Leading chiral logarithms to the hyperfine splitting of the hydrogen and muonic hydrogen

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We study the hydrogen and muonic hydrogen within an effective field theory framework. We perform the matching between heavy baryon effective theory coupled to photons and leptons and the relevant effective field theory at atomic scales. This matching can be performed in a perturbative expansion in α, $1/m_p$, and the chiral counting. We then compute the $O(m_{l_i}^3\alpha^5/m_p^2\times\text{logarithms})$ contribution (including the leading chiral logarithms) to the hyperfine splitting and compare with experiment. They can explain about 2/3 of the difference between experiment and the pure QED prediction when setting the renormalization scale at the ρ mass. We give an estimate of the matching coefficient of the spin-dependent proton-lepton operator in heavy baryon effective theory.

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I. INTRODUCTION

Years have passed since the advent of QCD. After numerous attempts to understand QCD by using several models, more studies now move towards trying to parametrize the QCD properties in a model independent way with the help of different systematics that are usually highlighted by the specific kinematic situation under study. One could hope that this approach may bring some light on the understanding of QCD or at least provide some consistency check between different models. Therefore, it becomes important to be able to relate as many observables as possible in a model independent framework. Effective field theories (EFTs) may play an important role in this approach.

Within the above philosophy, the study of hydrogen (ep) and muonic hydrogen (μp), in particular of the high precision measurement of different splittings, can provide accurate determinations of some hadronic parameters related to the proton elastic and inelastic electromagnetic form factor, like the proton radius and magnetic moment, polarization effects, etc.

In the ep and μp we are basically testing the proton with different probes (e, μ, γ). They correspond to the simplest possible probes since they are pointlike particles and the interaction is perturbative (the analogy with deep inelastic scattering is evident and it has already been used since long ago [1–4] in order to obtain some of these hadronic parameters from dispersion relations). They also provide the first natural step towards more complicated systems like exotic or heavy (muonic) atoms.

The ep and μp systems are, in a first approximation, states weakly bound by the Coulomb interaction and their typical binding energy and relative momentum are $E \sim m_{e(\mu)}\alpha^2$ and $|\mathbf{p}| \sim m_{e(\mu)}\alpha$, respectively. We will switch off the weak interactions in this work. Therefore, the ep and μp systems become stable, and C, P, and T are exact symmetries of these systems. In any case, several different scales are involved in their dynamics: For the ep system they are $\dots$, $m_e\alpha^2$, $m_e\alpha$, m_e, $\Delta m = m_n - m_p$, $\Delta = m_\Delta - m_p$, m_π, m_p, m_ρ, Λ_χ, $\dots$, that we will group and name in the following way.

(i) $m_e\alpha^2$: ultrasoft (US) scale.

(ii) $m_e\alpha$: soft scale.

(iii) $\mu_{ep} = m_e m_p/(m_e + m_p)$, $\Delta\, m = m_n - m_p$, m_e: hard scale.

(iv) m_μ, $\Delta = m_\Delta - m_p$, m_π: pion scale.

(v) m_p, m_ρ, Λ_χ: chiral scale.

For the μp system they are $\dots$, $m_\mu\alpha^2$, $m_\mu\alpha$, m_μ, $\Delta m = m_n - m_p$, m_e, $\Delta = m_\Delta - m_p$, m_π, m_p, m_ρ, Λ_χ, $\dots$, that we will group and name in the following way.

(vi) $m_\mu\alpha^2$: US scale.

(vii) $\Delta m = m_n - m_p$, m_e, $m_\mu\alpha$: soft scale.

(viii) $\mu_{\mu p} = m_\mu m_p/(m_\mu, + m_p)$, m_μ, $\Delta = m_\Delta - m_p$, m_π: hard/pion scale.

(ix) m_p, m_ρ, Λ_χ: chiral scale.

By doing ratios with the different scales, several small expansion parameters can be built. Basically, this will mean that the observables, the spectrum in our case, can be written, up to large logarithms, as an expansion, in the case of the ep, in α, m_e/m_π, and m_π/m_p, and in the case of the μp, in α and m_μ/m_p. It will also prove convenient sometimes to use the reduced mass $\mu_{\mu(e)p}$, since it will allow to keep (some of) the exact mass dependence at each order in α. In order to be more precise, the ep energy will be expanded in the following way (up to logarithms):

$$E(ep) = -\frac{\mu_{ep}\alpha^2}{2n^2}(1 + c_2\alpha^2 + c_3\alpha^3 + \cdots), \tag{1}$$

where

$$c_n = \sum_{i,j=0}^{\infty} c_n^{(i,j)}\left(\frac{m_e}{m_\pi}\right)^i\left(\frac{m_\pi}{m_p}\right)^j + \cdots, \tag{2}$$

and $c_n^{(i,j)}$ are functions of dimensionless quantities of $O(1)$ like μ_{ep}/m_e, m_μ/m_π, etc.

For the μp things work analogously,

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$$E(\mu p)=-\frac{\mu_{\mu p}\alpha^2}{2n^2}(1+c_1\alpha+c_2\alpha^2+c_3\alpha^3+\cdots), \quad (3)$$

where c_1 does not depend on hadronic quantities, only on $m_\mu\alpha/m_e$, and $(n\geqslant 2)$,

$$c_n=\sum_{j=0}^{\infty} c_n^{(j)}\left(\frac{m_\pi}{m_p}\right)^j+\cdots, \quad (4)$$

where $c_n^{(j)}$ are functions of dimensionless quantities of $O(1)$ like $\mu_{\mu p}/m_\mu$, m_μ/m_π, $m_\mu\alpha/m_e$, etc.

Let us stress that the coefficients c_n can be expanded in the ratio m_π/m_p, i.e., in the chiral/heavy-baryon expansion (m_p should also be understood as Λ_χ).

