Deuteron polarizability shifts and the deuteron matter radius

J. Martorell
Departament d'Estructura i Constituents de la Materia, Facultat Física, Universitat de Barcelona, Barcelona 08028, Spain

D. W. L. Sprung
Department of Physics and Astronomy, McMaster University, Hamilton, Ontario, Canada L8S 4M1

D. C. Zheng
Department of Physics, University of Arizona, Tucson, Arizona 85721
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New laser-based measurements of the isotope shift between hydrogen and deuterium by Hänsch *et al.* should determine accurately the deuteron matter radius r_m. We study the contribution of the deuteron polarizability energy shifts to the isotopic difference and show that this is known in a model independent way. We also make accurate determinations of the asymptotic normalization and the electric dipole polarizability by exploiting empirical linear relations between them and r_m^2.

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I. INTRODUCTION

Recent high accuracy measurements of the Lamb shift in the hydrogen isotopes [1,2] have opened new possibilities for high precision tests of quantum electrodynamics and also for accurate determination of the charge distributions of the hydrogen isotopes' nuclei. Thus the proton radius has been recently redetermined [3] using the measured Lamb shift in hydrogen and, after careful consideration of all relevant QED contributions, found to be in good agreement with the Mainz value extracted from electron scattering experiments [4]. In a similar vein, the isotopic difference in the Lamb shifts for hydrogen and deuterium has been analyzed [5,6]. Here a value for the mass radius of the deuteron has been extracted which contradicts the accepted analysis [7] of low momentum transfer electron scattering experiments designed specifically for the determination of the radius. It was soon realized [6,8,9] that this new value would solve the long-standing contradiction between theory and experiment pointed out by Klarsfeld *et al.* [7], which had put in question the ability of realistic nonrelativistic NN potential models to reproduce simultaneously the ensemble of low energy data for the two nucleon system.

A detailed theory of the hydrogen-deuterium isotopic shift has recently been given in Ref. [6], reviewing the evaluation of the different contributions and their corresponding uncertainties. Among these, the polarization of the deuteron by the presence of the orbital electron is relatively small, and has been treated only with very simple models of nuclear structure: with separable NN interactions of the Yamaguchi type in Ref. [8] and with a square well potential for the S-wave component in Refs. [5] and [6]. Consequently, the error estimates quoted for that contribution are conservatively large. In view of the forecast increase in experimental accuracy and improvements in the computation of the remaining contributions, we believe that it is timely to make a more careful assessment of the value of this polarizability contribution and of the errors that affect its calculation. This is the main purpose of this paper. The evaluation of this contribution will be analyzed in detail in Sec. II, avoiding some of the approximations introduced in [6] so as to obtain a more accurate estimate. Then the corresponding expressions will be evaluated using a representative set of realistic NN interactions.

Section III of the paper is devoted to a redetermination of the deuteron radius using the new estimates for the polarizability contribution. We show that there is a very precise proportionality relation between the polarizability and the mass radius squared, and that this allows one to reduce the contribution of this term to the error in the extracted matter radius. We then reexamine the compatibility between the predictions of realistic NN interactions, the experimental value of the scattering length and the matter radius thus determined. We finally present in Sec. IV relations between the radius and two other outer quantities, the total asymptotic normalization and deuteron polarizability, and determine corresponding values for them using the value for the mass radius.

II. POLARIZABILITY ENERGY SHIFTS

The most naive description of the deuterium atom considers its nucleus to be merely a static source of the Coulomb field. This picture neglects the changes in the structure of the nucleus induced by the interaction with the bound electron, and the corresponding changes in the total energy of the atom, usually called nuclear polarizability energy shifts. They can be conveniently com-

puted using perturbation theory. The explicit expressions for these shifts have been given independently in Refs. [5,6,8]. For convenience we shall follow the derivations of Refs. [5,6] and use their notation and conventions. In the Appendix we analyze the expressions used in Ref. [8] and show under which approximations they become equivalent to the previous ones.

A. The longitudinal energy shift

The dominant contribution to the shift is that due to the exchange of a Coulomb photon. As shown in Ref. [5], this longitudinal contribution can be written as

$$\Delta E^{(l)} = \langle 0|P^{(l)}|0\rangle\,, \tag{1}$$

where $|0\rangle$ is the unperturbed deuteron ground state and $P^{(l)}$ is an electron matrix element obtained from second-order perturbation theory with the electron treated relativistically and neglecting the Coulomb interaction in (virtual) intermediate states. The explicit form of $P^{(l)}$ is given in Eq. (18) of Ref. [5]:

$$P^{(l)} = \alpha^2\phi(0)^2 \int \frac{d^3q}{(2\pi)^3}\left(\frac{4\pi}{q^2}\right)^2 \left(1-e^{\frac{i}{2}\mathbf{q}\cdot\mathbf{R}}\right)\left(\frac{E_q+m}{2E_q}\frac{1}{m-\Lambda-E_q}+\frac{E_q-m}{2E_q}\frac{1}{m+\Lambda+E_q}\right)\left(1-e^{\frac{-i}{2}\mathbf{q}\cdot\mathbf{R}'}\right), \tag{2}$$

