comes to the different logical consequences, the strong one has been introduced as in [5, 8], whereas the weak one as in [11]. Further discussion about both logical consequences may be found in [13].

Definition 2.1. An ***algebraic realization*** for quantum logic is given by a pair $\mathcal{A} = \langle \mathcal{L}, v\rangle$, where $\mathcal{L} = \langle L, \wedge, \vee, \neg, \mathbf{1}, \mathbf{0}\rangle$ is a orthomodular lattice and v is the so-called ***valuation function***, an homomorphism from $Form(X)$ into L such that:

$$\begin{array}{rccc} v: & Form(X) & \longrightarrow & L \\ & \neg\alpha & \longmapsto & \neg v(\alpha) \\ & \alpha \wedge \beta & \longmapsto & v(\alpha) \wedge v(\beta) \end{array}$$

or, more explicitly, such that maps the logical negation of the set of formulas to the orthocomplement of the orthomodular lattice, and the logical conjunction to the conjunction operation.

Furthermore, one could easily see that $v(\alpha \vee \beta) = v(\alpha) \vee v(\beta)$ and, in spite of the fact that we do not have defined the implication $(\rightarrow)$ for quantum logic, yet; it is interesting to point that, as it will be also expressed in terms of $\wedge$ and $\neg$, we will also have that $v(\alpha \rightarrow \beta) = v(\alpha) \rightarrow v(\beta)$.

Definition 2.2. Given a formula $\alpha \in Form(X)$, we will say that α is ***true for an algebraic realization*** $\mathcal{A} = \langle \mathcal{L}, v \rangle$ whenever $v(\alpha) = \mathbf{1}$, and we will write $\models_{\mathcal{A}} \alpha$.

Definition 2.3. Given a formula $\alpha \in Form(X)$, we will say that it is a ***logical truth of algebraic semantics*** if, for any algebraic realization $\mathcal{A}$, $\models_{\mathcal{A}} \alpha$. Similarly, we will denote it as $\overset{A}{\models} \alpha$.

The natural step we should follow now is to define the concept of logical consequence but it will not be that easy, since further comments must be made. The problem here is that there are two main proposals for this:

Definition 2.4. Given a set of formulas $\Gamma \subseteq Form(X)$ and a formula $\alpha \in Form(X)$, we will distinguish two possible logical consequences in algebraic semantics:

(i) α is an ***algebraic strong logical consequence*** of Γ if and only if, for any algebraic realization $\mathcal{A} = \langle \mathcal{L}, v \rangle$ and $\forall p \in L$,

$$\text{if } p \leq v(\beta) \ \ \forall \beta \in \Gamma, \text{ then, } p \leq v(\alpha)$$

and we will write $\Gamma \overset{A}{\models}_S \alpha$.

(ii) α is an ***algebraic weak logical consequence*** of Γ if and only if, for any algebraic realization $\mathcal{A} = \langle \mathcal{L}, v \rangle$,

$$\text{if } v(\Gamma) \subseteq \{\mathbf{1}\}, \text{ then, } v(\alpha) = \mathbf{1}$$

or, equivalently,

$$\text{if } \models_{\mathcal{A}} \beta \ \ \forall \beta \in \Gamma, \text{ then, } \models_{\mathcal{A}} \alpha$$

and we will write $\Gamma \overset{A}{\models}_W \alpha$.

An unusual thing happens here when consulting the literature related to quantum or orthomodular logic: some authors call the strong and weak logical consequences inversely to how they are defined here and in many other sources (namely [13]). Moreover, $\models_S$ is a strengthening of $\models_W$ in the sense that:

$$\text{if } \Gamma \models_S \alpha, \text{ then, } \Gamma \models_W \alpha$$

This is why we have decided to define logical consequences in this way. Additionally, whenever we deal with lattices for which distributivity holds, then, these two logical consequences will be equivalent (as happen in the classic case).

Hence, after having defined the notion of algebraic semantics of a logic, we can go back to the discussion of Section 1.1.3 about the connection between the logic of classical physics and classical propositional calculus.

The key point is that the algebraic semantics for classical propositional calculus is given by the two-element Boolean algebra, $\mathcal{B}_0$, and a homomorphism that maps formulas from $Form(X)$ into $\mathcal{B}_0 = \{\mathbf{0}, \mathbf{1}\}$. Hence, all related concepts such as truth for a given realization and logical consequence are defined identically (with the difference that here we will not have to make any distinction between strong and weak logical consequence given that distributivity holds).

Under these circumstances, it is important to realize that in the classical case every formula is either true or false, given that the valuation function can only be equal to either **1** or **0**.

Moreover, it is possible to prove that the logical truths arising from considering as algebraic realizations any Boolean agebra coincide with those arising from the two-element one. In particular, it is a consequence of the following result:

Proposition 2.1. *Let $\mathbb{B}$ be the class of all Boolean algebras, $Form(X)$ be the set of formulas of classical logic and $\vDash$ and $\mathcal{A} = \langle \mathcal{B}, v \rangle$ be the algebraic logical consequence and an algebraic realization of the classical propositional calculus, respectively. Then, for any formula $\alpha \in Form(X)$,*

$$\vDash_{\mathcal{A}_0} \alpha, \ \ \forall \mathcal{A}_0 = \langle \mathcal{B}_0, v \rangle \iff \vDash_{\mathcal{A}} \alpha, \ \ \forall \mathcal{A} = \langle \mathcal{B}, v \rangle \text{ such that } \mathcal{B} \in \mathbb{B}.$$

Therefore, considering general Boolean algebras gives rise to exactly the same logical truths and, furthermore, the same logical consequences as restricting the algebraic realizations to two-element Boolean algebras. Consequently, as we have seen that classic events conform a Boolean algebra (independently of considering $\langle \mathcal{F}(\Sigma), \cap, \cup, ', \mathbf{1}, \mathbf{0} \rangle$ or $\langle \mathcal{P}(\Sigma), \cap, \cup, ', \mathbf{1}, \mathbf{0} \rangle$), then they will give rise to classical propositional calculus.

2.2 Kripkean semantics for quantum logic

As we have already introduced, another way to construct the semantics of quantum logic is based on what is known as Kripke semantics, that were initially constructed by S. Kripke for modal logic. Nevertheless, H. Dishkant was able in [7] to adapt this semantics to quantum logic, giving rise to a new approach somehow closer to a physical point of view. From now on and unless the contrary may be indicated, we will refer to Kripkean semantics for quantum logic as Kripkean semantics. All general definitions of this section are presented in the more standard presentation of [5]. Nevertheless, the treatment of the preclusive complement is somehow closer to the one of [4].

First of all, let us define some basic notions that will be of great importance in the construction of Kripkean semantics.

Definition 2.5. A ***Kripkean orthoframe*** is a couple $\langle W, R \rangle$ composed by:

- W: A non-empty set whose elements will be named and understood as ***possible worlds***.
- R: A reflexive and symmetric binary relation within W. It is also known as the ***accessibility relation***.

In our case, given a system S, we will obviously identify the set of possible worlds of our logic, W, with all possible quantum states of S. Taking into account the postulates of quantum mechanics, these states will be the unitary vectors, $|\psi\rangle$, of the Hilbert space, $\mathcal{H}$, related to the system.

When it comes to R, the accessibility relation between worlds, its choice must be based on quantum mechanics foundations bearing in mind that, from an intuitive point of view, if $w_1 R w_2$, we shall say that w_2 is ***accessible*** to w_1 and understand that there is a positive probability that w_1 turns into w_2 after making a measurement, i.e., that the projection of w_1 over w_2's direction is nonzero. Hence, *we will take as accessibility relation the non-orthogonality between vectors*, denoted as $\not\perp$. It comes clear that the accessibility relation of the Kripkean realization for quantum logic will be reflexive and symmetric, but not generally transitive, this is why $\not\perp$ is known as a similarity relation and $\langle W, \not\perp\rangle$ as a similarity space.

In this point we should define the concept of proposition in the context of Kripkean semantics for quantum logic. Let us give its formal construction first, in order to make an interpretation of its physical interpretation afterwards.

Hence, we will define an operation within the power-set of W as follows:

$$\neg X := \{w \in W | w \perp w' \quad \forall w' \in X\},$$

where $\perp$ is the orthogonality relation between vectors.

Given that $\perp$ is irreflexive and symmetric, it is a preclusivity relation. This is why, $\neg$ is known as the ***preclusive complement***. Moreover, as seen in [4], it gives rise to the following closure operator:

$$\begin{array}{rccc} \neg\neg: & \mathcal{P}(W) & \longrightarrow & \mathcal{P}(W) \\ & X & \longmapsto & \neg\neg X = \neg(\neg X) \end{array}$$

given that it satisfies the following properties:

1. $X \subseteq \neg\neg X$.