In order to disentangle all the different scales mentioned above it is convenient to use EFTs. In order to obtain the relevant one for these systems we first need to decide what are the degrees of freedom we want to describe. In our case we want to describe the ep and μp systems at ultrasoft or smaller energies. Therefore, degrees of freedom with higher energies can (and will) be integrated out in order to obtain the EFT to describe these systems. One EFT that fulfills this requirement is potential NRQED (pNRQED) [5,6] (for some applications see [7] and see also [8,9]). pNRQED appears after integrating out the soft scale from NRQED [10] and it shares some similarities with the approach followed in Ref. [11]. We will obtain pNRQED by passing through different intermediate effective field theories after integrating out different degrees of freedom. The path that we will take is the following (in some cases, instead of this chain of EFTs one can use dispersion relations, or direct experimental data, in order to obtain the matching coefficients):

$$\text{HBET}\rightarrow(\text{QED})\rightarrow\text{NRQED}\rightarrow\text{pNRQED}.$$

This way of working opens the possibility to compute the observables of atomic physics with the parameters obtained from heavy baryon effective theory (HBET), which is much closer to QCD since it incorporates its symmetries automatically, in particular the chiral symmetry. Besides, it is the matching with HBET that will allow us to relate the matching coefficients used for ep with the ones used in μp. HBET [12] describes systems with one heavy baryon: the proton, the neutron, or the delta [13] at the pion mass scale. The chiral scale explicitly appears in the Lagrangian as an expansion in $1/\Lambda_\chi$ and $1/m_p$ and any other smaller scale remains dynamical in this effective theory. In short, HBET is a EFT defined with an UV cutoff ν such that $\nu\ll\Lambda_\chi$ but larger than any other dynamical scale in the problem.

In the μp, NRQED appears after integrating out the hard scale, whereas in the ep, NRQED appears after integrating out the pion and hard scales. In this last case one could pass through an intermediate theory (QED) defined by integrating out the pion scale, and profit from the fact that pion and hard scales are widely separated. Nevertheless, we will do the matching here in one step for simplicity.

pNRQED is obtained after integrating out the soft scale. We refer to [5,6] for further details. The above methodology allows one to compute (or parametrize in a model independent way) the coefficients c's in a systematic expansion in the different small parameters on which these systems depend.

It is the aim of this paper to use this procedure for the computation of the leading-logarithm hadronic contributions to the hyperfine splitting for both the ep and μp. This means to compute the spin-dependent piece of c_3 with $O[(m_e/m_p)^2\times\text{logarithms}]$ and $O[(m_\mu/m_p)^2\times\text{logarithms}]$ accuracy for the ep and μp, respectively.

II. EFFECTIVE FIELD THEORIES

In this section, we will consider the different EFTs that will be necessary for our calculation.

A. HBEFT

Our starting point is the SU(2) version of HBEFT coupled to leptons, where the delta is kept as an explicit degree of freedom. The degrees of freedom of this theory are the proton, neutron, and delta, for which the nonrelativistic approximation can be taken, and pions, leptons (muons and electrons), and photons, which will be taken to be relativistic.

Our first aim will be to present the effective Lagrangian of this theory. It corresponds to a hard cutoff $\mu\ll m_p$, Λ_χ, and is much larger than any other scale in the problem. The Lagrangian can be split into several sectors. Most of them have already been extensively studied in the literature, but some will be new. Moreover, the fact that some particles will only enter through loops, since only some specific final states are desired, will simplify the problem. The Lagrangian can be structured as

$$\mathcal{L}_{HBET}=\mathcal{L}_\gamma+\mathcal{L}_l+\mathcal{L}_\pi+\mathcal{L}_{l\pi}+\mathcal{L}_{(N,\Delta)}+\mathcal{L}_{(N,\Delta)l}+\mathcal{L}_{(N,\Delta)\pi}+\mathcal{L}_{(N,\Delta)l\pi}, \quad (5)$$

representing the different sectors of the theory. In particular, the Δ stands for the spin 3/2 baryon multiplet (we also use $\Delta=m_\Delta-m_p$, the specific meaning in each case should be clear from the context).

The Lagrangian can be written as an expansion in e and $1/m_p$. Our aim is to obtain the hyperfine splitting with $O[m_{l_i}^3\alpha^5/m_p^2\times(\ln m_q,\ln\Delta,\ln m_{l_i})]$ accuracy, where m_q stands for the mass of the light, u and d (or s), quarks and m_{l_i} for the mass of the lepton [the leading order contribution to the hyperfine splitting reads $E_F=(8/3)c_F^{(p)}m_{l_i}^2\alpha^4/m_p$, where $c_F^{(p)}$ is defined in Eq. (10)]. Therefore, we need, in principle, the Lagrangian with $O(1/m_p^2)$ accuracy. Let us consider the different pieces of the Lagrangian more in detail.

The photonic Lagrangian reads (the first corrections to this term scale like α^2/m_p^4)

$$\mathcal{L}_\gamma=-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}. \quad (6)$$

The leptonic sector reads ($iD_\mu=i\partial_\mu-eA_\mu$)

$$\mathcal{L}_l=\sum_i \bar{l}_i(i\not{D}-m_{l_i})l_i, \tag{7}$$

where $i=e,\mu$. We do not include the term

$$-\frac{eg_{l_i}}{m_p}\bar{l}_i\sigma_{\mu\nu}l_iF^{\mu\nu}, \tag{8}$$

since the coefficient g_{l_i} is suppressed by powers of α and the mass of the lepton. Therefore, it would give contributions beyond the accuracy we aim. In any case, any eventual contribution would be absorbed in a low energy constant.

The pionic Lagrangian $\mathcal{L}_\pi$ is usually organized in the chiral counting. From the analysis of Sec. II B we will see that the free pion propagators provide with the necessary precision. Therefore, we only need the free-particle pionic Lagrangian:

$$\mathcal{L}_\pi=(\partial_\mu\pi^+)(\partial^\mu\pi^-)-m_\pi^2\pi^+\pi^- +\frac{1}{2}(\partial_\mu\pi^0)(\partial^\mu\pi^0)-\frac{1}{2}m_\pi^2\pi^0\pi^0. \tag{9}$$

The one-baryon Lagrangian $\mathcal{L}_{(N,\Delta)\pi}$ is needed at $O(1/m_p^2)$. Nevertheless a closer inspection simplifies the problem. A chiral loop produces a factor $1/(4\pi F_0)^2\sim 1/m_p^2$. Therefore, the pion-baryon interactions are only needed at $O(m_\pi)$, the leading order, which is known [12,13].[1] For the explicit expressions we refer to these references.