where $\mathbf{q}$, m, and E_q are, respectively, the electron momentum, mass, and energy. They are related by $E_q = \sqrt{q^2+m^2}$. The electron wave function at the origin is written as $\phi(0)$, where for simplicity, the quantum numbers specifying the electron eigenstate are not explicitly written. The operators Λ, $\mathbf{R}$, and $\mathbf{R}'$ act on the deuteron wave function: $\mathbf{R}$ ($\mathbf{R}'$), is the relative coordinate between the neutron and the proton in the wave function on the left (right) of the expectation value, whereas $\Lambda \equiv H - E_0$, with H, E_0 the unperturbed deuteron Hamiltonian and ground-state energy, respectively. Starting from the above expression, Pachucki *et al.* [5] proceed to introduce several approximations leading to their much simpler final result, Eq. (21) of Ref. [5] and Eq. (6) below. Since this expression has been used in their numerical estimates, we shall now reexamine these approximations to better ascertain their accuracy. The first step in simplifying Eq. (2) is an expansion in $R = |\mathbf{R}|$ which is found in [5] to be equivalent to the dipole approximation. We reformulate this simplification as follows: first we note that, since the integration over angles is trivial, the factors

$$\left(1-e^{\frac{i}{2}\mathbf{q}\cdot\mathbf{R}}\right)\left(1-e^{-\frac{i}{2}\mathbf{q}\cdot\mathbf{R}'}\right) \tag{3}$$

in the integrand of Eq. (2) can be exactly replaced by

$$1-\frac{\sin(qR/2)}{(qR/2)}-\frac{\sin(qR'/2)}{(qR'/2)}+\frac{\sin(q|\mathbf{R}-\mathbf{R}'|/2)}{(q|\mathbf{R}-\mathbf{R}'|/2)}\,, \tag{4}$$

and next, by truncating the power series expansion of the $\sin(x)$ at order x^3, we recover the dipole approximation. However, it is easy to see that when one adds more terms to the expansion, the integral in Eq. (2) diverges. Therefore we have studied the accuracy of that approximation numerically, choosing representative values of Λ, $\mathbf{R}$, and $\mathbf{R}'$ and comparing the results of the integration over q with and without the dipole approximation. From such calculations we estimate that this approximation is accurate to a few parts per thousand. This rules out sizable contributions to $\Delta E^{(l)}$ from virtual excitations to deuteron states of multipolarities other than the dipole.

The simplified form found for $P^{(l)}$ is then Eq. (19) of Ref. [5], which we rewrite as

$$P^{(l)} = -2m\alpha^2\phi(0)^2\,\frac{\mathbf{R}\cdot\mathbf{R}'}{3}\int \frac{dq}{E_q}\frac{\Lambda+2E_q}{(\Lambda+E_q)^2-m^2}\,. \tag{5}$$

It is shown in the Appendix that, under similar approximations, the expression for the longitudinal energy shift used in Ref. [8] agrees with this result.

In a second step, a further expansion is performed in powers of m/Λ and truncated to lowest order. This leads to the simpler expression for $P^{(l)}$, Eq. (20) of [5]:

$$P^{(l)} = -2m\alpha^2\phi(0)^2\,\frac{1}{\Lambda}\left(1+\ln\frac{2\Lambda}{m}\right)\frac{\mathbf{R}\cdot\mathbf{R}'}{3}\,, \tag{6}$$

which is used in their numerical estimates. However, there is no discussion in that paper of the accuracy of this additional approximation, and in the review [6], half of the error in the estimate of $\Delta E_{\rm pol}$, given in their Eq. (49), is assigned to the neglected contribution of the terms of higher order in m/Λ. To avoid this source of error we shall use Eq. (5), without further approximation, for the evaluation of the longitudinal energy shift. In addition, we will compare these results with those given by Eq. (6) to estimate its accuracy.

The integration over q in Eq. (5) can be performed analytically using standard decomposition and change of variable techniques, with the result:

$$P^{(l)} = \alpha^2\phi(0)^2\left(\frac{\sqrt{\Lambda+2m}}{\sqrt{\Lambda}}\ln\frac{\sqrt{2m+\Lambda}-\sqrt{\Lambda}}{\sqrt{2m+\Lambda}+\sqrt{\Lambda}} - \frac{\sqrt{\Lambda-2m}}{\sqrt{\Lambda}}\ln\frac{\sqrt{\Lambda}-\sqrt{\Lambda-2m}}{\sqrt{\Lambda}+\sqrt{\Lambda-2m}}\right)\frac{\mathbf{R}\cdot\mathbf{R}'}{3}\,. \tag{7}$$

For completeness we note that by expanding in powers of $z \equiv m/\Lambda$ one finds that

$$P^{(l)} \simeq -\alpha^2 \phi(0)^2 \frac{\mathbf{R}\cdot\mathbf{R}'}{3}\left\{2z\left[1+\left(1+\frac{z^2}{2}\right)\ln\frac{2}{z}\right] +O(z^3)\right\}, \tag{8}$$

from which Pachucki *et al.*'s final expression, Eq. (6), is recovered by performing an additional truncation.

B. The transverse energy shift

Repeating the derivation in [6], but avoiding the expansion in powers of z, we find

$$P^{(t)} = m\alpha^2\phi(0)^2\Big[-\lambda\sqrt{\lambda^2-2\lambda}\ln|\sqrt{\lambda^2-2\lambda}-\lambda+1| +\lambda\sqrt{\lambda^2+2\lambda}\ln|\sqrt{\lambda^2+2\lambda}-\lambda-1| +2\lambda+2\lambda\ln(2\lambda)\Big]\frac{\mathbf{R}\cdot\mathbf{R}'}{6}, \tag{9}$$

where $\lambda \equiv \Lambda/m$. For large λ this reduces to the simpler form written explicitly in Eq. (47) of [6]:

$$P^{(t)} \simeq -\alpha^2\phi(0)^2\frac{m}{\Lambda}\left(-\frac{5}{6}+\ln\frac{2\Lambda}{m}\right)\frac{\mathbf{R}\cdot\mathbf{R}'}{6}. \tag{10}$$

Since the transverse contribution is one order of magnitude smaller than the longitudinal one, we have found Eq. (10) sufficiently accurate for our purposes.