2. $X \subseteq Y$ implies that $\neg\neg X \subseteq \neg\neg Y$.

3. $\neg\neg(\neg\neg X) = \neg\neg X$.

Hence, we can define the set of propositions of Kripkean semantics as follows:

Definition 2.6. Any set of worlds $X \subseteq \mathcal{P}(W)$ is a ***proposition*** if $X = \neg\neg X$.

Additionally, further results may be obtained from this definitions if we consider the operations of conjunction and set-theoretic intersection within $\mathbb{P}$. Moreover, it must by pointed that by the conjunction of two propositions X and Y, we mean the smallest propositions containing both of them.

Proposition 2.2. *Given any Kripkean orthoframe $\langle W, \not\perp\rangle$, let $\mathbb{P} = \{X \subseteq W | X = \neg\neg X\}$ be the set containing all its propositions. Then,* *(i) If $X \in \mathbb{P}$, then $\neg X \in \mathbb{P}$.*

(ii) If $X_i \in \mathbb{P}$ for all $i \in I$, then $\bigcap_{i\in I} X_i \in \mathbb{P}$.

(iii) If $X, Y \in \mathbb{P}$, then $X \vee Y \in \mathbb{P}$. In particular, $X \vee Y = \neg\neg(X \cup Y)$.

(iv) $W, \varnothing \in \mathbb{P}$.

Proof. (i) Given $X \in \mathbb{P}$, then $\neg\neg(\neg X) = \neg(\neg\neg X) = \neg X$. Therefore, $\neg X \in \mathbb{P}$.

(ii) Let $\{X_i\}_{i\in I}$ be a family of propositions. On the one side, given that $\neg\neg$ is a closure operator, then $\bigcap_{i\in I} X_i \subseteq \neg\neg\left(\bigcap_{i\in I} X_i\right)$.

On the other side, as for any $j \in I$, $\bigcap_{i\in I} X_i \subseteq X_j$, then $\neg\neg\left(\bigcap_{i\in I} X_i\right) \subseteq \neg\neg X_j = X_j$, as $X_j \in \mathcal{P}$. Thus, $\neg\neg\left(\bigcap_{i\in I} X_i\right) \subseteq \bigcap_{i\in I} X_i$, and we can conclude that $\bigcap_{i\in I} X_i = \neg\neg\left(\bigcap_{i\in I} X_i\right) \in \mathbb{P}$.

(iv) Given $X \in \mathbb{P}$, by (i) $\neg X \in \mathbb{P}$, and by (ii) $X \cap \neg X \in \mathbb{P}$. Hence, let us assume that there exists a world $w \in X \cap \neg X$. If so, $w \in X$ and $w \in \neg X$ and, consequently $x \perp x$, which is a contradiction. Hence, $\varnothing = X \cap \neg X \in \mathbb{P}$.

On the other side, we can assert that $\neg\varnothing \in \mathbb{P}$. Moreover, by the definition of the preclusive complement, $\neg\varnothing = \{w \in W | w \perp w' \ \ \forall w' \in \varnothing\} = W$, and $W = \neg\varnothing \in \mathbb{P}$.

(iii) Let $X, Y \in \mathbb{P}$ and let us consider another proposition $T \in \mathbb{P}$ such that $X, Y \subseteq T$ (it will exist given that $W \in \mathbb{P}$). Then, $X \cup Y \subseteq T$, and, therefore, $\neg\neg(X \cup Y) \subseteq \neg\neg T = T$. Additionally, as $X, Y \subseteq \neg\neg(X \cup Y)$, we can conclude that $X \vee Y = \neg\neg(X \cup Y)$. Thus, $X \vee Y = \neg\neg(X \cup Y) = \neg\neg(\neg\neg(X \cup Y)) = \neg\neg(X \vee Y)$ and, consequently, $X \vee Y \in \mathbb{P}$.

□

We will not prove it but it is straightforward to check the following corollary (cf. [4]).

Corollary 2.3. *Given any Kripkean orthoframe $\langle W, \not\perp\rangle$, let $\mathbb{P} = \{X \subseteq W | X = \neg\neg X\}$ be the set containing all its propositions. Then, $\langle \mathbb{P}, \cap, \vee, \neg, W, \varnothing\rangle$ is an orthocomplemented lattice.*

After all this results, we are prepared to define the main concept of Kripkean semantics:

Definition 2.7. Given a Kripkean orthoframe $\langle W, \not\perp\rangle$, a ***Kripkean realization*** for quantum logic, is a quaternary $\mathcal{K} = \langle W, \not\perp, \Pi, \rho\rangle$ such that:

Π is a set of propositions such that it contains $\varnothing$ and W, is closed under the preclusive complement and the set-theoretic intersection and satisfies orthomodular law, i.e., for all $X, Y \in \Pi$,

$$X \subseteq Y \text{ implies that } X \vee (Y \cap \neg X) = Y.$$

ρ is an homomorphism that maps formulas of the formal language into Π and satisfies the following properties:

$$\begin{array}{rccc} \rho: & Form(X) & \longrightarrow & \Pi \\ & \neg\alpha & \longmapsto & \neg\rho(\alpha) \\ & \alpha \wedge \beta & \longmapsto & \rho(\alpha) \wedge \rho(\beta) \end{array}$$

It can be regarded as the Kripkean analogue of the valuation function that we have defined for algebraic realizations.

After having defined all this concepts, let us interpret them from a physical point of view in order to justify their presence in quantum logic. As we said in the beginning of this section, the set of possible worlds will be directly identified with the set of all quantum states, i.e., with the set of unitary vectors of the Hilbert space related to the system under consideration. Hence, the propositions of $\mathbb{P}$ should coincide with the quantum events that we have defined in Section 1.2.2. In order to see this, let us recall the following lemma from [5]:

Lemma 2.4. *Given any Kripkean orthoframe* $\langle W, \not\perp \rangle$, *let* $\mathbb{P} = \{X \subseteq W | X = \neg\neg X\}$. *Then,*

$$X = \{w \in W | \forall w' \in W : \textit{ if } w' \in \neg X, \textit{ then } w' \perp w\}.$$

Hence, a qualitative analysis of this alternative definition makes clear that, as seen in [5], the propositions of Kripkean semantics will consist of all closed linear vector spaces generated by the quantum states belonging to each proposition.

When it comes to the operations related to the set of propositions, namely $\cap$, $\vee$ and $\neg$, its interpretation also emerges quite naturally from the foundations of quantum mechanics. When it comes to $\cap$, its choice is quite obvious, given that if a set X of states satisfies a certain property, and another set Y satisfies another proposition, then, the quantum states satisfying both propositions will trivially coincide with $X \cap Y$. The interpretation of $\vee$ is also very straightforward, leading to its link with the sum of closed linear subspaces.

Finally, the preclusive complement is perhaps the most interesting case, since its choice is based on the way of computing the probability that a certain proposition p holds for a given world w or, equivalently, for a given state $|\psi\rangle$: as the third postulate of quantum mechanics asserts, we have to measure the length of the projection of $|\psi\rangle$ over the subspace associated to p, i.e., over the closed linear subspace that conforms the quantum event related to p. Hence, given any proposition, those states for which it will hold are precisely the ones belonging to the associated closed linear subspace, since the length of the projection over this subspace will trivially be the length of the unitary vector, and consequently, the probability that the proposition holds will be equal to one.

Furthermore, as Kripkean propositions coincide with quantum events, let X be a quantum event. Then, the set of all quantum events whose probability that the quantum event X is satisfied will be conformed by the quantum states that are orthogonal to the closed linear subspace X, i.e., that are orthogonal to all states of X, as it implies the alternative definition of the preclusive complement given by Lemma 2.4. Following the denominations of Kripke semantics, we will say that $\neg X$ consists of the worlds of W that are ***inaccessible*** to all worlds of X (given that $\forall w \in \neg X$, $\nexists w' \in X$ s.t. $w \not\perp w'$).

Similarly, given any formula $\alpha \in Form(X)$, $\rho(\alpha)$ represents the unique proposition of $\mathbb{P}$ such that α holds for all of its quantum states (and for no others). Considering all the development until here, the physical interpretation of ρ is somehow predictable: $\rho(\alpha)$ will actually be the set of all those states (worlds) such that the probability that α holds is equal to 1 for any of them. Hence, $\rho(p)$ will obviously coincide with the closed linear subspace related to the quantum event.

Now that we have justified from a physical point of view the construction of Kripkean realizations for quantum logic, we are prepared to define the notions of truth for a given realization, logical truth and logical consequence.