Therefore, we only need the one-baryon Lagrangian $\mathcal{L}_{(N,\Delta)}$ at $O(1/m_p^2)$ coupled to electromagnetism. This would be a NRQED-like Lagrangian for the proton, neutron (of spin 1/2), and delta (of spin 3/2). The neutron is actually not needed at this stage. The relevant term for the proton reads

$$\delta\mathcal{L}_{(N,\Delta)}=N_p^\dagger\left\{iD_0+\frac{\mathbf{D}_p^2}{2m_p}+\frac{\mathbf{D}_p^4}{8m_p^3}-eZ_p\frac{c_F^{(p)}}{2m_p}\boldsymbol{\sigma}\cdot\mathbf{B} -ieZ_p\frac{c_S^{(p)}}{8m_p^2}\boldsymbol{\sigma}\cdot(\mathbf{D}_p\times\mathbf{E}-\mathbf{E}\times\mathbf{D}_p)\right\}N_p, \tag{10}$$

where $iD_p^0=i\partial_0+Z_peA^0$, $i\mathbf{D}_p=i\boldsymbol{\nabla}-Z_pe\mathbf{A}$. For the proton $Z_p=1$. We have not included a term like

$$\frac{c_D^{(p)}}{m_p^2}N_p^\dagger[\boldsymbol{\nabla}\cdot\mathbf{E}]N_p. \tag{11}$$

We could have done so but it may also be eliminated by some field redefinitions. In any case it would give contribution to the spin-independent terms so we will not consider it further in this work.

As for the delta (of spin 3/2), it mixes with the nucleons at $O(1/m_p)$ [$O(1/m_p^2)$ are not needed in our case]. The only relevant interaction in our case is the p-Δ^+-γ term, which is encoded in the second term of

$$\delta\mathcal{L}_{(N,\Delta)}=T^\dagger(i\partial_0-\Delta)T+\frac{eb_{1,F}}{2m_p}(T^\dagger\boldsymbol{\sigma}^{(3/2)}_{(1/2)}\cdot\mathbf{B}\ \ \boldsymbol{\tau}^{3(3/2)}_{(1/2)}N +\mathrm{H.c.}), \tag{12}$$

where T stands for the delta 3/2 isospin multiplet, N for the nucleon 1/2 isospin multiplet and the transition spin/isospin matrix elements fulfill (see [14])

$$\boldsymbol{\sigma}^{i(1/2)}_{(3/2)}\boldsymbol{\sigma}^{j(3/2)}_{(1/2)}=\frac{1}{3}(2\delta^{ij}-i\epsilon^{ijk}\boldsymbol{\sigma}^k),$$

$$\boldsymbol{\tau}^{a(1/2)}_{(3/2)}\boldsymbol{\tau}^{b(3/2)}_{(1/2)}=\frac{1}{3}(2\delta^{ab}-i\epsilon^{abc}\boldsymbol{\tau}^c). \tag{13}$$

The baryon-lepton Lagrangian provides new terms that are not usually considered in HBET. The relevant term in our case is the interaction between the leptons and the nucleons (actually only the proton):

$$\delta\mathcal{L}_{(N,\Delta)l}=\frac{1}{m_p^2}\sum_i c_{3,R}^{pl_i}\bar{N}_p\gamma^0N_p\ \ \bar{l}_i\gamma_0l_i +\frac{1}{m_p^2}\sum_i c_{4,R}^{pl_i}\bar{N}_p\gamma^j\gamma_5N_p\ \ \bar{l}_i\gamma_j\gamma_5l_i. \tag{14}$$

The above matching coefficients fulfill $c_{3,R}^{pl_i}=c_{3,R}^p$ and $c_{4,R}^{pl_i}=c_{4,R}^p$ up to terms suppressed by m_{l_i}/m_p, which will be sufficient for our purposes.

Let us note that with the conventions above, N_p is the field of the proton (understood as a particle) with positive charge if l_i represents the leptons (understood as particles) with negative charge. This finishes all the needed terms for this paper, since the other sectors of the Lagrangian would give subleading contributions.

B. NRQED(μ)

In the muon-proton sector, by integrating out the m_π scale, an effective field theory for muons, protons, and photons appears. In principle, we should also consider neutrons, but they play no role at the precision we aim. The effective theory corresponds to a hard cutoff $\nu\ll m_\pi$, and therefore pions and deltas have been integrated out. The Lagrangian is equal to the previous case but without pions and deltas and with the following modifications: $\mathcal{L}_l\to\mathcal{L}_e+\mathcal{L}_\mu^{(NR)}$ and

[1]Actually terms that go into the physical mass of the proton and into the physical value of the anomalous magnetic moment of the proton $\mu_p=c_F^{(p)}-1$ should also be included (at least in the pure QED computations), and will be assumed in what follows. For our computation these effects would be formally subleading. In any case, their role is just to bring the *bare* values of m_0 and μ_0 to their physical values. Therefore, once the values of m_p and μ_p are measured by different experiments, they can be distinguished from the effects we are considering in this paper.