C. Zero-range estimates

It was shown long ago [10,11] that the zero-range approximation is a very useful, and sometimes remarkably accurate, model to describe deuteron properties dominated by the long-range part of the wave function. Explicit calculations, to be presented later, show that this is the case for the polarizability shifts. We therefore present estimates obtained with this simple model, to gain a qualitative understanding of which quantities influence the value of the $\Delta E_{\rm pol}$. We were guided in this study by the discussion and findings of Friar and Fallieros [10] on the dipole electric polarizability,

$$\alpha_E \equiv \frac{\alpha}{6}\left\langle 0\left|\frac{\mathbf{R}\cdot\mathbf{R}'}{H-E_0}\right|0\right\rangle = \frac{\alpha}{6}\sum_N \frac{|\langle 0|\mathbf{R}|N\rangle|^2}{E_N+E_B}, \tag{11}$$

where E_B=$-E_0$ is the deuteron ground-state binding energy, and $|N\rangle$ are intermediate states with spin S=1 and isospin T=1. It is understood that both the ground-state and intermediate-state wave functions in the above equation contain only the spatial component.

When $P^{(l)}$ is approximated by Eq. (6), the energy shift $\Delta E^{(l)}$ can be written in the form used in Ref. [5]:

$$\Delta E^{(l)} = -4m\alpha\phi(0)^2\,\alpha_E\left(1+\ln\frac{2\bar{E}}{m}\right), \tag{12}$$

where $\bar{E}$ represents the average excitation energy for the intermediate states $|N\rangle$. Similarly, for the transverse shift, with Eq. (10) for $P^{(t)}$, we can write

$$\Delta E^{(t)} = -m\alpha\phi(0)^2\,\alpha_E\left(-\frac{5}{6}+\ln\frac{2\bar{E}}{m}\right). \tag{13}$$

Working in the zero-range approximation for the ground state

$$|0\rangle_{0,\rm radial} = \frac{u(r)}{r} = \frac{A_S e^{-\kappa r}}{r} \tag{14}$$

and neglecting the forces in the odd waves

$$|N\rangle_{0,\rm radial} = 2kj_1(kr), \tag{15}$$

Friar and Fallieros found that the dipole electric polarizability has an analytic expression:

$$\alpha_E^0 = \frac{\alpha\mu A_S^2}{32\kappa^5}, \tag{16}$$

where μ is the reduced mass in fermis, and $\kappa = \sqrt{2\mu E_B}/\hbar c$.

With the zero-range approximation and Eq. (6) for $P^{(l)}$, we have found an analytic expression for $\Delta E^{(l)}$:

$$\Delta E_0^{(l)} = -4m\alpha\phi(0)^2\,\alpha_E^0\left(1+\ln\frac{2E_B}{m}+2\ln 2-\frac{7}{12}\right), = -4m\alpha\phi(0)^2\,\alpha_E^0\left(1+\ln\frac{2\bar{E}_0}{m}\right), \tag{17}$$

where in the second line we have written the result in the form of Eq. (12) by defining $\bar{E}_0$ as

$$\bar{E}_0 = 4e^{-7/12}\,E_B = 4.966\ \text{MeV}. \tag{18}$$

In Ref. [5] a square well potential was used to estimate $\bar{E}$ = 4.915 MeV, which is close to the analytic prediction ($\bar{E}_0$) of the zero-range approximation. The values predicted for the electric dipole polarizability are also similar. The zero-range results for α_E^0 are given in the first row of Table I and will be discussed later.

We note finally that in the zero-range approximation, the expression for the ground-state mass radius is also very simple:

$$r_{m,0}^2 = \frac{A_S^2}{16\kappa^3}. \tag{19}$$

Combining Eqs. (16)–(19) one sees that the polarizability shift $\Delta E_0^{(l)}$ and the squared mass radius $r_{m,0}^2$ are proportional in the zero-range approximation. We shall show below that this is still true to a great extent in the more accurate estimates of the next subsection.

D. Results for realistic *NN* interactions

From the zero-range estimates we expect that the main source of model dependence in the predicted energy shifts will be linked to the variation of the mass radius. Hence

TABLE I. Calculated results for the deuteron electric dipole polarizability: α_E^0 for the zero-range approximation, α_E^S for the deuteron S-wave-only approximation, $\Delta\alpha_E^D$ for the D-wave correction, $\Delta\alpha_E^V$ for the correction due to the odd-parity forces, and α_E for the full result. The deuteron asymptotic normalization A_S, the ratio $\eta=A_D/A_S$, the mass radius r_m, and $\ln(2\bar{E}/m)$ are also listed.

Potential	RSC	RHC	HW	TRS	NjmNR	NjmR	Reid93
$\alpha_E^0(\mathrm{fm}^3)$	0.6269	0.6308	0.6359	0.6426	0.6372	0.6373	0.6385
$\alpha_E^S(\mathrm{fm}^3)$	0.6205	0.6237	0.6282	0.6348	0.6299	0.6308	0.6318
$\Delta\alpha_E^D(\mathrm{fm}^3)$	0.0050	0.0049	0.0052	0.0051	0.0047	0.0047	0.0046
$\Delta\alpha_E^V(\mathrm{fm}^3)$	-0.0019	-0.0021	-0.0025	-0.0022	-0.0018	-0.0019	-0.0017
$\alpha_E(\mathrm{fm}^3)$	0.6236	0.6265	0.6309	0.6377	0.6328	0.6336	0.6347
$A_S(\mathrm{fm}^{-1/2})$	0.8776	0.8803	0.8852	0.8883	0.8847	0.8845	0.8853
η	0.0262	0.0259	0.0264	0.0262	0.0252	0.0252	0.0251
$r_m(\mathrm{fm})$	1.9569	1.9600	1.9672	1.9751	1.9671	1.9675	1.9686
$\ln(2\bar{E}/m)$	2.9638	2.9631	2.9637	2.9623	2.9624	2.9621	2.9618