Definition 2.8. Given a formula $\alpha \in Form(X)$, we will say that α is ***true for a Kripke realization*** $\mathcal{K} = \langle W, \not\perp, \Pi, \rho \rangle$ whenever $\rho(p) = W$, and we will write: $\vDash_{\mathcal{K}} \alpha$.

Again, working with Kripke realizations makes the interpretation of quantum logic easier from a physical point of view. When it comes to truth of a proposition for a given realization, it can be interpreted as the fact that the proposition holds for any world (quantum state) of the realization (Hilbert space).

Moreover, we can extend this notion to the logical truth, where the given proposition holds for any world of any Kripkean realization.

Definition 2.9. Given a formula $\alpha \in Form(X)$, we will say that it is a ***logical truth of Kripkean semantics*** if, for any Kripke realization $\mathcal{K}$, $\vDash_{\mathcal{K}} \alpha$. Similarly, we will denote it as: $\overset{K}{\vDash} \alpha$.

It is important to realize that whenever a world $w \in W$ and a formula $\alpha \in Form(X)$ satisfy that $w \in \rho(\alpha)$, then it must be understood as the fact that α holds in the world w.

Finally and differently to the case of algebraic realizations, when it comes to the logical consequence for the Kripkean semantics, just one logical consequence is usually considered, in particular, the analogue of our algebraic strong logical consequence. Nevertheless, we will also define the analogue of the algebraic weak logical consequence and in the following section we will show that it is properly defined, giving rise to an equivalence theorem between both semantics.

Definition 2.10. Given a set of formulas $\Gamma \subseteq Form(X)$ and a formula $\alpha \in Form(X)$, we will distinguish two possible logical consequences in Kripkean semantics:

(i) α is a ***Kripkean strong logical consequence*** of Γ if and only if, for any

Kripke realization $\mathcal{K} = \langle W, \not\perp, \Pi, \rho \rangle$ and $\forall w \in W$,

$$\text{if } w \in \bigcap_{\beta \in \Gamma} \rho(\beta), \text{ then, } w \in \rho(\alpha)$$

and we will write: $\Gamma \overset{K}{\vDash}_S \alpha$.

(ii) α is a ***Kripkean weak logical consequence*** of Γ if and only if, for any Kripke realization $\mathcal{K} = \langle W, \not\perp, \Pi, \rho \rangle$,

$$\text{if } \rho(\beta) = W \ \ \forall \beta \in \Gamma, \text{ then, } \rho(\alpha) = W$$

or, equivalently,

$$\text{if } \vDash_{\mathcal{K}} \beta \ \ \forall \beta \in \Gamma, \text{ then, } \vDash_{\mathcal{K}} \alpha$$

and we will write: $\Gamma \overset{K}{\vDash}_W \alpha$.

An interesting remark related to this definitions is that whenever a formula $\alpha \in Form(X)$ is a either strong or weak logical consequence of a set of formulas $\Gamma \subseteq Form(X)$, then, α will hold in any world where all the formulas of Γ hold.

2.3 Equivalence between algebraic and Kripkean semantics

Now that we have already introduced two different semantics for quantum logic: the algebraic one, which is better from a mathematical point of view, and the Kripkean one, closer to quantum physics; we ought to see their equivalence, or, at least, under what conditions do they coincide.

As we have already said in the introduction of this chapter, when we talk about equivalence between semantics we are referring to the invariance of logical deduction between them and, in particular, the equality of the formulas that are logically true for each semantic. This is obvious if one bears in mind that logical truths can be regarded as formulas that are logical consequences of $\varnothing$, i.e., without any hypothesis.

Before start dealing with the equivalence between algebraic and Kripkean realizations, we should see some previous results. First of all and in the same line as in [5], let us define the canonical transformation between algebraic and Kripkean realizations for quantum logic:

Definition 2.11. The ***canonical transformation of an algebraic realization*** for quantum logic, $\mathcal{A} = \langle \langle L, \wedge, \vee, \neg, \mathbf{1}, \mathbf{0} \rangle, v \rangle$, is the Kripkean realization $\mathcal{K}^{\mathcal{A}} = \langle W, \not\perp, \Pi, \rho \rangle$ such that:

- $W = L \setminus \{\mathbf{0}\}$.
- $w \not\perp w'$ if and only if, $w \not\leq \neg w'$; for any $w, w' \in W$.

- $\Pi = \{(p] \mid p \in L\}$, where $(p] = \{q \in L \setminus \{\mathbf{0}\} \mid q \leq p\}$.
- $\rho(\alpha) = (v(\alpha)]$, for any $\alpha \in Form(X)$.

To show that $\mathcal{K}^{\mathcal{A}}$ is a proper Kripkean realization for quantum logic we will need the following lemma, which can be found in [5].

Lemma 2.5. *Given a orthocomplemented lattice $\mathcal{L} = \langle L, \wedge, \vee, \neg, \mathbf{1}, \mathbf{0}\rangle$, then it is a orthomodular lattice if and only if $\forall p, q \in L$,*

$$p \leq q \iff p \wedge \neg(p \wedge q) = \mathbf{0}.$$

Lemma 2.6. *Given an algebraic realization $\mathcal{A}$, its canonical transformation, $\mathcal{K}^{\mathcal{A}}$ is a Kripkean realization for quantum logic.*

Proof. Given an algebraic realization $\langle\langle L, \wedge, \vee, \neg, \mathbf{1}, \mathbf{0}\rangle, v\rangle$, let us firstly prove that $\not\perp$ is a proper accessibility relation for quantum logic:

- Symmetry: $\forall w, w' \in W$, $w \not\perp w'$ implies that $w' \not\perp w$ or, equivalently, $\forall p, q \in L \setminus \{\mathbf{0}\}$, $p \not\leq \neg q$ implies that $q \not\leq \neg p$.

  It is a direct consequence of the following:

  $$p \leq \neg q \iff p \wedge \neg q = p \iff \neg p = \neg(p \wedge \neg q) = \neg p \vee q \iff q \leq \neg p.$$

- Reflexivity: $\forall w \in W$, $w \not\perp w$.

  Let us assume that there exists a world $w \in W$ such that $w \perp w$. Hence, $w \leq \neg w$, which implies that $w = w \wedge \neg w = \mathbf{0} \notin W$!

Thus, removing the bottom of the lattice from the set of worlds makes $\not\perp$ reflexive and, furthermore, it also implies that $\varnothing \in \Pi$, since $\varnothing = (\mathbf{0}] \in \Pi$. Additionally, $W = (\mathbf{1}] \in \Pi$.

It is straightforward to check that for any $p, q \in L$ the following will hold:

- $\neg(p] = (\neg p]$.
- $(p] \cap (q] = (p \wedge q]$.
- $(p] \vee (q] = (p \vee q]$.

Consequently, Π is closed under the preclusive complement, the disjunction and the set-theoretic intersection. One could also observe that $\Pi \subseteq \mathbb{P}$, given that $\neg\neg(p] = (\neg\neg p] = (p]$ for any $p \in L$.

Hence, we can easily conclude that $\langle \Pi, \cap, \vee, W, \varnothing\rangle$ is a orthocomplemented lattice, and so, by Lemma 2.5, to prove that it is an orthomodular lattice is equivalent to showing that for all $(p], (q] \in \Pi$,

$$(p] \subseteq (q] \text{ if and only if } (p] \cap \neg((p] \cap (q]) = \varnothing,$$

or, equivalently,

$$p \leq q \text{ if and only if } (p \wedge \neg(p \wedge q)] = (\mathbf{0}].$$

which is a direct consequence of the orthomodularity of $\langle L, \wedge, \vee, \neg, \mathbf{1}, \mathbf{0}\rangle$ together with Lemma 2.5.

Finally, we still have to see that ρ is properly defined, i.e., that for all formulas $\alpha, \beta \in Form(X)$, $\rho(\neg\alpha) = \neg\rho(\alpha)$ and $\rho(\alpha \wedge \beta) = \rho(\alpha) \wedge \rho(\beta)$:

$\wedge$: $\rho(\alpha \wedge \beta) = (v(\alpha \wedge \beta)] = (v(\alpha) \wedge v(\beta)] = (v(\alpha)] \wedge (v(\beta)] = \rho(\alpha) \wedge \rho(\beta)$.

$\neg$: $\rho(\neg\alpha) = (v(\neg\alpha)] = (\neg v(\alpha)] = \neg(v(\alpha)] = \neg\rho(\alpha)$.

