Treball final de grau

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Lie Groups, Lie Algebras, Representations and the Eightfold Way

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Abstract

Lie groups and Lie algebras are the basic objects of study of this work. Lie studied them as continuous transformations of partial differential equations, emulating Galois work with polynomial equations. The theory went much further thanks to Killing, Cartan and Weyl and now the wealth of properties of Lie groups makes them a central topic in modern mathematics. This richness comes from the merging of two initially unrelated mathematical structures such as the group structure and the smooth structure of a manifold, which turns out to impose many restrictions. For instance, a closed subgroup of a Lie group is automatically an embedded submanifold of the Lie group. Symmetries are related to groups, in particular continuous symmetries are related to Lie groups and whence, by Noether's theorem, its importance in modern physics.
In this work, we focus on the Lie group - Lie algebra relationship and on the representation theory of Lie groups through the representations of Lie algebras. Especially, we analyze the complex representations of Lie algebras related to compact simply connected Lie groups. With this purpose, we first study the theory of covering spaces and differential forms on Lie groups. Finally, an application to particle physics is presented which shows the role played by the representation theory of $\mathsf{SU}(3)$ on flavour symmetry and the theory of quarks.

Resum

Els grup de Lie i les àlgebres de Lie són els objectes bàsics d'estudi d'aquest treball. Lie els va estudiar com a transformacions contínues d'equacions en derivades parcials, emulant Galois amb les equacions polinòmiques. La teoria va anar molt més enllà gràcies a Killing, Cartan i Weyl. Actualment, la riquesa de propietats del grups de Lie els converteix en un tema central a les matemàtiques. Aquesta riquesa ve de la unió de dues estructures matemàtiques inicialment no relacionades com són l'estructura de grup i l'estructura diferenciable d'una varietat. Aquesta unió imposa moltes restriccions com, per exemple, que un subgrup tancat d'un grup de Lie és automàticament una subvarietat incrustada del grup de Lie.[1] Les simetries es relacionen amb els grups, en particular les simetries contínues es relacionen amb els grups de Lie i d'aquí, pel teorema de Noether, la seva importància a la física.
En aquest treball, ens centrem en la relació grup de Lie - àlgebra de Lie i en la teoria de representacions de grups de Lie a través de les representacions d'àlgebres de Lie. En concret, estudiem les representacions complexes d'àlgebres de Lie relacionades amb els grups de Lie compactes i simplement connexos. Amb aquesta finalitat, estudiem prèviament la teoria d'espais recobridors i les formes diferencials en grups de Lie. Es presenta finalment l'aplicació de la teoria de representacions de $\mathsf{SU}(3)$ a la física de partícules pel cas de la simetria de sabor i la teoria de quarks.

[1] Per incrustada ens referim a *embedded*.

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Contents

1 Introduction

1.1 Motivation

Lie groups are objects rich in algebraic, geometric and analytic structure. They are named after Sophus Lie (1842-1899) a Norwegian mathematician who was the first to work on them setting their theoretical foundation. Lie studied transformations of partial differential equations which took solutions to solutions, just in a similar way as Galois did with polynomial equations. As explained in [8], Lie worked first geometrically, together with Klein, and then more analytically following Jacobi's work. After his death, Lie's theory was improved, above all, by Killing, Cartan and Weyl, leading to what we understand now as Lie groups.

Lie groups go hand in hand with their corresponding Lie algebras, which preserve part of the information encoded in the Lie group. For instance, simply connected Lie groups are in one-to-one correspondence with their Lie algebras and have equivalent representations. Since the underlying structure of a Lie algebra is a vector space, it is usually easier to study Lie algebras than Lie groups. Consider in this regard the fact that all *semisimple* Lie algebras over the complex numbers have been classified.

As a student of Mathematics and Physics, Lie groups constitute a topic which fits my interests as they are an interesting mathematical topic with many applications to Physics.

1.2 Structure

As we will see, Lie groups are smooth manifolds and groups, at the same time. This fact has deep implications in the structure of the smooth manifold, one of the most remarkable being that the vector space of left invariant smooth vector fields is isomorphic to its tangent space at the identity, and both spaces have a Lie algebra structure.

We first study Lie groups and matrix Lie groups. Next, we turn to the matrix exponential to begin to grasp the properties that will be seen later for the Lie algebra. After studying Lie algebras as abstract algebraic structures, as sets of left invariant vector fields and as sets of matrices, we shortly study the theory of covering spaces, define universal coverings and consider the universal covering space of a Lie group. With these tools, we move on to the relationship between Lie groups and Lie algebras and prove the Closed subgroup theorem.

Subsequently, we introduce the representation theory of groups, first on finite groups, then extended to Lie groups with the help of the Haar measure associated to a volume form. Following that, we examine the basic properties of Lie group representations. The case of the double covering of $\mathsf{SO}(3)$ is undertaken along with the representations of $\mathsf{SU}(2)$ and $\mathsf{SU}(3)$ through the complex Lie algebra representations of $\mathsf{sl}(2;\mathbb{C})$ and $\mathsf{sl}(3;\mathbb{C})$.

Finally, we present the Eightfold Way and its historical context as well as its mathematical background.

Throughout this work, K will denote any of the fields $\mathbb{R}$ or $\mathbb{C}$ and $\mathcal{C}^\infty$ will stand for infinitely differentiable (i.e., smooth).

2 Lie Groups

2.1 Definition and properties

Definition 2.1. A ***Lie group*** G is a differentiable manifold with a group structure such that the map

$$\begin{array}{ccc} G \times G & \longrightarrow & G \\ (x,y) & \mapsto & xy^{-1} \end{array} \tag{2.1}$$

is smooth.

The identity element will be usually denoted e (for German *Einselement*).

Remark 1: The definition is often stated differently, in a more intuitive sense, requiring that the group product $\mu(x,y) = xy$ and the inverse $i(x) = x^{-1}$ are smooth operations. Both definitions are equivalent. Using the fact that the composition of smooth maps is smooth, $\mu(x, i(y)) = \mu(x, y^{-1}) = xy^{-1}$ and vice versa, $x \to (e, x) \to x^{-1}$ is $\mathcal{C}^\infty$ and $(x, y) \to (x, y^{-1}) \to xy$ is also $\mathcal{C}^\infty$.

Remark 2: Smoothness implies continuity and hence a Lie group is also a topological group (a topological space with a group structure such that the previous operation is continuous). It can be proved that a certain inverse also holds. This is known as Hilbert's fifth problem, or at least one of its interpretations, and it was solved by Montgomery-Zippin [15] and Gleason [5] in 1952, as it is explained by T. Tao in [23]: Every *locally euclidean* topological group is isomorphic as a topological group to a Lie group. Another interpretation (or generalisation) of the problem is the Hilbert-Schmidt conjecture, which deals with locally compact topological groups and their action over manifolds, and it has been proven for the case of 3-dimensional manifolds in 2013 by J. Pardon [17].

Definition 2.2. A ***Lie subgroup*** H of a Lie group G is a subgroup of G which has a smooth structure making it into a Lie group and an immersed submanifold of G, with the immersion being a group homomorphism.

Every subgroup H of G which is also an embedded submanifold of G is a Lie subgroup since the restriction of the operation $(g_1, g_2) \to g_1 g_2^{-1}$ to $H \times H$ maps to H and it will be also smooth because H is an embedded submanifold.

Moreover, every open submanifold H which is also a subgroup of G will automatically be a Lie subgroup, since open submanifolds are embedded submanifolds of G. Besides, H will also be closed. This comes from the fact that H is the complement of the union of its own cosets, which are open subsets. Thus,

Proposition 2.3. *Any open subgroup of a Lie group is closed.*

Later we will prove a theorem by Élie Cartan that states that every (topologically) closed subgroup of a Lie group is an embedded submanifold (and hence a Lie group).

Definition 2.4. Let $g \in G$, the ***left translation*** by g and the ***right translation*** by g are respectively the diffeomorphisms of $G \to G$ defined for all $h \in G$ by

$$\begin{aligned} L_g(h) &= gh \\ \text{and} \quad R_g(h) &= hg. \end{aligned}$$

These operations, which can be seen as actions of the group on himself, are indeed diffeomorphisms, their inverses are left and right translation by g^{-1}. For every pair of elements

$h, g \in G$, there are two diffeomorphisms of $G \to G$ which have g as the image of h, namely $L_{gh^{-1}}$ and $R_{h^{-1}g}$. We see now why the cosets of an open subgroup are open, since they are related by the diffeomorphisms L_g and R_g.

Definition 2.5. A Lie group G is said to be ***compact*** if it is compact in the usual topological sense as a smooth manifold.

Definition 2.6. The ***identity component*** of G, denoted G_0, is the path connected component of G which contains the unit element $e \in G$.

Remark: A Lie group, being a smooth manifold, is locally path-connected. Hence, is path connected if and only if it is connected.
For a Lie group G, a neighbourhood U of its identity will generate a subgroup, namely the intersection of all subgroups which contain the neighbourhood U or, equivalently, the set of elements which can be expressed as a *word* formed by elements of U and their inverses. Furthermore,

Proposition 2.7. *Let G a connected Lie group and $W \subset G$ an open neighbourhood of the identity. The subgroup generated by W is G.*

Proof. Let V be an open subset of W containing e such that $V = V^{-1} := \{g^{-1};\, g \in V\}$, for instance we can take $V = W \cap W^{-1}$. Let H be the subgroup generated by V. It is open, since $g \in H$ implies that $gV \subset H$ and therefore H is a union of open subsets. Moreover, as we have seen, it is also closed because each of its cosets is open. Since G is connected and $e \in H$, H is non-empty, so $H = G$. □

Corollary 2.8. *G_0 is the only connected open subgroup of G.*

Definition 2.9. Let G and H be Lie groups. A map $F : G \to H$ is a ***Lie group homomorphism*** if it is smooth and also a group homomorphism. If, in addition, F is a diffeomorphism, which implies that it has an inverse that is also a Lie group homomorphism, then it is called a ***Lie group isomorphism***. In the latter case, G and H are said to be ***isomorphic Lie groups***.

Remark: We will see (Theorem 6.12) that the definition can be stated in a less restrictive way, becayse a continuous map between Lie groups is automatically a smooth map. Hence, requiring F to be a continuous homomorphism would be enough to show that F is a Lie group homomorphism.
Now, we see one of the first important properties of Lie groups, which holds thanks to the existence of left translations.

Proposition 2.10. *Let $F : G \to H$ be a Lie group homomorphism. Then, the rank of d_gF is the same for all $g \in G$. That is, every Lie group homomorphism has constant rank.*

Proof. Let e and e' be the unit elements of G and H, respectively. Let $g \in G$, since F is a homomorphism,

$$F(L_g(h)) = F(gh) = F(g)F(h) = L_{F(g)}(F(h)).$$

Taking differentials at both sides,

$$dF_g \circ (dL_g)_e = (dL_{F(g)})_{e'} \circ dF_e$$

L_g and $L_{F(g)}$ are diffeomorphisms so dF_g and dF_e have the same rank, for any $g \in G$. □

Since a bijection of constant rank between smooth manifolds has smooth inverse, one has

Corollary 2.11. *A bijective Lie group homomorphism is a Lie group isomorphism.*

Definition 2.12. A ***one-parameter subgroup*** of G is a Lie group homomorphism $\gamma : \mathbb{R} \to G$, with $\mathbb{R}$ seen as a Lie group under addition. The image of the homomorphism is a Lie subgroup.

As we will see, most Lie groups can be realized as matrix groups, and hence we will make use of the next definition.

Definition 2.13. The ***matrix space*** $M_n(K)$ is the set of all $n \times n$ matrices with entries in K. $M_n(K)$ can be identified with K^{n^2} and we can use the standard notion of convergence of K^{n^2} in $M_n(K)$. Being explicit, let $\{A_m\}_m$ be a sequence of matrices in $M_n(K)$. We say that the sequence of matrices $\{A_m\}_m$ ***converges*** to $A \in M_n(K)$ if $(A_m)_{jk}$ converges to A_{jk} as $m \to \infty$ for all $1 \leq j, k \leq n$.