we have determined the values of $\Delta E^{(l)}$ and $\Delta E^{(t)}$ for a selection of realistic potentials that cover a range of values of r_m that enclose the expected experimental value. This selection contains conventional old potentials and also NN interactions newly fitted to experimental data and including a much more solid theoretical input. The most representative in the first class are Reid's potentials (RSC and RHC) [12], the improved Hamada-Johnston potential of Humberston and Wallace (HW) [13], and the supersoft core potential of de Tourreil, Rouben, and Sprung (TRS) [14]. The more recent potentials include the relativistic and nonrelativistic potentials NijmII (denoted by NjmNR and NjmR) given by the Nijmegen group [15], and a new Reid-like potential (Reid93) from the same group, also with relativistic kinematics. The results for this selection will be discussed in the following. We have also used other potentials that do not modify the conclusions drawn below.

For a given potential we have computed the wave functions for the deuteron bound state and the $^3P_0$, $^3P_1$, and $^3P_2$-$^3F_2$ partial waves. In this calculation a step size of 0.003 fm was found to give the required accuracy. The partial waves were determined up to the c.m. energy 1000 MeV with a step size of 2.5 MeV. We checked our results against published values of the ground-state properties, and with the computed values of the deuteron dipole polarizability given in Refs. [10,16]. In Table I we give, in a format similar to Table I of Ref. [10], our calculated results for the electric dipole polarizability α_E, its zero-range approximation α_E^0, the deuteron S-wave-only approximation α_E^S, the D-wave correction $\Delta\alpha_E^D$, and the correction due to the odd-parity forces $\Delta\alpha_E^V$. We also list the asymptotic normalization A_S, the ratio $\eta=A_D/A_S$, and the deuteron mass radius r_m. The results for $\ln(2\bar{E}/m)$, as defined by Eq. (12) or (13), are also shown for the sake of comparison. The energy shifts $\Delta E^{(l)}$ and $\Delta E^{(t)}$ corresponding to the use of Eq. (6) for $P^{(l)}$ and Eq. (10) for $P^{(t)}$ (as in Refs. [5,6]) can be easily evaluated using the results for α_E and $\ln(2\bar{E}/m)$. We prefer, however, to use Eq. (7) for $P^{(l)}$ to calculate $\Delta E^{(l)}$.

The total energy shift ΔE_{pol} is obtained by adding the longitudinal shift $\Delta E^{(l)}$ to the transverse shift $\Delta E^{(t)}$. Since the quantities of interest are the contributions to the $1S$-$2S$ transition, one has to take the difference of the expectation values for these two states. Since

$$\phi_{nS}(0)^2 = \frac{1}{n^3}\frac{1}{\pi a_B^3}, \tag{20}$$

this amounts to using a factor

$$-\frac{7}{8}\frac{1}{\pi a_B^3} \tag{21}$$

for the contribution of the electron wave functions to the difference. The computed results are shown in Table II. The trends are similar for all potentials considered: the zero-range approximation already gives a very accurate estimate, $\simeq 100.8\%$ of the full result, whereas the use of the true S wave function reduces that to about 99.4%. Adding the D wave causes an increase to $\simeq 100.4\%$ and finally switching on the interaction in the odd waves brings in again a reduction, of $\simeq 0.4\%$. Thus there is a sizable cancellation between these contributions that makes the zero-range prediction surprisingly reliable. The reduction produced by the inclusion of the full S wave can be easily predicted using a model wave function of the Hulthén type. We have not found any simple explanation for the smallness of the changes induced by the inclusion of the D component or the interaction in the odd waves. Still, the smallness of all these effects makes the prediction of the polarizability shift more reliable since it is mainly determined by the very basic quantities that appear in the zero-range analytic expression and very little by the specific details of each NN potential. This is confirmed

TABLE II. Total polarizability shift for the $1S$-$2S$ transition (in kHz) calculated for seven representative NN potentials.

Potential	RSC	RHC	HW	TRS	NjmNR	NjmR	Reid93
Zero range	19.17	19.29	19.45	19.65	19.49	19.49	19.53
S wave	18.93	19.02	19.16	19.36	19.21	19.23	19.26
S+D waves	19.11	19.20	19.35	19.54	19.38	19.41	19.44
Full	19.05	19.14	19.28	19.48	19.33	19.35	19.38
f_p	4.98	4.98	4.98	4.99	4.99	5.00	5.00

in the last line of Table II, where the ratio to the corresponding deuteron mass radius squared, $f_p = \Delta E_{\rm pol}/r_m^2$, is shown. It is seen to be constant to a very good approximation. Since our selection of potentials covers a range of mass radii that (we can be confident) encloses the experimental value, we take the two extreme values in the table to have a conservative prediction of the value and uncertainty of the total polarizability shift:

$$\Delta E_{\rm pol}^{(1S\text{-}2S)} = 19.27(22)\ \text{kHz}\,. \tag{22}$$

This value is compatible with the estimate in the review [6], which uses a simple square well potential for the neutron-proton interaction: 19(2) kHz, but the error there has been overestimated by an order of magnitude and our result is based on a more realistic description of the deuteron structure. The other estimate presently available is that of Ref. [8] for the polarizability shift in the $1S$ state, which when converted into a $\Delta E_{\rm pol}^{(1S\text{-}2S)}$ leads to values $\simeq$ 17.9 kHz somewhat lower than those above. However, these results have been obtained for very simple separable models of the NN interaction, and thus should not be directly compared to our calculations. We believe that the analysis presented in the Appendix is more significant: it shows that by introducing suitable approximations the analytical expressions used here and in [8] become equivalent.