$\mathcal{L}_{(N,\Delta)l} \to \mathcal{L}_{Ne} + \mathcal{L}_{N\mu}^{(NR)}$, where it is made explicit that the muon has become unrelativistic. Any further difference goes into the matching coefficients, in particular into the matching coefficients of the baryon-lepton operators. In summary, the Lagrangian reads

$$\mathcal{L}_{NRQED(\mu)} = \mathcal{L}_{\gamma} + \mathcal{L}_{e} + \mathcal{L}_{\mu}^{(NR)} + \mathcal{L}_{N} + \mathcal{L}_{Ne} + \mathcal{L}_{N\mu}^{(NR)}, \quad (15)$$

where

$$\mathcal{L}_{\mu}^{(NR)} = l_{\mu}^{\dagger}\left\{ iD_{\mu}^{0} + \frac{\mathbf{D}_{\mu}^{2}}{2m_{\mu}} + \frac{\mathbf{D}_{\mu}^{4}}{8m_{\mu}^{3}} + eZ_{\mu}\frac{c_{F}^{(\mu)}}{2m_{\mu}}\boldsymbol{\sigma}\cdot\mathbf{B} + ieZ_{\mu}\frac{c_{S}^{(\mu)}}{8m_{\mu}^{2}}\boldsymbol{\sigma}\cdot(\mathbf{D}_{\mu}\times E - E\times\mathbf{D}_{\mu})\right\} l_{\mu} \quad (16)$$

and

$$\mathcal{L}_{N\mu}^{NR} = \frac{c_{3,NR}^{pl_{\mu}}}{m_{p}^{2}} N_{p}^{\dagger}N_{p} l_{\mu}^{\dagger} l_{\mu} - \frac{c_{4,NR}^{pl_{\mu}}}{m_{p}^{2}} N_{p}^{\dagger}\boldsymbol{\sigma}N_{p} l_{\mu}^{\dagger}\boldsymbol{\sigma} l_{\mu}, \quad (17)$$

with the following definitions: $iD_{\mu}^{0} = i\partial_{0} - Z_{\mu}eA^{0}$, $i\mathbf{D}_{\mu} = i\boldsymbol{\nabla} + Z_{\mu}e\mathbf{A}$ and $Z_{\mu}=1$. $\mathcal{L}_{e}$ stands for the relativistic leptonic Lagrangian [Eq. (7)] and $\mathcal{L}_{Ne}$ for Eq. (14), both for the electron case only. A term of the type

$$-\frac{eg_{l_e}}{m_{\mu}}\bar{l}_{e}\sigma_{\mu\nu}l_{e}F^{\mu\nu} \quad (18)$$

is not taken into account because of the same reason as in Sec. II A.

C. QED(e)

After integrating out scales of $O(m_{\pi})$ in the electron-proton sector, an effective field theory for electrons coupled to protons (and photons) appears. Again, we should also consider neutrons, but they play no role at the precision we aim. This effective theory has a cutoff $\nu \ll m_{\pi}$ and pions, deltas, and muons have been integrated out. The Lagrangian reads

$$\mathcal{L}_{QED(e)} = \mathcal{L}_{\gamma} + \mathcal{L}_{e} + \mathcal{L}_{N} + \mathcal{L}_{Ne}. \quad (19)$$

This Lagrangian is similar to the previous subsection but without the muon.

D. NRQED(e)

After integrating out scales of $O(m_{e})$ in the electron-proton sector, we still have an effective field theory for electrons coupled to protons and photons. Nevertheless, now, the electrons are nonrelativistic. The Lagrangian is quite similar to the one in Sec. II B but without a light fermion and with the replacement $\mu \to e$. The Lagrangian reads

$$\mathcal{L}_{NRQED(e)} = \mathcal{L}_{\gamma} + \mathcal{L}_{e}^{(NR)} + \mathcal{L}_{N} + \mathcal{L}_{Ne}^{(NR)}. \quad (20)$$

E. pNRQED

After integrating out scales of $O(m_{l_i}\alpha)$ one ends up in a Schrödinger-like formulation of the bound-state problem. We refer to [5,6] for details. The pNRQED Lagrangian for the ep (the nonequal mass case) can be found in Appendix B of the second reference in [6] up to $O(m\alpha^5)$. The pNRQED Lagrangian for the μp is similar except for the fact that light fermion (electron) effects have to be taken into account. The explicit Lagrangian and a more detailed analysis of this case will be presented elsewhere. For the purposes of this paper, we only have to consider the spin-dependent delta potential,

$$\delta V = 2\frac{c_{4,NR}}{m_{p}^{2}}\mathbf{S}^{2}\delta^{(3)}(\mathbf{r}), \quad (21)$$

which will contribute to the hyperfine splitting.

III. FORM FACTORS: DEFINITIONS

It will turn out convenient to introduce some notation before performing the matching between HBET and NRQED. We first define the form factors, which we will understand as pure hadronic quantities, i.e., without electromagnetic corrections.

Our notation is based on the one of Ref. [15]. We define $J^{\mu} = \Sigma_{i} Q_{i}\bar{q}_{i}\gamma^{\mu}q_{i}$, where $i=u,d$ (or s). The form factors are then defined by the following equation:

$$\langle p',s|J^{\mu}|p,s\rangle = \bar{u}(p')\left[F_{1}(q^{2})\gamma^{\mu} + iF_{2}(q^{2})\frac{\sigma^{\mu\nu}q_{\nu}}{2m}\right]u(p), \quad (22)$$

where $q=p'-p$ and F_1, F_2 are the Dirac and Pauli form factors, respectively. The states are normalized in the following (standard relativistic) way:

$$\langle p',\lambda'|p,\lambda\rangle = (2\pi)^{3}2p^{0}\delta^{3}(\mathbf{p}'-\mathbf{p})\delta_{\lambda'\lambda} \quad (23)$$

and

$$u(p,s)\bar{u}(p,s) = \frac{\not{P}+m_{p}}{2m_{p}}\frac{1+\gamma_{5}\not{s}}{2}, \quad (24)$$

where s is an arbitrary spin four vector obeying $s^{2}=-1$ and $P\cdot s=0$.

The form factors could be (analytically) expanded as

$$F_{i}(q^{2}) = F_{i} + \frac{q^{2}}{m^{2}}F_{i}' + \cdots \quad (25)$$

for very low momentum. Nevertheless, we will be interested instead in their nonanalytic behavior in q since it is the one that will produce the logarithms.

