Final research project

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Iteration of holomorphic functions in the complex plane

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Abstract

When a holomorphic function is iterated, it generates a dynamic system on the complex plane. In this project, we describe both the local and global behavior of the different orbits of a rational map on the complex plane (or the Riemann sphere). We mainly concentrate in the study of the dynamical plane (where initial conditions and orbits live) although we briefly discuss one parameter families of polynomials and their bifurcation loci, like the well known Mandelbrot set. Towards the end, we experiment with a singular perturbation of a family of cubic polynomials and explore the drastic changes that occur in the topology of their Julia sets.

Preface

The main motivation for doing this project is my interest in dynamical systems and fractal-like structures, aside from the fact that the courses I enjoyed the most studying are "Complex analysis" and "Mathematical models and dynamic systems". It has evolved a lot from the initial idea to what it has been in the end, as new subjects of interest kept appearing while studying the originally planed topics.

Contents

1 INTRODUCTION

All throughout the history of mankind, both the admiration and interrogation about natural phenomena and the structure of natural objects have been a common trait. Most of the ancient civilizations were already fascinated by classical geometrical forms, such as circles and triangles, and tried to give a rational and reasoned description of natural objects taking as a starting point these classical forms. Two of the most renowned examples of their interest and knowledge in classical geometry are the *The Elements*, a geometry treaty written by Euclid in the IV century b.C, or the famous welcoming sentence written in Plato 's academy : "*Let no one ignorant of geometry enter*". However, nature often lacked the most estimated properties of classical geometry like symmetry, proportion and harmony, making it impossible to describe the world around us by the only use of classical figures, provided not extreme simplifications were made, and also leading to errors, such as the assumption that celestial orbits were perfectly circular. Although it it is true that in the following centuries some of these errors were corrected, only classical geometry forms were considered when addressing these problems. It was not until the XIX century, when mathematicians like Weierstrass[1], Koch[2] or Cantor[3] explored objects whose properties were not explained by traditional geometry; these objects are known nowadays as fractals. Their studies, due to the lack of technological means of visualization, and the lack of mathematical knowledge and background in this area, made their works very abstract.

In the XX century, Poincaré[4], Schröder[5], Fatou or Julia did the first incursions in the world of iteration in both the euclidean and the complex plane, discovering what we call dynamical systems, some of which behaved in a chaotic way, that is, similar initial conditions gave place to very different outcomes when iterated. These dynamical systems, sometimes generated geometrical objects with properties similar to the fractals studied by Cantor or Koch. However, the lack of technology, specially computers, proved a liability, as they were not able

[1]Karl Weierstrass (1815 - 1897) was a German mathematician, who made very important contributions to the field of analysis, like the definitions of continuity and limit of a function or the proof of the mean value theorem.

[2]Niels Helge von Koch (1870 - 1924) was a Swedish mathematician, and although he made contributions to number theory, his most known result is the description of the fractal known as Koch snowflake.

[3]Georg Cantor (1845 - 1918) was a German mathematician. He invented set theory, and among other results, he announced the theory of transfinite number, which was widely criticized at the time.

[4]Henri Poincaré (1854 - 1912) was a French mathematician and theoretical physicist. He is considered by many "The Last Universalist", since he made important contributions to all fields of mathematics. Among his many results are the Poincaré Conjecture, which remained unproven until 2002, and laying the foundations of chaos theory.

[5]Ernst Schröder (1841 - 1902) was a German mathematician, mostly known by his works in algebraic logic.

to see the objects they were facing. The appearance of computers, halfway into the XX century, and with them the capacity of doing millions of numerical operations by second and to visualize some of the obtained results, was the top expression of the technological revolution which had started a century earlier with the industrial revolution. These objects, far from being only an academical curiosity, kept appearing when studying natural phenomena and the universe that surrounds us, such as snow flakes, thunderbolts or drought-produced cracks, substituting the interest for classical geometrical objects, and opening a new world of possibilities. However, the balance was lost again, and many researchers classified as fractals a lot of objects which, although they presented one of the main properties of a fractal object, self-similarity, were not fractals.

When referring to iterative methods, Newton's method is probably the best known of them, and it is used to compute approximate solutions, either real or complex, of an equation

$$f(z) = 0$$

provided f is differentiable. It was described by Isaac Newton in his 1669 book " *De analysi per aequationes numero terminorum infinitas*". Although Newton's method and other iterative procedures (some of which were variations of Newton's method) had been already used earlier, the first mention of iteration of holomorphic functions is found in a study made by Schröder about Newton's method. The greatest difference between his work and the previous ones is the consideration of the method as the iteration of the map

$$N(z) = z - \frac{f(z)}{f'(z)}$$

in the complex plane. In doing so, he observed that the iterating holomorphic functions could be very useful towards the objective of finding better root-finding methods; in fact, Schröder proved the result known as Fixed Point Theorem, which explained why Newton's method worked, seeing that the roots of $f(z)$ are attracting fixed points of $N(z)$, when considered as a dynamical system.

Another problem which was studied by Schröder, and also by Cayley[6], was that of the iteration of Newton's Method far from the roots of the original function f. It could be seen as the first discernment between local and global theory in dynamical systems. They considered polynomials of degree 2, and tried to divide the complex plane, taking as differentiation criteria to which root of this polynomial did a point, under iteration, converge to. In general, if α is an attracting fixed point of the map $\phi(z)$, the basin of attraction of α is defined as the set

[6]Arthur Cayley (1821 - 1895) was a British mathematician. He postulated the Cayley - Hamilton theorem and gave the modern definition of a group.

of points of the plane which produce a convergent sequence to α; therefore, the problem they faced was to determine which points belong to each of the two basin of attractions. They achieved success for polynomials of degree 2, but were not able to do the same for polynomials of degree 3 or more. Looking back, it is easy to see why, as it was later seen that the boundaries between the basins of attraction for these polynomials of degree 3 or more are complicated fractal curves, nowadays known as Julia sets. Other mathematicians such as Koenigs, Böttcher and Siegel made further development and contribution to the local theory.

It was not until the works of Pierre Fatou and Gaston Julia in the decades of 1910-1920 when global theory was systematically studied. In particular, they studied the iteration of holomorphic functions in the Riemann sphere, that is, $\mathbb{C} \cup \{\infty\}$. Fatou proved that, under certain conditions, which are quite general, there is always an invariant set, $\mathbf{J}$, of initial conditions in the complex plane which is perfect. The most important innovation in the works of Fatou and Julia is the use of the theory of normal families to divide the Riemann Sphere into two completely invariant sets of different behavior: the Fatou set, $\mathbf{F}$, and the Julia Set, $\mathbf{J}$. Intuitively, points of the Fatou set are those whose dynamics are somewhat stable, that is, the ones which, when iterated, behave the same as their neighboring points. The complementary of the Fatou set is the Julia set, and it is composed by orbits with chaotic dynamics (in a certain sense); when referring to Newton's method, orbits in the Julia set do not converge to any of the roots of the function. Fatou and Julia were intrigued by the fact that the Julia set, in many cases, looked like a very peculiar and complicated set, even more if we compare it to the geometrical objects known and studied at the time. Actually, Fatou had already described some cases for which $\mathbf{J}$ was totally disconnected and perfect, but he also proved other properties which showed the geometrical complexity of this set. Although these two mathematicians described in a wide and detailed way both the Fatou set and the Julia set, they also left a lot of open problems, such as the classification of connected components of the Fatou set.

A more ambitious goal than studying the dynamic space of a particular holomorphic map, is to consider one (or more) parameter families of such functions. The best known example of this is the Mandelbrot Set, which is the parameter space of the family of polynomials $z^2 + c$, $c \in \mathbb{C}$, which has somewhat become an iconic symbol for fractal-like objects and dynamical systems, and is an active focus of investigation to this day.

Different families and classes of maps can be found in the literature which present particular interesting topological as invariant objects in their dynamical planes. A particularly rich source of examples is given by families of singularly perturbated simple maps; the first mathematician to consider such phenomena was Curtis T. McMullen in 1988, who introduced a perturbation to the map $z \mapsto z^2$, obtaining the family of functions

$$f_c(z) = z^2 + \frac{c}{z^2}$$

for c small enough. He proved that for such small values of c, the Julia set of f consisted of a Cantor set of quasicircles (non differentiable Jordan curves), many of which were not part of the boundary of any particular Fatou component. Later on, Devaney, Look and other authors studied similar singular perturbations and located topologically interesting Julia sets, homeomorphic to Sierpinski carpets, and other uncommon topological objects, some of which had never appeared outside of a purely theoretical environment.

In the very last section of this manuscript we explore a new singular perturbation obtained by adding a pole on the superattracting fixed point of certain cubic polynomials. We explore numerically..........

STRUCTURE

This project is divided in five sections.

In the first section, we list a number of important results of complex analysis and topology which will be used later on, so as not to list them every time we need them and interrupt the flow of the explanation.

The second section is a study of the local theory of holomorphic dynamic systems. We study the behavior of orbits in neighborhoods of periodic points, and also give some results about conjugations.

In the third section, we study the global behavior of these dynamical systems, focusing mainly in the study of critical points and Fatou components.

Both the second and third section are a study of the phase space of holomorphic maps. In the fourth section, however, we introduce the parameter space of a family of holomorphic maps, giving special emphasis to the quadratic family and the Mandelbrot Set.

While these four sections are focused on the selection, understanding and writing of existing literature, the last one is dedicated to a more personal approach of the subject. We introduce the concept of Milnor cubic polynomials, after which we consider the family which results from adding a pole singularity at $z = 0$ for these polynomials.

In this manuscript, not all the included results are proven, either because I thought that the proof did not give any extra insight on the concept, or because they were above the intended level of this project. However, should the reader be interested, all proofs can be found in [3], [5] or [13].

Throughout the work, there are some words accompanied by the symbol $(*)$. These words are included in a glossary at the end, listed in alphabetical order, intended to explain some concepts which may not be familiar to everyone.

There is also some brief biographic data of the authors that appear in the text, as it is both important and interesting to have a context for the results given, and their authors.

2 PRELIMINARIES AND TOOLS

In this section we include a selection of results that will be used as tools throughout the manuscript, but are not part of the goals of this project.

The first block of definitions and statements deal with normal families of holomorphic functions.

Definition 2.0.1 (Normal family). *A set, $\boldsymbol{F}$, of holomorphic functions defined in a complete metric space $\boldsymbol{X} \subset \mathbb{C}$ with values in another complete metric space $\boldsymbol{Y} \subset \mathbb{C}$, is said to be* normal *if every sequence of functions in $\boldsymbol{F}$ contains a subsequence which converges uniformly on compact subsets of $\boldsymbol{X}$ to an holomorphic function from $\boldsymbol{X}$ to $\boldsymbol{Y}$.*

The following theorem gives a convenient way to check whether a family of maps is normal or not.

Theorem 2.0.2 (Montel's[7] Theorem). *Let $\boldsymbol{F}$ be a family of holomorphic functions on a domain D. If there are two fixed values omitted by every $f \in \boldsymbol{F}$, then $\boldsymbol{F}$ is a normal family.*

Here, we will prove a weaker version of the theorem. In order to do so we will first present some useful results.

Theorem 2.0.3 (Arzela[8]-Ascoli[9]). *A set $\boldsymbol{F} \subset C(X, Y)$, where $\boldsymbol{X}$ is a compact topological space, $\boldsymbol{Y}$ is a complete metric space, and C(X, Y) is the space of continuous functions between $\boldsymbol{X}$ and $\boldsymbol{Y}$, is normal iff the following conditions are satisfied:*

1. *for each $z \in \boldsymbol{X}$, $\{f(z), f \in F\}$ has a compact closure in $\boldsymbol{Y}$.*
2. *$\boldsymbol{F}$ is equicontinuous$^{(*)}$ at each point of $\boldsymbol{X}$.*

Theorem 2.0.4 (Cauchy's[10] formula). *Let f be an analytic function over a closed path$^{(*)}$ γ, and consider $z_0 \notin \gamma$. Then*

[7] Paul Montel(1876 - 1975) was a French mathematician, whose main field of study was complex analysis and holomorphic functions. His most important contribution was the systematic development of the notion of normal family of functions.

[8] Cesare Arzela(1847 - 1912) was an Italian mathematician, whose main contributions to mathematics are in the field of the theory of functions. In particular, he is mostly known for his generalization of the characterization of continuous functions, given previously by Giulio Ascoli.

[9] Giulio Ascoli(1843 - 1896) was an Italian mathematician. He made contributions to the theory of functions of real variable and to Fourier Series. One of his most known results, is the concept of equicontinuity.

[10] Augustin-Louis Cauchy(1789 - 1857) was a French mathematician, considered by many a pioneer of analysis. He was one of the first mathematicians to prove calculus theorems rigorously. Although he found results in various areas, one of his most important achievements was the foundation of complex analysis.

$$f(z_0) = \frac{1}{2\pi i \cdot I_\gamma(z_0)} \cdot \oint_\gamma \frac{f(w)}{w - z_0} \cdot dw$$

where $I_\gamma(z_0)$ is the winding number$^{(*)}$ of γ with respect to z_0.

Theorem 2.0.5 (Weaker version of Montel's theorem)**.** *A family $\boldsymbol{F}$ of holomorphic functions on a domain $\boldsymbol{D}$ is normal iff $\boldsymbol{F}$ is locally bounded, i.e, for each compact set $\boldsymbol{K} \subset \boldsymbol{D}$, there is a constant M such that*

$$|f(z)| \leq M, \; \forall z \in \boldsymbol{D} \text{ and } \forall f \in \boldsymbol{F}$$

Proof. Firstly, suppose that $\boldsymbol{F}$ is normal but not locally bounded, i.e, there is a compact set $\mathbf{K} \subset \mathbf{D}$ such that $\sup\{f(z), z \in \mathbf{K}, f \in \boldsymbol{F}\} = \infty$. Equivalently, there is a a sequence $\{f_n\}$ in $\boldsymbol{F}$ such that $\sup\{f_n(z), z \in \mathbf{K}\} \geq k$, $\forall k \in \mathbb{N}$. Since $\boldsymbol{F}$ is normal, there is an holomorphic function in $\mathbf{D}$, f, and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \mapsto f$. However this gives us that $\sup\{|f_{n_k}(z) - f(z)|, z \in K\} \mapsto 0$ when $k \mapsto \infty$. If $|f(z)| \leq M$ for $z \in \mathbf{K}$, then

$$|f_{n_k}| \leq |f_{n_k} - f(z) + f(z)| \leq \sup\{|f_{n_k} - f(z)|, z \in \mathbf{K}\} + M$$

and since the right hand side converges to M, we reach contradiction.

Now suppose $\boldsymbol{F}$ is locally bounded. We see that the first condition of theorem 2.0.3 is clearly satisfied, and therefore it suffices to show that $\boldsymbol{F}$ is equicontinuous to prove that it is normal. We fix a point $a \in \mathbf{D}$ and $\epsilon > 0$, and as $\boldsymbol{F}$ is locally bounded, there is $r > 0$ and $M > 0$ such that $\overline{B}(a, r) \subset \mathbf{D}$ and $|f(z)| \leq M$, $\forall z \in B(a, r)$, $\forall f \in \boldsymbol{F}$. Let $|z - a| \leq \frac{1}{2} \cdot r$ and $f \in \boldsymbol{F}$, then, by using Cauchy's Formula with $\gamma(t) = a + r \cdot e^{it}$, $0 \leq t \leq 2 \cdot \pi$

$$|f(a) - f(z)| \leq \frac{1}{2 \cdot \pi} \left| \int_\gamma \frac{f(w) \cdot (a-z)}{(w-a) \cdot (w-z)} dw \right| \leq \frac{|z-a|}{2\pi} \cdot long(\gamma) \cdot sup_{w \in \gamma} \left| \frac{f(w)}{(w-a) \cdot (w-z)} \right| \leq$$

$$\leq |z - a| \cdot M \cdot r \cdot sup_{w \in \gamma} \left| \frac{1}{(w-a) \cdot (w-z)} \right| \leq |z - a| \cdot M \cdot sup_{w \in \gamma} \left| \frac{1}{(w-z)} \right| \leq \frac{2 \cdot M}{r} \cdot |a - z|$$

Letting $\delta \leq min\{\frac{1}{2} \cdot r, \frac{r}{2 \cdot M} \cdot \epsilon\}$, it follows that $|a - z| < \delta$ gives $|f(a) - f(z)| < \epsilon \; \forall f \in \boldsymbol{F}$. ☐

The second block of results deals with properties of univalent maps(i.e. holomorphic and injective, and therefore conformal).

Theorem 2.0.6 (Area Theorem)**.** *Let $g(z) = \frac{1}{z} + b_0 + b_1 \cdot z + \cdots$ be univalent in Δ (with a pole$^{(*)}$ singularity at $z = 0$). Then $\sum_{n=0}^{\infty} n \cdot |b_n^2| \leq 1$.*

Proposition 2.0.7 (Dimensioning of a_2)**.** *If $f = z + \sum_{n=2}^{\infty} a_n \cdot z^n$ is a $conformal^{(*)}$ function in the open unit disk, then $|a_2| \leq 2$.*

Proof. We define $g(z) = \frac{1}{\sqrt{f(z^2)}} = \frac{1}{z} - a_2 \cdot \frac{z}{2} + \cdots$. We have that g is univalent, because if $g(z_1) = g(z_2)$, then $f(z_1^2) = f(z_2^2) \Leftrightarrow z_1^2 = z_2^2 \Leftrightarrow z_1 = \pm z_2$. But g is an odd function, so $z_1 = z_2$. All in all, we can apply the area theorem to g(z), with $b_1 = \frac{-a_2}{2}$, and therefore we get $|a_2| \leq 2$, in order to have $\sum_{n=0}^{\infty} n \cdot |b_n^2| \leq 1$. □

Theorem 2.0.8 (Koebe's[11] one quarter theorem)**.** *Let* $\boldsymbol{S}$ *be the collection of conformal functions in the open unit disk,* Δ*, such that* $f(0) = 0$ *and* $f'(0) = 1$*,* $\forall f \in \boldsymbol{S}$*. If* $f \in \boldsymbol{S}$*, then* $f(\Delta) \supset D(0, \frac{1}{4})$*.*

Proof. Fix a point $c \in \mathbb{C}$ and suppose $f \neq c$ in Δ. Then

$$\frac{c \cdot f(z)}{c - f(z)} = z + (a_2 + \frac{1}{c}) \cdot z^2 + \cdots$$

belongs to $\boldsymbol{S}$. Applying 2.0.7 twice, we obtain

$$\frac{1}{|c|} \leq |a_2| + |a_2 + \frac{1}{c}| \leq 2 + 2 = 4. \Box$$

Theorem 2.0.9 (Riemann[12] Mapping Theorem)**.** *Let* $\boldsymbol{D}$ *be a simply connected region which is not the whole plane* $\mathbb{C}$*, and let* $a \in \boldsymbol{D}$*. There is a unique analytic function,* $f : \boldsymbol{D} \mapsto \mathbb{C}$ *which satisfies :*

1. $f(a) = 0$ *and* $f'(a) > 0$.
2. f *is one-one (injective).*
3. $f(\boldsymbol{D}) = \{z \in \mathbb{C}; |z| \leq 1\}$.