Note that to determine the values in Table II and Eq. (22) we have used Eq. (7) for $P^{(l)}$ and Eq. (10) for $P^{(t)}$. The use of the simplified $P^{(l)}$ in Eq. (6) produces a small systematic shift that begins to be relevant at the present level of accuracy: for the Reid93 potential, the results in the last column of Table II would be decreased by 0.36% to 19.46, 19.20, 19.37, and 19.32 kHz for the four cases considered, so that there is a difference of $\simeq$ 0.07 kHz with the more accurate values. The situation is similar for the other potentials. The difference is reduced to only $\simeq -0.02$ kHz when the more accurate expression of Eq. (8) is used for $P^{(l)}$.

III. DETERMINATION OF THE MATTER RADIUS

We determine the sum of the finite-size (FS) and polarizability shifts by subtracting from the experimental value of the $1S$-$2S$ transition given in [2], the remaining theoretical contributions to the isotope shift listed and evaluated in Sec. II of [6]. Since the expression for the $\alpha^6 m^2/M$ recoil correction given in that reference has been recently reexamined [20] and modified, we correct the contribution of this term accordingly, and have

$$\Delta E_{\rm FS}^{(1S\text{-}2S)} + \Delta E_{\rm pol}^{(1S\text{-}2S)} = -5182\ \text{kHz} \tag{23}$$

with an error of $\sqrt{22^2+14^2} \simeq 26$ kHz, obtained by adding quadratically the experimental error and that due to the reduced mass correction. At this stage one can subtract the predicted $\Delta E_{\rm pol}^{(1S\text{-}2S)}$ given in Eq. (22), to determine a value for $\Delta E_{\rm FS}^{(1S\text{-}2S)}$ and obtain from it the deuteron charge radius, as done in Ref. [6]. Given that at present the error $\delta(\Delta E) = 26$ kHz is much larger than the estimated 0.22 kHz in Eq. (22) this is sufficiently accurate. However, since it is anticipated that in the near future improvements in the analysis of the Lamb shift experiments will allow one to reduce $\delta(\Delta E)$ to $\simeq 1$ kHz, we present in the following a more careful discussion of the determination of the deuteron radius with a view to minimizing further the contribution to the error due to the uncertainty in the polarizability shift.

The finite-size effect is written [6]

$$\Delta E_{\rm FS}^{(1S\text{-}2S)} = -\frac{7\pi}{12} e^2 \phi_{1S}(0)^2 \langle r^2 \rangle\,, \tag{24}$$

with

$$\langle r^2 \rangle = r_{\rm ED}^2 + r_n^2 + \frac{3\hbar^2}{4m_p^2 c^2}\,, \tag{25}$$

where the neutron charge radius is known experimentally: $r_n^2 = -0.1192(18)$ fm^2 and the *zitterbewegung* or Darwin-Foldy (DF) contribution is $3\hbar^2/(4m_p^2c^2) = 0.03317$ fm^2 with negligible error. The deuteron electric radius $r_{\rm ED}$ still contains relativistic and meson exchange current (MEC) effects, and possibly a contribution from the quark degrees of freedom, and these must be removed before we can obtain a value for the matter radius r_m, which is the quantity that can be compared to potential models. Kohno [17] calculated that the MEC and relativistic effects give $\delta r_m^2 \equiv r_{\rm ED}^2 - r_m^2 = 0.0143$ fm^2. From Fig. 8 of Ref. [18] one can read off values of 0.014, 0.004, and -0.005 fm^2 for three representative covariant calculations (after subtracting out the DF contribution). Thus the indications are that δr_m^2 is small. Wong [19] estimates a small negative contribution of -0.005 fm^2 from quark exchange effects, to be included in it; his review should be consulted for a more thorough discussion. Putting these numbers together, we will estimate that

$$\delta r_m^2 \approx 0.0 \pm 0.010\ \text{fm}^2\,. \tag{26}$$

We emphasize that the Darwin-Foldy correction is included separately in Eq. (25), both because it *is* known, and because it is a part of the physical proton radius rather than a deuteron structure effect.

The finite-size effect has to be complemented by the much smaller polarizability shift computed in the previous section, where we have shown that the model dependent relation

$$\Delta E_{\rm pol}^{(1S\text{-}2S)} = f_p r_m^2\,, \tag{27}$$

with $f_p = 4.99(2)$ kHz fm^{-2} is satisfied by all potentials considered. In order to combine these equations, we replace r_m^2 by $r_{\rm ED}^2$ in Eq. (27) to get

$$\Delta E_{\rm FS}^{(1S\text{-}2S)} + \Delta E_{\rm pol}^{(1S\text{-}2S)} = -\xi\left(r_{\rm ED}^2 + r_n^2 + \frac{3\hbar^2}{4m_p^2c^2}\right) + f_p(r_{\rm ED}^2 - \delta r_m^2)\,, \quad (28)$$

where we have defined

$$\xi = \frac{7\pi}{12}\,\alpha\,\hbar c\,\phi_{1S}(0)^2\,. \quad (29)$$

Explicit calculation then gives $\xi = 1369.50\,\text{kHz fm}^{-2}$, to be compared to the much smaller f_p indicated above. The electric radius is therefore

$$r_{\rm ED}^2 = \frac{\Delta E_{\rm FS}^{(1S\text{-}2S)} + \Delta E_{\rm pol}^{(1S\text{-}2S)} + \xi\left(r_n^2 + \frac{3\hbar^2}{4m_p^2c^2}\right) + f_p\delta r_m^2}{f_p - \xi}\,, \quad (30)$$

and substituting numerical values, we get $r_{\rm ED}^2 = 3.884$ fm^2, and then from Eq. (25) $\langle r^2\rangle = 3.798$ fm^2, in agreement with the value 3.796(19) determined in [20].