After all, we can conclude that $\mathcal{K}^{\mathcal{A}}$ is a Kripkean realization for quantum logic. □

Lemma 2.7. *For any algebraic realization $\mathcal{A} = \langle \mathcal{L}, v\rangle$, there exists an associated Kripkean realization $\mathcal{K}^{\mathcal{A}} = \langle W, \not\perp, \Pi, \rho\rangle$ such that $\forall\alpha \in Form(X)$,*

$$\vDash_{\mathcal{A}} \alpha \textit{ iff } \vDash_{\mathcal{K}^{\mathcal{A}}} \alpha.$$

Proof. Given any formula $\alpha \in Form(X)$ and any algebraic realization for quantum logic $\mathcal{A}$. Let $\mathcal{K}^{\mathcal{A}}$ be its canonical transformation. Thus,

$$\vDash_{\mathcal{A}} \alpha \iff v(\alpha) = \mathbf{1} \iff \rho(\alpha) = (\mathbf{1}] = W \iff \vDash_{\mathcal{K}^{\mathcal{A}}} \alpha.$$

□

Definition 2.12. The ***canonical transformation of a Kripkean realization*** for quantum logic, $\mathcal{K} = \langle W, \not\perp, \Pi, \rho\rangle$, is the algebraic realization $\mathcal{A}^{\mathcal{K}} = \langle\langle L, \wedge, \vee, \neg, \mathbf{1}, \mathbf{0}\rangle, v\rangle$ such that:

- $L = \Pi$, $\mathbf{0} = \varnothing$ and $\mathbf{1} = W$.
- $p \leq q$ if and only if $p \subseteq q$, for any $p, q \in L$.
- $\neg p = \{w \in W \mid w \perp w', \ \ \forall w' \in p\}$, for any $p \in L$.
- $v(\alpha) = \rho(\alpha)$, for any $\alpha \in Form(X)$.

And exactly as we did for the previous case, we still need to show the following:

Lemma 2.8. *Given a Kripkean realization $\mathcal{K}$, its canonical transformation, $\mathcal{A}^{\mathcal{K}}$ is an algebraic realization for quantum logic.*

Proof. To prove it, we just need to see that $\langle L, \wedge, \vee, \neg, \mathbf{1}, \mathbf{0}\rangle$ is a orthomodular lattice and v an homomorphism from $Form(X)$ into $\langle L, \wedge, \vee, \neg, \mathbf{1}, \mathbf{0}\rangle$.

However, in comparison with the canonical transformation of an algebraic realization, this case results quite trivial, given that $\langle L, \wedge, \vee, \neg, \mathbf{1}, \mathbf{0}\rangle = \langle \Pi, \cap, \vee, W, \varnothing\rangle$.

Hence, bearing in mind Corollary 2.3 and given that Π, by definition, contains W and $\varnothing$, is closed under the preclusive complement and the set-theoretic intersection and satisfies the orthomodular property, we can conclude that it is an orthomodular lattice.

Finally, as ρ is an homomorphism from $Form(X)$ into $\Pi = L$, so will be v. $\square$

Lemma 2.9. *For any Kripkean realization $\mathcal{K} = \langle W, \not\perp, \Pi, \rho\rangle$, there exists an associated algebraic realization $\mathcal{A}^{\mathcal{K}} = \langle \mathcal{L}, v\rangle$ such that $\forall \alpha \in Form(X)$,*

$$\vDash_{\mathcal{K}} \alpha \textit{ iff } \vDash_{\mathcal{A}^{\mathcal{K}}} \alpha.$$

Proof. Given any formula $\alpha \in Form(X)$ and any Kripkean realization for quantum logic $\mathcal{K}$. Let $\mathcal{A}^{\mathcal{K}}$ be its canonical transformation. Thus,

$$\vDash_{\mathcal{K}} \alpha \iff \rho(\alpha) = W \iff v(\alpha) = \mathbf{1} \iff \vDash_{\mathcal{A}^{\mathcal{K}}} \alpha.$$

$\square$

Theorem 2.10 (Theorem of equivalence of logical truths)**.** *Let $\Gamma \subset Form(X)$ be any set of formulas and $\alpha \in Form(X)$ an arbitrary one. Then,*

$$\overset{K}{\vDash} \alpha \textit{ iff } \overset{A}{\vDash} \alpha.$$

Proof. Let us prove both implications separately:

- $\underline{\Rightarrow}$: We want to see that, given any algebraic realization $\mathcal{A}$, $\vDash_{\mathcal{A}} \alpha$ holds. By hypothesis, we have that $\overset{K}{\vDash} \alpha$ and, in particular, that $\vDash_{\mathcal{K}^{\mathcal{A}}} \alpha$. Thus, by Lemma 2.7, we can conclude that $\vDash_{\mathcal{A}} \alpha$ is satisfied.

- $\underline{\Leftarrow}$: Similarly, given any Kripkean realization $\mathcal{K}$, as by hypothesis $\overset{A}{\vDash} \alpha$ holds, then so does $\vDash_{\mathcal{A}^{\mathcal{K}}} \alpha$. Hence, recalling Lemma 2.9, we can conclude that $\vDash_{\mathcal{K}} \alpha$ holds.

$\square$

To sum up, until here we have seen that there is a very close relationship between algebraic and Kripkean realizations that can be constructed by means of canonical transformations. Moreover, this correspondence has let us prove the equivalence between the logical truths of each semantics.

However, further results should be obtained in order to state the total equivalence between algebraic and Kripkean semantics, in particular, the invariance of logical consequence has to be proved. Nevertheless, this must be done considering either the strong logical consequence or the weak one. As we introduced before defining the logical consequence for Kripkean realizations in Section 2.2, the semantics for quantum logics (especially the Kripkean ones) have been basically developed in terms of the strong logical consequence. This is why the equivalence between both of them has just been proved for this particular case.

Consequently, we will just explicitly show the proof of the case related to the equivalence between the algebraic and the Kripkean weak logical consequences.

Theorem 2.11 (Weak equivalence theorem)**.** *Let $\Gamma \subset Form(X)$ be any set of formulas and $\alpha \in Form(X)$ an arbitrary one. Then,*

$$\Gamma \overset{K}{\vDash}_W \alpha \textit{ iff } \Gamma \overset{A}{\vDash}_W \alpha.$$

Proof.

$$\begin{aligned} &\Gamma \overset{K}{\vDash}_W \alpha \iff \text{For any Kripkean realization } \mathcal{K}, \\ &\vDash_{\mathcal{K}} \beta \;\; \forall \beta \in \Gamma \text{ implies that } \vDash_{\mathcal{K}} \alpha \overset{\text{Theorem 2.10}}{\iff} \\ &\overset{\text{Theorem 2.10}}{\iff} \text{For any algebraic realization } \mathcal{A}, \; \vDash_{\mathcal{A}} \beta \\ &\forall \beta \in \Gamma \text{ implies that } \vDash_{\mathcal{A}} \alpha \iff \Gamma \overset{A}{\vDash}_W \alpha. \end{aligned}$$

□

As we said, the equivalence between algebraic and Kripkean semantics can also be proved for the case that considers the strong logical consequence, giving rise to the following theorem:

Theorem 2.12 (Strong equivalence theorem)**.** *Let $\Gamma \subset Form(X)$ be any set of formulas and $\alpha \in Form(X)$ an arbitrary one. Then,*

$$\Gamma \overset{K}{\vDash}_S \alpha \textit{ iff } \Gamma \overset{A}{\vDash}_S \alpha.$$

The proof of the strong equivalence theorem, as we have named it, is due to P. Minari and can be found in [16].

As we have seen that both semantics are equivalent, from now on we will just write $\vDash_S$ instead of $\overset{K}{\vDash}_S$ or $\overset{A}{\vDash}_S$, and $\vDash_W$ instead of $\overset{K}{\vDash}_W$ or $\overset{A}{\vDash}_W$. Furthermore, we will refer to them as strong and weak logical consequences, avoiding the previous adjectives "algebraic" and "Kripkean".

Chapter 3

An implication for quantum logic

As we have already said in the beginning of Chapter 2, defining an implication connective for quantum logic is not as easy as it has been for the disjunction one. This is a consequence of the fact that the classical implication does not have a proper behaviour and independently of the connective that we finally consider, some anomalies are going to take place.

Since we are not going to deal with any logical consequence throughout this chapter, the results obtained will hold for $\vDash_S$ and $\vDash_W$. Thus, we will generally refer to the semantics under consideration as $\vDash$.

It is widely accepted that two basic properties that should be expected to be accomplished by any implication connective, $\rightarrow$, for all $\alpha, \beta \in Form(X)$ are the following:

Reflexivity: $\vDash \alpha \rightarrow \alpha$.