Definition 2.14. The ***general linear group*** over a field K, denoted $\mathsf{GL}(n; K)$, is the group of all $n \times n$ invertible matrices with entries on K.

2.2 Examples of Lie groups

1. $\mathbb{R}^{\mathbf{n}}$ and $\mathbb{C}^{\mathbf{n}}$. The Euclidean space $\mathbb{R}^n$ is a Lie group under addition because addition is a smooth operation. Similarly, $\mathbb{C}^n$ is a Lie group under addition (of dimension $2n$).

2. $\mathbb{C}^*$. The non-zero complex numbers form a Lie group under multiplication of dimension 2. It can be identified with $\mathsf{GL}(1, \mathbb{C})$. Similarly, $\mathbb{R}^*$ forms a Lie group under multiplication of dimension 1 which is identifiable with $\mathsf{GL}(1, \mathbb{R})$.

3. $\mathbb{S}^{\mathbf{1}}$. The unit circle is a Lie group with the multiplication induced from that of $\mathbb{C}^*$.

4. Let G be an arbitrary Lie group and $H \subset G$ a subgroup which is also an open subset of G. Then, H is a smooth manifold which has a group operation which is smooth and hence H is a Lie group with the inherited group and manifold structure.

5. $\mathbf{M_n(K)}$. The matrix space is a Lie group under addition since it can be identified with $\mathbb{R}^{n^2}$ when $K = \mathbb{R}$ or $\mathbb{R}^{2n^2}$ for $K = \mathbb{C}$.

6. $\mathsf{GL}(\mathbf{n}; \mathbf{K})$. $\mathsf{GL}(n; K)$ is a Lie group. It is an open subset of $M_n(K)$ and hence, a smooth manifold. Furthermore, it is a group under matrix multiplication and the entries of the product of a matrix multiplication are polynomials, so it is a smooth operation. Finally, inversion is smooth because the determinant of invertible matrices does not vanish.

7. Given two Lie groups, G and H, their direct product $G \times H$ is also a Lie group with the group structure given by componentwise multiplication

$$(g, h)(g', h') = (gg', hh').$$

8. The n-torus $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ is a n-dimensional abelian Lie group.

2.3 Matrix Lie groups

Definition 2.15. A ***matrix Lie group*** is a subgroup G of $\mathsf{GL}(n;K)$ which is a closed subset of $\mathsf{GL}(n;K)$. For matrix Lie groups, we will denote the identity element by I.

2.3.1 Examples of matrix Lie groups

1. $\mathsf{SL}(\mathbf{n};\mathbf{K})$. The special linear group, $\mathsf{SL}(n;K)$, is the group of $n\times n$ matrices in $\mathsf{GL}(n;K)$ having determinant $+1$. Since the determinant is a continuous function, it is a closed subgroup of $\mathsf{GL}(n;K)$.

2. $\mathsf{O}(\mathbf{n})$. $\mathsf{O}(n;\mathbb{R})$ is the orthogonal group, written $\mathsf{O}(n)$ because it is generally considered over $\mathbb{R}$. It is defined as $\mathsf{O}(n)=\{A\in M_n(\mathbb{R})|AA^\top=I\}$. That is, the column vectors of all $A\in\mathsf{O}(n)$ are orthonormal with respect to the standard metric in $\mathbb{R}^n$. Thus, A is orthogonal if and only if it preserves the inner product on $\mathbb{R}^n$

$$(A\vec{x})\cdot(A\vec{y})=\vec{x}^\top A^\top A\vec{y}=\vec{x}^\top\vec{y}=\vec{x}\cdot\vec{y}.$$

   Since $\det A=\det A^\top$ we have

$$\det AA^\top=(\det A)^2=\det I=1.$$

   So, an orthogonal matrix has determinant ±1. $\mathsf{O}(n)$ is also easily seen to be a group and a subgroup of $\mathsf{GL}(n;\mathbb{R})$. It is closed in $\mathsf{GL}(n;\mathbb{R})$ because the matrix product and the transpose operation are continuous and hence it is a matrix Lie group.

3. $\mathsf{SO}(\mathbf{n})$. $\mathsf{SO}(n):=\mathsf{O}(n)\cap\mathsf{SL}(n;\mathbb{R})$ is the subgroup of matrices in $\mathsf{O}(n)$ with determinant one and it is called the special orthogonal group. It is also a matrix Lie group. Geometrically, the elements of $\mathsf{SO}(n)$ are rotations and the elements of $\mathsf{O}(n)$ are rotations and reflections. (We have seen that the matrices in $\mathsf{O}(n)$ are inner product-preserving linear operators in $\mathbb{R}^n$. That is, isometries which leave the origin fixed. Moreover, the matrices of $\mathsf{SO}(n)$ have determinant $+1$ and hence preserve the orientation).

4. $\mathsf{U}(\mathbf{n})$. The unitary group, $\mathsf{U}(n)$, is the set of all $n\times n$ complex matrices whose column vectors are orthonormal with respect to the standard hermitian product on $\mathbb{C}^n$. Denoting $A^\dagger=\overline{(A)}^\top$ the conjugate transpose matrix of A (its adjoint), the definition can be written as

$$I=A^\dagger A=\sum_{i=1}^n(\overline{A})^\top_{ji}A_{ik}=\sum_{i=1}^n\overline{A_{ij}}A_{ik}=(\delta_{jk})_{jk}.$$

   An equivalent definition is to say that A is unitary if and only if it preserves the standard hermitian product on $\mathbb{C}^n$, $\langle x,y\rangle=\sum_i\overline{x_i}y_i$ for $x,y\in\mathbb{C}^n$. It can be checked as we did for orthogonal matrices.
   For any matrix A, $\det A^\dagger=\det(\overline{A}^\top)=\det\overline{A}=\overline{\det A}$. So, if A is a unitary matrix,

$$\det(A^\dagger A)=\det A^\dagger\det A=\overline{\det A}\det A=|\det A|^2=\det I=1$$

   and then $\det A=e^{i\theta}$ for some $\theta\in\mathbb{R}$.
   From $A^\dagger A=I$, we see that a matrix is unitary if and only if

$$A^\dagger=A^{-1}.$$

In particular, every unitary matrix is invertible and since $(A^\dagger)^\dagger = A = (A^{-1})^\dagger$ the inverse of a unitary matrix is also unitary.
From $\overline{(AB)} = (\overline{A})(\overline{B})$ and $(AB)^\top = B^\top A^\top$ we have $(AB)^\dagger = B^\dagger A^\dagger$. Hence, if A and B are unitary, then

$$(AB)^\dagger(AB) = B^\dagger A^\dagger AB = I.$$

We have just seen that $\mathsf{U}(n)$ is, in effect, a group.
For the same argument used for $\mathsf{O}(n)$, it is closed in $\mathsf{GL}(n;\mathbb{C})$ and hence it is a matrix Lie group.

5. $\mathsf{SU}(\mathbf{n})$. The special unitary group, $\mathsf{SU}(n)$, is the subgroup of $\mathsf{U}(n)$ of matrices with determinant one.

6. $\mathsf{SP}(\mathbf{n};\mathbb{C})$. The set of $2n \times 2n$ matrices which preserve the antisymmetric bilinear form on $\mathbb{C}^{2n}$

$$\omega(x,y) = \sum_{j=1}^{n}(x_j y_{n+j} - x_{n+j} y_j)$$

is the complex symplectic group, $\mathsf{SP}(n;\mathbb{C})$. If

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

then

$$\omega(x,y) = \sum_{j=1}^{n}\left(x_j \sum_{i=1}^{2n} \Omega_{ji} y_i\right) := (x, \Omega y).$$

A matrix A preserves it if

$$(x,\Omega y) = \omega(x,y) = \omega(Ax,Ay) = (Ax,\Omega Ay) = (x, A^\top \Omega A y).$$

So A preserves the form if and only if

$$\Omega = A^\top \Omega A \iff -\Omega A^\top \Omega = A^{-1}; \quad \Omega^{-1} = -\Omega = \Omega^\top.$$

Taking the determinant of any of both formulas, we get $(\det A)^2 = 1$. In fact, it is always $+1$[2]. In particular, since $\Omega \in \mathsf{SP}(n;\mathbb{C})$, $\det(\Omega) = 1$.
It is a closed subgroup of $\mathsf{GL}(2n;\mathbb{C})$; therefore, it is a matrix Lie group.

As explained in [14], the group $\mathsf{SP}(n;\mathbb{R})$ arises from the study of the Hamilton equations for a system with n degrees of freedom. These are, writing $z = (q_1,\dots,q_n,p_1,\dots,p_n)$,

$$\dot{z} = \Omega\, dH(z,t)$$

where q_i are the configuration variables, p_i their canonically conjugate momentum and H is the Hamiltonian. This is the same as writing the more typical expression

$$\dot{q}_i = \frac{\partial H}{\partial p_i}; \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

7. $\mathsf{SP}(n)$. The compact symplectic group, $\mathsf{SP}(n)$, is defined as

$$\mathsf{SP}(n) = \mathsf{SP}(n;\mathbb{C}) \cap \mathsf{U}(2n).$$

It is the group of $2n \times 2n$ matrices that preserve at the same time the inner product and the bilinear form ω. The group $\mathsf{SP}(n)$ can be seen as the unitary group over the quaternions.

[2]See [18], where there is a proof without the use of Pfaffians.

2.3.2 Examples of subgroups of $\mathsf{GL}(n;\mathbb{C})$ which are not matrix Lie groups:

1. $\mathsf{GL}(n;\mathbb{Q})$ is a subgroup of $\mathsf{GL}(n;\mathbb{C})$ but it is not closed. To see it, for $m \in \mathbb{N}$ consider the matrix

$$A_m = \begin{pmatrix} (1+\frac{1}{m})^m & & 0 \\ & \ddots & \\ 0 & & (1+\frac{1}{m})^m \end{pmatrix} \xrightarrow[m\to\infty]{} A = \begin{pmatrix} e & & 0 \\ & \ddots & \\ 0 & & e \end{pmatrix}.$$

All matrices of this form are in $\mathsf{GL}(n;\mathbb{Q})$ but their limit $A = \lim_{m\to\infty} A_m$ is not in $\mathsf{GL}(n;\mathbb{Q})$.

2. For some fixed $a \in \mathbb{R}\setminus\mathbb{Q}$, consider the set

$$G = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{ita} \end{pmatrix} \mid t \in \mathbb{R} \right\}.$$

Clearly, G is a subgroup of $\mathsf{GL}(n;\mathbb{C})$. Since for $t = 0$, we have that $I_2 \in G$, to get the inverse matrix we only have to make the change $t \to -t$ and the product of two matrices of this form is of this form.
Now, $-I_2 \notin G$ since $e^{it} = -1$ implies that $t = (2m+1)\pi$ for some integer m, and $(2m+1)a$ with a irrational cannot be an odd multiple of π. For some well chosen $n \in \mathbb{Z}$, $t = (2n+1)\pi$, we can make ta arbitrarily close to some odd integer multiple of π. This is possible because the set $\left\{e^{i2\pi ma} \text{ with } m \in \mathbb{Z}\right\}$ is dense in $\mathbb{S}^1$. So we can find a sequence of matrices in G which converges to $-I_2$. It follows that G is not closed.

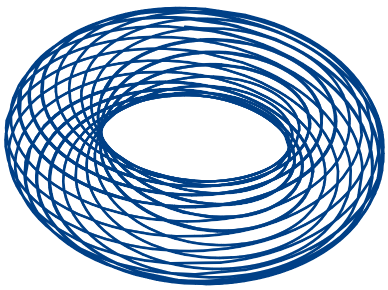

Figure 1: G is dense in $\mathbb{S}^1 \times \mathbb{S}^1$

Remark: The definition of a matrix Lie group is motivated by Theorem 6.17, which states that every closed subgroup of a Lie group is an embedded Lie subgroup. However, there are groups of matrices, such as the line on the torus with irrational slope, which are immersed and not embedded Lie subgroups of $\mathsf{GL}(n;\mathbb{C})$ (cf. [24]). Thus, our definition of matrix Lie group does not include every Lie group of matrices, only the embedded ones.