A. Error analysis

The first two terms in the numerator of Eq. (30) have absolute errors $\delta(\Delta E) = 26$ kHz and $\xi\cdot\delta(r_n^2)_{\rm exp} = 2.47$ kHz, so that the latter is as large as the transverse polarizability shift. When the improvements in the Lamb shift experiments allow a reduction of $\delta(\Delta E)$ to $\simeq 1$ kHz, the main source of error in extracting the deuteron electric radius will be that associated with the neutron charge radius. In contrast, the relative smallness of f_p ensures that the lack of knowledge of MEC and other such effects does not at present impinge on the determination of the electric radius. The relative error contributed by the numerator is found to be 0.54×10^{-2}.

The contribution of the denominator to the error is due to the estimated $\delta(f_p) = 0.02$ kHz fm^{-2}, giving a contribution to the relative error of 1.46×10^{-5} which is negligible compared to that from the numerator at present and will remain so when the error in the latter is reduced by an order of magnitude. Therefore we find that $\delta(r_{\rm ED}^2)/r_{\rm ED}^2 = 0.54\times10^{-2}$ which gives

$$r_{\rm ED}^2 = 3.884(21)\ \text{fm}^2\,,$$
$$r_{\rm ED} = 1.971(5)\ \text{fm}\,. \quad (31)$$

This is the value for the electric radius extracted from the experimental values of the Lamb shift and *the neutron radius.* Note that with this method the uncertainty of the polarizability shift plays a negligible role.

Using the estimate of Eq. (26), the matter radius r_m will have the same numerical value as $r_{\rm ED}$ but the compounded errors in Eq. (31) will be increased to (23) and (6), respectively. In our earlier work we used Kohno's value which made the matter radius 0.003 fm smaller than the electric radius. It should be clear that knowledge of δr_m^2 has only a small influence on the extraction of $r_{\rm ED}$, but it is absolutely vital in taking the final step to get r_m.

B. Comparison with other determinations

The value of r_m obtained above confirms the findings of Refs. [6,8,9], and is in contradiction with the previously accepted value based on low momentum transfer electron scattering data: in Ref. [7] we found $r_m = 1.950(3)$ fm. In a recent review Wong [19] has reanalyzed all existing electron scattering data and concludes that $r_m = 1.9505(20)$ fm, in good agreement with our analysis. The most recent set of electron scattering data, taken by the Saclay group at higher momentum transfers [18], seems to favor, however, a higher value for the mass radius: $r_m = 1.961(7)$ fm according to [21] or $r_m = 1.966(8)$ fm according to the more exhaustive analysis of [19]. Both of these estimates are in better agreement with the result in Eq. (31). It is thus an open question whether some unaccounted systematic error in the low momentum data can account for this discrepancy. Wong discounts this possibility. Alternatively, it has been suggested that the effect of dispersion corrections, not included in the analysis of the electron scattering data, could increase the extracted radius. The calculation of these corrections turns out to be nontrivial [16], and it is not possible at present to reach any firm conclusion on their relevance. The analysis of the Lamb shift experiments is free from these uncertainties, and therefore is at present a more reliable method to determine the mass radius of the deuteron. In addition, a look at Fig. 1 of Ref. [11] shows that this value solves the previous incompatibility between theoretical models of the NN interaction and the experimental values of scattering length and mass radius. Using the experimental value for a_t, one can read off from that figure a value $r_m \simeq 1.969(4)$ fm. A value $r_m = 1.967$ fm is derived in a similar but more careful analysis by Wong in his recent review [19]. These values overlap very well with that of Eq. (31), so that one must conclude that there is no contradiction between estimates based on the low energy properties of realistic NN interactions and the mass radius extracted from the Lamb shift experiments.

IV. DETERMINATION OF THE ELECTRIC POLARIZABILITY AND THE ASYMPTOTIC NORMALIZATION

The zero-range expressions suggest also that the ratio α_E/r_m^2 is approximately constant. This is indeed

true for the values in Table I. It is found that this ratio ranges from the lowest value, for RSC, of 0.1629 fm to the largest, for Reid93, of 0.1638 fm, so that a conservative estimate applying to potentials in that range of values of r_m is

$$\frac{\alpha_E}{r_m^2} = 0.1634(5)\,\text{fm}. \tag{32}$$

Combining this result with the previously extracted value for r_m^2, Eq. (31) and compounding the corresponding errors we find

$$\alpha_E = 0.635(6)\ \text{fm}^3\,. \tag{33}$$

This should be compared to the experimentally determined value [24]: $\alpha_E = 0.70(5)$ fm^3, and to that extracted in Ref. [26] from deuteron photoabsorption data: 0.61(4). An estimate similar to ours, but based on the proportionality between α_E and A_S, has also been given in [10]: $\alpha_E = 0.632(3)$ fm^3. There is thus good compatibility among all these estimates.

It was noted by Friar and Fallieros [10] that the zero-range value for α_E is already a very good estimate. This can also be seen by comparing the first and fifth rows of Table I. It is worth remarking that the additional changes due to the inclusion of the full S wave function, the D wave function, and the odd-parity forces are strongly correlated to those shown in Table II: the ratios of the corresponding changes are practically independent of the potential and quite similar: $\simeq 38$, $\simeq 37$, and $\simeq 31$ kHz fm^{-3}, respectively. This indicates that there should be a zero-range relation similar to Eq. (16) describing the bulk of these small changes.