Modus ponens for logical truths: $\vDash \alpha$ and $\vDash \alpha \rightarrow \beta$, implies that $\vDash \beta$.

As it is shown in [5], it is possible to prove a characterization for well-behaved implication connectives:

Proposition 3.1. *If for any formulas $\alpha, \beta \in Form(X)$ and for any algebraic realization $\mathcal{A} = \langle\langle L, \wedge, \vee, \neg, \mathbf{1}, \mathbf{0}\rangle, v\rangle$, $\vDash_{\mathcal{A}} \alpha \rightarrow \beta$ holds if and only if $v(\alpha) \leq v(\beta)$, then $\rightarrow$ satisfies reflexivity and modus ponens properties.*

Likewise, it can be equivalently formulated in terms of Kripkean semantics:

Proposition 3.2. *If for any formulas $\alpha, \beta \in Form(X)$ and for any Kripkean realization $\mathcal{K} = \langle W, \not\perp, \Pi, \rho\rangle$, $\vDash_{\mathcal{K}} \alpha \rightarrow \beta$ holds if and only if $\rho(\alpha) \subseteq \rho(\beta)$, then $\rightarrow$ satisfies reflexivity and modus ponens properties.*

Obviously, $\leq$ makes reference to the partial order relation of the lattice and $\subseteq$ to the set-theoretic inclusion.

Again, working with Kripkean semantics let us make an easier interpretation of quantum logic. In this case, we can infer from Proposition 3.2 that any implication connective $\to$ will satisfy reflexivity and modus ponens whenever it accomplishes that, $\alpha \to \beta$ holds for any world of W if and only if in any world where α holds, so will do β, which gives rise to a very intuitive notion of implication.

Another important thing to realize is that once we find an implication connective, $\to$, that satisfies this propositions, then it will be satisfied independently of the logical consequence that we consider, i.e., independently of considering the semantics $\vDash_S$ or $\vDash_W$. This is a consequence of the fact that both reflexivity and modus ponens only involve the notion of logical truth, which, as we have already said, is independent of the logical consequence considered.

In this point we have enough tools to discuss why the implication connective of classical propositional calculus, $\alpha \overset{CPC}{\to} \beta = \neg\alpha \vee \beta$, does not work well for quantum logic. This is a consequence of the existence of algebraic realizations $\mathcal{A} = \langle\langle L, \wedge, \vee, \neg, \mathbf{1}, \mathbf{0}\rangle, v\rangle$ such that for some formulas $\alpha, \beta \in Form(X)$, $\vDash \alpha \overset{CPC}{\to} \beta$ hold, i.e., $v(\neg\alpha \vee \beta) = \mathbf{1}$, while in the same time $v(\alpha) \not\leq v(\beta)$. Hence, by Proposition 3.1, $\overset{CPC}{\to}$ does not behave good enough to be the implication connective of quantum logic. In particular, as S. Smets remarks in [20], the orthomodular lattice of Figure 2 is a suitable counterexample of this fact.

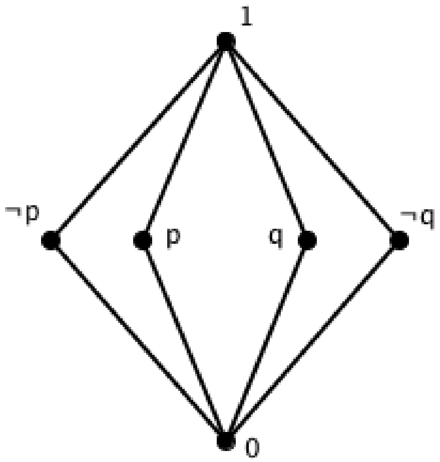

Figure 2: It represents the diagram of the orthomodular lattice known as the ***Chinese lantern***. Taking into account that the edges represent the inclusion relation, i.e., the points below are less than or equal to the ones above them if they are connected with an edge. Thus, one can trivially see that $\neg p \vee q = \mathbf{1}$ while $p \not\leq q$.

After having seen that $\overset{CPC}{\to}$ is not useful for quantum logic, to try to find a proper implication connective may look like a matter of luck. However, thanks to the thorough analysis about orthomodular lattices conducted by G. Kalmbach in [11], it is possible to determine a small set of implication connectives that satisfy reflexivity and modus ponens. In particular, she was able to prove that there were only five possible implication connectives that accomplish those properties and can be defined in terms of $\wedge$ and $\neg$.

Before getting into her proof, one should notice that, as $\to$ will be expressed in terms of $\wedge$ and $\neg$, then, as we did observe in Section 2.1, $v(\alpha \to \beta) = v(\alpha) \to v(\beta)$.

Thus, what she did was to consider the orthomodular lattice absolutely freely generated by two elements, namely $v(\alpha)$ and $v(\beta)$, and she studied all possible binary operations and showed that the only operations, $\to$, that satisfied the following property (equivalent to the one of Proposition 3.1):

$$v(\alpha) \leq v(\beta) \text{ if and only if } v(\alpha) \to v(\beta) = v(\alpha \to \beta) = \mathbf{1}, \qquad (*)$$

were the following five connectives:

Sasaki hook: $v(\alpha \to_1 \beta) = \neg v(\alpha) \vee (v(\alpha) \wedge v(\beta))$.

Dishkant implication: $v(\alpha \to_2 \beta) = v(\beta) \vee (\neg v(\alpha) \wedge \neg v(\beta)) = v(\neg\beta \to_1 \neg\alpha)$.

Kalmbach implication: $v(\alpha \to_3 \beta) = (\neg v(\alpha) \wedge v(\beta)) \vee (v(\alpha) \wedge v(\beta)) \vee (\neg v(\alpha) \wedge \neg v(\beta))$.

Non-tollens implication: $v(\alpha \to_4 \beta) = (\neg v(\alpha) \wedge v(\beta)) \vee (v(\alpha) \wedge v(\beta)) \vee ((\neg v(\alpha) \vee v(\beta)) \wedge \neg v(\beta)) = v(\neg\beta \to_3 \neg\alpha)$.

Relevance implication: $v(\alpha \to_5 \beta) = (\neg v(\alpha) \wedge v(\beta)) \vee (\neg v(\alpha) \wedge \neg v(\beta)) \vee (v(\alpha) \wedge (\neg v(\alpha) \vee v(\beta)))$.

It comes clear, then, that independently of the implication connective, $\to_i$, that we consider, we will have that, by virtue of $(*)$, for all $\alpha, \beta \in Form(X)$ and for any algebraic interpretation $\mathcal{A} = \langle\langle L, \wedge, \vee, \neg, \mathbf{1}, \mathbf{0}\rangle, v\rangle$,

$$\vDash_{\mathcal{A}} \alpha \to_i \beta \text{ if and only if } v(\alpha) \leq v(\beta).$$

Furthermore, it is also important to realize that all implication connectives give rise to equivalence connectives in the sense that $\alpha \leftrightarrow_i \beta := \alpha \to_i \beta \wedge \beta \to_i \alpha$ satisfies the following property for any algebraic interpretation $\mathcal{A}$, formulas $\alpha, \beta \in Form(X)$ and $i \in \{1, ..., 5\}$:

$$\vDash_{\mathcal{A}} \alpha \leftrightarrow_i \beta \text{ if and only if } v(\alpha) = v(\beta).$$

This fact is a direct consequence of $(*)$, given that:

$$\vDash_{\mathcal{A}} \alpha \leftrightarrow_i \beta \iff v(\alpha \leftrightarrow_i \beta) = \mathbf{1} \iff v(\alpha \to_i \beta) \wedge v(\beta \to_i \alpha) = \mathbf{1} \iff$$

$$\iff \begin{cases} v(\alpha \to_i \beta) = \mathbf{1} \\ v(\beta \to_i \alpha) = \mathbf{1} \end{cases} \overset{(*)}{\iff} \begin{cases} v(\alpha) \leq v(\beta) \\ v(\beta) \leq v(\alpha) \end{cases} \iff v(\alpha) \wedge v(\beta) = v(\alpha) = v(\beta).$$

Another key point is that if distributivity held, then all the implication connectives, $\to_1, ..., \to_5$; would be the same. In particular, they would coincide with the one of classical propositional calculus, $\overset{CPC}{\to}$, as one could expect.

As we said in the beginning of this section, no matter what implication connective of those five we chose, it will give rise to some anomalies with respect to the classical case. Hence, after the results of G. Kalmbach, there has been an attempt to choose the best possible option regarding to its particular properties.