3 The matrix exponential and its properties

Definition 3.1. The ***exponential of a square matrix***, $A \in M_n(K)$, is defined by the power series

$$e^A := \sum_{n=0}^{\infty} \frac{A^n}{n!}. \tag{3.1}$$

The series is convergent for all $A \in M_n(K)$ and e^A is a continuous function of A. This can be seen using the norm of the matrix together with the properties of the real exponential. Furthermore, since matrix multiplication is a smooth operation,

Proposition 3.2. *The exponential map is an smooth map of $M_n(K)$ into $M_n(K)$.*

Proposition 3.3 (Properties). *Let $X, Y \in M_n(K)$, $C \in \mathsf{GL}(n; K)$. Then,*

1. $e^0 = I$.
2. $(e^X)^\dagger = e^{X^\dagger}$ *(for $X \in M_n(\mathbb{R})$, $X^\dagger = X^\top$).*
3. e^X *is invertible and* $(e^X)^{-1} = e^{-X}$.
4. *If X and Y commute, then* $e^{X+Y} = e^X e^Y = e^Y e^X$.
5. $e^{CXC^{-1}} = Ce^X C^{-1}$.

Proof. The third and fourth properties are the only ones not so obvious. The third follows from the fourth and the fourth can be seen using the binomial formula, *which can only be used when X and Y commute*

$$e^X e^Y = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{X^k}{k!} \frac{Y^{n-k}}{(n-k)!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} X^k Y^{n-k} = \sum_{n=0}^{\infty} \frac{(X+Y)^n}{n!} = e^{X+Y}.$$

□

From item 3 we see that the map $\exp : M_n(K) \to M_n(K)$ actually maps $M_n(K)$ into $\mathsf{GL}(n; K)$, and hence Proposition 3.2 can be improved. Item 5 gives a way to compute the exponential of a matrix from its Jordan form.

Proposition 3.4. *Let X be a $n \times n$ complex matrix. Then e^{tX} is a smooth curve in $M_n(\mathbb{C})$ and*

$$\frac{d}{dt} e^{tX} = X e^{tX} = e^{tX} X. \tag{3.2}$$

In particular, $\frac{d}{dt} e^{tX}\big|_{t=0} = X$.

The validity of the proposition comes from the convergence of the series. In particular, every element of the matrix e^{tX} is given by a convergent power series in t, and we can differentiate it.

Definition 3.5. For a square matrix $A \in M_n(K)$, its ***matrix logarithm*** is defined by

$$\ln A = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(A-I)^n}{n} \tag{3.3}$$

whenever the series converges.

Next two theorems give us information about the matrix exponential and logarithm: the domain of the logarithm and the image of the exponential. A proof of them can be found in [7].

Theorem 3.6. *The function* $\ln A$ *is defined and it is a continuous function for all* $A \in M_n(\mathbb{C})$ *with* $\|A - I\| < 1$. *In this case,* $e^{\ln A} = A$. *Furthermore, for all* $X \in M_n(\mathbb{C})$ *with* $\|X\| < \ln 2$, $\left\|e^X - I\right\| < 1$ *and* $\ln e^X = X$.

Theorem 3.7. *Every square invertible matrix can be expressed as* e^X *for some* $X \in M_n(\mathbb{C})$.

This is, the exponential of complex matrices is surjective over $\mathsf{GL}(n;\mathbb{C})$. For real matrices it is not true (for $n = 1$ it already fails).

Theorem 3.8 (Lie product formula). *For all* $X, Y \in M_n(\mathbb{C})$,

$$e^{X+Y} = \lim_{m\to\infty} \left(e^{\frac{X}{m}} e^{\frac{Y}{m}}\right)^m . \tag{3.4}$$

Proof. We have that

$$e^{\frac{X}{m}} e^{\frac{Y}{m}} = I + \frac{X}{m} + \frac{Y}{m} + O\left(\frac{1}{m^2}\right).$$

Therefore,

$$\lim_{m\to\infty} e^{\frac{X}{m}} e^{\frac{Y}{m}} = I.$$

So there exists an $m_0 \in \mathbb{N}$ such that $\left\|e^{\frac{X}{m}} e^{\frac{Y}{m}} - I\right\| < 1$ for all $m \geq m_0$. Hence, we can take the logarithm for such m.

$$\ln\left(e^{\frac{X}{m}} e^{\frac{Y}{m}}\right) = \ln\left(I + \frac{X}{m} + \frac{Y}{m} + O\left(\frac{1}{m^2}\right)\right) = \frac{X}{m} + \frac{Y}{m} + O\left(\frac{1}{m^2}\right).$$

The last equality is obtained from the Taylor series of the logarithm. Now taking the exponential on both sides,

$$\left(e^{\frac{X}{m}} e^{\frac{Y}{m}}\right) = \exp\left(\frac{X}{m} + \frac{Y}{m} + O\left(\frac{1}{m^2}\right)\right) \Longrightarrow \left(e^{\frac{X}{m}} e^{\frac{Y}{m}}\right)^m = \exp\left(X + Y + O\left(\frac{1}{m}\right)\right).$$

Whence,

$$e^{X+Y} = \lim_{m\to\infty} \left(e^{\frac{X}{m}} e^{\frac{Y}{m}}\right)^m .$$

□

Proposition 3.9. *For any* $A \in \mathbb{C}$,

$$\det e^A = e^{\mathrm{tr} A}. \tag{3.5}$$

Proof. This result can be seen as a corollary of Liouville's theorem on Differential Equations. Consider the linear homogeneous differential equation, where $A : I \subset \mathbb{R} \to \mathbb{C}^n \times \mathbb{C}^n$,

$$x' = A(t)x.$$

If $M(t)$ is a matrix solution, Liouville's theorem states that

$$\det M(t) = \det M(t_0) e^{\int_{t_0}^t \mathrm{tr} A(s) ds}.$$

Now, setting the Cauchy problem with initial time $t = 0$ and initial value Id, and with our matrix A

$$\left\{\begin{array}{lcl} x' & = & Ax \\ x(0) & = & Id \end{array}\right. \Rightarrow \left\{\begin{array}{lcl} M(t) & = & e^{tA} \\ M(0) & = & Id \end{array}\right.$$

Setting $t = 1$, Liouville's theorem gives

$$\det M(1) = \det e^A = \det M(0) e^{\int_0^1 \operatorname{tr} A ds} = e^{\operatorname{tr} A}.$$

□

Definition 3.10. A function $A : \mathbb{R} \to \mathsf{GL}(n; \mathbb{C})$ is called a ***one-parameter subgroup*** of $\mathsf{GL}(n; \mathbb{C})$ if

1. A is continuous,
2. $A(0) = I$,
3. $A(t+s) = A(t)A(s)$ for all $t, s \in \mathbb{R}$.

This definition is consistent with the one given for general Lie groups.

Theorem 3.11. *If $A(\cdot)$ is a one-parameter subgroup of $\mathsf{GL}(n; \mathbb{C})$, there exists a unique $X \in M_n(\mathbb{C})$ such that*

$$A(t) = e^{tX}.$$

The proof of this result, which can be found in [7], is based in the fact that near enough to the identity, the exponential is injective with a continuous inverse, the logarithm. This yields to the property that every matrix $A(t_0)$ in a suitable neighbourhood of the identity has a unique square root in the neighbourhood, given by $\exp(\frac{1}{2}\ln(A(t_0)))$. Writing $X = \frac{1}{t_0}\ln(A(t_0))$, so that $t_0 X = \ln(A(t_0))$. Hence, $\exp(t_0 X) = A(t_0)$ and it belongs to the neighbourhood, just as $\exp(t_0 X/2) = A(t_0/2)$, the unique square root. This can be repeated and we obtain for any $k \in \mathbb{Z}$ $\exp(t_0 X/2^k) = A(t_0/2^k)$ and for $m \in \mathbb{Z}$, $\exp(t_0 X/2^k)^m = A(mt_0/2^k)$. The result is valid for a dense set of $\mathbb{R}$, the numbers of the form $mt_0/2^k$, and for continuity, $\mathbb{R}$. Uniqueness comes from taking the derivative at $t = 0$.

4 Lie Algebras

4.1 Definition and properties

Lie algebras are usually defined independently of Lie groups. However, one of the main examples are the Lie algebras associated to Lie groups.

Definition 4.1. Let K be a field, a (finite-dimensional) K-***Lie algebra*** is a (finite-dimensional) K-vector space, $\mathfrak{g}$, with a map

$$\begin{array}{rccc} [\cdot,\cdot] : & \mathfrak{g} \times \mathfrak{g} & \to & \mathfrak{g} \\ & (X, Y) & \to & [X, Y] \end{array}$$

with the following properties for all $X, Y, Z \in \mathfrak{g}$

1. $[\cdot,\cdot]$ is K-bilinear.
2. $[\cdot,\cdot]$ is antisymmetric: $[X, Y] = -[Y, X]$.

3. The Jacobi identity holds,

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0.$$

We say that two elements X and Y of $\mathfrak{g}$ commute if $[X,Y]=0$. We say that a Lie algebra is commutative if $[X,Y]=0$ for all $X,Y\in\mathfrak{g}$.

Remark: This operation, called the bracket or Lie bracket operation, is usually not associative. The Jacobi identity can be seen as a substitute for the associative property:

$$[X,[Y,Z]]-[[X,Y],Z]=-[Y,[Z,X]].$$

As an example, the vector space $\mathbb{R}^3$ forms a Lie algebra defining the Lie bracket of two vectors as its cross product

$$[u,v]=u\times v \text{ for } u,v\in\mathbb{R}^3.$$

Definition 4.2. A ***subalgebra*** of a real or complex Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h}$ of $\mathfrak{g}$ such that $[H_1,H_2]\in\mathfrak{h}$ for all $H_1,H_2\in\mathfrak{h}$.

Next definition is analogous to the one for Lie groups.

Definition 4.3. If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, then a linear map $\phi:\mathfrak{g}\to\mathfrak{h}$ is called a ***Lie algebra homomorphism*** if $\phi([X,Y])=[\phi(X),\phi(Y)]$ for all $X,Y\in\mathfrak{g}$. If, in addition, ϕ is bijective, then it is called a ***Lie algebra isomorphism***. In the latter case, if $\mathfrak{g}=\mathfrak{h}$, then it is called a Lie algebra automorphism.

Definition 4.4. For a fixed $X\in\mathfrak{g}$, we can define a linear map $\mathrm{ad}_X:\mathfrak{g}\to\mathfrak{g}$ by

$$\begin{array}{rrcl}\mathrm{ad}_X: & \mathfrak{g} & \to & \mathfrak{g}\\ & Y & \to & \mathrm{ad}_X(Y)=[X,Y]\end{array}$$

Linearity comes from the bilinearity of the bracket operation. The map $X\to\mathrm{ad}_X$ is called the ***adjoint map*** or adjoint representation and can be seen as the linear map

$$\begin{array}{rrcl}\mathrm{ad}: & \mathfrak{g} & \to & \mathrm{End}(\mathfrak{g})\\ & X & \to & \mathrm{ad}_X:\mathfrak{g}\to\mathfrak{g}\end{array}$$

Notice that $\mathrm{ad}_X(Y)$ is just the same as $[X,Y]$, but it is useful to see it this way. Rewriting the Jacobi identity,

$$\mathrm{ad}_X([Y,Z])+[Y,-\mathrm{ad}_X(Z)]+[Z,\mathrm{ad}_X(Y)]=0.$$

We see that ad_X can be considered as a *derivation* with respect to the Lie bracket:

$$\mathrm{ad}_X([Y,Z])=[\mathrm{ad}_X(Y),Z]+[Y,\mathrm{ad}_X(Z)].$$

Let $\varphi,\psi\in\mathrm{End}(\mathfrak{g})$, its Lie bracket is defined by $[\varphi,\psi]=\varphi\circ\psi-\psi\circ\varphi$. Thus, for the images of the adjoint map it is $[\mathrm{ad}_X,\mathrm{ad}_Y]:=\mathrm{ad}_X\mathrm{ad}_Y-\mathrm{ad}_Y\mathrm{ad}_X$ and it is easy to see that it is again a Lie algebra.