We also introduce the Sachs form factors

$$G_{E}(q^{2}) = F_{1}(q^{2}) + \frac{q^{2}}{4m^{2}}F_{2}(q^{2}),$$

$$G_{M}(q^{2}) = F_{1}(q^{2}) + F_{2}(q^{2}). \quad (26)$$

We will also need the forward virtual-photon Compton tensor

$$T^{\mu\nu}=i\int d^4x e^{iq\cdot x}\langle p,s|TJ^{\mu}(x)J^{\nu}(0)|p,s\rangle, \quad (27)$$

which has the structure ($\rho=q\cdot p/m$)

$$\begin{aligned}T^{\mu\nu}=&\left(-g^{\mu\nu}+\frac{q^{\mu}q^{\nu}}{q^2}\right)S_1(\rho,q^2)+\frac{1}{m_p^2}\left(p^{\mu}-\frac{m_p\rho}{q^2}q^{\mu}\right)\\&\times\left(p^{\nu}-\frac{m_p\rho}{q^2}q^{\nu}\right)S_2(\rho,q^2)-\frac{i}{m_p}\epsilon^{\mu\nu\rho\sigma}q_{\rho}s_{\sigma}A_1(\rho,q^2)\\&-\frac{i}{m_p^3}\epsilon^{\mu\nu\rho\sigma}q_{\rho}[(m_p\rho)s_{\sigma}-(q\cdot s)p_{\sigma}]A_2(\rho,q^2), \quad (28)\end{aligned}$$

depending on four scalar functions. It is usual to consider the Born approximation of these functions. They read

$$S_1^{\rm Born}(\rho,q^2)=-2F_1^2(q^2)-\frac{2(q^2)^2\ \ G_{\rm M}^2(q^2)}{(2m_p\rho)^2-(q^2)^2}, \quad (29)$$

$$S_2^{\rm Born}(\rho,q^2)=2\ \frac{4m_p^2q^2\ \ F_1^2(q^2)-(q^2)^2\ \ F_2^2(q^2)}{(2m_p\rho)^2-(q^2)^2}, \quad (30)$$

$$A_1^{\rm Born}(\rho,q^2)=-F_2^2(q^2)+\frac{4m_p^2q^2\ \ F_1(q^2)G_{\rm M}(q^2)}{(2m_p\rho)^2-(q^2)^2}, \quad (31)$$

$$A_2^{\rm Born}(\rho,q^2)=\frac{4m_p^3\rho\ \ F_2(q^2)G_{\rm M}(q^2)}{(2m_p\rho)^2-(q^2)^2}. \quad (32)$$

In the following section, we will use the results of Ji and Osborne [16]. Their notation relates to ours in the following way (for the spin-dependent terms): $S_1^{JO}=A_1/m_p^2$ and $S_2^{JO}=A_2/m_p^3$. Note, however, that $A_1^{\rm Born}$ above is different from the $\bar{S}_1$ definition in Ref. [16] by the F_2^2 term.

IV. MATCHING

The matching between HBET and NRQED can be performed in a generic expansion in $1/m_p$, $1/m_{\mu}$, and α. We have two sort of loops: chiral and electromagnetic. The former are always associated to $1/(4\pi F_0)^2$ factors, whereas the latter are always suppressed by α factors. Any scale left to get the dimensions right scales with m_{π}. In our case we are only concerned in obtaining the matching coefficients of the lepton-baryon operators of NRQCD with $O[\alpha^2\times(\ln m_q,\ln\Delta,\ln m_{l_i})]$ accuracy. Therefore, the piece of the Lagrangian we are interested in reads

$$\delta\mathcal{L}=\frac{c_{3,NR}^{pl_i}}{m_p^2}N_p^{\dagger}N_pl_i^{\dagger}l_i-\frac{c_{4,NR}^{pl_i}}{m_p^2}N_p^{\dagger}\boldsymbol{\sigma}N_pl_i^{\dagger}\boldsymbol{\sigma}l_i, \quad (33)$$

where we have not specified the lepton (either the electron or the muon). In what follows, we will assume that we are doing the matching to NRQED(μ). Therefore, we have to keep the whole dependence on m_{l_i}/m_{π}. The NRQED(e) case can then be derived by expanding m_e versus m_{π}. In principle, a more systematic procedure would mean to go through QED(e). Nevertheless, for this paper, as we do it will turn out to be the easier.

In principle, the contributions scaling with $1/m_{\mu}$ are the more important ones. Nevertheless, they go beyond the aim of this paper. This is specially so as far as we are only interested in logarithms and the spin-dependent term $c_{4,NR}^{pl_i}$. Its general expression at $O(\alpha^2)$ reads (an infrared cutoff larger than $m_l\alpha$ is understood and the expression for the integrand should be generalized for an eventual full computation in D dimensions)

$$\begin{aligned}c_{4,NR}^{pl_i}=&-\frac{ig^4}{3}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2}\frac{1}{k^4-4m_{l_i}^2k_0^2}\\&\times\left\{A_1(k_0,k^2)(k_0^2+2k^2)+3k^2\frac{k_0}{m_p}A_2(k_0,k^2)\right\},\end{aligned} \quad (34)$$

consistent with the expressions obtained long ago as in Ref. [3]. This expression has been obtained in the Feynman gauge. It correctly incorporates the whole dependence on the lepton mass. Therefore, the same expressions are valid for the hydrogen and muonic hydrogen.

In principle one should also consider contributions with one, two, or three electromagnetic current insertions in the hadronic matrix elements instead of only two as in Eqs. (27) and (34). Nevertheless, only the above contribute to the order of interest.

Within the EFT framework the contribution from energies of $O(m_{\rho})$ or higher in Eq. (34) are encoded in $c_{4,R}^{pl_i}\simeq c_{4,R}^{p}$ (analogously for c_3). The contribution from energies of $O(m_{\pi})$ are usually split into three terms (actually this division is usually made irrespective of the energy which is being integrated out): pointlike, Zemach, and polarizability corrections. They will be discussed further later. From the point of view of chiral counting the three terms are of the same order. Therefore, at the order of interest we can divide $c_{4,NR}$ in the following way:

$$c_{4,NR}^{pl_i}=c_{4,R}^{p}+\delta c_{4,pointlike}^{pl_i}+\delta c_{4,Zemach}^{pl_i}+\delta c_{4,pol.}^{pl_i}. \quad (35)$$

Indeed, a similar splitting is usually done for $c_{3,NR}^{pl_i}$:

$$c_{3,NR}^{pl_i}=c_{3,R}^{p}+\delta c_{3,pointlike}^{pl_i}+\delta c_{3,Zemach}^{pl_i}+\delta c_{3,pol.}^{pl_i}. \quad (36)$$

Let us stress at this point that we are only interested in the logarithms. Therefore, we do not need to take care of the finite pieces. This will significantly simplify the calculation.