Proof. Here we will only prove the uniqueness. Suppose there is an analytic function, g, with the same properties as f. Then, $f \cdot g^{-1} : \mathbf{D} \mapsto \mathbf{D}$ is analytic and bijective, and it satisfies $f \cdot g^{-1}(0) = f(a) = 0$ so, by applying Koenig's linearization theorem, we get that there is a constant c, $|c| = 1$ and $f \cdot g^{-1}(z) = c \cdot z$, $\forall z \in \mathbf{D}$. This gives $f = c \cdot g$, and therefore $0 < f'(a) = c \cdot g'(a)$, and since $g'(a) > 0$, we get c = 1, and f = g. □

Theorem 2.0.10 (Carathéodory[13])**.** *Let* $\boldsymbol{D}$ *be a simply connected*$^{(*)}$ *domain in* $\overline{\boldsymbol{C}}$ *whose boundary has at least two points. Then* $\delta\boldsymbol{D}$ *is locally connected if and only if the Riemann mapping* $\psi : \Delta \mapsto \boldsymbol{D}$ *extends continuously to the closed disk* $\overline{\Delta}$*.*

[11] Paul Koebe(1882 - 1945) was a Germany mathematician. His only field of study was complex numbers, and his most important result is the uniformization of Riemann Surfaces.

[12] Bernhard Riemann(1826 - 1866) was a German mathematician who made key contributions to analysis, number theory and differential geometry. Some of his most important results are the rigorous formulation of the Riemann integral, the introduction of Riemann surfaces, and Riemann's hypothesis in number theory.

[13] Constantin Carathéodory(1873 - 1950) was a Greek mathematician. He made contributions to the theory of functions of a real variable, the calculus of variations and measure theory.

Proposition 2.0.11. *A conformal mapping of Δ on to itself has the form*

$$w = e^{i\cdot\theta} \cdot \frac{(z-a)}{(1-\bar{a}\cdot z)},\ 0 \leq \theta \leq 2\pi,\ |a| < 1$$

Theorem 2.0.12. *Every conformal automorphism, g, of $\overline{\mathbb{C}}$ can be expressed as a Möbius transformation*(*)

$$g(z) = \frac{a\cdot z+b}{c\cdot z+d}$$

where $a, b, c, d \in \mathbb{C}$, and $ad - bc \neq 0$. Every non-identity automorphism of $\overline{\mathbb{C}}$ either has two distinct fixed points or one double fixed point in $\overline{\mathbb{C}}$.

Proposition 2.0.13. *If A and B are two conformal self-maps of the open unit disk Δ which commute, and A is not the identity, then B belongs to the one-parameter subgroup generated by A.*

Proof. There are three cases to be considered:

1. Suppose A has a fixed point in Δ. We may assume the fixed point is $z_0 = 0$, so that $A(z) = e^{i\cdot\theta} \cdot z$. Then $e^{i\cdot\theta} \cdot B = (AB)(0) = (BA)(0) = B(0)$. Since A is not the identity, $B(0) = 0$, and B has the form $e^{i\cdot\theta} \cdot z$.

2. Suppose A has two fixed points on $\{|z| = 1\}$, which are different. We can map this problem to the right half-plane, with the fixed points going to 0 and ∞, and $A(z) = \lambda\cdot z$ for some $\lambda > 0$, $\lambda \neq 1$. As above, either B fixes 0 and ∞, or interchanges them. In the second case, $B(z) = \frac{\mu}{z}$ for some $\mu > 0$ and does not commute with A. Therefore, B fixes these points and $B(z) = \mu \cdot \zeta$.

3. Suppose A has one fixed point on $\{|z| = 1\}$. Again, we map the problem to the right half-plane, with ∞ fixed. Then we have $A(z) = z + \lambda \cdot i$ for some real $\lambda \neq 0$ and $B(z) = z + i \cdot \mu$, for some $\mu \in \mathbb{R}$.

Möbius transformations corresponding to these three cases are called elliptic, hyperbolic, and parabolic respectively. Every such transformation preserving the disk is one of these three cases.

Theorem 2.0.14 (Argument Principle)**.** *Let f be a meromorphic*(*) *map defined in a connected open set $\Omega \subset \mathbb{C}$ enclosed by a closed path γ. Let M be the number of zeros of f in γ and let P be the number of poles of f in γ, with each zero and pole counted as many times as its multiplicity and order respectively. Then*

$$M - P = \frac{1}{2\pi i} \cdot \int_\gamma \frac{f'(z)dz}{f(z)}$$

Now we will consider hyperbolic metric, which is very important in complex dynamics, as every holomorphic function is a contraction with respect to this metric (this result is known as the Schwarz-Pick lemma).

Definition 2.0.15 (Hyperbolic metric)**.**

By 2.0.12., a conformal self-map of Δ is of the form

$$w = e^{i\cdot\theta} \cdot \frac{(z-a)}{(1-\overline{a}\cdot z)},\ 0 \leq \theta \leq 2\pi,\ |a| < 1$$

And operating

$$|\frac{dw}{dz}| = \frac{1-|z|^2}{1-|w|^2}$$

and therefore

$$d\rho = \frac{2\cdot|dz|}{1-|z|^2} = \frac{2\cdot|dw|}{1-|w|^2}$$

is invariant under the mapping. We call the metric $d\rho$ *hyperbolic metric.*

Definition 2.0.16 (Riemann Surface)**.** *A connected complex analytic manifold of complex dimension one is called a Riemann Surface.*

Proposition 2.0.17 (Uniformization theorem)**.** *There are only three Riemann surfaces, up to isomorphism. That is, any simple Riemann surface is conformally isomorphic to*

1. *the whole plane* $\mathbb{C}$*.*
2. *the open unit disc,* Δ*.*
3. *the Riemann sphere* $\overline{\mathbb{C}}$*, consisting of* $\mathbb{C}$ *together with a point at infinity.*

Theorem 2.0.18. *Let* $\boldsymbol{D}$ *be a domain in* $\mathbb{C}$*, and for* $z \in \boldsymbol{D}$*, let* $\delta(z)$ *denote the distance from* z *to* $\partial\boldsymbol{D}$*. Then, if* $\boldsymbol{D}$ *is simply connected*

$$\frac{1}{2} \cdot \frac{|dz|}{\delta z} \leq d_{\rho_{\boldsymbol{D}}}(z) \leq 2 \cdot \frac{|dz|}{\delta(z)}$$

3 INTRODUCTION AND LOCAL THEORY

3.1 INTRODUCTORY RESULTS

Given a continuous function $f : \mathbb{C} \mapsto \mathbb{C}$, the set of all the iterates, $\{f^n(z_0)\}$, $n \geq 0$, for the various points $z_0 \in \mathbb{C}$ is a dynamical system where a "force", f, for each period, n, changes z_0 into $f^n(z_0)$. From this point of view, the main question is where will z_0 be when iterated by f many times. Mathematically, our aim is to study the asymptotic behavior of $f^n(z_0)$ when $n \mapsto \infty$ for every $z_0 \in \mathbb{C}$. In this same line of thinking we may also ask ourselves about the previous states of a given point; Due to the nature of functions, the question about the future states of an initial seed is uniquely determined (that is, if we can find the solution), but, on the other hand, the question about the previous states of a point does not have, in general, a unique answer, unless the map, f, is globally invertible.

To accomplish our goal we first characterize some types of points which have a special behavior and characteristics:

Definition 3.1.1 (Fixed point)**.** *An element $z \in \mathbb{C}$ is said to be a fixed point of $f : \mathbb{C} \mapsto \mathbb{C}$ if $f(z) = z$.*

Definition 3.1.2 (Periodic point)**.** *An element $z \in \mathbb{C}$ is said to be a periodic point of period p of f if $f^p(z) = z$ and $f^n(z) \neq z$, $\forall n < p$, $n, p \in \mathbb{N}$. In this case, the orbit of z_0, called periodic orbit, is given by the finite set $\{z_0, z_1, \ldots, z_{p-1}\}$.*

There is a strong relation between the fixed points of f^p and the periodic points of f: a fixed point of f^p is either a fixed point of f, or a periodic point of period d of f, for some $d|p$.

Definition 3.1.3 (Preperiodic point)**.** *An element $z \in \mathbb{C}$ is said to be a preperiodic point of f if z is not periodic, but a point in its orbit is periodic.*

We have only considered two types of special orbits, but, taking benefit from the special nature of periodic points, we can determine, in some cases, how will nearby points (points in a neighborhood of the periodic point) behave when iterated by the same function f.

Definition 3.1.4 (Classification of fixed points)**.** *Given a fixed a point, z_0, of an holomorphic function f, we call $\lambda = f'(z_0)$ its multiplier. We can classify the fixed point according to λ:*

1. *if $|\lambda| < 1$, the fixed point is attracting (if $\lambda = 0$, we say the fixed point is superattracting).*

2. *if* $|\lambda| > 1$*, the fixed point is repelling.*

3. *if* $|\lambda| = 1$ *and* $\lambda^n = 1$*, for some* $n \in \mathbb{Z}$*, the fixed point is* rationally neutral*.*

4. *if* $|\lambda| = 1$ *and* $\lambda^n \neq 1$*, for every* $n \in \mathbb{Z}$*, the fixed point is* irrationally neutral*.*

As its name states, an attracting point is a fixed point that attracts nearby points when they are iterated by f. Indeed, if $|\lambda| < \rho < 1$ then, in a neighborhood of z_0, $|f(z) - z_0| < \rho \cdot |z - z_0| \Rightarrow |f^n(z) - z_0| < \rho^n \cdot |z - z_0|$ and therefore $f^n(z)$ uniformly converges$^{(*)}$ to z_0 on such neighborhood; this concept opposes to that of a repelling fixed point. In the case of $|\lambda| = 1$, the nature of the fixed point is not as obvious, and requires a further study to determine the behavior of nearby points.

Definition 3.1.5 (Basin of attraction)**.** *Given an attracting fixed point* $z_0 \in \mathbb{C}$*, we define its basin of attraction as* $A(z_0) = \{z \in \mathbb{C}$ *such that* f^n *is defined for all* $n \geq 1$ *and* $f^n(z) \to z_0$*, when* $n \to \infty$ $\}$*, where* f^n *is the n-fold iterate of f.*

Observation. $A(z_0)$ is open, because it is the union of the backwards iterates $f^{-n}(D(z_0, \epsilon))$, for a given small $\epsilon > 0$.

Definition 3.1.6 (Classification of periodic points)**.** *Given a periodic orbit* $\{\overline{z_0 z_1 \ldots z_{p-1}}\}$ *for the function f, we may see* z_0 *as a fixed point of* $g(z) = f^p(z)$*, and therefore we define the multiplier of the periodic orbit*

$$g'(z_0) = \prod_{i=0}^{i=p-1} f'(z_i)$$

which determines its local nature.

The same results given for a fixed point stand true for a periodic orbit, using the new concept of multiplier that we have just defined. Therefore, periodic orbits can be classified the same way as fixed points.

3.2 LOCAL THEORY

In some cases, it might be difficult to determine the basin of attraction of an attracting fixed point for a given function, or the dynamics of points in a neighborhood of periodic points. To solve this problem, we can sometimes use other simpler functions that we already know the behavior of, and obtain results of the original functions from them. We do this by using conjugations.

Definition 3.2.1 (Conformal conjugations)**.** *We say that a function $f : \boldsymbol{U} \subset \mathbb{C} \longmapsto \boldsymbol{U}$ is conformally conjugate to $g : \boldsymbol{V} \subset \mathbb{C} \longmapsto \boldsymbol{V}$ if there is a conformal map$^{(*)}$ $\phi : \boldsymbol{U} \longmapsto \boldsymbol{V}$ such that*

$$\phi(f(z)) = g(\phi(z)), \forall z \in \boldsymbol{U}$$

We may consider f and g to be the same map expressed in different coordinates. Note that ϕ maps fixed points to fixed points, and the multipliers associated to those fixed points remain equal by ϕ. It also maps a basin of attraction by f to a basin of attraction by g.

This definition also implies that the iterates f^n and g^n are also conjugated by ϕ, i.e : $g^n = \phi \cdot f^n \cdot \phi^{-1}$.

We have only defined, and will only be considering, conformal conjugations. However, there are also other types of conjugations: if instead of considering a conformal map $\phi : \mathbf{U} \mapsto \mathbf{V}$ conjugating f and g, we consider a map $\phi : \mathbf{U} \mapsto \mathbf{V}$ which is continuous (C^1) we get a continuous (C^1) conjugation.

These other conjugations also map fixed points to fixed points, periodic orbits to periodic orbits, ... However, the difference between them and a conformal conjugation is that the latter does not alter the multiplier of the periodic orbits.

Example: The polynomial $P(z) = z^2 - 2$ has a superattracting fixed point at $z = \infty$. Consider the conformal map $h(z) = \frac{z+1}{z}$ of $\{z > 1\}$ onto $\mathbb{C}[-2, 2]$. We see that $P(h(z)) = h(z)^2 - 2 = h(z^2)$, which gives $h^{-1} \cdot P \cdot h = z^2$, and thus P is conjugate to z^2 on $\{|z| > 1\}$. Therefore, the dynamics of $P(z)$ on $\mathbb{C}[-2, 2]$ are the same as those of z^2 on $\{|z| > 1\}$. Since the iterates of any z, $|z| > 1$ under z^2 tend to ∞, so do the iterates under P of any $z \in \mathbb{C}[-2, 2]$. Evidently, $[-2, 2]$ is invariant under P, so the basin of attraction of ∞ for P is $A(\infty) = \overline{\mathbb{C}}[-2, 2]$.

Theorem 3.2.2 (Koenigs[14] linearization theorem)**.** *Suppose f has an attracting fixed point at z_0, with its multiplier, λ, satisfying $0 < |\lambda| < 1$ (not superattracting). Then, there is a conformal map $\zeta = \phi(z)$ of a neighborhood of z_0 onto a neighborhood of 0, which conjugates $f(z)$ to the linear function $g(\zeta) = \lambda \cdot \zeta$. Furthermore, the conjugating function is unique, up to multiplication by a nonzero scalar factor.*

Proof. We suppose $z_0 = 0$ and define $\phi_n = \lambda^{-n} \cdot f^n(z)$. Then, ϕ_n satisfies

[14]Gabriel Xavier Paul Koenigs(1858 - 1931) was a French mathematician who worked on analysis and geometry.

$$\phi_n \cdot f = \lambda^{-n} \cdot f^{n+1} = \lambda \cdot \phi_{n+1}$$

Therefore, if $\phi_n \mapsto \phi$, then $\phi \cdot f = \lambda \cdot \phi$, so $\phi \cdot f \cdot \phi^{-1} = \lambda \cdot \zeta \Rightarrow \phi$ is a conjugation. To show convergence note that for $\delta > 0$ small,

$$|f(z) - \lambda \cdot z| \leq C \cdot |z|^2, \, |z| \leq \delta$$

Thus $|f(z)| \leq |\lambda| \cdot |z| + C \cdot |z|^2 \leq (|\lambda| + C \cdot \delta) \cdot |z|$ and, by induction, with $|\lambda| + C \cdot \delta < 1$,

$$|f^n(z)| \leq (|\lambda| + C \cdot \delta)^n \cdot |z|, \, |z| \leq \delta$$

We choose $\delta > 0$ so small that $\rho = \frac{(|\lambda|+C\cdot\delta)^2}{|\lambda|} < 1$, and we obtain

$$|\phi_{n+1}(z) - \phi_n(z)| = |\frac{f^n(f(z)) - \lambda \cdot f^n(z)}{\lambda^{n+1}}| \leq \frac{C \cdot |f^n(z)|^2}{|\lambda|^{n+1}} \leq \frac{\rho^n \cdot C \cdot |z|^2}{|\lambda|}$$

for $|z| \leq \delta$. Therefore $\phi_n(z)$ converges uniformly for $|z| \leq \delta$, and the conjugation exists. Now we will prove the uniqueness:

Suppose there are two such conjugations, ϕ_1 and ϕ_2. Then, the composition

$$\phi_2 \cdot \phi_1^{-1}(z) = a_1 \cdot z + a_2 \cdot z^2 + \dots, \text{ for some } a_i \in \mathbb{C}, \, i \in \mathbb{N}$$

would commute with the map $s(z) = \lambda \cdot z$, and comparing the coefficients of the two resulting power series we get $\lambda \cdot a_n = a_n \cdot \lambda^n$, $\forall n \in \mathbb{N}$. Since λ is not zero nor a root of the unity, $a_2 = a_3 = \dots = 0$, and therefore $\phi_2 \cdot \phi_1^{-1}(z) = a_1 \cdot z$, or equivalently $\phi_2 = a_1 \cdot \phi_1$. □

Proposition 3.2.3 (The repelling case)**.** *The existence of a conjugating map for a repelling fixed point follows from the attracting case, because if $f(z_0) = z_0$, $f'(z_0) = \lambda > 1$, we know that $f'(z) \neq 0$, $\forall z \in \boldsymbol{U}$, where $\boldsymbol{U}$ is a neighborhood of z_0. Therefore, there is a branch of f^{-1}, $g(z)$, in $\boldsymbol{U}$ for which z_0 is a fixed point. Applying the inverse function theorem, we know that $g'(z_0) = \frac{1}{\lambda} = \mu$, $\mu < 1$, and there is a conformal conjugation $\zeta = \phi(z)$ such that*

$$\zeta g(z) = \mu \cdot \zeta = \frac{\zeta}{\lambda}$$

And this same map also conjugates $f(z)$ to $\lambda \cdot \zeta$.

In the case of superattracting fixed points we can also prove the existence of a conjugation.

Proposition 3.2.4 (Böttcher[15] coordinates)**.** *Suppose f has a superattracting fixed point at z_0*

[15] Lucjan Böttcher(1872 - 1937) was a Polish mathematician. His most important result was in the iteration of rational mappings of the Riemann sphere.

$$f(z) = z_0 + a_p(z - z_0)^p + \cdots,\ a_p \neq 0,\ p \geq 2.$$

Then there is a conformal map $\zeta = \phi(z)$ of a neighborhood of z_0 onto a neighborhood of 0 which conjugates f(z) to ζ^p. Furthermore, the conjugation is unique, up to a multiplication by a (p-1)th root of the unity.