Proposition 4.5. *If $\mathfrak{g}$ is a Lie algebra, then $\mathrm{ad}:\mathfrak{g}\to End(\mathfrak{g})$ is a Lie algebra homomorphism.*

Proof. We only have to see that

$$\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y].$$

Let $Z \in \mathfrak{g}$, then

$$\mathrm{ad}_{[X,Y]}(Z) = [[X,Y],Z]$$

whereas

$$[\mathrm{ad}_X, \mathrm{ad}_Y](Z) = \mathrm{ad}_X \mathrm{ad}_Y(Z) - \mathrm{ad}_Y \mathrm{ad}_X(Z) = [X,[Y,Z]] - [Y,[X,Z]].$$

The equality of both expressions comes from the Jacobi identity. □

Definition 4.6. If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are Lie algebras, the ***direct sum*** of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is the vector space $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, with Lie bracket given by

$$[(X_1, X_2), (Y_1, Y_2)] = ([X_1, Y_1], [X_2, Y_2]).$$

If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are subalgebras, $\mathfrak{g}$ decomposes as the Lie algebra direct sum of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ if, as vector spaces, $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $[\mathfrak{g}_1, \mathfrak{g}_2] = 0$, i.e. $[X,Y] = 0$ for all $X \in \mathfrak{g}_1$, $Y \in \mathfrak{g}_2$.

Definition 4.7. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, $X_1, \ldots, X_N$ a basis for $\mathfrak{g}$. Then the unique constants $c_{jkl} \in K$ such that

$$[X_j, X_k] = \sum_{l=1}^{N} c_{jkl} X_l.$$

are called the ***structure constants*** of $\mathfrak{g}$ for the basis $X_1, \ldots X_N$.

4.2 Lie algebras of Lie groups

Recall that Lie brackets of smooth vector fields on smooth manifolds are defined. The Lie bracket sends smooth vector fields to smooth vector fields and it has the following coordinate expression,

$$[X,Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

It can also be seen as a derivation, $[X,Y] := \mathcal{L}_X Y$ is called the Lie derivative of Y along the flow generated by X.

Now, we want to define the Lie algebra associated to a Lie group, we will see that its elements are *left invariant vector fields* on G.

Definition 4.8. A vector field X on G is called ***left invariant*** if for each $g \in G$,

$$dL_g \circ X = X \circ L_g.$$

Since L_g is a diffeomorphism, left invariance can be expressed with the pushforward of the vector field

$$(L_g)_* X = X.$$

Proposition 4.9. *Let G be a Lie group and $\mathfrak{g}$ the set of its left invariant vector fields. Then:*

1. *Left invariant vector fields are smooth.*
2. *The Lie bracket of two left invariant vector fields is itself a left invariant vector field.*
3. $\mathfrak{g}$ *is a real vector space and the map* $\alpha : \mathfrak{g} \to T_eG$ *defined by* $\alpha(X) = X_e$ *is an isomorphism of* $\mathbb{R}-$*vector spaces between* $\mathfrak{g}$ *and the tangent space* T_eG *to* G *at the identity. Hence,* $\dim \mathfrak{g} = \dim T_eG = \dim G$.
4. $\mathfrak{g}$ *forms a Lie algebra under the Lie bracket operation on vector fields.*

Proof. From the fact that, if $F : G \to G$ is a diffeomorphism, its pushforward acts on Lie brackets like

$$F_*[X_1, X_2] = [F_*X_1, F_*X_2]$$

we have proved item 1, since

$$(L_g)_*[X, Y] = [(L_g)_*X, (L_g)_*Y] = [X, Y].$$

The linearity of α is clear. The fact that $\mathfrak{g}$ is a real vector space comes from $(L_g)_*(aX + bY) = a(L_g)_*X + b(L_g)_*Y$ for $a, b \in \mathbb{R}$. It is injective, if $\alpha(X) = X_e = 0$ for some $X \in \mathfrak{g}$, the left invariance of X implies that $X_g = d(L_g)_e(X_e) = 0$ for all $g \in G$, so $X = 0$. To see that it is a surjection, let $v \in T_eG$ and define a vector field on G by

$$v^L|_g = d(L_g)_e(v).$$

It is left invariant,

$$d(L_h)_g \left(v^L|_g\right) = d(L_h)_g \circ d(L_g)_e(v) = d(L_h \circ L_g)_e(v) = d(L_{hg})_e(v) = v^L|_{hg}.$$

So $v^L \in \mathfrak{g}$ and $\alpha(v^L) = v^L|_e = v$ because L_e is the identity map of G. Thus, α is surjective and we have proved the second item.
To see item 3, let $X \in \mathfrak{g}$ and let $f \in \mathcal{C}^\infty(G)$, if we show that $Xf \in \mathcal{C}^\infty(G)$ we are done. Now,

$$(Xf)g = X_g f = d(L_g)_e X_e f = X_e(f \circ L_g)$$

and this function is a composition of smooth maps.
Since left invariant vector fields are smooth, their Lie bracket is defined and it is also a left invariant vector field. Hence, $\mathfrak{g}$ is a Lie algebra. □

We have seen that the set of smooth left invariant vector fields on a Lie group G, denoted $\mathfrak{X}_{li}(G)$, together with the Lie bracket operation, forms a Lie algebra which as a vector space is isomorphic to T_eG. From this, we have the definition,

Definition 4.10. Let G be a Lie group. We define the ***Lie algebra*** of G, denoted $\mathfrak{g}$, as the Lie algebra of left invariant vector fields on G.

Remark: The Lie algebra $\mathfrak{g}$ of a Lie group G can be realized equivalently as the tangent space to the identity, T_eG, with the Lie bracket inherited from the isomorphism as vector spaces between $\mathfrak{g}$ and T_eG.

Now, since every left invariant vector field on a Lie group is smooth, any basis of the Lie algebra will give us a left invariant global frame. Recalling that a manifold is parallelizable if it admits a global frame, we get the following important property.

Corollary 4.11. *Every Lie group is parallelizable.*

This result is applicable for the case of studying the spheres $\mathbb{S}^n$ and looking for which $n \in \mathbb{N}$ the manifold has a differentiable structure and an operation which makes it into a Lie group. Since $\mathbb{S}^n$ is only parallelizable for $n = 0, 1, 3, 7$, we can restrict the study to these cases. It is interesting to note that these cases are, respectively, the unit elements of the real numbers, the complex numbers, the quaternions and the octonions. It turns out that the only one which is not a Lie group is $\mathbb{S}^7$, which is related with the fact that the octonions are not a group with the multiplication. The case $\mathbb{S}^3$ will be studied later.

4.3 Lie algebras of matrix Lie groups

Definition 4.12. The ***Lie algebra of a matrix Lie group*** G, denoted $\mathfrak{g}$, is the set of all matrices X such that $e^{tX} \in G$ for all $t \in \mathbb{R}$.

Theorem 4.13. *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. For $X, Y \in \mathfrak{g}$, the following results hold.*

1. $AXA^{-1} \in \mathfrak{g}$ $\forall A \in G$.
2. $sX \in \mathfrak{g}$ $\forall s \in \mathbb{R}$.
3. $X + Y \in \mathfrak{g}$.
4. $XY - YX \in \mathfrak{g}$.

Proof. Using the properties of the exponential in (3.3) and the Lie product formula (3.8) we have the first three properties proven. The proof of the fourth comes from the facts that if two matrix-valued functions of $t \in \mathbb{R}$, $A(t), B(t)$, are smooth, then $A(t)B(t)$ is smooth and

$$\frac{d}{dt}[A(t)(B(t)] = \frac{dA}{dt}B(t) + A(t)\frac{dB}{dt}$$

and that, as we have seen in 3.4,

$$\left.\frac{d}{dt}e^{tX}\right|_{t=0} = X.$$

Now,

$$\left.\frac{d}{dt}\left(e^{tX}Ye^{-tX}\right)\right|_{t=0} = (XY)e^0 + (e^0Y)(-X) = XY - YX.$$

For item 1, $\left(e^{tX}Ye^{-tX}\right) \in \mathfrak{g}$, for all $t \in \mathbb{R}$. For items 2 and 3, we see that $\mathfrak{g}$ is a real subspace of $M_n(\mathbb{C})$, so it is a closed subset of $M_n(\mathbb{C})$. We have, then

$$XY - YX = \lim_{h\to 0} \frac{e^{hX}Ye^{-hX} - Y}{h} \in \mathfrak{g}.$$

□

Defining the bracket operation as $[X, Y] = XY - YX$, one easily checks that it is bilinear, antisymmetric and the Jacobi identity holds. Thus, the Lie algebra of a matrix Lie group is a Lie algebra in the sense of Definition 4.1.

Noticing that for every $X \in \mathfrak{g}$, e^X and the identity matrix $I = e^0$ are connected by the path $t \to e^{tX}$, for $t \in [0, 1]$, we have the following proposition.

Proposition 4.14. *Let G be a matrix Lie group and X an element of its Lie algebra. Then, e^X is an element of the identity component G_0 of G.*

Next examples are the Lie algebras associated to the examples given for matrix Lie groups.

4.3.1 Examples

1. $\mathsf{gl}(\mathbf{n};\mathbf{K})$. The Lie algebra of $\mathsf{GL}(n;K)$, denoted $\mathsf{gl}(n;K)$, is the space $M_n(K)$. For every matrix $X \in M_n(K)$, e^X is invertible and with coefficients on K.
2. $\mathsf{sl}(\mathbf{n};\mathbf{K})$. If X is any matrix with $\det(e^{tX}) = 1$ for all $t \in \mathbb{R}$, then, for proposition 3.9, $t \cdot \text{trace}(X) = i2\pi n$ for all $t \in \mathbb{R}$ for some $n \in \mathbb{Z}$. Which means that $\text{trace}(X) = 0$. $\mathsf{sl}(n;K) := \{X \in M_n(K) | \text{trace}(X) = 0\}$ is the Lie algebra of $\mathsf{SL}(n;K)$.
3. $\mathsf{o}(\mathbf{n})$. Given the matrix $X \in M_n(\mathbb{R})$, e^{tX} is orthogonal if and only if $(e^{tX})^\top = (e^{tX})^{-1} = e^{-tX} = e^{tX^\top}$, the last two equalities come from 3.3. If $-tX = tX^\top$ the equalities hold. And if it holds for all t, then
$$-X = X^\top.$$
So the Lie algebra of $\mathsf{O}(n)$ is $\mathsf{o}(n) := \{X \in M_n(\mathbb{R}) \text{ s.t. } -X = X^\top\}$.
4. $\mathsf{so}(\mathbf{n})$. The property $-X = X^\top$ is the same as saying that the matrix is antisymmetric. In particular, this implies that the trace of the matrix is zero. And we have seen that if the trace of X is zero, then $\det(e^{tX}) = 1$ for all t. So, every matrix in $\mathsf{o}(n)$ is also in $\mathsf{so}(n)$, the Lie algebra of $\mathsf{SO}(n)$ and by definition, every matrix in $\mathsf{so}(n)$ is in $\mathsf{o}(n)$. Thus, $\mathsf{o}(n)$ and $\mathsf{so}(n)$ are equal.
5. $\mathsf{u}(\mathbf{n})$. e^{tX} is unitary if and only if
$$(e^{tX})^\dagger = (e^{tX})^{-1}.$$
Which, using properties of the exponential, is equivalent to
$$e^{tX^\dagger} = e^{-tX}.$$
This holds if $X^\dagger = -X$. And if the equality holds for all t, then the condition is necessary. Thus, the Lie algebra of $\mathsf{U}(n)$ is the set of those $X \in M_n(\mathbb{C})$ with the property $X^\dagger = -X$. It is denoted $\mathsf{u}(n)$.
6. $\mathsf{su}(\mathbf{n})$. $X^\dagger = -X$ doesn't imply that its trace is zero, so the Lie algebra of $\mathsf{SU}(n)$ is the subset of $\mathsf{u}(n)$ of matrices with trace zero, denoted $\mathsf{su}(n)$.
7. $\mathsf{sp}(\mathbf{n};\mathbf{C})$. The Lie algebra of $\mathsf{SP}(n;\mathbb{C})$, denoted $\mathsf{sp}(n;\mathbb{C})$, is the space of matrices $X \in M_n(\mathbb{C})$ which verify $e^{tX} \in \mathsf{SP}(n;\mathbb{C})$. That is, those which satisfy $-\Omega(e^{tX})^\top\Omega = (e^{tX})^{-1}$. Now, the next equalities are a straightforward computation,
$$-\Omega(e^{tX})^\top\Omega = -\Omega e^{tX^\top}\Omega = e^{-tX}.$$
This has to hold for all real t, so
$$-\Omega\frac{d}{dt}e^{tX^\top}\Omega = \frac{d}{dt}e^{-tX}$$

also has to hold for all real t, and for $t = 0$ we have

$$-\Omega X^\top \Omega = -X \iff \Omega X^\top \Omega = X.$$

Then, $\mathsf{sp}(n; \mathbb{C})$ is the set of complex matrices for which this last equality holds.