Furthermore, an agreement has been reached around the choice of $\to_1$, which is widely known as the ***Sasaki hook*** (given that it was firstly studied by U. Sasaki), as the best implication connective for quantum logic. M. L. dalla Chiara and R. Giuntini justify this choice arguing that it is the only one that satisfies a weaker form of the ***import-export condition*** (cf. [5]). This condition ensures, when is generally satisfied by an orthomodular lattice, that the distributive property holds, i.e., that the orthomodular lattice is a Boolean algebra. Obviously the Sasaki hook does not satisfy the import-export condition, given that quantum logic is precisely based on the lack of the distributive property. Nevertheless, it is the only implication connective amongst the five given by G. Kalmbach that satisfies the following weakening of that condition for all formulas $\alpha, \beta, \gamma \in Form(X)$ and for any algebraic interpretation $\mathcal{A} = \langle\langle L, \wedge, \vee, \neg, \mathbf{1}, \mathbf{0}\rangle, v\rangle$:

$$\text{If } v(\alpha) = (v(\alpha) \wedge \neg v(\beta)) \vee (v(\alpha) \wedge v(\beta)), \text{ then:}$$

$$v(\gamma) \wedge v(\alpha) \leq v(\beta) \text{ iff } v(\gamma) \leq v(\alpha) \to v(\beta).$$

However, some other important properties do fail, too, namely, and just to put some examples the first three axioms of the axiomatic system purposed by G. Frege for classical propositional calculus, do not hold for the Sasaki hook. Hence, for all $\alpha, \beta, \gamma \in Form(X)$, the following formulas are not logical truths (cf. [5]):

1. $\alpha \to_1 (\beta \to_1 \alpha)$.
2. $(\alpha \to_1 (\beta \to_1 \gamma)) \to_1 ((\alpha \to_1 \gamma) \to_1 (\alpha \to_1 \gamma))$.
3. $(\alpha \to_1 (\beta \to_1 \gamma)) \to_1 (\beta \to_1 (\alpha \to_1 \gamma))$.

In spite of the fact that the Sasaki hook is possibly the most spread implication connective for quantum logic; as M. Pavičić and N. Megill analyse throughout [17], there are many other proposals of more elaborated logics for quantum physics based on different implication connectives and made by other authors such as G. Kalmbach, also in [11], who used the Kalmbach implication, $\to_3$, to create a propositional calculus, or H. Dishkant in 1974 using the Dishkant implication, $\to_2$, to construct a first-order predicate logic.

In the following section we are going to study the proposals of G. Kalmbach on the one side, and M. L. dalla Chiara and R. Giuntini on the other; who construct two different quantum propositional calculus using different logical consequences, implication connectives and methods of defining the syntax of the logic.

Chapter 4

Syntax for quantum logic

Before dealing with the final step of our construction, some discussion about our development may be very handy in order to understand better in where we are and how should be our next step in order to give rise to a proper quantum logic.

First of all, we have defined the formal language on which quantum logic is going to be based, $Form(X)$, and we have given an interpretation to its propositions and formulas, which have been understood as quantum events. This assumptions, together with the foundations of quantum mechanics, have let us state that quantum logic will not match with classical propositional calculus. Once we have seen it, we have been able to define appropriate semantics for quantum logic bearing in mind physical properties of quantum events. Thus, we have reached the notions of logical truth and logical consequence from to different but equivalent approaches (one algebraic and the other one based on Kripkean realizations), giving rise to two different semantics, namely $\vDash_S$ and $\vDash_W$.

Moreover, as we have introduced in the beginning of Chapter 2, we still need to give appropriate syntax for quantum logic. Differently to the construction process of its semantics, now we will not have to consider any possible interpretation from a physical point of view of our definitions. In contrast, we will just have to make this new syntax of quantum logic match with the semantics under consideration, and so it will automatically incorporate all the anomalies of quantum physics. But what do we mean with making it match with the semantics under consideration?

Similarly to the semantics, the syntax of a logic settles a way of discerning whether a given formula of $Form(X)$ is "true" or not. Furthermore, this can also be done from several approaches and probably, amongst all of them, the most classical one would be by settling a set of axioms and some rules of inference that, by means of a deductive process, will give rise to what are known as ***theorems***, the syntactical analogues of logical truths. Must be remembered that theorems such as 2.11 and 2.12 are actually called ***metatheorems*** in order to avoid any possible confusion with the theorems of the syntax of the logic that we have just introduced. Nevertheless, we will recall this definitions more rigorously afterwards, since the key point here is that, independently of the way of reaching the theorems of the syntax, one should try to make them coincide with the logical truths of the semantics and, furthermore, to make the semantic logical consequence match with its syntactical analogue.

For example, if we take a look at the classic case, i.e., at the classical propositional calculus, we have already observed that its semantics are given by Boolean algebras. On the other side, there are many different proposals of syntaxes, namely the one made by G. Frege by giving a set of six axioms (the first three have already been introduced in the end of the previous chapter and seen to be incompatible with the Sasaki hook) and one rule of inference (modus ponens). Nevertheless, what is important to realize is that for the classic case there exists a total correspondence between its semantics and its syntax, and so one could wonder whether this correspondence does also exist for the different proposals of quantum logic.

Therefore, the aim of this section is to define proper syntactical systems for quantum logic and, accordingly to the second section where we have defined the semantics $\vDash_S$ and $\vDash_W$, here we are going to give its related syntaxes. In the first place, we will define the one based on $\vDash_S$ which uses as implication connective the Sasaki hook ($\rightarrow_1$). It was firstly purposed by R. I. Goldblatt in [9] and later developed by M. L. dalla Chiara and others (cf. [5, 8]). Particularly, they construct this syntax without any axiom but a set of twelve rules and using a kind of natural deduction, introducing all connectives by means of appropriate rules. After having introduced it, we will state some fundamental metatheorems concerning the strong correspondence of her syntax with $\vDash_S$.

Finally, we will repeat the same schema for the semantics $\vDash_W$ and the implication connective $\rightarrow_3$ (the Kalmbach implication). More specifically, we will introduce the syntax purposed by G. Kalmbach in [11], which, differently to he one of M. L. dalla Chiara and R. Giuntini, will be given by means of a larger set of axioms and only one rule of inference.

4.1 Strong quantum propositional calculus

As we have already pointed, the semantics on which the syntax of M. L. dalla Chiara and R. Giuntini is based are $\vDash_S$. Let us recall that its specificity is that it is defined by means of the strong logical consequence (see Definition 2.4). This is why we have called ***strong quantum propositional calculus*** the propositional calculus that arises from her construction

Thus, let us introduce her syntax respecting her own notation, which is somewhat genuine and can be found in [4, 5, 8].

Definition 4.1. Given $\Gamma \subseteq Form(X)$ and $\alpha \in Form(X)$, whenever α may be inferred from Γ we will denote it as $\Gamma \vdash \alpha$ and we will call this a ***configuration***.

Definition 4.2. Given $\Gamma, \Gamma_1, ..., \Gamma_n \subseteq Form(X)$ and $\alpha, \alpha_1, ..., \alpha_n \in Form(X)$. Whenever a configuration $\Gamma \vdash \alpha$ holds if $\Gamma_1 \vdash \alpha_1, ..., \Gamma_n \vdash \alpha_n$ also hold, then we will denote it as:

$$\frac{\Gamma_1 \vdash \alpha_1, ..., \Gamma_n \vdash \alpha_n}{\Gamma \vdash \alpha}$$

and we will call this a ***rule***. Moreover, the configurations $\Gamma_1 \vdash \alpha_1, ..., \Gamma_n \vdash \alpha_n$ are called ***premisses*** and $\Gamma \vdash \alpha$ is the ***conclusion*** of the rule.

To make the notation clearer, given $\Gamma \subseteq Form(X)$ and $\alpha, \beta \in Form(X)$ we will write $\Gamma \cup \alpha \vdash \beta$ instead of $\Gamma \cup \{\alpha\} \vdash \beta$ and $\alpha \vdash \beta$ instead of $\{\alpha\} \vdash \beta$.

It is important to remark that a rule can have infinite premisses or none at all. This last particular case plays a role which is close to the one of axioms. Furthermore, this particular rules will be called ***improper rules*** and will be denoted as:

$$\Gamma \vdash \alpha \text{ instead of } \frac{\varnothing}{\Gamma \vdash \alpha}.$$

Given these points we can assert that the syntax of M. L. dalla Chiara and R. Giuntini will not have any axiom but some proper and improper rules from which, by means of a deductive process that is going to be defined now, the theorems of the syntax will be derived.

Thus, The set of twelve rules of the syntax of strong quantum propositional calculus that hold for all $\Gamma, \Gamma' \subseteq Form(X)$ and for all $\alpha, \beta, \gamma \in Form(X)$ will be comprised of:

Identity: $\Gamma \cup \alpha \vdash \alpha$.