Proof. Suppose $z_0 = 0$. For z small enough, there is a constant $C > 1$ such that $|f(z)| \leq C \cdot |z|^p$. By induction, writing $f^{n+1} = f^n \cdot f$, and using $p \geq 2$, we obtain

$$|f^n(z)| \leq (C \cdot |z|)^{p^n},\ z \leq \delta$$

and therefore $\lim_{n\to+\infty} f^n(z) = 0$.

If we change variables by setting $w = c \cdot z$, where $c^{p-1} = \frac{1}{a_p}$, we get $f(w) = w^p + \cdots$, and therefore we can assume $a_p = 1$. Our aim is to find a conjugating map $\phi(z) = z + \cdots$ such that $\phi(f(z)) = \phi(z)^p$, which is equivalent to the condition $\phi \cdot f \cdot \phi^{-1} = \zeta^p$. Let

$$\phi_n(z) = f^n(z)^{p^{-n}} = (z^{p^n} + \cdots)^{p^{-n}} = z \cdot (1 + \cdots)^{p^{-n}}$$

which is well defined in a neighborhood of the origin. Every ϕ_n in the succession satisfies

$$\phi_{n-1} \cdot f = (f^{n-1} \cdot f)^{p^{-n+1}} = \phi_n^p$$

So, if $\lim_{n\to+\infty} \phi^n = \phi$, then ϕ satisfies $\phi \cdot f = \phi^p$, and is a solution. To show that $\{\phi_n\}$ converges, we write $f^{n+1} = f \cdot f^n$ and note that

$$\tfrac{\phi_{n+1}}{\phi_n} = (\tfrac{\phi_1 \cdot f^n}{f^n})^{p^{-n}} = (1 + o(|f^n|))^{p^{-n}} = 1 + o(p^{-n}) \cdot o(|z|^{p^n} \cdot C^{p^n}) = 1 + o(p^{-n})$$

if $|z| \leq \frac{1}{C}$. Therefore the product

$$\textstyle\prod_{n=1}^{\infty} \frac{\phi_{n+1}}{\phi_n}$$

converges uniformly for $|z| \leq c < \frac{1}{C}$, and this implies $\{\phi_n\}$ converges. Hence, ϕ exists. □

We have proved the existence of conjugations for attracting, repelling and superattracting fixed points, so all that is left to consider is the case where $|\lambda| = 1$, that is, $\lambda = e^{2\pi i\theta}$. The two options to be considered are:

1. θ is rational (parabolic case).

2. θ is irrational (irrational case).

Suppose $\lambda = e^{2\pi i\theta}$, with θ rational. As θ is rational, $\theta = \frac{p}{q}$, for some p, $q \in \mathbb{Z}$, we may consider f^q, which satisfies $\lambda = 1$, because if ϕ is a conformal conjugation between f and g, it is also a conformal conjugation of f^q and g^q.

Definition 3.2.5 (Order or multiplicity of a fixed point)**.** *The order of z_0 as a fixed point of $f : \boldsymbol{U} \mapsto \mathbb{C}$ is the order or multiplicity of z_0 as a zero of $f(z) - z$, that is, the degree of the first non-vanishing term of the Taylor 's expansion of f - Id at z_0(in any set of local coordinates centered at z_0).*

Considering the Taylor's expansion, we therefore take maps of the form

$$f(z) = z + a \cdot z^{m+1} + o(z^{m+2}),\ m > 0 \text{ and } a \neq 0$$

where m is the order of z_0.

Definition 3.2.6 (Attracting and repelling petals)**.** *Suppose f is defined and univalent in a neighborhood, $\boldsymbol{U}$, of the origin. An open set $\boldsymbol{P} \subset \boldsymbol{U}$ is called an attracting petal for f at the fixed point if*

1. $f(\overline{\boldsymbol{P}}) \subset \boldsymbol{P} \cup \{0\}$.
2. $\bigcap_n f^n(\overline{\boldsymbol{P}}) = \{0\}$.

An open set $\boldsymbol{P} \subset \boldsymbol{U}$ is called a repelling petal for f at the fixed point if $\boldsymbol{P}$ is an attracting point for $f^{-1} : f(\boldsymbol{U}) \mapsto \boldsymbol{U}$, where f^{-1} denotes the branch of the inverse of f fixing the origin.

Theorem 3.2.7 (Parabolic Flower Theorem)**.** *Suppose f has a parabolic fixed point with multiplier $\lambda = 1$ at the origin of multiplicity m + 1. Then, there are 2m petals, $\{\boldsymbol{P}_j\}_{j=1}^{j=2m}$, numbered cyclically around the origin and such that $\boldsymbol{P}_j$ is attracting or repelling according to whether j is even or odd. Each petal, $\boldsymbol{P}_j$, intersects only its two immediate neighbors, $\boldsymbol{P}_{j-1}$ and $\boldsymbol{P}_{j+1}$, and is disjoint from the rest. The petals can be chosen so that the union*

$$\boldsymbol{P}_1 \cup \boldsymbol{P}_2 \cup \cdots \cup \boldsymbol{P}_{2m} \cup \{0\}$$

form an open neighborhood of the origin.

If the iterates of a point $z \neq 0$ converge to 0, it needs to belong to one of the attracting petals, **P**, from an iterate onward. We then say that the orbit converges to 0 through **P**. Therefore, it makes sense to consider the (parabolic) basin of attraction of **P**.

Definition 3.2.8 (Parabolic basin of attraction)**.** *If z_0 is a parabolic fixed point of f with multiplier $\lambda = 1$, and $\boldsymbol{P}$ is an attracting petal at z_0, we define the parabolic basin of attraction of z_0 associated to $\boldsymbol{P}$ as*

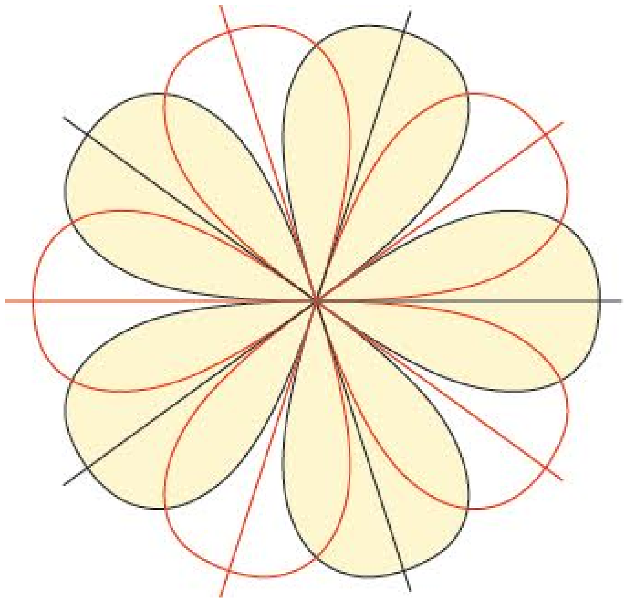

Figure 1: Distribution of the invariant petals around a parabolic point with multiplicity 5.

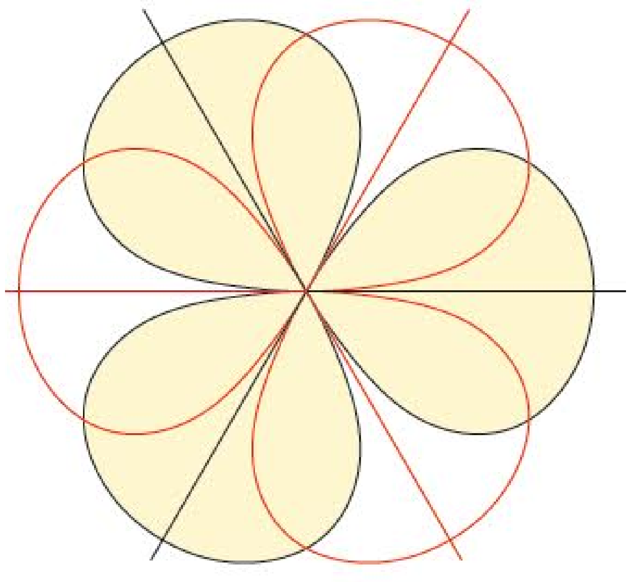

Figure 2: Distribution of the invariant petals around a parabolic point with multiplicity 3.

$$A_p = \{z \in S \setminus \bigcup_{n>0} f^{-n}(z_0) | f^n(z) \mapsto z_0 \textit{ through } \boldsymbol{P} \textit{ when } n \mapsto \infty\}$$

If the parabolic point has multiplicity m + 1, then it has exactly m disjoint parabolic basins.

Although we can not find a conjugation to its linear part as we have for attracting, repelling and superattracting fixed points, some kind of linearization is possible inside each petal.

Theorem 3.2.9 (Fatou[16] Coordinates)**.** *For every attracting and for every repelling petal* $\boldsymbol{P}$*, there is a conformal embedding*

$$\phi : \boldsymbol{P} \mapsto \mathbb{C}$$

mapping 0 to ∞*, called the Fatou coordinate in* $\boldsymbol{P}$*, which conjugates f to the translation* $z \mapsto z+1$ *on* $\boldsymbol{P} \cap f^{-1}(P)$*.*

Consider now the case where $\lambda = e^{2\pi i\theta}$, with θ irrational. Our aim is to find a solution to the Schröder[17] equation $\phi(f(z)) = \lambda\phi(z)$, normalized by $\phi'(0) = 1$. If we write $h = \phi^{-1}$, the equation becomes

$$f(h(\zeta)) = h(\lambda \cdot \zeta),\ h'(0) = 1$$

Theorem 3.2.10 (Uniqueness of solutions)**.** *A solution to the Schröder equation in any disk* $\{|\zeta| > r\}$ *is unique.*

Proof. Suppose $h(\zeta_1) = h(\zeta_2) \Rightarrow h(\lambda^n \cdot \zeta_1) = h(\lambda^n \cdot \zeta_2)$ for all $n > 0$, $n \in \mathbb{N}$. Since $\{\lambda^n\}$ is dense in the unit circle (as θ is irrational), $h(\zeta_1 \cdot e^{i\cdot\theta}) = h(\zeta_2 \cdot e^{i\cdot\theta})$ for every θ. This implies $h(\zeta_1 \cdot z) = h(\zeta_2 \cdot z)$ for $|z| < 1$, and since $h'(0) = 1$, $\zeta_1 = \zeta_2$. □

Theorem 3.2.11 (Existence of solution)**.** *A solution h of the Schröder equation exists if and only if the sequence of iterates* $\{f^n\}$ *is uniformly bounded in some neighborhood of the origin.*

Proof. If h exists, then $f^n(z) = h(\lambda^n \cdot h^{-1}(z))$ is obviously bounded. On the other hand, if $|f^n(z)| \leq M$, for some $M \in \mathbb{R}$ and $\forall n \geq 1$, we can define:

$$\phi_n(z) = \frac{1}{n} \sum_{j=0}^{j=n-1} \lambda^{-j} \cdot f^j(z)$$

[16]Pierre Fatou(1878 - 1929) was a French mathematician and astronomer. He made important contributions to the field of analysis. As seen in this project later, he gives name to the Fatou Set.

[17]Ernst Schröder(1841 - 1902) was a German mathematician, mostly known by his work in algebraic logic.

Then $\{\phi_n\}$ is a uniformly bounded sequence of analytic functions, and therefore contains a convergent subsequence. Since $\phi_n \cdot f = \lambda \cdot \phi_n + o(\frac{1}{n})$, any limit of the sequence ϕ_n satisfies $\phi \cdot f = \lambda \cdot \phi$. From $\lambda = f'(0)$, we get $\phi_n'(0) = 1$ and so $\phi'(0) = 1$. All in all, $h = \phi^{-1}$ is a solution of the Schröder equation. □

Theorem 3.2.12 (Condition for non existence of solutions)**.** *There exists a $\lambda = e^{2\pi i\theta}$ so that the Schröder equation has no solution for any polynomial f.*

Proof. We will use a reduction to absurdity argument. Let $f(z) = z^d + \cdots + \lambda \cdot z$, and suppose there is a conjugation h defined on $\Delta(0, \delta)$. We consider the d^n fixed points of f^n, which are the roots of

$$f^n(z) - z = z^{d^n} + \cdots + (\lambda^n - 1) \cdot z$$

One root is $z = 0$, and we label the others $z_1, \ldots, z_{d^n-1}$. Since $f^n(z) = h(\lambda^n \cdot h^{-1}(z))$ has only one zero on $\Delta(0, \delta)$, we get $z_j \notin \Delta(0, \delta)$ for $j = 1, \ldots, d^n - 1$. Therefore

$$\delta^{d^n} \leq \prod_{j=1}^{j=d^n-1} |z_j| = |1 - \lambda^n|$$

We now construct a λ for which this is impossible, therefore contradicting the hypothesis of the existence of h. Let $\{q_n\}$ be a sequence of integers such that $q_1 < q_2 < \ldots$, and let $\theta = \sum_{k=1}^{\infty} 2^{-q_k}$ and $\lambda = e^{2\cdot\pi\cdot i\cdot\theta}$. Then

$$|1 - \lambda^{2^{q_k}}| \approx 2^{q_k - q_{k+1}}$$

and taking logarithms

$$q_{k+1} \leq C(\delta) \cdot d^{2^{q_k}}$$

If we define inductively the q_k's to grow very rapidly, say with $log(q_{k+1}) \geq k \cdot 2^{q_k}$, the inequality does not hold for any d and $\delta > 0$. □

Definition 3.2.13 (Cremer Point)**.** *Let $f : \mathbb{C} \mapsto \mathbb{C}$ be an holomorphic function, and let z_0 be a fixed point of f. If its multiplier, λ, satisfies $|\lambda| = 1$, $\lambda = e^{2\cdot\pi\cdot i\cdot\theta}$, with θ irrational, and there is not a solution to the Schröder equation for λ, we say that z_0 is a Cremer Point.*

Definition 3.2.14 (Diophantine numbers)**.** *A real number θ is Diophantine if it can be badly approximated by rational numbers, that is if there exist $c > 0$ and $\mu < \infty$ so that*

$$|\theta - \frac{p}{q}| \geq \frac{c}{q^\mu}$$

If we write $\lambda = e^{2\pi i\theta}$, this condition is equivalent to

$$|\lambda^n - 1| \geq c \cdot n^{1-\mu},\ n \geq 1$$

Proposition 3.2.15 (Existence of Diophantine numbers)**.** *For a fixed $\mu > 2$, the condition $|\theta - \frac{p}{q}| \geq \frac{c}{q^\mu}$ holds for almost every real number θ. Indeed, if E is the set of $\theta \in [0,1]$, such that $|\theta - \frac{p}{q}| < \frac{1}{q^\mu}$ infinitely often, then*

$$|E| \leq \textstyle\sum_{q=n}^{\infty} 2 \cdot q^{-\mu} \cdot q = o(n^{2-\mu}) \mapsto 0$$

In particular, almost all real numbers are Diophantine.

The first theorem about the existence of linearizable points is due to Siegel, and it was an important breakthrough at the time. Since then, the conditions used by Siegel have been improved by other mathematicians such as Herman[18] or Bruno[19]. We conclude this section by stating Siegel's original theorem.

Theorem 3.2.16 (Siegel[20])**.** *If θ is Diophantine, and if f has a fixed point at 0 with multiplier $e^{2\pi i\theta}$, then there is a solution to the Schröder equation. In other words, f can be conjugated near 0 to multiplication by $e^{2\pi i\theta}$.*

[18]Michael Herman(1942 - 2000) was a French-American mathematician. He was one of the most known figures in dynamical systems. One of his most important results was the introduction of Herman Rings in 1979.

[19]Alexander Bruno(1940) is a Russian mathematician who has made contributions to the normal forms theory.

[20]Carl Ludwig Siegel(1896 - 1981) was a German mathematician. His main field of work was number theory and celestial mechanics. His most known results are his contribution to the Thue-Siegel-Roth theorem for Diophantine approximation and the Siegel Mass Formula for quadratic forms. He is considered one of the most important mathematicians in the XX century.

4 GLOBAL THEORY

We will now study the phase space of holomorphic functions as a whole, not just locally, so we can determine the behavior of points anywhere on the plane.

4.1 GENERAL RESULTS

Definition 4.1.1 (Fatou and Julia[21] Set)**.** *Let $\boldsymbol{S}$ be a Riemann surface, $f : \boldsymbol{S} \mapsto \boldsymbol{S}$ a non-constant holomorphic mapping, and let $f^n : \boldsymbol{S} \mapsto \boldsymbol{S}$ be its n-fold iterate. Given a fixed $z_0 \in \boldsymbol{S}$, if there exists a neighborhood of z_0, $\boldsymbol{U}$, so that the sequence of iterates $\{f^n\}_n$ restricted to $\boldsymbol{U}$ forms a normal family, then we say that z_0 is a normal point, or that it belongs to the Fatou Set of f. On the other hand, if such neighborhood does not exist, we say that z_0 belongs to the Julia Set of f.*

Therefore, by definition, the Julia Set is a closed set and the Fatou Set is open. We may think of the Fatou Set as the set or orbits which are in some sense stable, and the Julia Set as the set of chaotic orbits.

Example: Consider the map $f : \overline{\mathbb{C}} \mapsto \overline{\mathbb{C}}$, $f(z) = z^2$. The entire open disk, Δ, is contained in the Fatou Set of f, since successive iterates on any compact subset converge uniformly to zero. Likewise, the set $\overline{\mathbb{C}} \setminus \Delta$ is contained in the Fatou Set of f, since the iterates of f converge to ∞. On the other hand, if z_0 is a point of the unit circle, in any neighborhood of this point, any limit of iterates would have a jump discontinuity as we cross the unit circle. This shows that $\mathbf{J} = S^1$.

However, not all Julia Sets are as easy to compute and draw as this one. The next example shows a more complicated Julia Set:

Example: Consider the map $f : \overline{\mathbb{C}} \mapsto \overline{\mathbb{C}}$, $f(z) = z^2 - 1$. In this case, f has two repelling fixed points, $z_\pm = \frac{1 \pm \sqrt{5}}{2}$ and an attracting two cycle : $\{0, -1\}$ (actually, it is a superattracting cycle). The Julia Set corresponding to this map is actually a fractal, and it has infinitely many connected components.

We present some properties of both sets to further characterize them.

[21]Gaston Julia(1893 - 1978) was a French mathematician who devised the formula for the Julia Set. His work gained a lot of popularity with the use of the computer for mathematical research, as it could then be visualized.