8. $\mathsf{sp}(\mathbf{n})$. The Lie algebra of $\mathsf{SP}(n)$ is the space of complex matrices X such that $\Omega X^\top \Omega = X$ and $X^\dagger = -X$.

Before going further, we will take a look at covering spaces. We do so in order to see that every Lie group has a *universal covering space* which is also a Lie group. This fact will have implications which will be useful for the study of the relationship between Lie groups and Lie algebras.

5 Covering spaces

In this section X will denote a path connected and locally path connected topological space and I the compact metric space $[0, 1] \subset \mathbb{R}$.

Definition 5.1. A ***covering space*** of X is a pair $(\tilde{X}, p)$ consisting of a space $\tilde{X}$ and a continuous map $p : \tilde{X} \to X$ such that for all $x \in X$ there is an open path connected neighbourhood, U_x, and for each path connected component U_i of $p^{-1}(U_x)$, the map $p|_{U_i} : U_i \to U_x$ is a homeomorphism.

Remark: This definition requires $p^{-1}(U_x)$ to be non-empty i.e. p is a surjection. Every neighbourhood U satisfying the condition stated is called an ***elementary neighbourhood***. The map p is called a ***projection***. It is easily seen that if $(\tilde{X}, p)$ is a covering space of X, then p is a local homeomorphism[3]. This comes from the fact that the path connected components of an open set of a locally path connected space are open. A local homeomorphism is an open map, so if $(\tilde{X}, p)$ is a covering space, then p is an open map.

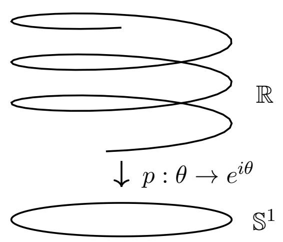

Figure 2: $\mathbb{R}$ is a covering space of $\mathbb{S}^1$

Definition 5.2. If $(\tilde{X}, p)$ is a covering space and $f : A \to X$ is a continuous map, a ***lift of*** f is a continuous map $\tilde{f} : A \to \tilde{X}$ such that $p \circ \tilde{f} = f$.

Now let's introduce some useful lemmas and theorems. Some of the next results are direct and some more arduous, we will not prove any of them though. Proofs can be found in many introductory book of Algebraic Topology (we have followed [13]).

[3] See definition A.2

Lemma 5.3 (Unique lifting property)**.** *Let $(\tilde{X}, p)$ be a covering space of X and let Y be a space which is connected. Given any two continuous maps $f_0, f_1 : Y \to \tilde{X}$ such that $p \circ f_0 = p \circ f_1$, the set $\{y \in Y | \ f_0(y) = f_1(y)\}$ is either empty or all of Y.*

Lemma 5.4. *Let $(\tilde{X}, p)$ be a covering space of X and let $g_0, g_1 : I \to \tilde{X}$ be paths in $\tilde{X}$ which have the same initial point. If $pg_0 \sim pg_1$, then $g_0 \sim g_1$. In particular, g_0 and g_1 have the same terminal point.*

Lemma 5.5. *If $(\tilde{X}, p)$ is a covering space of X, then the sets $p^{-1}(x)$ have the same cardinal number for all $x \in X$.*

This cardinal number is called the ***number of sheets*** of the covering space.

Theorem 5.6. *Let $(\tilde{X}, p)$ be a covering space of X, $\tilde{x}_0 \in \tilde{X}$, and $x_0 = p(\tilde{x}_0)$. Then, the induced homomorphism $p_* : \pi(\tilde{X}, \tilde{x}_0) \to \pi(X, x_0)$ is injective.*

Theorem 5.7. *Let $(\tilde{X}, p)$ be a covering space of X, $\tilde{x}_0 \in \tilde{X}$. Then, the subgroups $p_*\pi(\tilde{X}, \tilde{x})$ for $\tilde{x} \in p^{-1}(x_0)$ are exactly a conjugacy class of subgroups of $\pi(X, x_o)$.*

Notation: (X, x) is a pointed space and $f : (Y, y) \to (X, x)$ means that $f : Y \to X$ and $f(y) = x$.

Theorem 5.8. *Let $(\tilde{X}, p)$ be a covering space of X, Y a connected, locally path connected space, $y_0 \in Y$, $\tilde{x}_0 \in \tilde{X}$ and $x_0 = p(\tilde{x}_0)$. Given a map $\varphi : (Y, y_0) \to (X, x_0)$, there exists a lifting $\tilde{\varphi} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ if and only if $\varphi_*\pi(Y, y_0) \subset p_* \ \pi(\tilde{X}, \tilde{x}_0)$.*

Definition 5.9. Let $(\tilde{X}_1, p_1)$, $(\tilde{X}_2, p_2)$ covering spaces of X. A ***homomorphism of*** $(\tilde{X}_1, p_1)$ ***into*** $(\tilde{X}_2, p_2)$ is a continuous map $\varphi : \tilde{X}_1 \to \tilde{X}_2$ such that the diagram

$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "\varphi"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

commutes. If there is an homomorphism ψ of $(\tilde{X}_2, p_2)$ into $(\tilde{X}_1, p_1)$ such that $\psi\varphi$ and $\varphi\psi$ are identity maps then we say that the two spaces are ***isomorphic***.

Lemma 5.10. *Let $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ be covering spaces of X and $\tilde{x}_i \in \tilde{X}_i$, $i = 1, 2$ points such that $p_1(\tilde{x}_1) = p_2(\tilde{x}_2)$. Then, there exists a homomorphism φ of $(\tilde{X}_1, p_1)$ into $(\tilde{X}_2, p_2)$ such that $\varphi(\tilde{x}_1) = \tilde{x}_2$ if and only if $p_{1*}\pi(\tilde{X}_1, \tilde{x}_1) \subset p_{2*}\pi(\tilde{X}_2, \tilde{x}_2)$*

Theorem 5.11. *Two covering spaces $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ of X are isomorphic if and only if for any two points $\tilde{x}_i \in \tilde{X}_i$, $i = 1, 2$ such that $p_1(\tilde{x}_1) = p_2(\tilde{x}_2) = x_0$ the subgroups $p_{1*}\pi(\tilde{X}_1, \tilde{x}_1)$ and $p_{2*}\pi(\tilde{X}_2, \tilde{x}_2)$ belong to the same conjugacy class in $\pi(X, x_0)$.*

Lemma 5.12. *Let $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ covering spaces of X and let φ be a homomorphism of the first covering space into the second. Then, $(\tilde{X}_1, \varphi)$ is a covering space of $\tilde{X}_2$.*

Let $(\tilde{X}, p)$ be a covering space of X such that $\tilde{X}$ is simply connected. If $(\tilde{X}', p')$ is another covering space of X then, by Lemma 5.10 there exists a homomorphism φ of $(\tilde{X}, p)$ onto $(\tilde{X}', p')$ and, by Lemma 5.12, $(\tilde{X}, \varphi)$ is a covering space of $\tilde{X}'$. Hence, $\tilde{X}$ can be a covering space of any covering of X and this leads to the definition:

Definition 5.13. A simply connected covering space $(\tilde{X}, p)$ of X is called a ***universal covering space***.

Remark: By Theorem 5.11, any two universal covering spaces of X are isomorphic. Now we want to see the necessary conditions for X to have a universal covering space: Let $(\tilde{X}, p)$ be a universal covering space of X, $x \in X$ an arbitrary point, $\tilde{x} \in \tilde{X}$ a point of $p^{-1}(x)$, U an elementary neighbourhood of x and V the component of $p^{-1}(U)$ which contains $\tilde{x}$. We then have the following commutative diagram

$$\begin{tikzcd} \pi(V,\tilde{x}) \arrow[r] \arrow[d, "(p|V)_*"] & \pi(\tilde{X},\tilde{x}) \arrow[d, "p_*"] \\ \pi(U,x) \arrow[r, "i_*"] & \pi(X,x) \end{tikzcd}$$

First of all, since $(p|V)$ is a homeomorphism of V onto U, $(p|V)_*$ is an isomorphism. Moreover, noting that $\pi(X, x) = \{1\}$ and that the diagram is commutative, we obtain that i_* is a trivial homomorphism, i.e., $i_*(\pi(U, x)) = \{1\}$. Thus, the space X has the property stated in the next definition.

Definition 5.14. We say that a space X is ***semilocally simply connected*** if every point $x \in X$ has a neighbourhood U such that the homomorphism $\pi(U, x) \xrightarrow{i_*} \pi(X, x)$ is trivial. This is, any loop in U can be shrunk to a point within X.

Every *nice* topological space is semilocally simply connected, so it is not a very restrictive condition. Spaces which are not semilocally simply connected are often considered *pathological*, an example would be the *Hawaiian earring*. For instance, all manifolds and manifolds with boundary are semilocally simply connected. Going further, one can can find the sufficient conditions for a more general case,

Theorem 5.15. *Let X be a topological space which is connected, locally path connected and semilocally simply connected. Then, given any conjugacy class of subgroups of $\pi(X, x)$, there exists a covering space $(\tilde{X}, p)$ of X corresponding to the given conjugacy class (i.e. such that $p_*\pi(\tilde{X}, \tilde{x})$ belongs to the conjugacy class).*

and as a result,

Corollary 5.16. *Suppose that X is a connected, locally path connected and semilocally simply connected topological space. Then, X has a universal cover.*

The existence and uniqueness (up to homeomorphism) of the universal covering space are guaranteed by this result. Furthermore, for smooth manifolds,

Corollary 5.17. *If X is a connected smooth manifold, X has a universal covering space.*

Comment: These properties of the universal covering lead to a different definition of a simply connected space, used in texts as [1]. A simply connected space is defined as a connected, locally connected space which admits only covering spaces that are isomorphic to the trivial covering space. However, this definition is not equivalent to ours, as it is shown in [4], and it is not used here.

We are interested in universal covers of Lie groups which are Lie groups, so we have to go a little further. The next results can be found in [11]. For a topological covering map here we mean a covering map in the previous sense. We will use another kind of covering map, a ***smooth covering map***, which is a covering map that is a local diffeomorphism. The next result shows that every covering map over a smooth manifold can be converted into a smooth covering map.