Transitivity: $\dfrac{\Gamma \vdash \alpha, \Gamma' \cup \alpha \vdash \beta}{\Gamma \cup \Gamma' \vdash \beta}$.

$\wedge$-elimination: $\begin{cases} \Gamma \cup \alpha \wedge \beta \vdash \alpha, \\ \Gamma \cup \alpha \wedge \beta \vdash \beta. \end{cases}$

$\wedge$-introduction: $\begin{cases} \dfrac{\Gamma \vdash \alpha, \Gamma \vdash \beta}{\Gamma \vdash \alpha \wedge \beta}, \\ \dfrac{\Gamma \cup \{\alpha, \beta\} \vdash \gamma}{\Gamma \cup \alpha \wedge \beta \vdash \gamma}. \end{cases}$

Absurdity: $\dfrac{\alpha \vdash \beta, \alpha \vdash \neg\beta}{\varnothing \vdash \neg\alpha}$.

Weak double negation: $\Gamma \cup \alpha \vdash \neg(\neg\alpha)$.

Strong double negation: $\Gamma \cup \neg(\neg\alpha) \vdash \alpha$.

Duns Scotus: $\Gamma \cup \alpha \wedge \neg\alpha \vdash \beta$.

Contraposition: $\dfrac{\alpha \vdash \beta}{\neg\beta \vdash \neg\alpha}$.

Orthomodularity: $\alpha \wedge \neg(\alpha \wedge \neg(\alpha \wedge \beta)) \vdash \beta$.

We are ready to define, by means of two concepts, how is going to be the deduction process of this syntax.

Definition 4.3. A ***derivation*** of strong quantum propositional calculus will be a finite sequence of configurations such that any element of the sequence is the conclusion of, either an improper rule, or a proper rule whose premises are previous elements of the sequence.

Definition 4.4. Given $\Gamma \subseteq Form(X)$ and any formula $\alpha \in Form(X)$, we will say that α is ***derivable*** from Γ if there exists a derivation such that $\Gamma \vdash \alpha$ is the last element of the sequence. Moreover, we will denote it as $\Gamma \vdash_S \alpha$.

We are finally in position to define rigorously the syntactical analogue of the logical truths that we have seen in the semantics, i.e., the theorems:

Definition 4.5. Given a formula $\alpha \in Form(X)$, we will say that it is a ***theorem*** of strong quantum propositional calculus if $\varnothing \vdash_S \alpha$, and in that case we will simply write $\vdash_S \alpha$.

Before dealing with the connection between the logical truths and the theorems that arise from the twelve rules that have been settled, or, more generally, between the strong logical consequence and derivability, let us focus on a fundamental metatheorem of classical propositional calculus, the ***deduction theorem***.

It states that given $\Gamma \subseteq Form(X)$ and two formulas $\alpha, \beta \in Form(X)$, then:

$$\Gamma \cup \{\alpha\} \vdash_{CPC} \beta \text{ implies that } \Gamma \vdash_{CPC} \alpha \overset{CPC}{\to} \beta,$$

where $\vdash_{CPC}$ makes reference to the classical analogue of $\vdash_S$.

It is important to realize that this metatheorem is of great significance, given that it guarantees that, in practise, to prove that a formula α can be inferred from another formula β, it is enough to see that the conditional $\alpha \overset{CPC}{\to} \beta$ holds. An immediate question arises: does deduction theorem generally hold for quantum logic or, at least, for some of its constructions? A very strong negative answer has been given by J. Malinowski in [12], where he has been able to prove that no logic based on orthomodular lattices may satisfy the deduction theorem. Moreover, he showed that if it was satisfied, then distributivity would have to hold and the logic would be Boolean and, therefore, it would coincide with the classical case.

However, weaker forms of the deduction theorem have been proved for strong quantum propositional calculus. As seen in [4,5,8], a weakening of this metatheorem related to the strong logical consequence can be enunciated as follows:

Theorem 4.1. *Given any formulas $\alpha, \beta \in Form(X)$, then,*

$$\alpha \vdash_S \beta \textit{ if and only if } \vdash_S \alpha \to_1 \beta.$$

Moreover, we could easily prove another similar weakening:

Theorem 4.2. *Given any formulas $\alpha, \beta \in Form(X)$, then,*

$$\vdash_S \alpha \textit{ and } \vdash_S \alpha \to_1 \beta \textit{ implies that } \vdash_S \alpha \to_1 \beta.$$

Hence, considering the Sasaki hook results in recovering similar forms of deduction theorem. Additionally, we can complete our construction of strong quantum propositional calculus by enunciating two metatheorems that show that its semantics, $\models_S$, and its syntax, $\vdash_S$, are equivalent, giving rise to the same notions of consequence and truth. Thus, we can state the following metatheorems (cf. [4,5,8]):

Theorem 4.3 (Strong soundness theorem). *Given* $\Gamma \subseteq Form(X)$ *and* $\alpha \in Form(X)$*, then,*

$$\Gamma \vdash_S \alpha \text{ implies that } \Gamma \models_S \alpha.$$

Theorem 4.4 (Strong completeness theorem). *Given* $\Gamma \subseteq Form(X)$ *and* $\alpha \in Form(X)$*, then,*

$$\Gamma \models_S \alpha \text{ implies that } \Gamma \vdash_S \alpha.$$

In conclusion, we have seen that considering the strong logical consequence together with the Sasaki hook gives rise to a propositional calculus that can be syntactically defined in a proper way, in the sense that both approaches give rise to analogous interpretations of the formal language of the logic.

4.2 Weak quantum propositional calculus

The aim of this section is to construct the syntax purposed by G. Kalmbach in [11], which considers the Kalmbach implication, $\to_3$, and the weak logical consequence, $\models_W$. Therefore, we will call this propositional calculus ***weak quantum propositional calculus***.

As we have already introduced, this syntax, differently to the one that we have just seen, is given in a very standard way. It settles a set of fifteen formulas of $Form(X)$ as axioms and, by means of one rule of inference, gives a Hilbert-style deductive method to derive the theorems. Additionally, we will understand why G. Kalmbach decided to use the Kalmbach implication, $\to_3$, instead of the Sasaki hook. Finally, some comments about completeness and soundness theorems will be made, as well as about the weaker forms of the deduction theorem that are satisfied.

Hence, let us define its main related concepts in the same way as E. Mendelson in [15]:

Definition 4.6. Given a formal language $Form(X)$, some formulas will be considered ***axioms***.

Definition 4.7. A ***rule of inference*** is a relation among the formulas of $Form(X)$, such that there exists a unique positive integer i that accomplishes that, given any set of i formulas of $Form(X)$, there is an effective method to determine whether this formulas are in the relation R to another formula $\alpha \in Form(X)$. In that case, we will say that α ***follows from*** the given i formulas by virtue of R.

Thus, we are prepared to construct the syntax of G. Kalmbach. Let us start by giving its fifteen axioms and, in order to make the notation clearer, write $\alpha * \beta$

instead of $(\alpha \wedge \beta) \vee (\neg\alpha \wedge \neg\beta)$. Hence, given any formulas $\alpha, \beta, \gamma \in Form(X)$, the following are *axioms* of weak quantum propositional calculus:

A1. $\alpha * \alpha$.

A2. $\neg(\alpha * \beta) \vee (\neg(\beta * \gamma) \vee (\alpha * \gamma))$.

A3. $\neg(\alpha * \beta) \vee (\neg\beta * \neg\beta)$.

A4. $\neg(\alpha * \beta) \vee ((\alpha \wedge \gamma) * (\beta \wedge \gamma))$.

A5. $(\alpha \wedge \beta) * (\beta \wedge \alpha)$.

A6. $(\alpha \wedge (\beta \wedge \gamma)) * ((\alpha \wedge \beta) \wedge \gamma)$.

A7. $(\alpha \wedge (\alpha \vee \beta)) * \alpha$.

A8. $(\neg\alpha \wedge \alpha) * ((\neg\alpha \wedge \alpha) \wedge \beta)$.

A9. $\alpha * \neg\neg\alpha$.

A10. $\neg(\alpha \vee \beta) * (\neg\alpha \wedge \neg\beta)$.

A11. $\alpha \vee (\neg\alpha \wedge (\alpha \vee \beta)) * (\alpha \vee \beta)$.

A12. $(\alpha * \beta) * (\beta * \alpha)$.

A13. $\neg(\alpha * \beta) \vee (\neg\alpha \vee \beta)$.

A14. $(\neg\alpha \vee \beta) \rightarrow_3 (\alpha \rightarrow_3 (\alpha \rightarrow_3 \beta))$.