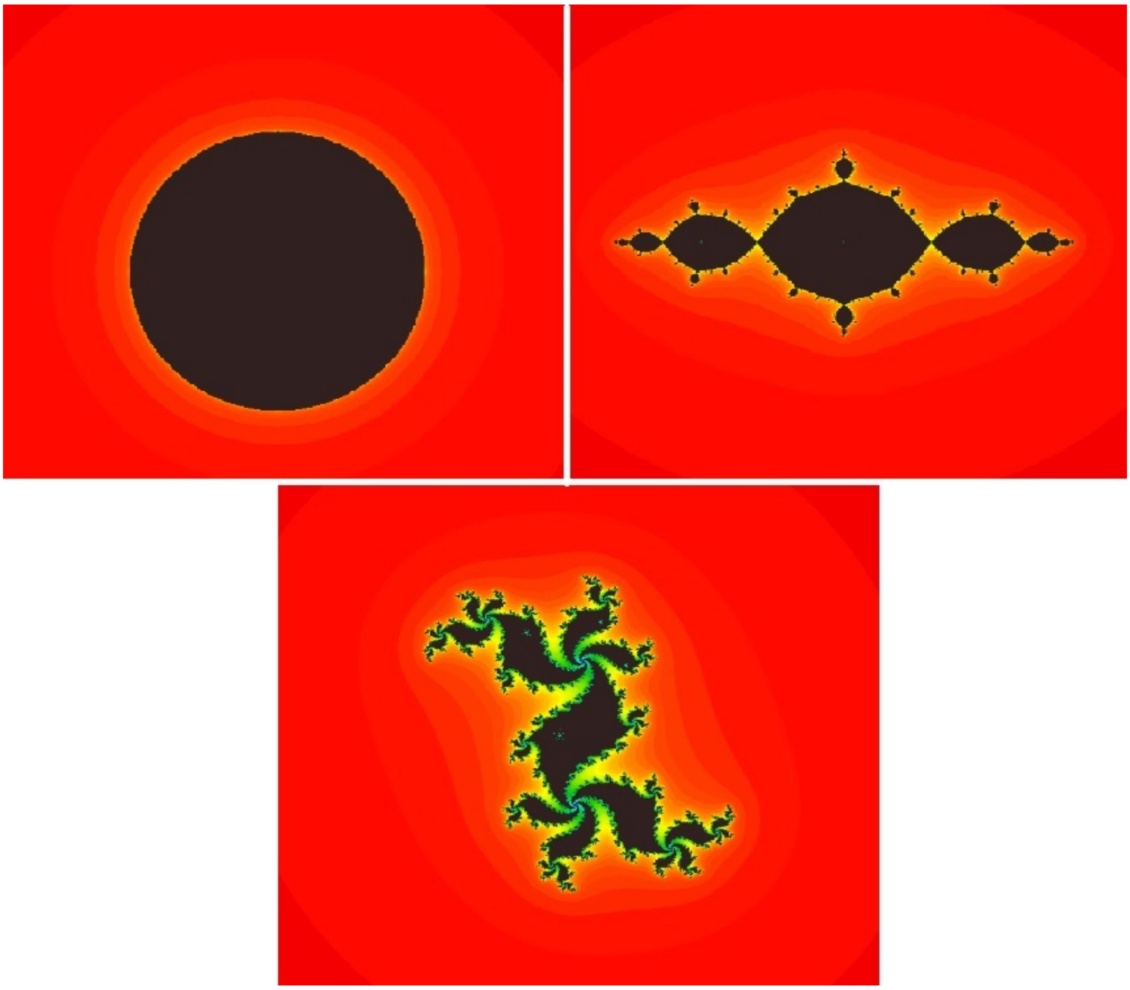

Figure 3: The Julia Sets of $f(z) = z^2$ (top left), $f(z) = z^2 - 1$ (top right) and $h(z) = z^2 + (0.3, 0.5)$ (bottom center)

Definition 4.1.2 (Grand Orbit)**.** *We call grand orbit of a point z under $f : \boldsymbol{S} \mapsto \boldsymbol{S}$ the set GO(z,f), consisting of all the points $z' \in \boldsymbol{S}$ whose orbits eventually intersect the orbit of z. It is clear that z and z' have the same grand orbit iff $f^m(z) = f^n(z')$ for some $n, m \in \mathbb{Z}$.*

Proposition 4.1.3 (Invariance of Julia Set)**.** *The Julia Set of a holomorphic map $f : \boldsymbol{S} \mapsto \boldsymbol{S}$ is fully invariant under f. That is, if z belongs to **J**(f), then the entire grand orbit GO(z,f) is contained in **J**(f).*

Proof. We need to show that $z \in \mathbf{J}(f)$ iff $f(z) \in \mathbf{J}(f)$. It is equivalent to prove that the Fatou Set is fully invariant under f, which is what we will do. We consider both implications:

1. If $z \in \mathbf{F}(f)$, then there is a neighborhood, **U**, of z for which $f^n_{|U}$ forms a normal family; in other words, $\{f^n\}_n$ contains a subsequence which converges in compact subsets of **U** to a holomorphic function g. Since f is a holomorphic map, it is, in particular,

open and therefore $f(\mathbf{U})$ is an open neighborhood of $f(z)$. If we consider $f^n_{|f(U)}$, it is also normal, because $f(f(z)) = f^2(z)$, $f^2(f(z)) = f^3(z)$, etc... And therefore if f^{n_k} converges in compact subsets of U to g, then $f^{n_{k-1}}$ converges in compact subsets of $f(U)$, and is therefore normal in $f(U)$. All in all, $f(z) \in \mathbf{F}(f)$.

2. Let $u \in \mathbf{S}$ such that $f(u) = z$, and $u \in \mathbf{F}(f)$. We want to see that $z \in \mathbf{F}(f)$. Again, because f is holomorphic, it is in particular continuous, and therefore $f^{-1}(\mathbf{U})$ is an open neighborhood of u. As above, we see that $f^2(z) = f^3(f^{-1}(z))$, $f^3(z) = f^4(f^{-1}(z))$, etc... Therefore, if $f^{n_{k+1}}$ converges in compact subsets of $f^{-1}(\mathbf{U})$ to g, then f^{n_k} converges in compact subsets of $\mathbf{U}$, and is therefore normal in $\mathbf{U}$. All in all, $z \in \mathbf{F}(f)$. □

Proposition 4.1.4. *For any $n > 0$, $n \in \mathbb{N}$, the set $\boldsymbol{J}(f^n) = \boldsymbol{J}(f)$.*

Proof. Again, it is equivalent to prove that $\mathbf{F}(f^n) = \mathbf{F}(f)$. We will prove both inclusions:

1. Suppose $z \in \mathbf{F}(f)$ and we want to see that $z \in \mathbf{F}(f^n)$. If $z \in \mathbf{F}(f)$, there is a neighborhood, $\mathbf{U}$, such that $f^p_{|U}$ is normal, that is f^{p_k} contains a subsequence which converges in compact subsets of $\mathbf{U}$ to a holomorphic function g. But, since every subsequence of f^{pn_k} is also a subsequence of f^{p_k}, the result is clear.

2. Suppose $z \in \mathbf{F}(f^n)$ and we want to see that $z \in \mathbf{F}(f)$. We know that if $\{f^p\}$ is normal, then $\{f^{p+j}\}$ is also normal, for every $j \in \mathbb{N}$(we have seen this in the previous theorem). We can express

$$\{f^p\} = \bigcup_r \{f^{rp}\} \cup \{f^{rp+1}\} \cup ... \cup \{f^{rp+(r-1)}\}$$

so any partial of $\{f^p\}$ must contain infinite terms of the form f^{rp}, f^{rp+1}, ..., or $f^{rp+(r-1)}$, as it can not have finitely many of all of them. Therefore, if a subsequence of f^{np} is convergent, there is also a convergent subsequence of f^p, which is what we wanted to prove.□

Theorem 4.1.5 (Periodic orbits and Fatou Set)**.** *Every attracting, periodic orbit of a map is contained in its Fatou Set. In fact, the whole basin of attraction, Ω, of that periodic orbit is contained in the Fatou Set. However, its boundary, $\partial\Omega$ is contained in the Julia Set, as is every repelling orbit.*

Proof. We only need to consider the case of a fixed point, $f(z_0) = z_0$, as the previous proposition states that $\mathbf{J}(f^n) = \mathbf{J}(f)$, for every $n > 0$, $n \in \mathbb{N}$ (in this particular case we take n to be the period of the orbit). If z_0 is attracting, it follows from Taylor's Theorem that the successive iterates of f, restricted to a neighborhood of z_0, converge uniformly to the constant function

$$g : \mathbf{S} \mapsto \mathbf{S}$$
$$z \mapsto z_0$$

We can apply this to any compact subset of the basin of attraction, Ω. On the other hand, if $z \in \overline{\Omega}$, but $z \notin \Omega$, it is clear that no sequence of iterates can converge to a continuous limit. Finally, if z_0 is repelling, then no sequence of iterates can converge uniformly near z_0, since the derivative $\frac{df^n(z)}{dz}$ takes the value λ^n (with $\lambda > 1$) and therefore diverges to infinity as $n \mapsto \infty$. □

Theorem 4.1.6. *If $z_0 \in \boldsymbol{J}(f)$, then the set of all iterated pre-images*

$$A = \{z : f^n(z) = z_0 \textit{ for some } n \geq 0,\ n \in \mathbb{N}\}$$

is everywhere dense in $\boldsymbol{J}$(f). In particular, it follows that the corresponding grand orbit, GO(z_0,f) is everywhere dense in $\boldsymbol{J}$(f).

This theorem is specially important, as it gives us an effective way of computing images of Julia Sets: starting with any $z_0 \in \mathbf{J}(f)$, we compute all pre-images of such point, that is, $A = \{z_1 \in \overline{\mathbb{C}}$ such that $f(z_1) = z_0\}$; then we compute all the pre-images of the points in A, and so on. Eventually, we will be coming arbitrarily close to every point of $\mathbf{J}$(f).

Corollary 4.1.7. *For generic$^{(*)}$ $z \in \boldsymbol{J}(f)$, the forward orbit*

$$\{z, f(z), f^2(z), \cdots\}$$

is everywhere dense in $\boldsymbol{J}$(f).

Proof. Let $\{B_j\}_n$ be a countable collection of open sets forming a basis for the topology of $\overline{\mathbb{C}}$. For each B_j which intersects $\mathbf{J}$(f), let U_j be the union of the iterated pre-images $f^{-n}(B_j)$ for $n \geq 0$, $n \in \mathbb{N}$. Then it follows from the previous theorem that the closure of $U_j \cap J(f)$ is equal to the entire Julia Set, $\mathbf{J}$(f), and the conclusion follows. □

The results given up to this point are valid for any function. From here we will focus on rational functions and give a deeper characterization of them.

4.2 RATIONAL FUNCTIONS

Let R(z) be rational, $R = \frac{P}{Q}$, where P and Q are polynomials with no common factors and $d = max(degP, degQ) \geq 2$.

Theorem 4.2.1. *The Julia Set is nonempty.*

Proof. Suppose $\mathbf{J} = \emptyset$. Then, $\{R^n\}$ is a normal family on all of $\overline{\mathbb{C}}$, and so there is a subsequence $\{n_k\}$, $k \in \mathbb{N}$, such that $R^{n_k} \mapsto f(z)$ for some analytic function $f : \overline{\mathbb{C}} \mapsto \overline{\mathbb{C}}$. Since f is analytic on all of $\overline{\mathbb{C}}$, it is a rational function, as it can not have infinitely many poles and zeros. If f is constant, the image of R^{n_k} is eventually contained in a small neighborhood of the constant value, which is impossible since R^{n_k} covers $\overline{\mathbb{C}}$. If f is not constant, eventually R^{n_k} has the same number of zeros as f(this follows from the argument principle), which is also impossible, since the number of zeros of R^n grows monotonically. □

Theorem 4.2.2. *The Julia Set, **J**, contains no isolated points, that is, **J** is a perfect set.*

Proof. Take $z_0 \in \mathbf{J}$ and $\mathbf{U}$ an open neighborhood of z_0. First assume z_0 is not periodic and choose z_1 with $R(z_1) = z_0$. Then $R(z_0) \neq z_1$ for all n. Since $z_1 \in \mathbf{J}$, backward iterates of z_1 are dense in $\mathbf{J}$, so there is a $\zeta \in \mathbf{U}$ with $R^m(\zeta) = z_1$. Thus $\zeta \in \mathbf{J} \cap \mathbf{U}$ and $\zeta \neq z_0$.

Next, suppose $R^n(z) = z_0$. Then z_0 would be a superattracting fixed point for R^n, contradicting $z_0 \in \mathbf{J}$. Hence there is $z_1 \neq z_0$ with $R^n(z_1) = z_0$. Furthermore $R^j(z_0) \neq z_1$ for all j, since otherwise it would hold for some $0 \leq j < n$ (by periodicity) and therefore $R^j(z_0) = R^{n+j}(z_0) = R^n(z_1) = z_0$, contradicting the minimality of n. As before, z_1 must have a preimage in $\mathbf{U} \cap \mathbf{J}$ which cannot be z_0. □

Theorem 4.2.3. *The Julia Set of a rational map of degree $d \geq 2$ is equal to the closure of its set of periodic points.*

Proof. We have just seen that $\mathbf{J}(f)$ has no isolated points, and therefore we can exclude finitely many points of $\mathbf{J}(f)$ without affecting the argument. Let z_0 be any point of $\mathbf{J}(f)$ which is not a fixed point nor a critical value, that is, we assume that there are d preimages z_1, ..., z_d which are distinct from each other and from z_0, where $d \geq 2$ is the degree of f. By the Inverse Function Theorem, we can find d holomorphic functions, $z \mapsto \phi_j(z)$, $0 \leq j \leq d-1$, which are defined throughout some neighborhood, $\mathbf{V}$, of z_0 and which satisfy $f(\phi_j(z)) = z$, with $\phi_j(z_0) = z_j$. We claim that for some $n > 0$, $n \in \mathbb{N}$, and for some $z \in \mathbb{N}$, the function $f^n(z)$ must take one of the three values z, $\phi_1(z)$ or $\phi_2(z)$. For otherwise, the family of holomorphic functions

$$g_n = \tfrac{(f^n(z)-\phi_1(z))\cdot(z-\phi_2(z))}{(f^n(z)-\phi_2(z))\cdot(z-\phi_1(z))}$$

on $\mathbf{V}$ would avoid the three values 0, 1 and ∞, and therefore be a normal family. But this is a contradiction, since then $\{f^n_{|V}\}$ would also be normal, contradicting the hypothesis that $\mathbf{V}$ intersects the Julia Set. Therefore, we can find $z \in \mathbf{V}$ so as to satisfy either $f^n(z) = z$ or $f^n(z) = \phi_j(z)$. Then, z is a periodic point of period n, or n + 1 respectively.

This shows that every point in $\mathbf{J}(f)$ can be approximated arbitrarily close by periodic points. □

We shall see later that all but finitely many points must repel, so, in fact, the Julia Set, $J(f)$, of a rational map of degree $d \geq 2$ is equal to the close of its set of repelling periodic points.

If R has an attracting fixed point z_0, its basin of attraction, $A(z_0)$ is contained in the Fatou Set. On the other hand, since the iterates of R do not converge to z_0 on the complement of $A(z_0)$, these iterates can not be normal on any open set meeting $\partial A(z_0)$.

Definition 4.2.4 (Siegel Disk)**.** *A simply connected component of the Fatou Set in which R is conjugate to an irrational rotation is called a Siegel Disk.*

Definition 4.2.5 (Herman Ring)**.** *A periodic component of period n, **U**, of the Fatou Set is called a Herman Ring if it is doubly connected(*) and R^n is conjugate to either a rotation on an annulus or to a rotation followed by an inversion.*

Theorem 4.2.6. *The Julia Set contains all repelling periodic points and all neutral periodic points which do not correspond to Siegel Disks. The Fatou Set contains all attracting periodic points and all neutral periodic points corresponding to Siegel Disks.*

Proof. As we have seen that $F(f) = F(f^n)$, we can assume without loss of generality that z_0 is a fixed point, instead of a periodic point. If z_0 is a neutral fixed point, by 3.2.11 we know that $\{f^n\}$ is bounded in a neighborhood of z_0 iff $z_0 \in F(f)$(and in this case, z_0 corresponds to a Siegel disk). If instead z_0 is a Cremer point, then there is not a solution to the Schröder equation for λ, where λ is the multiplier of the periodic point z_0, and therefore $\{f^n\}$ is not bounded in a neighborhood of z_0, hence $z_0 \in J(f)$.

If z_0 is an attracting fixed point, f^n is obviously bounded in a neighborhood of z_0, and f^n contains a subsequence which converges uniformly to the constant function $g(z) = z_0(z_0 \in F(f))$. If instead z_0 is a repelling fixed point, $|f^n(z_0)| \mapsto \infty$ when $n \mapsto \infty$, and therefore no limit function can exist for any subsequence($z_0 \in J(f)$). □

4.2.1 FATOU COMPONENTS, NON-REPELLING CYCLES AND CRITICAL POINTS

Here we will see the relation between the critical points of a map and its Fatou Set.

Definition 4.2.7 (Exceptional points)**.** *A point $z \in \overline{\mathbb{C}}$ is called exceptional under a map f if its grand orbit, $GO(z, f) \subset \overline{\mathbb{C}}$ is a finite set.*

Proposition 4.2.8 (Number of exceptional points)**.** *If f is a rational map of degree two or more, then the set $\mathcal{E}(f)$ of exceptional points can have, at most, two elements. These exceptional points, if they exist, must be critical points of f, and they must belong to the Fatou Set of f.*

Proof. First we must notice that f maps any grand orbit onto itself, and therefore any finite grand orbit must constitute a single periodic orbit under f. Each point in this orbit must be critical, since otherwise $f(z)$ would have two or more pre-images. We conclude then that such an orbit must be superattracting, and therefore contained in the Fatou Set.

If there were three different grand orbit finite points, then the union of the grand orbits of these points would form a finite set whose complement, **U**, in $\overline{\mathbb{C}}$ would be hyperbolic, with $f(\mathbf{U}) = \mathbf{U}$. By Montel's Theorem, the set of iterates of f restricted to **U** would be normal, and therefore both **U** and its complement would be contained in the Fatou Set, contradicting the previous theorem. □

Proposition 4.2.9. *If z_0 is an attracting or parabolic periodic point, then the immediate basin of attraction $A^*(z_0)$ contains at least one critical point.*

Proof. If z_0 is an attracting point, its multiplier, λ, satisfies $0 < |\lambda| < 1$. Let $U_0 = \phi^{-1}(\Delta(0, \epsilon))$ be a small disk, invariant under R, on which the analytic branch, f, of R^{-1} satisfying $f(z_0) = z_0$ is defined. The branch f maps U_0 into $A^* z_0$ and is one-to-one. Therefore, $U_1 = f(U_0)$ is simply connected, and $U_0 \subset U_1$, if U_0 is appropriately chosen. If we do not encounter any critical point, we keep on doing this process, constructing $U_{n+1} = f(U_n)$, $U_n \subset f(U_n)$, and extending f analytically to U_{n+1}. If this process does not end, we obtain a sequence $f^n : U_0 \mapsto U_n$ of analytic functions on U_0 which omits **J**, and is therefore normal on U_0. But this is impossible, since $z_0 \in U_0$ is a repelling fixed point for f. Then, eventually, we reach a U_n to which we can not extend f. Then there is a critical point $p \in A^*(z_0)$ such that $R(p) \in U_n$.