Proposition 5.18. *Suppose M is a connected smooth n-manifold, and $p : E \to M$ is a topological covering map. Then E is a topological n-manifold, and has a unique smooth structure such that p is a smooth covering map.*

The proof consists in, firstly, show that the covering space E is Hausdorff and second countable, inheriting it from M with the fact that they are locally homeomorphic, and then, with the coordinate charts and the elementary neighbourhoods one can build its unique smooth structure.

Corollary 5.19. *If M is a connected smooth manifold, it has a universal smooth covering.*

And now, the awaited final result that we were looking for.

Theorem 5.20 (Existence and Uniqueness of a Universal Covering Group)**.** *Let G be a connected Lie group. There exists a simply connected Lie group $\tilde{G}$, called the* ***universal covering group of*** *G, that admits a smooth covering map $p : \tilde{G} \to G$ that is also a Lie group homomorphism. Furthermore, the universal covering group is unique up to isomorphism.*

Proof. Let $\tilde{G}$ be the universal covering manifold of G and $p : \tilde{G} \to G$ the corresponding smooth covering map. The map $p \times p : \tilde{G} \times \tilde{G} \to G \times G$ is also a smooth covering map. We denote by $m : G \times G \to G$ the multiplication map and by $i : G \to G$ the inversion map. Let $\tilde{e} \in p^{-1}(e) \subset \tilde{G}$. Since $\tilde{G}$ is simply connected, it is connected and we can apply the Lemma 5.3 and we have that the map $m \circ (p \times p) : \tilde{G} \times \tilde{G} \to G$ has a unique lift $\tilde{m} : \tilde{G} \times \tilde{G} \to \tilde{G}$ such that $\tilde{m}(\tilde{e}, \tilde{e}) = \tilde{e}$ and $p \circ \tilde{m} = m \circ (p \times p)$. We have the following commutative diagram

$$\begin{array}{ccc} \tilde{G} \times \tilde{G} & \xrightarrow{\tilde{m}} & \tilde{G} \\ \big\downarrow{\scriptstyle p \times p} & & \big\downarrow{\scriptstyle p} \\ G \times G & \xrightarrow{m} & G \end{array}$$

From the fact that p is a local diffeomorphism and that $p \circ \tilde{m}$ is smooth, it follows that $\tilde{m}$ is smooth. With the same reasoning as before we get the commutative diagram for the inversion map

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\tilde{i}} & \tilde{G} \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p} \\ G & \xrightarrow{i} & G \end{array}$$

where $\tilde{i} : \tilde{G} \to \tilde{G}$ is the smooth lift of $i \circ p : \tilde{G} \to G$ which satisfies $\tilde{i}(\tilde{e}) = \tilde{e}$ and $p \circ \tilde{i} = i \circ p$. We define these two smooth operations as the multiplication and inversion maps in $\tilde{G}$. This is, $\tilde{m}(x, y) = xy$ and $\tilde{i}(x) = x^{-1}$, for all $x, y \in \tilde{G}$. With the commutative diagrams, we have that:

$$p(xy) = p(x)p(y) \ \text{ and } \ p(x^{-1}) = p(x)^{-1} \tag{5.1}$$

and so proving that $\tilde{G}$ with these operations is a Lie group we would have that p is a homomorphism. The proof that $\tilde{G}$ is a Lie group is quite simple. Using the unique lifting property, it is easily seen that $\tilde{e}$ is the neutral element, that $xx^{-1} = x^{-1}x = \tilde{e}$ and that the multiplication is associative.
Finally, the uniqueness announced is in the sense that if $\tilde{G}$ and $\tilde{G}'$ are both universal covering groups of G, with smooth covering maps p and p', then there exists a Lie group isomorphism $\Phi : \tilde{G} \to \tilde{G}'$ such that $p' \circ \Phi = p$. From the fact that G is a smooth manifold, we know that its universal cover is unique in the sense that we have a diffeomorphism

$\Phi : \tilde{G} \to \tilde{G}'$ such that $p' \circ \Phi = p$. It can be seen that Φ is also a homomorphism, this is, the following diagram

$$\begin{array}{ccc} \tilde{G} \times \tilde{G} & \xrightarrow{\Phi \times \Phi} & \tilde{G}' \times \tilde{G}' \\ \Big\downarrow{\scriptstyle \tilde{m}} & & \Big\downarrow{\scriptstyle \tilde{m}'} \\ \tilde{G} & \xrightarrow{\Phi} & \tilde{G}' \end{array}$$

is commutative. Due to the fact that $\Phi \circ \tilde{m}$ and $\tilde{m}' \circ (\Phi \times \Phi)$ are lifts of the map $m \circ (p \times p)$, if we fix $\Phi(\tilde{e}) = \tilde{e}'$, $\Phi \circ \tilde{m}$ and $\tilde{m}' \circ (\Phi \times \Phi)$ agree on $(\tilde{e}, \tilde{e})$ and for the unique lifting property (Lemma 5.3), are the same map. Finally, a diffeomorphic homomorphism is an isomorphism. $\square$

6 Lie Groups and Lie Algebras

A Lie group homomorphism, namely $F : G \to H$, maps the identity of $e \in G$ to the identity of $e' \in H$. Hence, the differential at the identity $dF_e : T_eG \to T_{e'}H$ is $\mathbb{R}$-linear. Since T_eG and $T_{e'}H$ are canonically isomorphic (as vector spaces) to the Lie algebras of G and H, $\mathfrak{g}$ and $\mathfrak{h}$, dF_e induces a linear transformation from $\mathfrak{g}$ to $\mathfrak{h}$. We denote this induced map by $dF : \mathfrak{g} \to \mathfrak{h}$. For $X \in \mathfrak{g}$, $dF(X)$ will be the unique left invariant vector field on H such that

$$dF(X)_{e'} = dF_e(X_e), \tag{6.1}$$

(see Proposition 4.9). Furthermore, this linear map is a Lie algebra homomorphism:

Theorem 6.1. *Let $F : G \to H$ be a Lie group homomorphism and X a left invariant vector field in G. The left invariant vector field $dF(X)$ on H is the unique left invariant vector field such that $[dF(X), dF(Y)] = dF([X, Y])$ for all left invariant vector fields Y in G. Hence,*

$$dF : \mathfrak{g} \to \mathfrak{h}$$

is a Lie algebra homomorphism.

Proof. Let $g, h \in G$, using the fact that F is a homomorphism, $F(gh) = F(g)F(h)$, we obtain that $L_{F(g)} \circ F = F \circ L_g$, so

$$\begin{aligned} dF(X)_{F(g)} &= (dL_{F(g)})_{e'} dF(X)_{e'} && \text{from left invariance} \\ &= (dL_{F(g)})_{e'} dF_e(X_e) && \text{from the value of } dF(X) \text{ at } e' \\ &= d(L_{F(g)} \circ F)_e(X_e) && \text{by the chain rule} \\ &= d(F \circ L_g)_e X_e && \text{from } F \text{ being a homomorphism} \\ &= dF_g(X_g) && \text{by the chain rule again.} \end{aligned}$$

The equality $dF(X)_{F(g)} = dF_g(X_g)$ holds for all $g \in G$ so $dF(X)$ and X are $F-$related[4]. Having seen this, the fact that that dF is a Lie algebra homomorphism comes from the fact that two pairs of $F-$related vector fields have $F-$related Lie brackets, known as the naturality of Lie brackets. This can be seen considering the way that $F-$related vector fields act on functions or with the expression of the Lie bracket as a Lie derivative. $\square$

We have seen that dF is the unique Lie algebra homomorphism which makes the following diagram commutative.

[4] See Definition A.7

$$
\begin{tikzcd}
\mathfrak{g} \arrow[r, "dF"] \arrow[d, "\cong"'] & \mathfrak{h} \arrow[d, "\cong"] \\
T_eG \arrow[r, "dF_e"] & T_{e'}H
\end{tikzcd}
$$

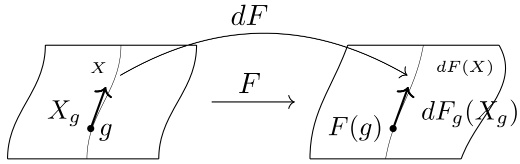

Figure 3: Sketch of Theorem 6.1

The identity map $\mathrm{Id} : G \to G$ induces the identity map on $\mathfrak{g}$. That is, $d(\mathrm{Id}) = \mathrm{Id}_{\mathfrak{g}}$. Moreover, the differential obeys the chain rule and hence, the induced map of a composition of homomorphisms, $d(F_1 \circ F_2)$, will be the composition of the induced maps $dF_1 \circ dF_2$. Isomorphic Lie groups have isomorphisms which go both ways. Whence,

Corollary 6.2. *Isomorphic Lie groups have isomorphic Lie algebras.*

Remark: In fact, we have seen that there is a functor from the category of Lie groups to the category of finite-dimensional real Lie algebras.

The converse of this result is not usually true but it is enough to require Lie groups to be simply connected,

Theorem 6.3. *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ and with G simply connected. Let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a homomorphism. Then there exists a unique homomorphism $\Phi : G \to H$ such that $d\Phi = \phi$.*

The proof can be found in [26]. As a result, we have

Corollary 6.4. *Simply connected Lie groups with isomorphic Lie algebras are isomorphic.*

To finish this section, we introduce a useful result, the proof of which is in [26] too.

Proposition 6.5. *Let G and H be connected Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, let $F : G \to H$ be a Lie group homomorphism. Then F is a smooth covering map if and only if $dF : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra isomorphism.*

6.1 The exponential map

First of all, we state two propositions regarding integral curves, vector fields and one-parameter subgroups. Recall that a complete vector field is a vector field which has every maximal integral curve defined for all $t \in \mathbb{R}$,

Proposition 6.6. *Every left invariant vector field on a Lie group is complete.*

The proof is based on the Uniform Time Lemma[5] applying the left translation over the identity to every g in the Lie group. Having seen this, next proposition's proof can be done using the naturality[6] of integral curves (cf. [11]),

Proposition 6.7. *The one-parameters subgroups of a Lie group G are the maximal integral curves $\gamma : \mathbb{R} \to G$ of left invariant vector fields with $\gamma(0) = e$, which implies that $\gamma'(0) \in T_eG \cong \mathfrak{g}$.*

[5] See A.12
[6] See A.13

Now, let G be a Lie group, $\mathfrak{g}$ its Lie algebra and d/dt a left invariant vector field on $\mathbb{R}$, i.e. an element of the Lie algebra of $\mathbb{R}$, we can define a homomorphism

$$\frac{d}{dt} \mapsto X$$

where X is an element of $\mathfrak{g}$. Since $\mathbb{R}$ is simply connected, by Theorem 6.3 this induces the unique homomorphism

$$\gamma : \mathbb{R} \to G$$

such that

$$d\gamma\left(\frac{d}{dt}\right) = X.$$

Since γ is a homomorphism from $\mathbb{R} \to G$, $\gamma(0) = e$, and γ is a one-parameter subgroup of G. It is determined by

$$\gamma'(0) = d\gamma_0\left(\left.\frac{d}{dt}\right|_0\right) = X_{\gamma(0)}$$

and hence it is the maximal integral curve generated by X such that $\gamma(0) = e$. Left invariant vector fields are determined by its value at the identity so every one-parameter subgroup is determined by its velocity in T_eG.

As we have seen, the matrix exponential is the tool which relates matrix Lie groups with their Lie algebras. This is the reason of the next definition

Definition 6.8. Let G be a Lie group with Lie algebra $\mathfrak{g}$. For any $X \in \mathfrak{g}$, the **exponential map**

$$\exp : \mathfrak{g} \to G$$

is defined by setting

$$\exp X = \gamma_X(1), \tag{6.2}$$

where γ_X is the one-parameter subgroup generated by X or, equivalently, the maximal integral curve of X starting at e.