The zero-range expressions and the more detailed analysis in Ref. [11] suggest that there may be a linear relation relating the asymptotic normalizations to the mass radius. We have therefore tried a fit of the following form:

$$A_S^2(1+\eta^2) = a r_m^2 + b\,, \tag{34}$$

where a and b are parameters, to the pairs of points $[r_m^2, A_S^2(1+\eta^2)\cdot(\kappa_{\text{exp}}/\kappa)^3]$, generated from several NN potentials. (The correction factor $(\kappa_{\text{exp}}/\kappa)^3$ is relevant only for potentials fitted to old values of κ.) One such fit is shown in Fig. 1. When only the data points corresponding to the seven potentials in Tables I and II are included in the fit, one finds $a = 0.2440$ and $b = -0.1617$. For a larger selection of potentials (those in Table I of Ref. [11]), the fit is very similar.

Combining Eq. (34) with the value of r_m from Eq. (31) one finds

$$A_S^2(1+\eta^2) = 0.7860(47)\ \text{fm}^{-1}\,, \tag{35}$$

and, since the contribution from the D state is almost negligible, we use this to extract a value for A_S. Choosing $\eta = 0.0252(1)$ from Ref. [15] we get

$$A_S = 0.8863(26)\ \text{fm}^{-1/2}\,, \tag{36}$$

which is compatible with the more precise value $A_S = 0.8838(4)$ obtained by Stoks *et al.* [24] from a direct analysis of low energy NN scattering data. (A similar estimate $A_S = 0.8843(10)$ fm$^{-1/2}$ has been given by the same group in [15].) Note that this solves the discrepancy pointed out in Ref. [24] between their value and that determined in [7] using the deuteron radius extracted from the electron scattering data, and thus confirms again the compatibility of low energy NN data with the Lamb shift experiment.

V. SUMMARY AND CONCLUSIONS

We have obtained accurate estimates for the polarizability shifts using realistic NN interactions. Our results are in agreement with earlier calculations using less sophisticated models, but the estimated accuracy is now

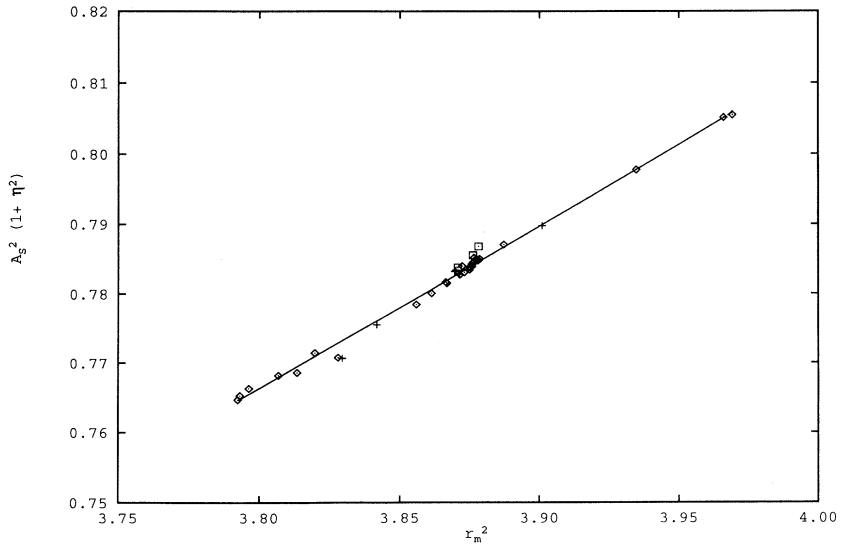

FIG. 1. The empirical linear relation between the squares of the total asymptotic normalization and mass radius. The crosses correspond to the seven potentials described in the text, and the diamonds to additional potentials indicated in Table I and Fig. 1 of Ref. [12]. The three square boxes are the Bonn potentials from Table A.1 of Machleidt's review [26].

considerably improved. Furthermore, for all the NN potentials studied, we have found a very precise proportionality relation between polarizability shifts and the mass radius squared. This has allowed us to present a method to determine r_m where the uncertainty in the polarizability has practically no effect on the estimated error.

The mass radius that we have determined confirms the discrepancy with determinations based on low momentum transfer electron scattering data, but is in good agreement with the systematics of low energy NN data and the linear relations predicted by potential models. With the same methods we have also extracted a value for the asymptotic normalization that is in agreement with the determination of the Nijmegen group. The compatibility of all these results makes it very tempting to blame the discrepancy, between the mass radius derived from low momentum electron scattering experiments and that determined here, on unaccounted systematic errors, or neglected dispersion corrections. A new effort to repeat those measurements, and to compute the missing theoretical corrections, would be very welcome to settle this matter with greater confidence.

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Note: After completion of the manuscript, Professor Friar kindly pointed out to us that J. Bernabeu and T.E.O. Ericsson, in a very pedagogical article, Z. Phys. A **309** 213 (1983) had already derived expressions analogous to those of Ref. [6] for the polarizability shifts, although they did not give any estimates for deuterium. In addition, we received a preprint from W. Leidemann and R. Rosenfelder, with calculations of the polarizability shifts based on a different set of realistic potentials. Their results appear to be compatible with ours within the quoted error bars, but they are somewhat lower. According to further private correspondence these authors attribute the main part of the difference to not having used the dipole approximation in their calculations. As indicated in the text our own estimates for the error in the dipole approximation are smaller than theirs.