If z_0 is periodic with period $n > 1$ and $|(R^n)'(z_0)| < 1$, this argument shows each component of $A^*(z_0)$ contains a critical point of R^n. Since $(R^n)'(z) = \prod R'(R^j(z))$, $A^*(z_0)$ must also contain a critical point of R. The same result is true for a parabolic basin, and can be proved using a similar argument. □

Theorem 4.2.10. *If **U** is a Siegel disk or a Herman ring, then the boundary of **U** is contained in the closure of the post-critical set of R, CL.*

Proof. Let **U** be a rotation domain that is invariant under R, and suppose that CL does not contain ∂U. Let **D** be an open disk disjoint from CL which meets ∂U. We also assume that **D** is disjoint from some open invariant subset $\mathbf{V} \neq \emptyset$ of **U**. We define f_n to be the branch of R^{-n} which maps $\mathbf{D} \cap \mathbf{U}$ to $\mathbf{U}$. Since $f_n(\mathbf{D})$ omits a periodic orbit of period $p \geq 3$, they form a normal family on $\mathbf{D} \cap \mathbf{U}$, and therefore a partial subsequence converges uniformly to a function g which is holomorphic. This functions, g can not be constant on $\mathbf{D} \cap \mathbf{U}$, since $g^n(\mathbf{D} \cap \mathbf{U})$ does not shrink to a single point, since R is a rotation of $\mathbf{U}$, and we know that a holomorphic function is either constant or open. Therefore, $g(\mathbf{D})$ contains an open set $\mathbf{W}$. Therefore, for n large enough, $f_{nk}(D) \subset \mathbf{W}' \subset \mathbf{W}$, for some open set $\mathbf{W}'$. Hence, $R^{nk}(\mathbf{W}') \subset \mathbf{D}$!! Because $\mathbf{W}'$ contains points of $\partial\mathbf{U} \subset \mathbf{J}(f)$.□

We can use similar arguments to show this result for Cremer points. A refinement of the results and arguments above is the Fatou-Shishikura inequality.

Theorem 4.2.11 (Fatou - Shishikura inequality)**.** *Let R be a rational function of degree $d \geq 2$. We note*

1. *$Att(R)$ = number of attracting cycles of R.*
2. *$Par(R)$ = number of immediate parabolic basins of R.*
3. *$Irr(R)$ = number of irrationally indifferent cycles of R.*
4. *$HR(R)$ = number of cycles of Herman rings of R.*

Then, $Att(R) + Par(R) + Irr(R) + HR(R) \leq 2d - 2$, and $HR(R) \leq d - 2$. This two inequalities are optimal.

Once we have seen the relation between the critical points of a map and its Fatou Set, we focus on the structure of these Fatou Set components. Since the Julia Set is invariant, the image of any component of the Fatou Set under R is a component of the Fatou Set.

Proposition 4.2.12. *Let **U** be a fixed component of F. There are several possibilities for the orbit of **U** under R*

1. *If $R(\boldsymbol{U}) = \boldsymbol{U}$, we call **U** a fixed component of F.*
2. *If $R^n(\boldsymbol{U}) = \boldsymbol{U}$ for some $n \geq 1$, $n \in \mathbb{N}$, we call **U** a periodic component of F.*
3. *If $R^m(\boldsymbol{U})$ is periodic for some $m \geq 1$, we call **U** a preperiodic component of F.*
4. *Otherwise, all $\{R^n(\boldsymbol{U})\}$ are distinct, and we call **U** a wandering domain.*

I.N Baker[22] proved that some entire functions do have wandering domains, and our aim is to show that this is not possible for rational functions.

A very special type of components are those which are fully or completely invariant

Proposition 4.2.13. *If $\boldsymbol{D}$ is a union of components of $\boldsymbol{F}$ which is completely invariant, then $\boldsymbol{J} = \partial \boldsymbol{D}$.*

Theorem 4.2.14. *If $\boldsymbol{U}$ is a completely invariant component of F, then $\partial \boldsymbol{U} = \boldsymbol{J}$, and every other component of F is simply connected. There are, at most, two completely invariant components of $\boldsymbol{F}$.*

Proof. If $\mathbf{U}$ is completely invariant, then $\partial \mathbf{U} = \mathbf{J}$, by the previous proposition . Moreover the sequence $\{R^n\}$ omits the open set $\mathbf{U}$ on $\overline{\mathbb{C}} \setminus \overline{\mathbf{U}}$, so $\{R^n\}$ is normal on such domain, and $\overline{\mathbb{C}} \setminus \overline{\mathbf{U}} \cup \mathbf{F}$. Since $\mathbf{U}$ is connected, each component of $\overline{\mathbb{C}} \setminus \overline{\mathbf{U}}$ is simply connected. If $\mathbf{U}$ is furthermore simply connected, then, since R is a d-to-1 mapping, $\mathbf{U}$ must contain d-1 critical points. Since there are only 2d - 2 critical points altogether, there can be, at most, two simply connected completely invariant components. □

Theorem 4.2.15. *The number of components of the Fatou Set can be 0, 1, 2 or ∞, and all cases occur.*

Proof. Suppose $\mathbf{F}$ has only finitely many components and let U_0 be one of them. Consider a chain of inverse images $R(U_{-1}) = U_0$, $R(U_{-2}) = U_{-1}$, $\ldots$. Eventually, we reach U_n such that $R(U_{-n}) = U_{-k}$, $0 \leq k < n$. Then, $U_0 = R^n(U_{-n}) = R^n(U_{-k}) = R^{n-k}(U_0)$. Therefore each component of U_0 of $\mathbf{F}$ is periodic, and since there are only finitely many, there is some $N \in \mathbb{N}$ such that $R^N(U) = U$ for every component U. Hence, every component is completely invariant by R^N, and by the preceding theorem there are at most two components.

To see there may also be infinite components, we choose $P(z) = \lambda \cdot z + z^2$ to correspond to a Siegel Disk. Let U_0 and U_∞ be the components corresponding to 0 and ∞ respectively. Since R is conjugate to a rotation of U_0, there are no critical points on U_0 and R is one-to-one on U_0. Therefore, U_0 has a different from itself preimage, U_1, which is also distinct from U_∞, so $\mathbf{F}$ has infinitely many components. □

Theorem 4.2.16 (Sullivan)**.** *A rational map has no wandering domains*

Once we know that it can not be a wandering domain, we get that every component of the Fatou Set is either periodic or preperiodic.

Definition 4.2.17 (Parabolic component)**.** *A periodic component of period n, $\boldsymbol{U}$, of the Fatou Set is called parabolic if there is a neutral fixed point on the boundary of $\boldsymbol{U}$, ζ, for R^n and with multiplier 1 such that all points in $\boldsymbol{U}$ converge to ζ under iteration by R^n.*

[22]Irvine Noel Baker(1932 - 2001) was an Australian mathematician.

The next theorem is due to Pierre Fatou, and was an important breakthrough at the time.

Theorem 4.2.18 (Classification of Fatou components)**.** *Suppose* $\boldsymbol{U}$ *is a periodic component of the Fatou Set* $\boldsymbol{F}$*. Then, exactly one of the following holds:*

1. $\boldsymbol{U}$ *is contained in a basin of attraction.*

2. $\boldsymbol{U}$ *is parabolic.*

3. $\boldsymbol{U}$ *is a Siegel Disk.*

4. $\boldsymbol{U}$ *is a Herman Ring.*

We may assume **U** is fixed by R, and since **J** has more than two points, **U** is hyperbolic. To prove this theorem, we will be using a set of propositions. From now on, let $\rho = \rho_{\mathbf{U}}$ denote the hyperbolic metric on **U**.

Proposition 4.2.19. *Suppose* $\boldsymbol{U}$ *is hyperbolic,* $f : \boldsymbol{U} \mapsto \boldsymbol{U}$ *is analytic and f is not an isometry with respect to the hyperbolic metric. Then, either* $f^n(z) \mapsto \partial\boldsymbol{U}$ *for all* $z \in \boldsymbol{U}$*, or else there is an attracting fixed point for f in* $\boldsymbol{U}$ *to which all orbits converge.*

Proof. Since f is not an isometry, $\rho(f(z), f(w)) < \rho(z, w)$, for all $z, w \in \mathbf{U}$. In particular, for any compact set $\mathbf{K} \cup \mathbf{U}$, there is a constant $k \in \mathbb{R}$, $k(\mathbf{K}) < 1$, such that

$$\rho(f(z), f(w)) < k \cdot \rho(z, w),\ z, w \in \mathbf{K}.$$

Suppose there is a $z_0 \in \mathbf{U}$ whose iterates $z_n = f^n(z_0)$ visit some compact subset of **U**, **L**, infinitely often. Take **K** to be a compact neighborhood $\mathbf{L} \cup f(\mathbf{L})$. Then $\rho(z_{m+2}, z_{m+1}) \leq k \cdot \rho(z_{m+1}, z)$ whenever $z_m \in \mathbf{L}$, and this occurs infinitely often, so $\rho(z_{n+1}, z_n) \mapsto 0$. Therefore, by continuity, any cluster point $\zeta \in \mathbf{L}$ of the sequence $\{z_n\}$ is fixed by f, and is actually an attracting fixed point, since $\rho(f(z), \zeta) \leq k \cdot \rho(z, \zeta)$ in some neighborhood of ζ. Since the iterates of f form a normal family, they converge on **U** to ζ. □

Proposition 4.2.20 (Analytic maps in hyperbolic components)**.** *Suppose* $\boldsymbol{U}$ *is hyperbolic,* $f : \boldsymbol{U} \mapsto \boldsymbol{U}$ *is analytic, and f is an isometry with respect to the hyperbolic metric. Then, exactly one of the following holds:*

1. $f^n(z) \mapsto \partial\boldsymbol{U}$*, for all* $z \in \boldsymbol{U}$ *.*

2. $f^m(z) = z$ *for all* $z \in \boldsymbol{U}$ *and some fixed* $m \geq 1$*,* $m \in \mathbb{N}$*.*

3. $\boldsymbol{U}$ *is conformally a disk, and f is conjugate to an irrational rotation.*

4. ***U** is conformally an annulus and f is conjugate to an irrational rotation or to a reflection followed by an irrational rotation.*

5. ***U** is conformally a punctured disk, and f is conjugate to an irrational rotation.*

Proof. Since f is an isometry with respect to the hyperbolic metric, f is a conformal self-map to **U**.

Suppose first **U** is simply connected. Let ϕ map **U** conformally to the open unit disk Δ. Then, $S = \phi \cdot f \cdot \phi^{-1}$ is a conformal self-map of Δ, a Möbius transformation. If S has fixed points on the unit circle, then $|S^n| \mapsto 1$ and (i) holds. If S has a fixed point in the disk, we may assume it is the origin, so S is a rotation and either (ii) or (iii) holds.

Now assume **U** is not simply connected. Let $\psi : \Delta \mapsto \mathbf{U}$ be the universal covering map, and let G be the associated group of covering transformations, that is the group of conformal self-maps, g, of Δ satisfying $\psi \cdot g = \psi$. The lift of f to the unit disk via ψ is a Möbius transformation F, which satisfies $\psi \cdot F = f \cdot \psi$. Let Γ be the group obtained by adjoining F to G.

We first assume that Γ is discrete (orbits accumulate only on $\partial\Delta$). Since no iterate f^k of f is the identity on **U**, no iterate F^k of F belongs to G. Γ being discrete implies $gF^K(0) \mapsto \partial\Delta$ uniformly in $g \in G$, so $f^k(z_0) \mapsto \partial\mathbf{U}$ and (i) holds.

Suppose now Γ is not discrete. Let $\overline{\Gamma}$ be the closure of Γ in the Lie[23] group$^{(*)}$ of conformal self-maps of Δ, and let Γ_0 be the connected component of $\overline{\Gamma}$ containing the identity. If $g \in G$ then also $FgF^{-1} \in G$, since

$$\psi \cdot (F \cdot g \cdot F^{-1}) = f \cdot \psi \cdot g \cdot F^{-1} = f \cdot \psi \cdot F^{-1} = f \cdot f^{-1} \cdot \psi = \psi.$$

It follows that $\overline{\Gamma}$, and hence Γ_0, also conjugates G to itself. Since G is discrete and Γ_0 is connected, $hgh^{-1} = g$ for all $h \in \Gamma_0$ and $g \in G$, and every $g \in G$ commutes with every $h \in \Gamma_0$.

Choose $h \in \Gamma_0$ which is not the identity. Since G commutes with h, it belongs to the one-parameter group generated by h. Since G is discrete and infinite, we conclude that G has the form $\{g^n\}_{-\infty}^{\infty}$. This means that the fundamental group of **U** is isomorphic to $\mathbb{Z}$, and **U** is doubly connected. Since **U** is hyperbolic, **U** can not be a punctured plane, and **U** is either an annulus or a punctured disk. All in all one of the (ii), (iv) or (v) must hold. □

Proposition 4.2.21 (Boundary of hyperbolic components)**.** *Suppose **U** is hyperbolic, $f : \boldsymbol{U} \mapsto \boldsymbol{U}$ is analytic on **U** and across $\partial\boldsymbol{U}$, and $f^n(z_0) \mapsto \partial\boldsymbol{U}$ for some $z_0 \in \boldsymbol{U}$. Then, there*

[23]Marius Sophus Lie (1842 - 1899) was a Norwegian mathematician. He created the theory of continuous symmetry and applied it to the study of differential equations and geometry.

is a fixed point $\zeta \in \partial \boldsymbol{U}$ for f such that $f^n(z) \mapsto \zeta$ for all $z \in \boldsymbol{U}$. Either ζ is an attracting fixed point, or ζ is a parabolic fixed point with multiplier $f'(\zeta) = 1$.

Proof. By 2.0.18, the spherical distances between the iterates z_n and z_{n+1} of z_0 tend to 0. Therefore the limit set of $\{z_n\}$ is a connected subset of $\partial\mathbf{U}$, and furthermore, by continuity of f, any limit point is a fixed point for f. Since the fixed points of f are isolated, we conclude that $z_n \mapsto \zeta$ for some fixed point $\zeta \in \partial\mathbf{U}$ of R. The orbit of every other $z \in \mathbf{U}$ also converges to ζ, since it remains at a bounded hyperbolic distance from the orbit of z_0. The fixed point ζ is not repelling, since $z_n \mapsto \zeta$.

Suppose $f'(\zeta) = e^{2\pi i\theta}$, where θ is rational. Then $\mathbf{U}$ is contained in the basin of attraction associated with one of the petals of ζ. Since the local rotation f at ζ induces a cyclic permutation of the petals at ζ, and since f leaves $\mathbf{U}$ invariant, in fact f induces the identity permutation and $f'(\zeta) = 1$.

Suppose $f'(\zeta) = e^{2\pi i\theta}$, where θ is irrational. Assume $\zeta = 0$. Let $z_0 \in \mathbf{U}$, and let $\mathbf{V}$ be a relatively compact subdomain of $\mathbf{U}$ such that $\mathbf{V}$ is simply connected and z_0 and $z_1 = f(z_0)$ belongs to $\mathbf{V}$. Since f is univalent near 0, and $f^n \mapsto 0$ uniformly on $\mathbf{V}$, we can assume each f^n is univalent on $\mathbf{V}$. Then

$$\phi_n(z) = \tfrac{f^n(z)}{f^n(z_0)},\ z \in \mathbf{V}.$$

is also univalent on $\mathbf{V}$, $\phi_n(z_0) = 1$, and $0 \notin \phi_n(\mathbf{V})$. Let ψ be the Riemann map from Δ to $\mathbf{V}$, $\psi(0) = z_0$. Then $h_n(\zeta) = \phi_n(\psi(\zeta)) - 1$ univalent on Δ, $h_n(0) = 0$, $h_n'(0) = \phi_n'(z_0) \cdot \psi'(0)$, and h_n omits -1. Therefore, the function $\frac{h_n}{h_n'(0)}$ belongs to S and omits $\frac{-1}{h_n'(0)}$. The Koebe one quarter theorem implies $|h_n'(0)| \leq 4$. Since S is a normal family, the sequence $\{\frac{h_n}{h_n'(0)}\}$ is normal on Δ, as is $\{h_n\}$. Consequently, $\{\phi_n\}$ is normal on $\mathbf{V}$.

We claim that all limit functions of $\{\phi_n\}$ are non-constant. For if $|\phi_n - 1| < \delta$, then $f^n(\mathbf{V})$ would be included in a narrow angle with vertex 0, and if this angle would be smaller than $\frac{\theta}{3}$ then since $f(z) \cong e^{i\theta} \cdot z$ near 0, $f^{n+1}(V)$ would be disjoint from $f^n(\mathbf{V})$, contrary to hypothesis. Thus $\phi_n'(z_0)$ is bounded away from 0. Applying again Koebe's one quarter theorem we deduce that there is $\rho > 0$ such that $\phi_n(\mathbf{V})$ contains a disk centered at 1 of radius ρ. Therefore, $f^n(\mathbf{V})$ contains a disk centered at z_n of radius $\rho|z_n|$.

Choose N so that the disks of radius $\frac{\rho}{2}$ centered at $e^{2\pi i m\theta}$, $0 \leq m \leq N$, $m \in \mathbb{N}$, cover an annulus containing the unit circle. Since $z_{m+1} = e^{2\pi i\theta} \cdot z_m + o(|z_m|)$, the disks centered at z_m of radius $\rho|z_m|$, $n \leq m \leq n+N$, cover an annulus containing z_n and z_{n+1} for n sufficiently large. Therefore, $\cup f^n(\mathbf{V})$ contains a punctured neighborhood of 0, and 0 is an isolated point of $\partial\mathbf{U}$. But then theorem 3.2.11 implies that f is conjugate to a rotation about 0, contradicting $f^n(z) \mapsto 0$. We conclude that $f'(\zeta)$ can not be irrational.