Theorem 6.9. *Let $X \in \mathfrak{g}$, the Lie algebra of the Lie group G. Then,*

a) $\exp(sX) = \gamma_X(s)$ *for all* $t \in \mathbb{R}$.

b) $\exp(t_1 + t_2)X = \exp(t_1X)\exp(t_2X)$.

c) $\exp(-tX) = (\exp tX)^{-1}$.

d) $(\exp X)^n = \exp(nX)$.

e) The exponential map is smooth.

f) $(d\exp)_0 : T_0\mathfrak{g} \to T_eG$ *is the identity map with the usual identifications of* $T_0\mathfrak{g}$ *and* T_eG *with* $\mathfrak{g}$.

g) The exponential map is a local diffeomorphism around the identities of $\mathfrak{g}$ *and* G.

h) If H *is another Lie group with Lie algebra* $\mathfrak{h}$ *and* $F : G \to H$ *is a Lie group homomorphism, the following diagram commutes*

$$\begin{array}{ccc} G & \xrightarrow{F} & H \\ {\scriptstyle\exp}\uparrow & & \uparrow{\scriptstyle\exp} \\ \mathfrak{g} & \xrightarrow{dF} & \mathfrak{h} \end{array}$$

Proof. Let $s \in \mathbb{R}$, we define $\tilde{\gamma}_{sX}(t) = \gamma_X(st)$ and it is just the integral curve of sX starting at e. Setting $t = 1$ we obtain the first item a)

$$\exp sX = \tilde{\gamma}_{sX}(1) = \gamma_X(s).$$

The validity of b) and c) comes from the fact that γ_X is a homomorphism. The fourth can be seen by induction for positive integers, and by induction and c) for negative integers. For e), we define a vector field V on $G \times \mathfrak{g}$ by

$$V_{(g,X)} = (X_g, 0) \in T_g G \oplus T_X \mathfrak{g} \cong T_{(X,g)}(G \times \mathfrak{g}),$$

where $g \in G$. Then V is a smooth vector field. The integral curve of V through (g, X) is

$$t \mapsto (g \exp tX, X)$$

since $L_g \circ \gamma_X$ is the unique integral curve of X which takes the value g at 0. That is, the local flow

$$\Phi_t(g, X) = (g \exp tX, X).$$

V is complete so we can take the value for $t = 1$ and with the projection $\pi_G : G \times \mathfrak{g} \to G$, at $g = e$ we get

$$\pi_G \circ \Phi_1(e, X) = \pi_G \circ (e \cdot \exp X, X) = \exp X$$

and this is a composition of smooth mappings. To prove f), we only have to observe that tX is a curve in $\mathfrak{g}$ whose tangent vector at $t = 0$ is X and the tangent vector of the curve in G $\exp tX$ at $t = 0$ is $\gamma'_X(0) = X_e$ and g) is a direct result from this. Finally, we need to prove that $F(\exp X) = \exp dF(X)$ for any $X \in \mathfrak{g}$. Let e' be the unit element of H. Consider the curve $t \mapsto F(\exp tX)$, it is smooth in H and its tangent at 0 is $dF_e(X_e)$. Besides, since it is a homomorphism from $\mathbb{R}$ to H, it is a one-parameter subgroup. Now, $t \to \exp t(dF(X))$ is the unique one-parameter subgroup of H whose tangent at 0 is $dF(X)_{e'}$. By the condition (6.1), we get that

$$F(\exp tX) = \exp t dF(X)$$

so, for $t = 1$,

$$F(\exp X) = \exp dF(X).$$

□

Let $A \in M_n(K)$, the map $t \mapsto e^{tA}$ of $\mathbb{R}$ into $\mathsf{GL}(n; K)$ is smooth and a homomorphism, as we have seen in section 3. Moreover, it is a one-parameter subgroup and its tangent vector at $t = 0$ is A. Whence, the exponential map for $\mathsf{GL}(n; K)$ is

$$\exp(A) = e^A \quad \text{for} \quad A \in \mathsf{gl}(n; K) \tag{6.3}$$

as expected. Moreover, for a matrix Lie group homomorphism $\Phi : G \to H$, we have the following identity

$$d\Phi(X) = \frac{d}{dt}\Phi(e^{tX})\bigg|_{t=0} \tag{6.4}$$

for any element X in the Lie algebra of the matrix Lie group. This is the usual way to compute $d\Phi$.

Remark: The exponential map is a diffeomorphism between the identities of the Lie group and Lie algebra so for connected Lie groups, by Proposition 2.7, one could say that the image of the exponential generates the Lie group. However, this does not mean that that the exponential is surjective and, in general, it is not true.

6.2 Subgroups and homomorphisms

Now that we have the exponential map defined, we can study deeper properties of Lie group homomorphisms as well as Lie subgroups of Lie groups and their associated Lie subalgebras.

Proposition 6.10. *Let G be a Lie group and H a Lie subgroup of G. The one-parameter subgroups of H are the one-parameter subgroups of G whose initial velocities lie on T_eH.*

The proof is done using the composition with the inclusion and uniqueness of one-parameter subgroups.
As a consequence, the exponential map of a Lie subgroup H of G with Lie algebra $\mathfrak{h}$ will be the restriction of the exponential map from the Lie algebra of $\mathfrak{g}$ of G to $\mathfrak{h}$. Furthermore, we have the following result (see [11] for a proof)

Proposition 6.11. *Let H a Lie subgroup of a Lie group G with respective Lie algebras $\mathfrak{h}$ and $\mathfrak{g}$. Then,*

$$\mathfrak{h} = \{X \in \mathfrak{g} \,|\, \exp tX \in H \text{ for all } t \in \mathbb{R}\}.$$

Remark: It is clear now the reason for the definition of Lie algebras of matrix Lie groups. Matrix Lie groups are subgroups of $\mathsf{GL}(n;K)$, which we have seen that is a Lie group, and its Lie algebra, $\mathsf{gl}(n;K)$, is just the space of matrices $M_n(K)$. Thus, once we have proven that closed groups of a Lie group are Lie subgroups, the definitions will be equivalent.
Now a previously announced result,

Theorem 6.12. *Let $\varphi : H \to G$ be a continuous homomorphism of Lie groups. Then φ is smooth.*

The validity of this result comes from the case $H = \mathbb{R}$ and the extension of the proof is simple (cf. [26]).

Theorem 6.13. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\varphi : \mathbb{R} \to G$ be a continuous homomorphism. Then it is smooth.*

Proof. The proof is quite similar to the one sketched in Theorem 3.11. It is enough to prove that φ is smooth in a neighbourhood of 0 and then composing with suitable left translations extend the smoothness to $\mathbb{R}$. Let V be a neighbourhood of $e \in G$ diffeomorphic with a starlike neighbourhood U of $0 \in \mathfrak{g}$ under the exponential map. By starlike we mean $tX \in U$ for $t \in [0,1]$ when $X \in U$ (recall that $\mathfrak{g}$ is a vector space). Let $U' = \{X/2 \,|\, X \in U\}$. Setting a $t_0 > 0$ such that $\varphi(t) \in \exp(U')$ for all $|t| \leq t_0$ and a positive integer n, there exist $X, Y \in U'$ such that they are the unique ones which hold

$$\exp X = \varphi(t_0/n) \text{ and } \exp Y = \varphi(t_0).$$

Then,

$$\exp(nX) = \varphi(t_0) = \exp(Y).$$

Now, let $1 \leq j < n$, for $j = 1$, $jX \in U'$. Assume $jX \in U'$ for some j, then $2jX \in U$ and in particular $(j+1)X \in U$. Thus, $\exp((j+1)X) = \varphi((j+1)t_0/n)$ and it belongs to $\exp(U')$ because $(j+1)t_0/n < t_0$. Since exp is injective on U, $(j+1)X \in U'$. We have proven by induction that $nX \in U'$ and the injectivity of exp on U implies injectivity on U'. Hence,

$$nX = Y.$$

For every positive integer $0 < m \leq n$ we have

$$\varphi(mt_0/n) = \varphi(t_0/n)^m = \exp(Y/n)^m = \exp(mY/n).$$

For every negative integer $-n \leq m < 0$ we will have the same using $\varphi(mt_0/n) = \varphi((-m)t_0/n)^{-1}$. The numbers m/n are dense in $[-1, 1]$. Then, by continuity,

$$\varphi(t) = \exp(tY/t_0) \text{ for } |t| \leq t_0.$$

We have that φ is smooth in a neighbourhood of 0. Therefore, φ is smooth. □

We do not have the tools to prove the next theorem (a prove can be found in [10]), but it is a result with important consequences.

Theorem 6.14 (Ado's theorem)**.** *Every finite-dimensional Lie algebra is isomorphic to a matrix Lie algebra.*

As we will see, this is the same as saying that every Lie algebra has a faithful finite-dimensional linear representation. The result does not hold for Lie groups (f.e., the metaplectic group). From this, we see that if $\mathfrak{g}$ is a Lie algebra, it is (or is isomorphic to) a subalgebra of $\mathsf{gl}(n, K)$ for some $n \in \mathbb{N}$. This, together with the next result, (see [11]),

Theorem 6.15. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. If $\mathfrak{h}$ is any subalgebra of $\mathfrak{g}$, then there is a unique connected Lie subgroup H of G whose Lie algebra is $\mathfrak{h}$.*

leads us to a deep result:

Theorem 6.16. *There is a one-to-one correspondence between isomorphism classes of simply connected Lie groups and isomorphism classes of Lie algebras.*

$\mathsf{gl}(n, \mathbb{R})$ is the Lie algebra of the Lie group $\mathsf{GL}(n, \mathbb{R})$, so because of this last theorem there is a connected Lie subgroup, G, of $\mathsf{GL}(n, \mathbb{R})$ with Lie algebra $\mathfrak{g}$. Moreover, if $\tilde{G}$ is the universal covering group of G, then by Proposition 6.5, $\mathfrak{g}$ and the Lie algebra of $\tilde{G}$ are isomorphic. Taking into account Corollary 6.4, we have proven Theorem 6.16.

6.3 The closed subgroup theorem

Theorem 6.17 (**Closed subgroup theorem** (Cartan, 1930))**.** *Let G be a Lie group and H a subgroup that is also a closed subset of G. Then H is an embedded Lie subgroup.*

Proof. We need to show that H is an embedded submanifold of G. Let $\mathfrak{g}$ be the Lie algebra of G, we define

$$\mathfrak{h} = \{\ X \in \mathfrak{g} \text{ s.t. } \exp tX \in H \text{ for all } t \in \mathbb{R}\}.$$

First, we show that $\mathfrak{h}$ is a linear subspace of $\mathfrak{g}$. The product by a scalar is clear, since tX belongs to H for all $t \in \mathbb{R}$ and $X \in \mathfrak{h}$. Now, if $X, Y \in \mathfrak{h}$, then $\exp \frac{t}{n}X$ and $\exp \frac{t}{n}Y$ are in H for all integer n. H is a group, so the product of both elements is in H and the product of n times the product too. Since H is closed, the limit for $n \to \infty$ will also be in H, and for the Lie product formula (3.8), which is also valid for Lie groups (and can be proved equivalently, c.f. [11]),

$$\lim_{n\to\infty} \left(\exp \frac{t}{n}X \exp \frac{t}{n}Y\right)^n = \exp t(X + Y) \in H.$$