APPENDIX A

In a series of papers Rosenfelder and his collaborators have developed tools for the determination of nuclear polarization shifts in different nuclear systems. Of particular interest to our case is Ref. [25], where the expressions later used in [8] are derived. Although the methods are similar to those of Pachucki *et al.* in [5] and [6], the two approaches use different notations and decompositions, so that it is useful to check that they lead to compatible results. We therefore begin by summarizing the main steps needed to recover Eq. (5) for the longitudinal energy shift starting from the expressions of Ref. [25]. From Eq. (14) of that reference:

$$\Delta E^{(l)} = -8\alpha^2\phi(0)^2 \int_0^\infty dq\; R_L\; f^2(q^2)\;, \tag{A1}$$

where $f^2(q^2)$ is the spatial Fourier transform of the nucleon charge density, given explicitly in [25]. Numerical calculation of Eq. (5) for several characteristic values of Λ shows that approximating $f^2(q^2) \simeq 1$ is accurate to about a part per thousand. For full relativistic kinematics for the electron, the expression for R_L is given in Eqs. (A1) and (A4) of [25]:

$$R_L = \int_0^\infty d\omega S_L(\omega, \mathbf{q}) g(\omega, q)\,, \tag{A2}$$

$$g(\omega, q) = \frac{1}{2E_q}\left[\frac{1}{(E_q - m)(\omega + E_q - m)} - \frac{1}{(E_q + m)(\omega + E_q + m)}\right] \tag{A3}$$

and

$$S_L(\omega, \mathbf{q}) = \sum_{n\neq 0} \delta[\omega - (E_n - E_0)]|\langle 0|e^{i\mathbf{q}\cdot\mathbf{r}}|n\rangle|^2\,, \tag{A4}$$

where $\mathbf{r}$ is the proton center-of-mass coordinate. Making now the dipole approximation: $\exp(i\mathbf{q}\cdot\mathbf{r}) \simeq 1 + i\mathbf{q}\cdot\mathbf{r}$ and performing the sum of the two terms in $g(\omega, q)$, one easily recovers Eq. (5).

The transverse contribution can be derived similarly, with some additional approximations. Starting again from Eq. (14) of [25], we write

$$\Delta E^{(t)} \simeq -8\alpha^2\phi(0)^2 \int_0^\infty dq\; (R_T + R_S)\,, \tag{A5}$$

where now, from Eqs. (A2) and (A3) of the same reference,

$$R_T + R_S = \int_0^\infty d\omega S_T(\omega, \mathbf{q})\left[-\frac{1}{4mq}\frac{\omega + 2q}{(\omega + q)^2} + \frac{1}{4m\omega}\left(\frac{1}{q} - \frac{1}{E_q}\right) + \frac{q^2}{4m^2} g(\omega, q)\right]. \tag{A6}$$

Under the assumption that $S_T(\omega, \mathbf{q})$ can be replaced by its low momentum behavior, as in Eq. (17) of [25],

$$S_T(\omega, \mathbf{q}) \simeq 2\frac{\omega^2}{q^2} S_L(\omega, \mathbf{q})\,, \tag{A7}$$

one gets an approximate expression for $R_T + R_S$ which, when substituted into Eq. (A5), leads to

$$P^{(t)} = -\alpha^2\phi(0)^2\left(\frac{\mathbf{R}\cdot\mathbf{R}'}{3}\right)\cdot\frac{\omega}{m} \times\int_0^\infty dq\,\frac{q(E_q-q)[(\omega+q)^2+2\omega E_q]}{E_q(\omega+q)^2[(E_q+\omega)^2-m^2]}. \quad \text{(A8)}$$

Neglecting m^2 compared to ω^2 in the denominator, and making the successive changes of variables, $q = m\sinh(x)$ and $y = \exp(-x)$, this can be rewritten as

$$P^{(t)} = -\alpha^2\phi(0)^2\left(\frac{\mathbf{R}\cdot\mathbf{R}'}{6}\right)\cdot\bar{\omega} \times\int_0^1 dy\,y\frac{(1-y^2)\left[\frac{1}{4}y^4+(\bar{\omega}^2-\frac{1}{2})y^2+2\bar{\omega}y+\frac{1}{4}\right]}{\left[(\bar{\omega}y+\frac{1}{2})^2-\frac{1}{4}y^4\right]^2}, \quad \text{(A9)}$$

where $\bar{\omega}\equiv\omega/m$. Finally, assuming again $\bar{\omega}\gg 1$,

$$P^{(t)} \simeq -\alpha^2\phi(0)^2\left(\frac{\mathbf{R}\cdot\mathbf{R}'}{6}\right)\cdot\bar{\omega} \times\int_0^1 dy\,y\frac{(1-y^2)(\bar{\omega}^2y^2+2\bar{\omega}y+\frac{1}{4})}{(\bar{\omega}y+\frac{1}{2})^4}. \quad \text{(A10)}$$

The latter integral can be easily performed and after dropping terms of $O(1/\bar{\omega})$ one finds

$$P^{(t)} = -\alpha^2\phi(0)^2\frac{\mathbf{R}\cdot\mathbf{R}'}{6}\frac{1}{\bar{\omega}}\left[-\frac{5}{6}+\ln(2\bar{\omega})\right], \quad \text{(A11)}$$

in agreement with the simplified result of Pachucki *et al.*, Eq. (10).

Finally, in [8] a contribution due to the momentum dependence of the potential is also estimated. It amounts to 0.31 kHz for the Yamaguchi interaction, whose momentum dependence is much larger than that of any realistic NN interaction. We have therefore chosen to omit that term in our calculations.

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