We obtain the following result for the pointlike contribution:

$$\delta c^{pl_i}_{4,pointlike} = \frac{3+2c_F-c_F^2}{4}\alpha^2 \ln\frac{m_{l_i}^2}{\nu^2}. \qquad (37)$$

Usually, in the literature, the computation of this type of contributions is made considering the proton pointlike and relativistic, i.e., using standard QED computations even at scales of $O(m_p)$. The fact that the proton has some anomalous magnetic moment that is due to hadronic effects or, in other words, the fact that proton has structure makes such theory no-renormalizable. This makes the result of the computation divergent. Within this philosophy, the result to the hyperfine splitting due to pointlike contributions in Ref. [3] was proportional to[2]

$$\frac{3+2c_F-c_F^2}{4}\alpha^2 \ln\frac{m_{l_i}^2}{m_p^2} + \frac{3}{4}(c_F-1)^2 \ln\frac{m_p^2}{\Lambda^2}, \qquad (38)$$

where Λ is the cutoff of this computation. This computation would make sense if the scales on which the structure of the nucleon appears were much larger than the mass of the nucleon (then Λ could run up to this scale). Unfortunately, this is not the case and the structure of the nucleon appears at scales of $O(m_p)$ or even before. Therefore, to compute loops at the scale of $O(m_p)$ assuming the proton to be pointlike produces problems (divergences), as we have seen. The procedure we use in this paper to deal with this issue is to work with effective field theories where the nucleon is considered to be nonrelativistic. In other words, only at scales much smaller than the mass of the nucleon it is a good approximation to consider the nucleon to be pointlike. The other usual method is to use some parametrization of the form factors fitted to the experimental data (see, for instance, [18,19]). This regulates the ultraviolet divergences, providing predictions for the hadronic correction to the hyperfine splitting. This is a very reasonable attitude in the cases where we are mainly interested in getting a number for the hadronic correction to the hyperfine splitting. Nevertheless, this is not the procedure we will follow in this paper, since our aim here is to gain as much understanding as possible of the structure of the proton from QCD and chiral symmetry. We want to understand how much of the coefficient can be understood from logarithms and energies of $O(m_\pi)$, for which a chiral Lagrangian can be used. This is something that could not be done with the standard form factors used to fit the experimental data, since they do not incorporate the correct momentum dependence at low energies due to chiral symmetry.

The Zemach correction is due to what is called the *elastic* contribution [Eq. (27) of Ref. [16]] for A_1 and analogously for A_2 (nevertheless A_2 does not appear to give a contribution). It reads

$$\delta c^{pl_i}_{4,Zemach} = (4\pi\alpha)^2 m_p \frac{2}{3}\int \frac{d^{D-1}k}{(2\pi)^{D-1}}\frac{1}{\mathbf{k}^4} G_E^{(0)} G_M^{(2)}. \qquad (39)$$

Equation (39) can be obtained directly from Eq. (34) or working directly with the nonrelativistic expressions and introducing the form factors. Either way, it is comforting to find the Zemach expression [1].

The upper index in G_E and G_M has to do with the chiral counting. $G_E^{(0)}=1$. It is illustrative to split the contribution to $G_M^{(2)}$ from u and d, and the Δ, $G_M^{(2)}=G_{M,u,d}^{(2)}+G_{M,\Delta}^{(2)}$, and analogously for the Zemach contribution

$$\delta c^{pl_i}_{4,Zemach} = \delta c^{pl_i}_{4,Zemach-u,d} + \delta c^{pl_i}_{4,Zemach-\Delta}. \qquad (40)$$

Another strong simplification comes from the fact that we are just searching for logarithms. Therefore, we are only interested in the behavior of the form factors for $m_p \gg k \gg m_\pi$ and not analytical in $\mathbf{k}^2$. In particular, for the logarithms, we are only interested in the linear behavior in $|\mathbf{k}|$. From Refs. [20,21] (also see [22]), we obtain

$$G_{M,u,d}^{(2)} \doteq \frac{m_p}{(4\pi F_0)^2}\frac{\pi^2}{12}|\mathbf{k}|[-3g_A^2], \qquad (41)$$

$$G_{M,\Delta}^{(2)} \doteq \frac{m_p}{(4\pi F_0)^2}\frac{\pi^2}{12}|\mathbf{k}|\left[\frac{-4g_{\pi N\Delta}^2}{3}\right], \qquad (42)$$

$$\delta c^{pl_i}_{4,Zemach-u,d} \simeq \frac{m_p^2}{(4\pi F_0)^2}\alpha^2 \frac{2}{3}\pi^2 g_A^2 \ln\frac{m_\pi^2}{\nu^2}, \qquad (43)$$

$$\delta c^{pl_i}_{4,Zemach-\Delta} \simeq \frac{m_p^2}{(4\pi F_0)^2}\alpha^2 \frac{8}{27}\pi^2 g_{\pi N\Delta}^2 \ln\frac{\Delta^2}{\nu^2}. \qquad (44)$$

It is remarkable that the above results are π-enhanced.

Just for completeness, we also give the expression for $\delta c^{pl_i}_{3,Zemach}$:

$$\delta c^{pl_i}_{3,Zemach} = 4(4\pi\alpha)^2 m_p^2 m_{l_i}\int \frac{d^{D-1}k}{(2\pi)^{D-1}}\frac{1}{\mathbf{k}^6} G_E^{(0)} G_E^{(2)}. \qquad (45)$$

This term appears to be finite to the order of interest (it produces no logarithms), and it agrees with the result obtained by Pachucki [23] at leading order.