To end the proof, we observe that since R^n has degree > 1, no power of R can coincide with the identity and case (ii) of 4.2.20 is ruled out. Since J has no isolated points, case (v) of 4.2.20 is also impossible for R. Finally, since there are no attracting periodic points in J, 4.2.21 produces a parabolic fixed point in ∂U whenever $f^n(z) \mapsto \partial U$. □

4.3 POLYNOMIALS

For $d \geq 2$, $g \in \mathbb{N}$, let $f \in pol_d$, that is

$$f(z) = a_d \cdot z^d + ... + a_1 \cdot z + a_0$$

with $a_d \neq 0$. Note that $f(\infty) = f^{-1}(\infty) = \infty$. Therefore, f has $d-1$ critical points when counted with multiplicity and one critical point of multiplicity $d-1$ at infinity. This point at infinity is superattracting, and its basin of attraction

$$A_f(\infty) = \{z \in \mathbb{C} | f^n(z) \mapsto \infty, \text{ when } n \mapsto \infty\}$$

is always connected, since f has no poles.

Definition 4.3.1. *The complement*

$$\boldsymbol{K}_f = \overline{\mathbb{C}} \setminus A_f(\infty)$$

is called the filled Julia Set of f, and is compact and completely invariant. The common boundary

$$\boldsymbol{J}_f = \partial A_f(\infty) = \partial \boldsymbol{K}_f.$$

is called the Julia Set, $\boldsymbol{J}_f$. Finally, the Fatou Set, $\boldsymbol{F}_f$, consists of the connected component $A_f(\infty)$ and all connected components of the interior of $\boldsymbol{K}_f$, if there is any.

Definition 4.3.2 (Green's function)**.** *Set $A_f^*(\infty) = A_f(\infty) \setminus \{\infty\}$ and $\boldsymbol{U}^* = \boldsymbol{U} \setminus \{\infty\}$. The Green's function for $\boldsymbol{K}_f$ is the real continuous harmonic function $g : A_f^*(\infty) \mapsto \mathbb{R}_+$ which extends $log|\phi| : \boldsymbol{U}^* \mapsto (log(r), \infty)$ as follows:*

$$g(z) = \begin{cases} log|\phi(z)| & if \quad z \in \boldsymbol{U}^* \\ \frac{1}{d^k} g(f^k(z)) & if \quad f^k(z) \in \boldsymbol{U}^* \end{cases}$$

Definition 4.3.3 (Equipotential)**.** *Since $g(f(z)) = d \cdot g(z)$ for $z \in \boldsymbol{U}^*$, the function g is well defined in all of $A_f^*(\infty)$. It may also be extended to the whole plane by defining $g \equiv 0$ on $\boldsymbol{K}_f$. For $\rho > 0$, the level set*

$$g_\rho = g^{-1}(\rho) = \{z \in A_f^*(\infty) | g(z) = \rho\}$$

is called the equipotential of potential ρ. If $e^\rho > r$, it is a simple closed curve which surrounds the Julia Set.

Let ϕ conjugate $P(z)$ to ζ^d near ∞, with $P(z) = z + o(1)$ at ∞. We have that $log|\phi(z)|$ coincides with Green's function, $G(z)$, for $A(\infty)$ with pole at ∞. The equation for $\phi(z)$ gives an equation for Green's function:

$$G(P(z)) = d \cdot G(z),\ z \in A(\infty)$$

Therefore, P maps level curves of G to level curves, increasing d-fold the value of Green's function so that this function provides a precise measure of the escape to ∞. The exterior $\{G > r\}$ of the level curve is invariant under P, and P maps it d-to-one onto $\{G > r \cdot d\}$. For r large enough, $\phi(z)$ is defined on $\{G > r\}$, and maps it conformally onto $\{|\zeta| > e^r\}$. The equation $\phi(z) = (\phi(P(z)))^{\frac{1}{d}}$ allows us to extend $\phi(z)$ to $\{G > \frac{r}{d}\}$, provided no critical point of P belongs to this domain.

Now, there are two cases to consider:

1. If there is no critical point of P in $A(\infty)$, we can continue ϕ indefinitely to all of $A(\infty)$, and ϕ maps $A(\infty)$ conformally onto the complement $\{|\zeta| > 1\}$ of the closed unit disk in the ζ-plane. In particular, $A(\infty)$ is simply connected, and the Julia set $\mathbf{J} = \partial A(\infty)$ is connected.

2. Otherwise, we extend ϕ until we reach a level line $\{G = r\}$ of Green's function that contains a critical point of P. This situation is then as follows: the domain $\{G > r\}$ is simply connected, and mapped by ϕ conformally onto $\{|\zeta| > e^r\}$. The domain forms several cusps at the critical point, and $\phi(z)$ approaches different values, as z approaches the critical point through different cusps. The level line $\{G = r\}$ consists of at least two simple closed curves which meet at the critical point. Each of these curves are of $\mathbf{J}$, or else G would be harmonic and positive, and therefore constant within the curve. Therefore, $\mathbf{J}$ is disconnected. In fact, in this case $\mathbf{J}$ has uncountably many connected components. This can be seen by noting that the critical points of G are the critical points of P and all their inverse iterates, and by following the splitting of level curves at each such critical point.

What we have stated here, can be summarized in the following theorem.

Theorem 4.3.4 (Connectivity of polynomial Julia Sets). *Let $f \in Pol_d$, then:*

1. *K_f is connected iff $Crit(f) \subset K_f$. In this case, the restriction of f to $\overline{\mathbb{C}} \setminus K_f$ is conformally conjugate to $z \mapsto z^d$ on $\overline{\mathbb{C}} \setminus \overline{D}$.*
2. *K_f is totally disconnected if $Crit(f) \subset A_f(\infty)$. In this case, J_f is a Cantor Set and $J_f = K_f$.*
3. *if, at least, one critical point of f belongs to A_f, then both K_f and J_f are disconnected and have uncountably many connected components.*

As there is a superattracting fixed point at infinity, it follows from the Böttcher coordinates theorem that there exists a neighborhood, $\mathbf{U}$, of ∞ and $\phi : \mathbf{U} \mapsto \overline{\mathbb{C}}$ a local conformal conjugation which conjugates f to z^d in $\mathbf{U}$. If the polynomial f is monic, we call the conjugation determined by the condition $\frac{f(z)}{z} \mapsto 1$ when $z \mapsto \infty$ *Böttcher coordinate* of f around ∞.

Assume f is monic, and let $\phi : \mathbf{U} \mapsto \overline{\mathbb{C}} \setminus \overline{D}$ be the Böttcher coordinate which conjugates f to $z \mapsto z^d$ in $\mathbf{U}$, and which is mapped outside of D_r for the maximum $r \geq 1$. If $r = 1$, by the previous theorem K_f is connected, and if K_f is disconnected, then $r > 1$ and $\partial\mathbf{U}$ contains a critical point.

Definition 4.3.5 (External rays). *Suppose $\boldsymbol{K}_f$ is connected. Then, for $t \in \mathbb{T} = \mathbb{R} \setminus \mathbb{Z}$, the curve*

$$R_f(t) = \{z \in \mathbb{C} \setminus \boldsymbol{K}_f | arg(\phi(z)) = 2 \cdot \pi \cdot t\}$$

is called the external ray of argument t. Note that $R_f(t)$ is mapped bijectively under f to $R_f(d \cdot t)$ with arguments in $\mathbb{T}$. In particular, if $d^p \cdot t \equiv t (mod\ 1)$, then the ray $R_f(t)$ is p-periodic.

Definition 4.3.6 (Landing point). *If the limit*

$$\gamma(t) = lim \phi^{-1}(r \cdot e^{2 \cdot \pi \cdot i \cdot t}),\ r \mapsto 1^+$$

exists, we say that $R_f(t)$ lands at the point $\gamma(t)$, which necessarily belongs to the Julia Set. All rays of rational argument land at a repelling or parabolic periodic point. If $R_f(t)$ lands at $\gamma(t)$, then $R_f(d \cdot t)$ lands at $f(\gamma(t))$. Therefore, the landing points of a k-cycle of rays, must form a periodic orbit of a period dividing k.

Conversely, every repelling or parabolic point z_0 is the landing point of, at least, one periodic ray. If k is the period of z_0, only finitely many rays, say q', land at z_0 and these rays are all *periodic and of the same period. They are transitively permuted by f^k, and this permutation must preserve their circular order, since f is a local homeomorphism at z_0; therefore, the permutation, must send each ray to the one which is p' further counterclockwise for some $p' < q'$. We call $\frac{p}{q}$ the combinatorial rotation number of f at z_0, with $\frac{p}{q} = \frac{p'}{q'}$ with the lowest common terms.*

There are polynomials for which not all rays land. However, Fatou showed that the set of arguments $t \in \mathbb{T}$ for with $R_f(t)$ does not land measures zero.

Theorem 4.3.7 (Continuous landing of rays)**.** *For any polynomial f with connected Julia Set, the following conditions are equivalent:*

1. *every external ray R_t lands at a point $\gamma(t)$ which depends continuously on the argument t.*
2. *the Julia Set is locally connected.*
3. *the Filled Julia Set is locally connected.*
4. *the inverse of the Böttcher coordinate $\phi^{-1} : \mathbb{C} \setminus \overline{\boldsymbol{D}} \mapsto \mathbb{C} \setminus \boldsymbol{K}_f$ extends continuously to $\partial \boldsymbol{D}$ and therefore induces a continuous parametrization $\gamma : \mathbb{T} \mapsto \boldsymbol{J}_f$ of the Julia Set, given as $\gamma(t) = \phi^{-1}(e^{2 \cdot \pi \cdot i \cdot t})$.*

5 QUADRATIC POLYNOMIALS AND THE MANDELBROT SET

In this section, we consider holomorphic families of rational maps

$$\{f_\lambda : \overline{\mathbb{C}} \mapsto \overline{\mathbb{C}},\ \lambda \in \Lambda\} \subset Rat$$

where the parameter space, Λ, is a complex manifold of dimension $n \geq 1$, $n \in \mathbb{N}$ so that the map $r : \Lambda \times \overline{\mathbb{C}} \mapsto \overline{\mathbb{C}}$, $r(\lambda, z) = f_\lambda(z)$ is holomorphic, that is, the coefficients of $f_\lambda(z)$ depend holomorphically on λ.

Given a quadratic polynomial, $P(z) = a \cdot z^2 + b \cdot z + d$, we can conjugate it by $z' = a \cdot z$ to $Q(z) = z^2 + \alpha \cdot z + \beta$. Furthermore, if we make this polynomial centered (the sum of its critical values is 0), we obtain a polynomial of the form $M_c = z^2 + c$, for some $c \in \mathbb{C}$, which is conjugated to the original polynomial $P(z)$. Apart from its simplicity, the study of the quadratic family is very interesting, as universal concepts arise when doing so.

Proposition 5.0.1. *Given $Q_c(z) = z^2 + c$, and $R = max\{2, |c|\}$, if a point z_0 satisfies $|z| \geq R$, $|z| > 2$, then*

$$lim\ Q_c^n(z) = \infty$$

Proof. If $|z| > R$,then $|z| > 2$ and $|z| > |c|$, and therefore

$$\tfrac{|z^2+c|}{|z|} \geq |z| - \tfrac{|c|}{|z|} \geq |z| - 1 = 1 + \epsilon,\ \epsilon > 0,\ \epsilon \in \mathbb{R}.$$

In the first inequality, we have used that $|a + b| \geq ||a| - |b||$. All in all, $|Q_c(z)| > |z|$ and the succession $|Q_c^n(z)|$ tends to infinity. In particular, this proposition shows that the basin of attraction of the infinity point always exists. □

Corollary 5.0.2. *(Localization criterium): The filled Julia Set, K_c, of the polinomial Q_c is contained in the disk of center 0 and radius R, where $R = max\{2, |c|\}$.*

Theorem 5.0.3. *If $P_c^n(0) \mapsto \infty$, then the Julia Set, J_c is totally disconnected. Otherwise, if $P_c^n(0)$ is bounded, the Julia Set is connected.*

Definition 5.0.4. *The set of parameters values c such that $P_c^n(0)$ is bounded is called Mandelbrot[24] Set, and denoted by M. Therefore, $c \in M$ iff 0 does not belong to the basin of attraction of the superattracting point ∞.*

[24]Benoît Mandelbrot(1924 - 2010) was a Polish mathematician, mostly known by his work with fractals. He used a great technological breakthrough on the 1970 's, the computer, to visualize the fractals already known at that time and to show his results.

Theorem 5.0.5 (Structure of Mandelbrot Set)**.** *The Mandelbrot Set is a closed, simply connected subset of of the disc* $\{|c| \leq 2\}$*, intersecting the real axis in the interval* $[-2, \frac{1}{4}]$*. Furthermore, M consists of the parameters c such that* $|P_c^n(0)| \leq 2$*,* $\forall n \geq 1$*.*

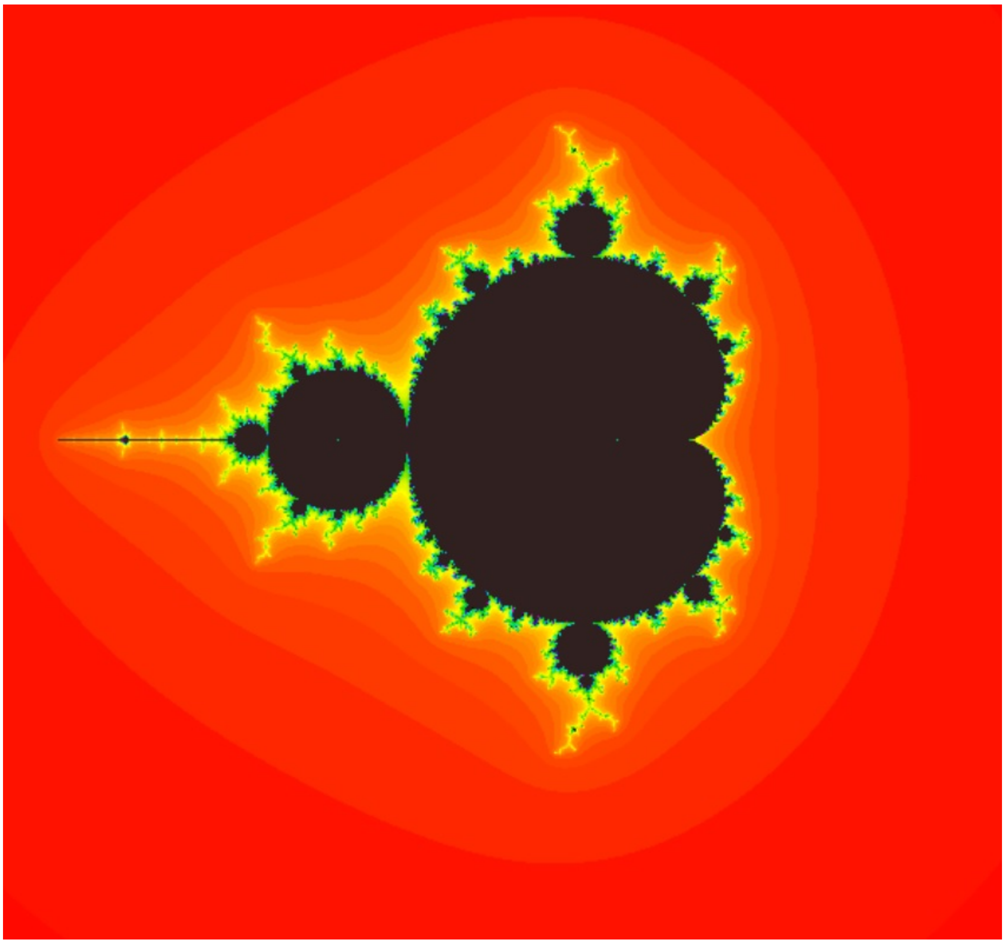

Figure 4: Mandelbrot Set

Proof. Suppose $|c| > 2$, and we will see by induction that $|P_c^n(0)| \geq |c| \cdot (|c|-1)^{2^{n-1}}$, and in this case $P_c^n(0) \mapsto \infty$, and therefore $c \notin M$.

1. $|P_c^1(0)| = |c|$, and $|c| \geq |c| \cdot 1 = |c|$, $n = 1$.

2. We suppose the statement is true up to a certain $n \in \mathbb{N}$, and we check if it is also true for the next natural number. $|P_c^{n+1}(0)| = |(P_c^n(0))^2 + c| \geq (|c| \cdot (|c|-1)^{2^{n-1}})^2 - |c| = (|c|^2 \cdot (|c|-1)^{2^n}) - |c|$, and $(|c|^2 \cdot (|c|-1)^{2^n}) - |c| \geq (|c| \cdot (|c|-1)^{2^n})$, because $|c| > 2$, and therefore $(|c|-1) > 1$.

To prove the second part of the theorem, suppose $|P_c^m(0)| = 2 + \delta$, for some $\delta > 0$, $m \geq 1$. We consider two different cases:

1. if $|c| = |P_c(0)| > 2$, then $c \notin M$
2. if $|c| \leq 2$, then $|P_c^{m+1}(0)| \geq (2+\delta)^2 - 2 \geq 2 + 4 \cdot \delta$, and by induction we get $|P_c^{m+k}(0)| \geq 2 + 4^k \cdot \delta \mapsto \infty$, and again $c \notin M$

From these second part of the theorem, it follows that M is closed.

As for the final statement, if c is real, $P_c(0) - x$ has no real roots if $c > \frac{1}{4}$, one root at $x = \frac{1}{2}$ if $c = \frac{1}{4}$ and two real roots for each $c < \frac{1}{4}$. Therefore, if $c > \frac{1}{4}$, $P_c^n(0)$ is increasing, and tends to infinity, since any finite limit should satisfy $P_c^n(0) = x$, for some $n \geq 1$, so $c \notin M$. Now, if f $c \leq \frac{1}{4}$, we consider $a = \frac{1+\sqrt{1-4 \cdot c}}{2}$ the larger root of $P_c(0) - x$, and if c also satisfies $c \geq -2$, then $a \geq |c| = P_c(0)$. Then, $P_c^n(0) \leq a$ implies $|P_c^{n+1}(0)| = |P_c^n(0)^2 + c| \leq a^2 + c = a$, and the sequence is bounded. All in all, $M \cap \mathbb{R} = [-2, \frac{1}{4}]$. □

Theorem 5.0.6. *For each λ, $|\lambda| < 1$, there is a unique $c = c(\lambda)$ such that P_c has a fixed point with multiplier λ. The values, c, for which P_c has an attracting fixed point form a cardioid $C \subset \boldsymbol{M}$, and $\partial C \subset \partial \boldsymbol{M}$. If $c \in C$, then J_c is a quasicircle$^{(*)}$.*

Proof. The fixed points of P_c are $z_c = \frac{1 \pm \sqrt{1-4c}}{2}$, with multiplier $\lambda(c) = 2 \cdot z_c$. Since $c = \frac{\lambda}{2} - \frac{\lambda^2}{4}$, the condition $|\lambda(c)| < 1$ corresponds in the c-plane to the cardioid:

$$C = \{\frac{\lambda}{2} - \frac{\lambda^2}{4} : \lambda < 1\}$$

It is a subset of $\mathbf{M}$ called the main cardioid of $\mathbf{M}$. Since the function $\frac{\lambda}{2} - \frac{\lambda^2}{4}$ is one-to-one in the closed unit disk, we get the uniqueness.