A15. $\neg(\alpha \rightarrow_3 \beta) \vee (\neg\alpha \vee \beta)$.

Moreover, the only rule of inference that will have this syntax is *modus ponens*: β follows from α and $\alpha \rightarrow_3 \beta$, for any $\alpha, \beta \in Form(X)$.

Under these circumstances, we can define the notions of theorem and being derivable as follows:

Definition 4.8. A ***deduction*** of $\alpha \in Form(X)$ from $\Gamma \subseteq Form(X)$ is a sequence of formulas $\alpha_1, ..., \alpha_n$ such that $\alpha_n = \alpha$ and every formula in the sequence either is an axiom, or belongs to Γ, or follows by virtue of modus ponens from previous formulas of the sequence. In that case, we will denote it as $\Gamma \vdash_W \alpha$.

Definition 4.9. A formula $\alpha \in Form(X)$ is a ***theorem*** if $\varnothing \vdash_W \alpha$ holds. In that case we will write $\vdash_W \alpha$.

After all, it is important to realize that a suitable syntax for classical propositional calculus is also given by a set of axioms and modus ponens as the only rule of inference. Nevertheless, it must be pointed that it is formulated by means of the classical implicative connective $\overset{CPC}{\rightarrow}$ instead of $\rightarrow_3$.

Thus, as for the syntax of M. L. dalla Chiara and R. Giuntini, this one also coincides with its related semantics in the sense that $\models_W$ and $\vdash_W$ are totally equivalent. As before, this result can be summarized in the following metatheorems whose proofs can be found in [11]:

Theorem 4.5 (Weak soundness theorem). *Given $\Gamma \subseteq Form(X)$ and $\alpha \in Form(X)$, then,*

$$\Gamma \vdash_W \alpha \textit{ implies that } \Gamma \models_W \alpha.$$

Theorem 4.6 (Weak completeness theorem). *Given $\Gamma \subseteq Form(X)$ and $\alpha \in Form(X)$, then,*

$$\Gamma \models_W \alpha \textit{ implies that } \Gamma \vdash_W \alpha.$$

As a consequence, we can conclude that weak quantum propositional calculus is also properly constructed, at least from the point of view that it is a sentential logic whose semantics and syntax are equivalent.

Additionally, it is interesting to comment the steps of her proof of theorems 4.5 and 4.6, given that it will let us understand from a richer point of view her construction of this syntax. Initially, she considers the first thirteen axioms (A1 to A13) and the rule of inference: β follows from α and $\neg\alpha \vee \beta$, or, equivalently, β follows from α and $\alpha \overset{CPC}{\rightarrow} \beta$ for all formulas $\alpha, \beta \in Form(X)$, which is nothing else than the rule of modus ponens applied to the classical implication connective.

To see that it is an adequate rule of inference we should see that $\{\alpha, \neg\alpha \vee \beta\} \models_W \beta$, which is straightforward to check, given that if $v(\alpha) = v(\alpha \overset{CPC}{\rightarrow} \beta) = \mathbf{1}$, then,

$$v(\beta) = \mathbf{0} \vee v(\beta) = \neg v(\alpha) \vee v(\beta) = v(\alpha \overset{CPC}{\rightarrow} \beta) = \mathbf{1}.$$

Thus, G. Kalmbach proves soundness and completeness theorems for the resulting syntax. Interestingly enough, $\alpha * \beta = (\alpha \wedge \beta) \vee (\neg\alpha \wedge \neg\beta) = (\alpha \overset{CPC}{\rightarrow} \beta) \wedge (\beta \overset{CPC}{\rightarrow} \alpha) = \alpha \overset{CPC}{\leftrightarrow} \beta$ and, additionally, one can observe that the axioms under consideration are closely related to the equations that define an orthomodular lattice (e.g., A11 is related to the orthomodular property). Thus, making use of all these facts, she is able to show that for all $\Gamma \subseteq Form(X)$ and $\alpha \in Form(X)$,

$$\Gamma \models_W \alpha \text{ if and only if } \Gamma \vdash_0 \alpha, \qquad (**)$$

where $\Gamma \vdash_0 \alpha$ is defined exactly as $\Gamma \vdash_W \alpha$ but considering A1 to A13 and the rule of inference applied to $\overset{CPC}{\rightarrow}$.

After seeing this, she manages to see that A14 and A15 are logical truths and she adds them to the set A1 to A13. Additionally, she considers the rule of inference applied to the implication connective $\rightarrow_3$ and she sees that it is equivalent to considering the one applied to $\overset{CPC}{\rightarrow}$, i.e., that $\{\alpha, \alpha \rightarrow_3 \beta\} \vdash_0 \beta$ and $\{\alpha, \alpha \overset{CPC}{\rightarrow} \beta\} \vdash_W \beta$.

Finally, she concludes that for all $\Gamma \subseteq Form(X)$ and $\alpha \in Form(X)$,

$$\Gamma \vdash_0 \alpha \text{ if and only if } \Gamma \vdash_W \alpha,$$

from where theorems 4.5 and 4.6 immediately follow by means of $(**)$.

Conclusions

In order to analyse from a richer point of view the results that we have obtained, let us take a glance at the historical process that has preceded this constructions and at the more recent discoveries that have succeeded them.

As A. Cabello suggests in [3], the true aim of quantum logic has been trying to identify the algebraic structure of the empirical propositions of quantum mechanics in order to develop a quantum logic based on those structures. Given that we have considered $\mathcal{C}(\mathcal{H})$ as an orthomodular lattice throughout this work, one could expect it to be a widely spread fact since the early birth of quantum logic in [1]. However, quantum logic has been approached from many different points of view, and the ones that we have shown here are two just representative constructions.

A clear example of this fact is that the first construction of quantum logic made by G. Birkhoff and J. von Neumann in 1936 was based on the assumption that quantum events not only had to satisfy the orthomodular, but also the modular one. The reason for this, as thoroughly studied in [19], was the way on how they understood the probabilities related to quantum mechanics. Consequently, they conceived modularity as a *sine qua non* condition. However, given that modularity does only hold for finite-dimensional Hilbert spaces, they encountered many complications, having to restrict the range of application of their logic. Some years after the publication of "The logic of quantum mechanics" J. von Neumann attempted to overcome this difficulties without meeting any succeed.

Afterwards, othomodularity was shown to be generally satisfied by $\mathcal{C}(\mathcal{H})$ for any Hilbert space, giving rise to a new interpretation of quantum logic as a orthomodular logic, i.e., a logic based on orthomodular lattices. Under these circumstances many different proposals appeared, and from that time quantum logic has been studied from many approaches.

Thus, we have presented a construction of two sentential logics based on those orthomodular lattices. In spite of the fact that quantum logic is surrounded by very deep questions, it must be observed that our propositional calculi satisfy very interesting notions from a physical point of view, such us tending to the Boolean logic when we deal with systems in the scale of energies of the classical physics, given that the uncertainty principle becomes negligible and distributivity holds.

Leaving aside the problem of having many different proposals of logics for a unique purpose, other deeper doubts about the validity of quantum logic appeared, namely the problem of considering the conjunction of two propositions related to operators that do no commute, i.e., that cannot be measured simultanously. In words of D. J. Foulis and others (from [8]):

> If, for a quantum-mechanical system, most pairs of observations are incompatible and cannot be made simultaneously, what experimental meaning can one attach to the meet $p \wedge q$ of two propositions?

As observed in [8] and further developed in [4, 5], it is possible to avoid this issue by partially defining the conjunction operation, or to put it differently, by restricting it to the cases where both observations are compatible.

Nevertheless, orthomodular logic was subsequently shown to be unsuitable for quantum events. It is as a consequence of the discovery that the algebraic structure of $\mathcal{C}(\mathcal{H})$ (henceforth called ***Hilbert lattice***) satisfies additional equations, namely the orthoarguesian laws, that are not generally satisfied by orthomodular lattices. Furthermore, as observed in [14], successive discoveries gave rise to more general infinite equational varieties satisfied by Hilbert lattices but not for orthomodular lattices.

Thus, we can conclude that orthomodular logic is not an adequate abstraction of quantum mechanics and, furthermore, we can deduce that further investigations should attempt to fully describe algebraically Hilbert lattices as well as construct a proper logic based on them.

After all, and considering the successive development that quantum logic has experienced; even if orthomodular logic and many other approaches have been shown to be inadequate for its original purpose, it must be accepted that this pursuit has been a source of intensive investigation of non-classical logics, giving rise to many interpretations that have shed new questions related to the quantum world. However, a question made in [5] arises:

> To what extent is it reasonable to look for the "right logic" of quantum theory?

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