The Zemach corrections (both spin dependent and spin independent) correctly incorporate the whole dependence on the lepton mass. Therefore, the same expressions are valid for the hydrogen and the muonic hydrogen.

Let us now consider the polarizability contributions. In the SU(2) case (and including the Δ) they should come from Eqs. (30) and (32)–(36) of Ref. [16]. In principle, in our

[2]Yet relativistic-type computations can be very useful sometimes, like in identifying some $\ln(m_{l_i}/m_p)\to\ln(m_{l_i}/\nu)$ logarithms and m_{l_i}/m_p corrections in an efficient way; see [17].

case, one should consider more diagrams besides those plotted in Ref. [16], like the ones due to the Wess-Zumino anomaly action.[3] Nevertheless, as already stated in the description of the pionic Lagrangian, it turns out that they do not contribute in our case. Finally, we split the polarizability contribution as follows [for the SU(2) case]:

$$\delta c_{4,pol.}^{pl_i} = \delta c_{4,pol.-\Delta}^{pl_i} + \delta c_{4,pol.-\pi N}^{pl_i} + \delta c_{4,pol.-\pi\Delta}^{pl_i}. \quad (46)$$

It is again a great simplification the fact that we are only searching for logarithms.

From Eqs. (32) and (35) of Ref. [16] we obtain (we also checked this result by doing the computation directly in the nonrelativistic limit)

$$\delta c_{4,pol.-\Delta}^{pl_i} = \frac{b_{1,F}^2}{18}\alpha^2 \ln\frac{\Delta^2}{\nu^2}, \quad (47)$$

where $b_1^F = G_1$ according to the definition in Ref. [16]. The consequences this result has in a large N_c analysis are remarkable enough. In the large N_c, $b_1^F = 3/(2\sqrt{2})\mu_V$ according to the Ji and Osborne definitions, where μ_V stands for the isovector magnetic moment. On the other hand, by using the results of Ref. [24] $\mu_p/\mu_n = -1$, one obtains (for practical purposes) $\mu_p = \mu_V/2$ in the large N_c. It follows that in this limit the role of the delta is to cancel *all* the μ_p contribution in Eq. (37) $[(3+2c_F - c_F^2)/4 = 1-\mu_p^2/4]$, which effectively becomes the result of a pointlike particle.

From Eqs. (30) and (34) of Ref. [16] we obtain

$$\delta c_{4,pol.-\pi N}^{pl_i} = -\frac{m_p^2}{(4\pi F_0)^2} g_A^2 \frac{\alpha^2}{\pi}\frac{8}{3} C \ln\frac{m_\pi^2}{\nu^2}, \quad (48)$$

where C is defined in the Appendix. From Eqs. (33) and (36) of Ref. [16] we obtain

$$\delta c_{4,pol.-\pi\Delta}^{pl_i} = \frac{m_p^2}{(4\pi F_0)^2} g_{\pi N\Delta}^2 \frac{\alpha^2}{\pi}\frac{64}{27} C \ln\frac{\Delta^2}{\nu^2}. \quad (49)$$

It is worth noting that Eqs. (48) and (49) cancel each other in the large N_c limit, since $g_{\pi N\Delta} = 3/(2\sqrt{2}) g_A$ in this case with the definitions of Ref. [16]. Moreover, they are suppressed by $1/\pi$ factors and the smallness of the numerical coefficient compared with the Zemach term. Let us note that Eqs. (47), (48), and (49) may bring some light on why the polarization term is much smaller than the Zemach term in a model-independent way, since we have an almost analytical result.

Our results can be summarized in Eqs. (37), (43), (44), (47), (48), and (49). The above computation has been performed in SU(2), it would be interesting to repeat the analysis for SU(3). Indeed, we can compute the Zemach correction due to the strange quark (if we do not consider the spin 3/2 baryons) by using the results of Ref. [25] for $G_{M,s}$:

$$G_{M,s}^{(2)} \doteq \frac{m_p}{(4\pi F_0)^2}\frac{\pi^2}{12}|\mathbf{k}|[5D^2 - 6DF + 9F^2]. \quad (50)$$

[3]We thank T. Hemmert for stressing this possibility to us.

We then obtain

$$\begin{aligned}\delta c_{4,Zemach-s}^{pl_i} &= (4\pi\alpha)^2 m_p \frac{2}{3}\int \frac{d^{D-1}k}{(2\pi)^{D-1}}\frac{1}{\mathbf{k}^4} G_E^{(0)} G_{M,s}^{(2)} \\ &\simeq -\frac{m_p^2}{(4\pi F_0)^2}\alpha^2 \frac{2}{9}\pi^2 \\ &\quad \times (5D^2 - 6DF + 9F^2)\ln\frac{m_K^2}{\nu^2}. \end{aligned} \quad (51)$$

In order to obtain a complete result, one should add the strange-related contribution to the Zemach correction due to the baryon spin 3/2 multiplet and to obtain the whole strange contribution to the polarizability. This would require us to have A_1 and A_2 for SU(3) and including the baryon spin 3/2 multiplet, which are unfortunately unknown. In any case, one may wonder whether, for the polarizability corrections, the large N_c cancellation would also hold in this case as well as the $1/\pi$ and numerical factor suppression, so that it would be a very tiny contribution as in the SU(2) case.