Let $\mathbf{W}$ be the component of the interior of $\mathbf{M}$ containing $\mathbf{C}$. By theorem 5.0.5 the polynomials $f_n(c) = P_c^n(0)$ are uniformly bounded in $\mathbf{W}$, and they converge to the attracting fixed point z_c on C. However, if $c \notin \overline{C}$, then z_c is a repelling fixed point, and we can not have $P_c^n(0) \mapsto z_c$ unless $P_c^n(0) = z_c$ for some n large enough. As this can occur only in a countable set, we conclude that $\mathbf{W} = \mathbf{C}$ □

Theorem 5.0.7. *Suppose there is an attracting cycle of length m for P_a. Then $\boldsymbol{a}$ belongs to the interior of $\boldsymbol{M}$. If $\boldsymbol{W}$ is the component of the interior of $\boldsymbol{M}$ containing $\boldsymbol{a}$, then P_c has an attracting cycle $\{z_1(c), ..., z_n(c)\}$ of length $\boldsymbol{m}$ for all $c \in \boldsymbol{W}$, where each $z_j(c)$, $j \in \{1, ..., m\}$ depends analytically on $\boldsymbol{c}$.*

Proof. Let $z_1(a)$ be an attracting periodic point of period $\mathbf{m}$ for $\mathbf{a}$. Applying the implicit function theorem to $Q(z, c) = P_c^m(z) - z$, we obtain an attracting periodic point $z_1(c)$ for P_c of period $\mathbf{m}$, which depends analytically on $\mathbf{c}$ in a neighborhood of $\mathbf{a}$. In particular, $\mathbf{a}$ belongs to the interior of $\mathbf{M}$, and we name $\mathbf{W}$ the component of $\mathbf{M}$ containing $\mathbf{a}$. The sequence $f_j(c) = P_c^{jm}(0)$ is bounded, and therefore normal, in **W**, and it converges at **a** to some point in the cycle $z_1(a)$, say to $z_1(a)$ itself.

Since $z_1(c)$ is attracting, $f_j(c)$ converges to $z_1(c)$ for **c** near **a**. By normality, $f_j(c)$ converges on **W** to some analytical function, $g(c)$ which satisfies

$$Q(g(c), c) = 0 \text{ near } \mathbf{a}(\text{and therefore in } \mathbf{W})$$

We know that $g(c)$ can be a repelling periodic point for a fixed $c \in \mathbf{W}$ only if $P_c^{jm}(0)$ actually coincides with $g(c)$ for j sufficiently large. This occurs a set which is at most countable. Since the multiplier, $\lambda(c)$, of the cycle is analytic, we conclude that $|\lambda| < 1$ on **W**, that is, the cycle of $g(c)$ is attracting for all $c \in \mathbf{W}$. □

The components of the interior of **M** associated with attracting cycles are called *hyperbolic components* of the interior of **M**. This name comes from the fact that P_c is hyperbolic when $c \notin \mathbf{M}$ or P_c has an attracting cycle. It is not known whether the hyperbolic components completely fill out the interior of **M**, but it do is known that they are dense in the interior of **M**.

Definition 5.0.8. *A point $c \in \boldsymbol{M}$ is called a Misiurewicz*[25] *point if 0 is strictly preperiodic, that is, $P_c^n(0) = P_c^k(0)$ for some $n > k > 0$, but $P_c^n(0) \neq 0$ for all $n \geq 1$*

A polynomial Q_c can have, at most, one attracting cycle in $\mathbb{C}$. It follows that Q_c is hyperbolic iff Q_c has an attracting cycle in $\mathbb{C}$, or if the orbit of the critical point is attracted to the superattracting fixed point at ∞. In this case, the c-value is called a hyperbolic parameter.

If Q_{c_0} has an attracting p-cycle in $\mathbb{C}$, it follows from the Implicit Function Theorem that this p-cycle moves holomorphically for parameter values close to c_0 and the cycle is still attracting. As a consequence, $c_0 \in int(\mathbf{M})$. The maximal open set containing c_0, Ω, of parameter for which a p-cycle exists and remains attracting, is called a hyperbolic component of **M**.

It is not known yet if all the components of int(**M**) are hyperbolic. However, there is a conjecture, which has been proved true for parameters in the real line:

Conjecture 5.0.9 (Hyperbolic Conjecture)**.** *The interior of **M** is equal to the union of all the hyperbolic components:*

$$int(\boldsymbol{M}) = \bigcup \Omega$$

for every Ω which is a hyperbolic component.

[25]Michael Misiurewicz(1948 -) is a Polish mathematician who is known for his contributions to dynamical systems and fractal geometry.

Definition 5.0.10 (Multiplier map). *Given a hyperbolic component, Ω, of period p, the multiplier map is defined as*

$$\Lambda : \Omega \mapsto \mathbb{D}$$

satysfing $\Lambda(c) = (Q_c^p)'(z(c))$, where $z(c)$, for each $c \in \Omega$, denotes a point in the attracting p-cycle so that $c \mapsto z(c)$ is holomorphic. It follows then that Λ is holomorphic.

Theorem 5.0.11 (Multiplier map). *Let Ω be a hyperbolic component of $\boldsymbol{M}$ of period p. Then, the multiplier map is a conformal isomorphism which extends continuously to $\Lambda : \overline{\Omega} \mapsto \overline{\mathbb{D}}$.*

It follows from this theorem that all hyperbolic components are topological discs, each with a center, that is, the unique parameter value for which the attracting cycle is superattracting, i.e. $\Lambda^{-1}(0)$.

Definition 5.0.12. *The boundary of Ω has a unique point, called the root of Ω, for which the multiplier of the p-cycle is exactly 1.*

Attached to every boundary parameter for which the multiplier of the cycle has derivative $e^{2\cdot\pi\cdot i\cdot\frac{r}{s}}$ we find a new hyperbolic component of period $p \cdot s$. This explains part of the fractal structure which we encounter near any point of the boundary of **M**.

Theorem 5.0.13 (Parameters in $\partial\mathbf{M}$). *For the boundary of the Mandelbrot Set, we have the following results:*

1. *Misiurewicz points are dense in $\partial\boldsymbol{M}$.*
2. *For each $c \in \partial\boldsymbol{M}$ there exists a sequence $\{c_n\}_n$ of centers of hyperbolic components such that $c_n \mapsto c$ as $n \mapsto \infty$.*
3. *If Q_c has a parabolic, Siegel or Cremer cycle, then $c \in \partial\boldsymbol{M}$.*

Of special interest are the parameter values of c for which P_c has a superattracting cycle. The critical point 0 must be in this set, so these are precisely the **c**'s for which 0 is periodic, that is, the solutions of $P_c^n(0) = 0$.

Theorem 5.0.14. *The values of $\mathbf{c} \in \boldsymbol{M}$ corresponding to superattracting cycles cluster on the entire boundary $\partial\boldsymbol{M}$. In particular, the interior of $\boldsymbol{M}$ is dense in $\boldsymbol{M}$.*

Proof. Let **U** be a disk that meets $\partial\mathbf{M}$ and such that $0 \notin \mathbf{U}$. Suppose **U** contains no value **c** for which 0 is periodic and consider the branch of $\sqrt{-c}$ defined on **U**. We have $P_c^n(0) \neq \sqrt{-c}$, or else $P_c^n(0) = 0$ and 0 is periodic. Therefore, $f_n(c) = \frac{P_c^n(0)}{\sqrt{-c}}$ omits the values 0, 1 and ∞ on **U**, hence is a normal sequence on **U**. But, since **U** meets $\partial\mathbf{M}$, it contains both values **c** with $f_n(c)$ bounded and with $f_n(c) \mapsto \infty$, so the sequence can not be normal.□

The most important conjecture about the Mandelbrot Set is about a topological property:

Conjecture 5.0.15 (MLC)**.** *The Mandelbrot Set,* $\boldsymbol{M}$*, is locally connected.*

This conjecture would imply the HC conjecture as well, and is the focus of many studies currently.

6 SINGULAR PERTURBATIONS OF CUBIC MILNOR POLYNOMIALS

In the previous section, we have studied the parameter space of the map

$$f(z) = z^2 + c,\ c \in \mathbb{C}$$

And now we want to take a look at the changes that this, and other families of polynomial maps, experience when we add a pole singularity.

6.1 SINGULAR PERTURBATIONS

Robert Devaney, Daniel Look and David Uminsky studied the behavior of the Julia Set of functions in the family of rational maps given by

$$F_\lambda(z) = z^n + \frac{\lambda}{z^d}$$

where $n, d \in \mathbb{Z}$ and $n \geq 2$, $d \geq 1$. For $\lambda = 0$, we get the map $z \mapsto z^n$, which has a known and well understood behavior, and for which the Julia set is also clear.

However, when $\lambda \neq 0$, we produce several modifications to the initial map:

1. The map F_λ now has degree $n + d$.
2. The origin is no longer a fixed point, but a pole instead.
3. There are $n + d$ new critical points, in addition to the original critical points at 0 and ∞. The orbits of the critical points at ∞ and the origin are fixed, and eventually fixed respectively, so their behavior is completely determined. As for the orbits of all the other critical points, they behave symmetrically with respect to rotation through angle $\frac{2 \cdot \pi}{n+d}$, so essentially we have only one additional "free" critical orbit for each of these maps.

For the family F_λ, the point at ∞ is still a superattracting fixed point, and therefore it still has an immediate basin of attraction, which we denote by B. However, this basin of attraction may consist of infinitely many disjoint preimages of the immediate basin of ∞. In particular, the component of the basin that contains 0 may be disjoint from B, and in this case we denote this component by T. The following result was established:

Theorem 6.1.1 (The escape Trichotomy). *Suppose the orbits of the free critical points of F_λ tend to ∞. Then*

1. *If one of the critical values lies in B, then $\boldsymbol{J}(F_\lambda)$ is a Cantor Set and $F_{\lambda_{|J(F_\lambda)}}$ is a one-sided shift on $n+d$ symbols. Otherwise, $\boldsymbol{J}(F_\lambda)$ is connected and the preimage T is disjoint from B.*

2. *If one of the critical points lies in T, then $\boldsymbol{J}(F_\lambda)$ is a Cantor set of simple closed curves(quasicircles).*

3. *If one of the critical values lies in a preimage of T, then $\boldsymbol{J}(F_\lambda)$ is a Sierpinski curve$^{(*)}$.*

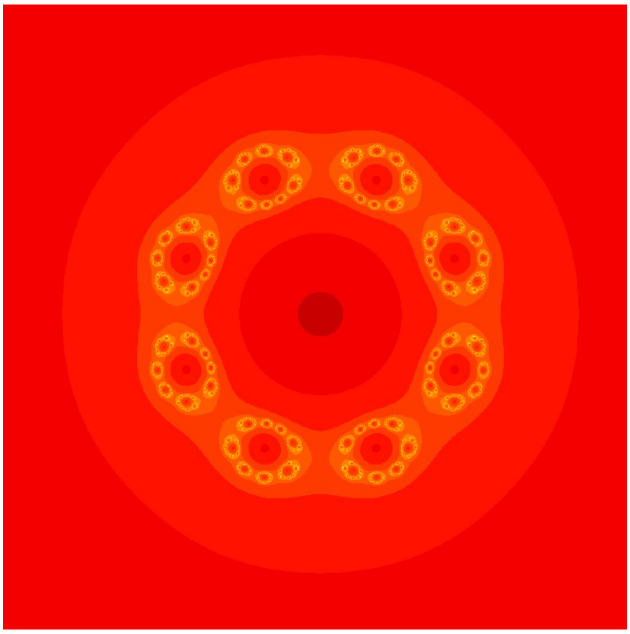

Figure 5: The Julia Set for $z^4 + \frac{0.3}{z^4}$, which is a Cantor Set.

The proofs to all the results found by Devaney, Look and Uminsky can be found in [8].

6.2 MILNOR POLYNOMIALS

Any cubic polynomial is of the form $P(z) = A\cdot z^3 + B\cdot z^2 + C\cdot z + D$, for some $A, B, C, D \in \mathbb{R}$. However, we can conjugate $P(z)$ to $S(z) = z^3 + \alpha\cdot z^2 + \beta\cdot z + \gamma$ by a linear map, making it monic. Furthermore, by an affine conjugacy, we can make $S(z)$ centered (the sum of its critical points is 0), therefore obtaining the condition $\alpha = 0$, so that we get the polynomial $R(z) = z^3 + \beta\cdot z + \gamma$. Its critical points are $x_\pm = \pm\frac{\sqrt{-12\beta}}{6}$, and if we write $\beta = -3\cdot a^2$ for some $a \in \mathbb{R}$, the critical points become $x_\pm = \pm a$. All in all, we have obtained $Q(z) = z^3 - 3\cdot a^2\cdot z + \gamma$, conjugated to the original polynomial $P(z)$. Their natural parameter space is $\mathbb{C}^2$, and one way to try to understand it is by taking dynamically meaningful one-dimensional slices. In

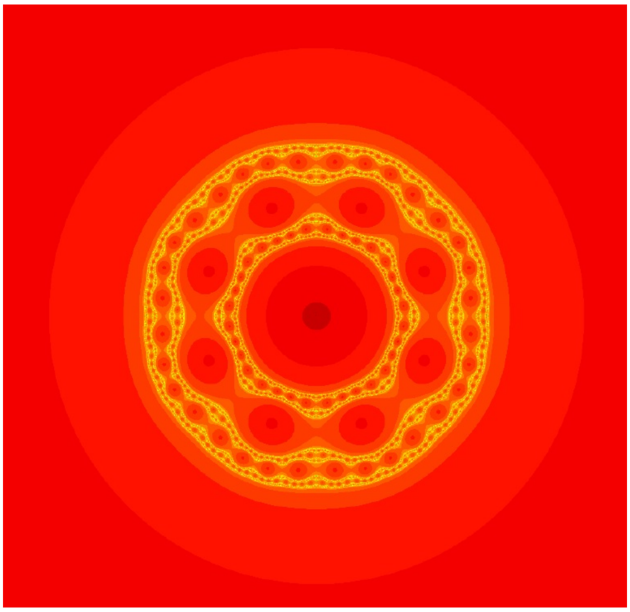

Figure 6: The Julia Set for $z^4 + \frac{0.04}{z^4}$, which is a Cantor Set of circles.

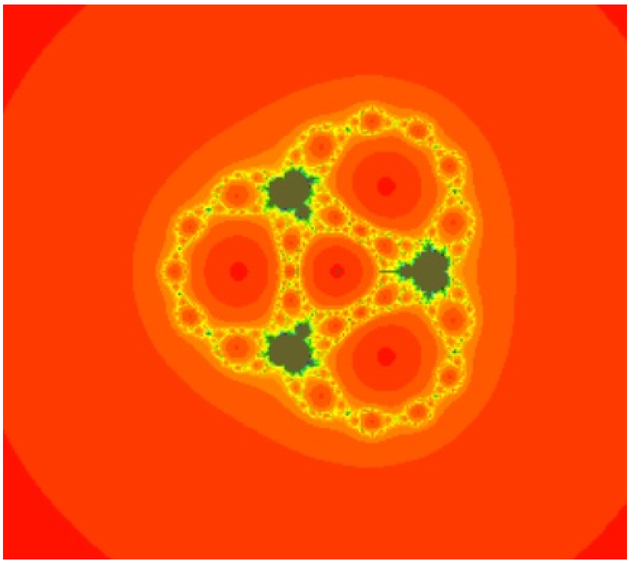

Figure 7: The parameter space for $z^4 + \frac{\lambda}{z^4}$.

[14], Milnor considers the slice formed by those polynomials for which a is a fixed point, which makes $b = 2 \cdot a^3 + a$, therefore resulting in polynomials of the from $P_a(z) = z^3 - 3 \cdot a^2 \cdot z + 2 \cdot a^3 + a$.

If we move the superattracting fixed point to 0 with the transformation $z \mapsto z - a$, and use the conjugation $z \mapsto \frac{-1}{3a} \cdot z$, we get $P_a(z) = 9a^2 \cdot z^2 \cdot (z-1)$. Finally, we can write $b = 9a^2$, so that the resulting polynomial has the form $P_b(z) = b \cdot z^2 \cdot (z-1)$. The new critical points are now located at $z = 0$ and $z = \frac{2}{3}$. We can observe that $P_b'(0) = 0$, and therefore 0 is actually a superattracting fixed point. We will denote by $A(0)$ the basin of attraction of this superattracting fixed point.

Similarly to the result shown at 4.3.4 we had for quadratic polynomials, we know that if $P_b^n(\frac{2}{3}) \mapsto \infty$ when $n \mapsto \infty$, then its Julia set is disconnected. On the other hand, if $P_b^n(\frac{2}{3})$ is bounded when $n \mapsto \infty$, then its Julia set is connected. We can summarize this result in the following two possibilities:

1. $\frac{2}{3} \in K(P)$.

2. $P_b^n(\frac{2}{3}) \mapsto \infty$ and $K(P)$ is disconnected, but not a Cantor Set.

In this same line of thinking, we can have three different situations when iterating the point $z = \frac{2}{3}$:

1. $\frac{2}{3} \in A(0)$. We say that the point $z = 0$ captures the orbit of $\frac{2}{3}$.

2. $P_b^n(\frac{2}{3}) \mapsto \infty$.

3. $\{P_b^n(\frac{2}{3})\}$ is bounded, but is not captured by the point $z = 0$.

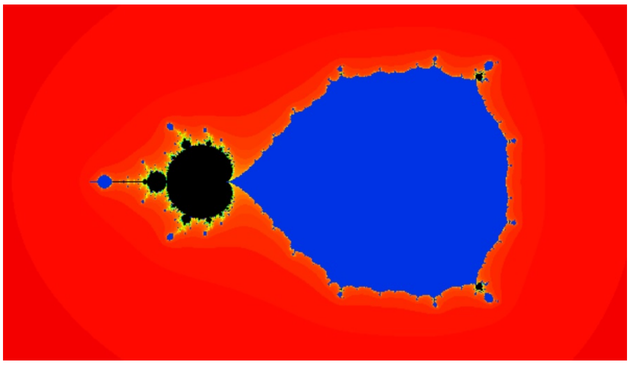

Figure 8: Parameter space for $b \cdot z^2 \cdot (z - 1)$.

In Figure 8, when drawing the parameter space for $P_b(z) = b \cdot z^2 \cdot (z-1)$, we paint in blue the values of a for which $P_b^n(\frac{2}{3})$ is captured by the superattracting fixed point $z = 0$, in black the values of a for which $P_b^n(\frac{2}{3})$ does not escape to infinity, but is not captured by $z = 0$ either, and in red(varying its tone depending on the number of iterations needed for the orbit to escape to infinity) the rest of the values of b.