Matching to pNRQED: Energy correction

With the above results one can obtain the leading hadronic contribution to the hyperfine splitting. It reads

$$E_{\rm HF} = 4\frac{c_{4,NR}^{pl_i}}{m_p^2}\frac{1}{\pi}(\mu_{l_i p}\alpha)^3. \quad (52)$$

By fixing the scale $\nu = m_\rho$ we obtain the following number for the total sum in the SU(2) case:

$$E_{\rm HF,logarithms}(m_\rho) = -0.031 \ \text{MHz}. \quad (53)$$

The absolute value of this number would increase for a larger value of ν and decrease for a smaller value. The main contribution to Eq. (53) comes from the Zemach and pointlike corrections:

$$E_{\rm HF,Zemach\text{-}u,d}(m_\rho) = -0.022 \ \text{MHz}, \quad (54)$$

$$E_{\rm HF,Zemach\text{-}\Delta}(m_\rho) = -0.004 \ \text{MHz}, \quad (55)$$

$$E_{\rm HF,pointlike}(m_\rho) = -0.003 \ \text{MHz}. \quad (56)$$

Equation (53) accounts for approximately 2/3 of the difference between theory (pure QED) [18] and experiment [26]:

$$E_{\rm HF}(QED) - E_{\rm HF}(\exp) = -0.046 \ \text{MHz}. \quad (57)$$

What is left gives the expected size of the counterterm. Experimentally what we have is $c_{4,NR} = -48\alpha^2$ and $c_{4,R}(m_\rho) \simeq -16\alpha^2$. This last figure gives the expected size of the counterterm of the Lagrangian. A more detailed analysis would require to work in an specific scheme (for instance MS) to fix the finite pieces. We expect to come back to this issue in the future. A point to stress is that this number is universal, i.e., the same for the electron and for the muon up to corrections suppressed by the ratio of the lepton mass versus the proton mass:

$$c_{4,R}^{pe}(m_\rho)\simeq c_{4,R}^{p\mu}(m_\rho). \tag{58}$$

This observation could be used in an eventual measurement of the hyperfine splitting of the muonic hydrogen.

The introduction of the partial SU(3) computation would worsen the above prediction by

$$E_{\mathrm{HF,Zemach\text{-}kaon}}(m_\rho)=0.003 \ \ \mathrm{MHz}, \tag{59}$$

bringing the total sum down to -0.027 MHz and $c_{4,R}(m_\rho)$ to $-20\alpha^2$.

V. CONCLUSIONS

We have performed a first exploratory study on the application of effective field theories emanated from chiral Lagrangians on atomic physics.[4] We have computed the $c_{4,NR}^{pl_i}$ matching coefficient of the NRQED Lagrangian for the e-p and μ-p sectors with $O(\alpha^2\times[\ln m_q,\ln\Delta,\ln m_{l_i}])$ accuracy. The hyperfine splitting of the hydrogen and muonic hydrogen has been computed with $O(m_{l_i}^3\alpha^5/m_p^2\times[\ln m_q,\ln\Delta,\ln m_{l_i}])$ accuracy. We note that our results include the complete expression for the leading chiral logarithms.

The difference between the experimental value of the hydrogen hyperfine splitting and the pure QED computation reads

$$E_{\mathrm{HF}}(QED)-E_{\mathrm{HF}}(\exp)=-0.046 \ \ \mathrm{MHz}, \tag{60}$$

whereas our theoretical prediction reads

$$E_{\mathrm{HF,logarithms}}(m_\rho)=-0.031 \ \ \mathrm{MHz}. \tag{61}$$

We are then able to obtain an estimate of $c_{4,R}^{p}(m_\rho)\simeq-16\alpha^2$ which is valid for both e-p and μ-p systems.

One could improve these results by performing the whole computation in the MS scheme or alike (in this case some of the expressions in this paper should be rewritten in D dimensions). It would follow that not only the logarithms but the finite pieces (in a specific scheme) of the matching coefficient would be obtained too. This would fix with a greater precision the value of $c_{4,R}^{p}(m_\rho)$ and, thus, the respective size of the effects due to the physics at scales of $O(m_\rho)$ and at scales of $O(m_\pi)$. This is important since the experimental number is precise enough to give an accurate number for $c_{4,R}^{p}(m_\rho)$, which could be used to test models at $O(m_\rho)$ scales. To perform the full computation in SU(3) would be also highly desirable. A partial SU(3) result brings Eq. (61) down to -0.027 MHz.

Our results may help to better understand the fact that the Zemach correction is much larger than the polarizability contribution, since we have (almost) analytical expressions for these contributions. The polarizability term [except Eq. (47)] vanishes in the large N_c and it is $1/\pi$ and numerical-factor suppressed with respect the Zemach terms. On the other hand, Eq. (47), in the large N_c limit, cancels *all* the μ_p contribution in Eq. (37), which effectively becomes the result of a point-like particle.

Several lines of research are worth pursuing. One is trying to compute $c_{3,NR}$ within HBET, since its numerical value could be obtained from measurements of the Lamb shift, and it is related to (and in a way defines) the proton radius. Another could be to consider more complicated atoms within this effective field theory formalism (see, for instance, [9] for helium).

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APPENDIX: CONSTANTS

$$\begin{gathered}F_0=92.5 \ \ \mathrm{MeV},\\ g_A=1.25,\\ m_\pi=140 \ \ \mathrm{MeV},\\ m_p=m_n=938 \ \ \mathrm{MeV},\\ \Delta=294 \ \ \mathrm{MeV},\\ g_{\pi N\Delta}=1.05,\\ b_1^F=3.86,\\ F=1/2,\\ D=3/4,\\ m_\rho=770 \ \ \mathrm{MeV}.\end{gathered} \tag{A1}$$

b_1^F and $g_{\pi N\Delta}$ have been obtained from the decays of the delta in the nonrelativistic limit (consistent with the accuracy of our calculation).

The values of F and D are consistent with the large N_c limit. Finally,

[4]The pionium, which has received quite attention recently [27–32], has also been studied within a similar nonrelativistic effective field theory philosophy [27,29,30,32]. Specially close to ours is the approach followed in [30]. The pionic hydrogen has also been studied using effective field theories very recently [33].

$$C=2\int_0^1\int_0^1\sqrt{1-y^2}\left\{-2x(2+y^2)+\frac{1}{y}\left[2\ (1-x)x(2+y^2)\ \sqrt{\frac{1}{x-x^2+x^2\ y^2}}\right.\right.$$
$$\left.\left.-3(1-2x)y^2\ \sqrt{\frac{x}{1-x(1-y^2)}}\right]\sinh^{-1}\left[\sqrt{\left(\frac{x}{1-x}\right)}\ y\right]dydx\right\}=-0.165\,037. \tag{A2}$$

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