6.3 SINGULAR PERTURBATIONS OF MILNOR CUBIC POLYNOMIALS

We add a pole singularity at $z = 0$ to the map $P_b(z)$, controlling the size of the perturbation with the parameter λ, obtaining a family of functions of the form

$$P_{b,\lambda}(z) = b \cdot z^2 \cdot (z-1) + \frac{\lambda}{z^3}$$

Our aim is to a do a numerical exploration, looking for interesting topological objects in form of Julia Sets. After adding a pole singularity, we now have a rational function of degree 6, with 9 critical points, one of which is the point at infinity with multiplicity 3. By computing $P'_{b,\lambda}(z)$, we see that its critical points satisfy

$$b \cdot z^6 - \frac{2}{3} \cdot b \cdot z^5 = \lambda$$

When $\lambda = 0$, this equation has 6 solutions: the origin with multiplicity 5, and $z = \frac{2}{3}$ with multiplicity 1. By continuity, for small enough values of $|\lambda|$ the 6 solutions of the equation become simple zeros of $P'_{b,\lambda}(z)$ which are approximately symmetrically distributed around the origin and $\frac{2}{3}$. Therefore, although we can not compute the exact value of the new critical points, we will iterate $z = \frac{2}{3}$, because for $|\lambda|$ small enough, the qualitative behavior of the orbit $P^n_{b,\lambda}(z)$ is the same.

The point at ∞ is a superattracting fixed point for $P_{b,\lambda}(z)$, and, in a neighborhood of infinity, $P_{b,\lambda}(z)$ is conjugate to $z \mapsto z^3$, so we have an immediate basin of attraction, B, at ∞. Since $P_{b,\lambda}(z)$ has a pole of order 3 at 0, there is an open neighborhood of 0 that is mapped 3 to 1 onto a neighborhood of ∞ in B. If B does not contain this neighborhood, then there is a disjoint open set T about 0 that is mapped 3 to 1 onto B. We call T the *trap door* since any point whose orbit eventually enters B must pass through T en-route to B. Since the degree of $P_{b,\lambda}(z)$ is 6, all points in the preimage of B lie either in B or in T.

In Figure 9 we see that, when we add a pole singularity at the superattracting fixed point $z = 0$ to $P_b(z)$, for $|\lambda|$ small, all the components previously painted in blue suffer a dramatic change. On the other hand, both the components painted in black or red, suffer little to no qualitative variation. This happens because this perturbation is only truly meaningful in a neighborhood of 0, and therefore all the values of b for which $P^n_{b,\lambda}(\frac{2}{3})$ does not visit such neighborhood of 0, have the same behavior as before. We can see three examples of very different parameter spaces, one of which is formed by Sierpisnki curves, and the other two which are Cantor Sets of circles, and the *trap doors* can also be observed, in form of red

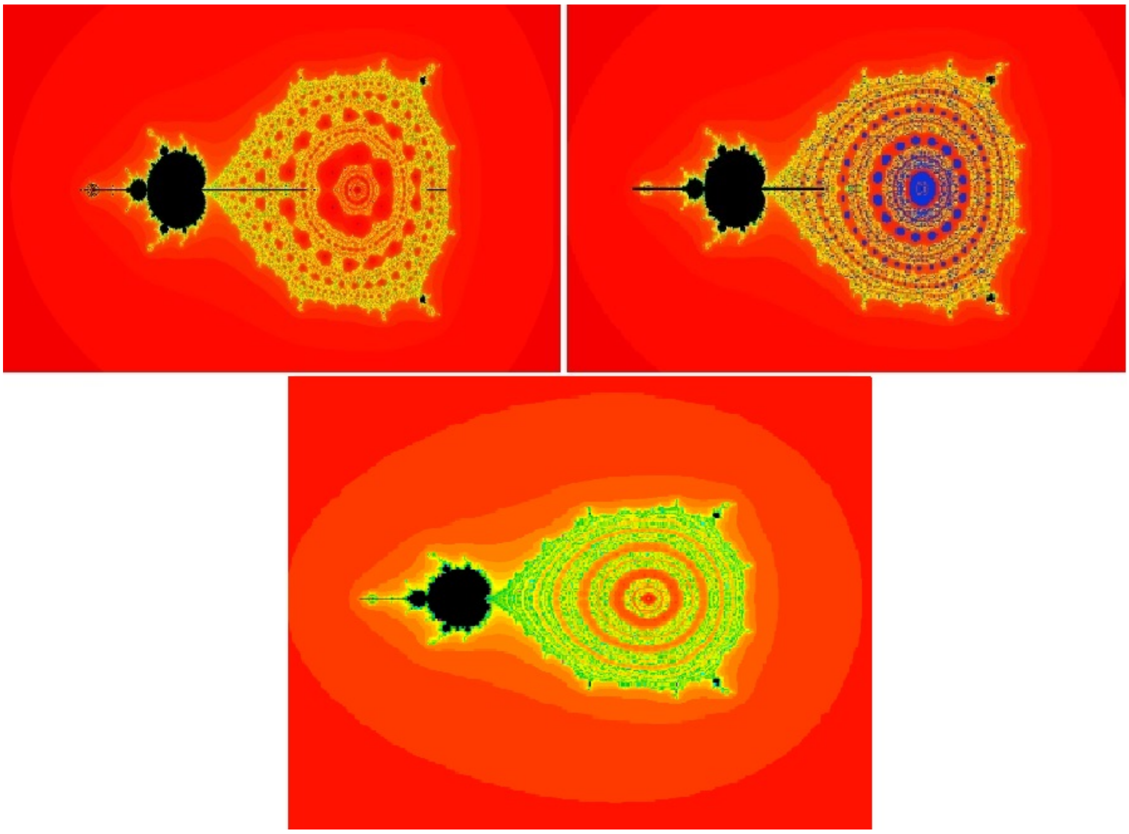

Figure 9: Comparison between the parameter space for $b \cdot z^2 \cdot (z-1) + \frac{10^{-5}}{z^3}$ (top left), $b \cdot z^2 \cdot (z-1) + \frac{10^{-7}}{z^3}$ (top right) and $b \cdot z^2 \cdot (z-1) + \frac{10^{-11}}{z^3}$ (bottom center).

disks around $z = 0$. This image also shows that the iteration of $z = \frac{2}{3}$ is a good enough approximation, because the Mandelbrot-like shapes appear well shaped and detailed.

In figure 10, similarly to the Escape Trichotomy established by Devaney, Look and Uminsky, we have included examples for each of the three type of Julia Sets described in that result: a Cantor Set, a Cantor Set of circles and a Sierpinski curve.

One of the main reasons for choosing Milnor Cubic Polynomials as the object of study when adding the pole singularity, was the fact that these functions already presented interesing Julia Sets before the addition of the perturbation, and we wanted to see if such behavior persisted afterward. In this direction, in figure 11 we see two examples of curious Julia Sets: a disconnected Julia set which is not a Cantor Set and a structure very similar to a Sierpinski carpet, but which can not be such object, because the boundaries of the complementary domains are not Jordan Curves. We also see that not all critical points need to act in a symmetrical way: here 4 of the critical points close to $z = 0$ form a periodic orbit of period 4, while the other converges to a fixed point.

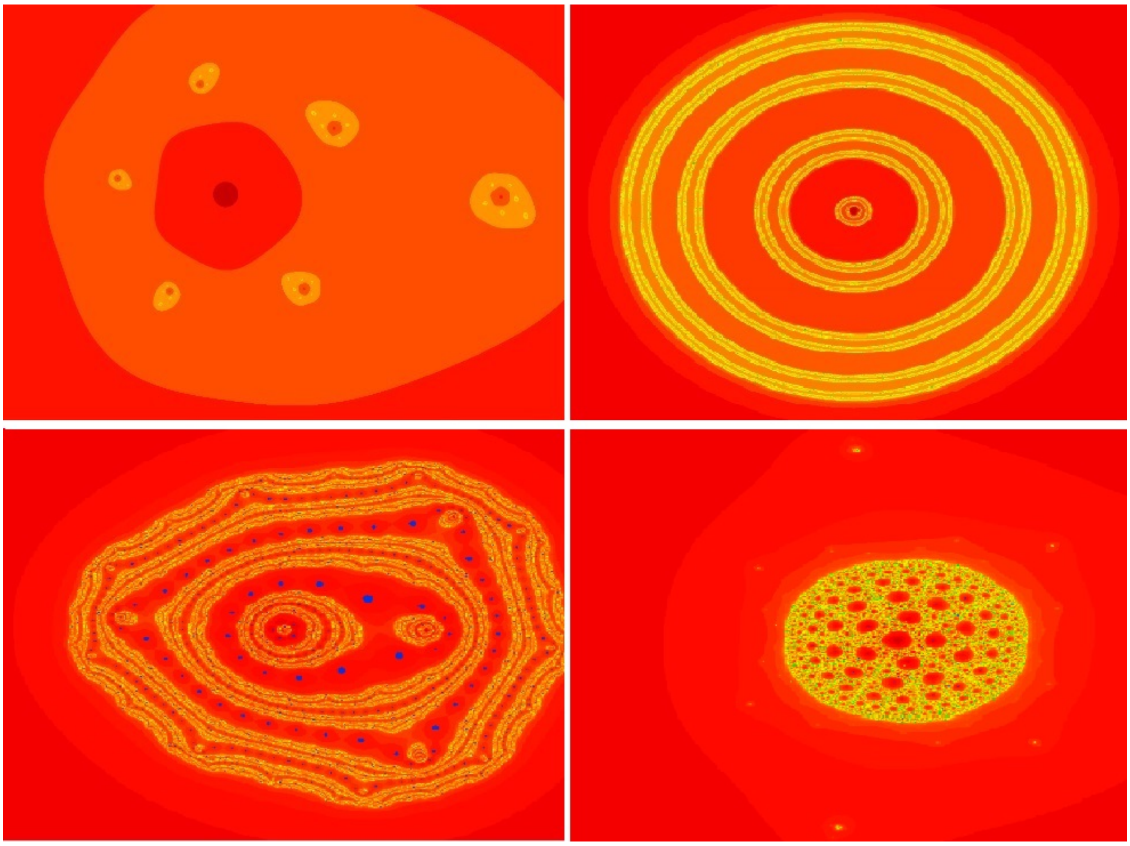

Figure 10: A Cantor set for $b \cdot z^2 \cdot (z-1) + \frac{0.01}{z^3}$, $b = 5+5i$ (top left), a Cantor set of circles for $b \cdot z^2 \cdot (z-1) + \frac{10^{-5}}{z^3}$, $b = 10^{-5}$ (top right) and $b \cdot z^2 \cdot (z-1) + \frac{10^{-5}}{z^3}$ (bottom right), and a Sierpinski carpet for $b \cdot z^2 \cdot (z-1) + \frac{10^{-5}}{z^3}$ (bottom right).

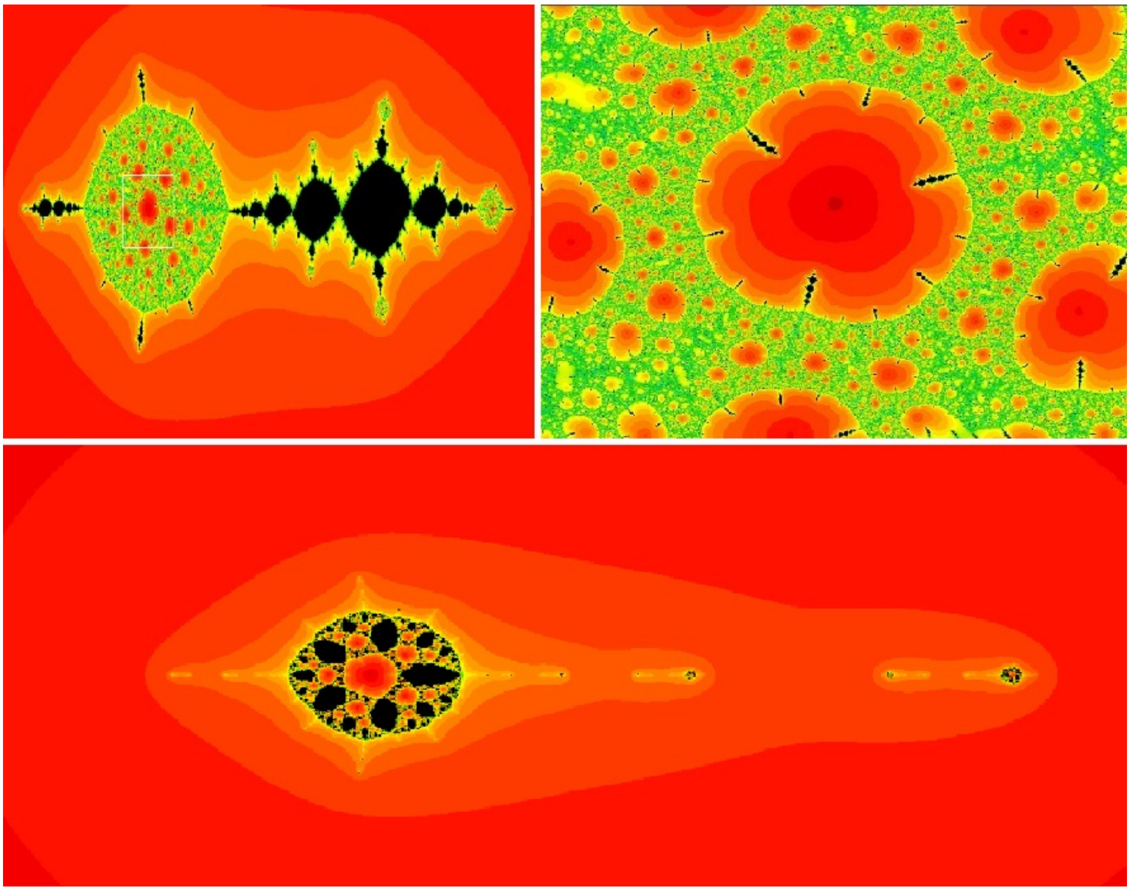

Figure 11: The Julia Set for $b \cdot z^2 \cdot (z-1) + \frac{5.32264 \cdot 10^{-06}}{z^3}$, $b = 5.55$ (top left), and a zoomed image of it (top right), and the Julia Set for $b \cdot z^2 \cdot (z-1) + \frac{10^{-5}}{z^3}$, $b = -8$ (bottom center).

7 GLOSSARY

Conformal map A map $f : U \mapsto V$ is said to be conformal at a point $u_0 \in U$ if it preserves oriented angles between curves through u_0 with respected to their orientation. In complex analysis, f is conformal $\Leftrightarrow$ it is holomorphic and $f'(z) \neq 0 \ \forall z \in U$.

Doubly connected A topological space, **X**, is said to be doubly connected if it is non-empty, path-connected, and its first two homotopy groups vanish identically, that is, if

$$\pi_i(X) \simeq 0, \ i = 1, 2.$$

where the left-hand side denotes the i-th homotopy group.

Equicontinuous Let X and Y be metric spaces and let F be a set of maps

$$f : X \longmapsto Y$$

We say that F is equicontinuous at a point $a \in X$ if $\forall \epsilon > 0$, there is some $\delta > 0$ such that if $d(x, a) < \delta$, then $d(f(x), f(a)) < \epsilon$, $x \in X$, $\forall f \in F$.

Generic point Given a topological space X, we say that a point $p \in X$ is generic if the closure of the singleton set $\{p\}$ is X.

Lie group A Lie group is a group which is also a a differential manifold, with the property that the group operations are compatible with the smooth structure.

Locally connected Let X be a topological space, and let $x \in X$. We say that X is locally connected at x if, for every open set V containing x, there exists a connected, open set U, $x \in U \subset V$.

Meromorphic function We say that a function $f : U \mapsto \mathbb{C}$ is meromorphic, if it is holomorphic in all of U except for a set of isolated points(the poles of the function) at each of which the function must have a Laurent series.

Möbius transformation Let $a \in \mathbb{C}$, $|a| \leq 1$. Then,

$$\phi_a(z) = \frac{z-a}{1-\overline{a} \cdot z}$$

is a Möbius transformation. ϕ_a is a conformal self-map of the unit disk for each a, and any conformal self-map of the unit disk to itself is a composition of a Möbius transformation with a rotation.

Path A path in a topological space X, is a continuous function from a closed interval to X, i.e, a continuous function $f : I = [a, b] \mapsto X$, for some $a, b \in \mathbb{R}$.

Pole Suppose $U \subset \mathbb{C}$ is an open set, $p \in U$ and $f : U \setminus p \mapsto \mathbb{C}$ is an holomorphic function. If there exists a holomorphic function $g : U \mapsto \mathbb{C}$ such that $g(p) \neq 0$ and a positive integer, $n \geq 0$, $n \in \mathbb{N}$ such that for all $z \in U \setminus p$, f can be expressed as

$$f(z) = \frac{g(z)}{(z-p)^n}$$

we say that p is a pole of f.

Quasicircle A quasicircle is the image of a circle by a quasiconformal mapping of the extended complex plane.

Quasiconformal mapping Let $f : \mathbb{C} \mapsto \mathbb{C}$, and we define the dilatation of the mapping f at the point z as

$$D_f(z) = \frac{|f_z|+|f_{\overline{z}}|}{|f_z|-|f_{\overline{z}}|} \geq 1$$

and we define the maximal dilatation of the mapping as

$$K_f = sup_z D_f(z)$$

We say that f is quasiconformal if is absolutely continuous on lines, and its maximal dilatation is finite.

Sierpinski curve A Sierpinski curve is a planar set that is homeomorphic to the Sierpinski carpet fractal.

Simply connected We say that a set is simply connected if it is path-connected, and it's fundamental group is the trivial group.

Uniform convergence A sequence of functions f_n, $n = 1, 2, 3, ...$ is said to be uniformly convergent to f for a set, **E**, of values of x if, $\forall \epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \epsilon, \, \forall n \geq n_0, \, \forall x \in \mathbf{E}$$

Winding number The winding number of a closed curve around a point is an integer showing the number of counterclockwise travels of the curve around the point. Formally, given a closed curve γ, and a point $z_0 \notin \gamma$, if such curve does not pass through the origin (if it does, we can redefine the coordinate system), we can express γ in polar coordinates, making it dependent of $r(t), \theta(t), 0 \leq t \leq 1$, so that, as the curve is closed, $\theta(1) - \theta(0) = 2\pi \cdot k$, for some $k \in \mathbb{Z}$. Finally we define the winding number as

$$winding\ number := \frac{\theta(1)-\theta(0)}{2\pi}$$

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