This holds for all $t \in \mathbb{R}$ and follows that $X+Y \in \mathfrak{h}$. So $\mathfrak{h}$ is a linear subspace of $\mathfrak{g}$.
Now, let U be a neighbourhood of $0 \in \mathfrak{g}$ such that the exponential is a diffeomorphism. By the definition of $\mathfrak{h}$, we have

$$\exp(U \cap \mathfrak{h}) \subset (\exp U) \cap H.$$

We want to see that the inclusion goes in the other way too. Assuming that it does not, let $\mathfrak{k}$ be the linear subspace of $\mathfrak{g}$ such that $\mathfrak{h} \oplus \mathfrak{k} = \mathfrak{g}$ as vector spaces. The map $\phi : \mathfrak{h} \times \mathfrak{k} \to G$ such that $\phi(X,Y) = \exp(X)\exp(Y)$ can be seen that is a diffeomorphism between suitable neighbourhoods of $(0,0)$ and e, since $d\phi|_{(0,0)} = \mathrm{id}_{\mathfrak{g}}$.
In order to see the inclusion, we want to show that there is an open neighbourhood of $0 \in \mathfrak{k}$, $U_{\mathfrak{k}}$ such that

$$H \cap \exp(U_{\mathfrak{k}} \setminus \{0\}) = \emptyset. \tag{6.5}$$

If this does not hold, then there exists a sequence $(X_j) \subset U_{\mathfrak{k}}$ with $\exp(X_j) \in H$ for $X_j \to 0$. Let $\|\cdot\|_{\mathfrak{k}}$ be a norm in $U_{\mathfrak{k}}$, the sequence $(X'_j) = \frac{X_j}{\|X_j\|_{\mathfrak{k}}}$ is in the compact unit ball. Hence, there exists a convergent partial sequence $(X'_{j_k}) \to Y \in \mathfrak{k}$, with $Y \neq 0$. Denoting $t_{j_k} = \|X_{j_k}\|_{\mathfrak{k}}$, as $t_{j_k} \to 0$, for every t we can choose integers $n_{j_k}(t)$ such that $t_{j_k} n_{j_k}(t) \to t$. This can be seen with these integers being the ones which make $t_{j_k} n_{j_k}(t)$ the closest possible value to t for each j_k. That is,

$$\left| n_{j_k}(t) - \frac{t}{t_{j_k}} \right| \leq 1 \Rightarrow |n_{j_k}(t) t_{j_k} - t| \leq t_{j_k} \to 0.$$

And we have, using the fact that H is closed,

$$\begin{aligned} &\exp(tY) = \exp(t \lim X'_{j_k}) = \exp(\lim n_{j_k}(t) t_{j_k} X'_{j_k}) = \exp(\lim n_{j_k}(t) X_{j_k}) = \\ &= \lim \exp(X_{j_k})^{n_{j_k}(t)} \in H \ \Rightarrow Y \in \mathfrak{h} \Rightarrow Y = 0 \end{aligned}$$

because 0 is the only element shared by $\mathfrak{h}$ and $\mathfrak{k}$. We have arrived to a contradiction with $Y \neq 0$. Thus, we can choose $U_{\mathfrak{k}}$ with this property and such that $\phi : U_{\mathfrak{h}} \times U_{\mathfrak{k}} \to G$ is a diffeomorphism with its image, $\mathrm{Im}(\phi|_{U_{\mathfrak{h}} \times U_{\mathfrak{k}}})$, which we denote W. W is an open neighbourhood of $e \in G$. For $x \in W \cap H$ we have that

$$x = \exp(X)\exp(Y)$$

with $X \in U_{\mathfrak{h}}$, $Y \in U_{\mathfrak{k}}$. $\exp(X) \in H$, so $\exp(Y) \in H \cap \exp(U_{\mathfrak{k}})$ and by our choice of $U_{\mathfrak{k}}$, $Y = 0$ and $x \in \exp(U_{\mathfrak{h}} \times \{0\})$. That is, we have proven that $(\exp U) \cap H \subset \exp(U \cap \mathfrak{h})$ for $U = U_{\mathfrak{h}} \times U_{\mathfrak{k}}$. Hence, there is a neighbourhood U of $0 \in \mathfrak{g}$ such that

$$\exp(U \cap \mathfrak{h}) = (\exp U) \cap H.$$

Finally, let $\iota : \mathfrak{g} \to \mathbb{R}^n$ be an isomorphism of vector spaces which sends $\mathfrak{h}$ to $\mathbb{R}^k$. We can build the composite map near the identity

$$\varphi = \iota \circ \exp^{-1} : \exp U \to \mathbb{R}^n$$

which is a smooth chart for G and $\varphi((\exp U) \cap H) = \iota(U \cap \mathfrak{h})$ is the slice of the chart obtained setting the last $n-k$ coordinates to zero. Now, let $h \in H$, $L_h(\exp U)$ is a diffeomorphism from $\exp U$ to a neighbourhood of h. H is invariant under L_h because $h \in H$, so

$$L_h(\exp U \cap H) = L_h(\exp U) \cap H$$

and $\varphi \circ L_h^{-1} = \varphi \circ L_{h^{-1}}$ is a chart for H in a neighbourhood of h. Thus, H is an embedded submanifold of G and therefore it is a Lie subgroup. $\square$

Applying this to matrix Lie groups, which are closed subgroups of the Lie group $\mathsf{GL}(n;K)$, we obtain that they are Lie subgroups of $\mathsf{GL}(n;K)$. Hence,

Corollary 6.18. *Every matrix Lie group is a Lie group.*

7 Representation Theory

Let V be a vector space over K and let $\mathsf{GL}(V)$ be the group of isomorphisms of V onto itself. When V is finite dimensional and we fix a basis of V, each linear map a is defined by a square invertible matrix (a_{ij}) of order n. In this case, $\mathsf{GL}(V)$ can be identified with $\mathsf{GL}(n,K)$.

Definition 7.1. Let G be a group, with identity element e and with composition $(s,t) \to st$ and let V be a vector space over K. A ***linear representation*** of G in V is a homomorphism $\rho : G \to \mathsf{GL}(V)$. If G is a Lie group, the homomorphism has to be continuous. If ρ is one-to-one, the representation is called ***faithful***.

The definition is the same as saying that we associate with each element $s \in G$ an element $\rho(s) \in \mathsf{GL}(V)$ in such a way that

$$\rho(st) = \rho(s) \cdot \rho(t), \qquad \rho(e) = I, \qquad \rho(s^{-1}) = \rho(s)^{-1} \quad \text{for } s,t \in G.$$

When ρ is given, it is said that V is a representation space of G or, directly, a representation of G. We will use the notation $\rho(s) = \rho_s$. The dimension n of V is called the ***degree*** of the representation.

Definition 7.2. Given a basis (e_i) of V, to every ρ_s we can associate a matrix $R_s = r_{ij}(s) \in \mathsf{GL}(n,K)$ so that

$$\det(R_s) \neq 0, \qquad R_{st} = R_s \cdot R_t = r_{ik}(st) = \sum_j r_{ij}(s) \cdot r_{jk}(t), \text{ for } s,t \in G.$$

Conversely, given invertible matrices $R_s = r_{ij}(s)$ satisfying the preceding identities, there is a corresponding linear representation ρ of G in V. We say we have given a ***representation of G in matrix form***.

Definition 7.3. Let ρ and ρ' be two representations of the same group G in vector spaces V and V'. These representations are said to be ***isomorphic*** if there exists a linear isomorphism $\tau : V \to V'$ which satisfies

$$\tau \circ \rho_s = \rho'_s \circ \tau \quad \forall s \in G.$$

When ρ and ρ' are given in matrix form by $\{R_s\}_{s\in G}$ and $\{R'_s\}_{s\in G}$, this means that there exists an invertible matrix $T \in \mathsf{GL}(n,K)$ such that

$$T \cdot R_s = R'_s \cdot T, \quad \forall s \in G.$$

Since τ is an isomorphism of vector spaces, V and V' have the same dimension and hence ρ and ρ' have the same degree.

Definition 7.4. Let $\rho : G \to \mathsf{GL}(V)$ be a linear representation and let W be a vector subspace of V. If $\rho_s(W) \subset W$ for all $s \in G$, we say that W is ***stable*** or ***invariant*** under the action of G.

Let W be a stable subspace of a representation $\rho : G \to \mathsf{GL}(V)$. The restriction ρ_s^W of ρ_s to W is then an isomorphism of W onto itself, and we have $\rho_{st}^W = \rho_s^W \cdot \rho_t^W$. Thus, $\rho^W : G \to \mathsf{GL}(W)$ is a linear representation of G in W and we can make the next definition

Definition 7.5. Let $\rho : G \to \mathsf{GL}(V)$ be a linear representation and let W be a vector subspace of V stable under the action of G. W is said to be a ***subrepresentation*** of V.

For finite groups, we have the next two results:

Theorem 7.6. *Let $\rho : G \to \mathsf{GL}(V)$ be a linear representation of a finite group G in V and let W be a vector subspace of V stable under G. Then there exists a complement W^0 of W in V*[7] *which is stable under G.*

Proof. We can assume that V is endowed with a hermitian scalar product $\langle\ ,\ \rangle$ invariant under G, $\langle \rho_s v, \rho_s w\rangle = \langle v, w\rangle$ for all $v, w \in V$ and $s \in G$. We can always get this by choosing an arbitrary hermitian product $(\ ,\ )$ in V and defining

$$\langle v, \omega\rangle = \sum_{t\in G} (\rho_t v, \rho_t w).$$

It is easy to see that $\langle\ ,\ \rangle$ is also a hermitian product and if W is invariant under G, its orthogonal complement $W^\perp$ is also invariant and we can take $W^0 = W\perp$. $\square$

It is worth noting that the invariance of the scalar product $\langle\ ,\ \rangle$ implies that the matrices of the representation ρ are unitary in a basis of orthogonal vectors.
Now, if $V = W \oplus W^0$ with W invariant under ρ, every element of V can be written as the sum of elements of W and W^0. Besides, since W and W^0 are invariant under ρ, every the representation ρ will determine representations of W and W^0. If W and W^0 are given in the matrix form R_s and R_s^0, the representation of $W \oplus W^0$ is given in matrix form by

$$\begin{pmatrix} R_s & 0 \\ 0 & R_s^0 \end{pmatrix}.$$

We can state the definition

Definition 7.7. Let G be a group and ρ^1, ρ^2 two representations of G acting on the vector spaces V_1, V_2. The ***direct sum of*** ρ^1 ***and*** ρ^2 is the representation of G, denoted $\rho^1 \oplus \rho^2$ acting on the space $V_1 \oplus V_2$ by

$$(\rho_s^1 \oplus \rho_s^2)(v_1, v_2) = (\rho_s^1(v_1), \rho_s^2(v_2)).$$

Definition 7.8. Let $\rho : G \to \mathsf{GL}(V)$ be a linear representation. We say that it is ***irreducible*** or ***simple*** if $V \neq \vec{0}$ and if no non-trivial vector subspace of V is stable under G.

The definition is equivalent to saying that V is not the direct sum of two representations. For finite groups, irreducible representations are the building blocks used to construct all the others.

Theorem 7.9. *Every representation of a finite group is a direct sum of irreducible representations*

[7]By saying that W^0 is a complement of W, we mean that $W \oplus W^0 = V$, where "$\oplus$" is the direct sum of subspaces.

Proof. The proof is done by induction. Let $\rho : G \to \mathsf{GL}(V)$ be a representation of degree n. For $n = 0$, the theorem holds. For $n \geq 1$, if the representation is irreducible, we are done. Otherwise, by the previous theorem, it can be decomposed as a direct sum of a proper stable subspace W and its complement W^0. Both have dimension smaller than n and by the induction hypothesis are direct sums of irreducible representations, so the same holds for V. □

Definition 7.10. Let $\rho^1 : G \to \mathsf{GL}(V_1)$ and $\rho^2 : G \to \mathsf{GL}(V_2)$ two representations of a group G. The ***tensor product of the representations*** is the representation $\rho := \rho^1 \otimes \rho^2$ of G acting on the tensor product of the vector spaces, $V_1 \otimes V_2$, by

$$\rho_s(x_1 \otimes x_2) = \rho_s^1(x_1) \otimes \rho_s^2(x_2) \text{ for } x_1 \in V_1,\, x_2 \in V_2,\, s \in G.$$

Recall that $M_m(K) \otimes M_n(K) \cong M_{mn}(K)$. In general, the tensor product of two irreducible representations is not irreducible. However, it can be decomposed as a direct sum of irreducible representations. This process is called Clebsch-Gordan theory and is widely used in physics.

Definition 7.11. Let $\rho : G \to \mathsf{GL}(V)$ be a representation of a group G and let V^* be the dual space of V. The ***dual representation*** of ρ is the representation $\rho^* : G \to \mathsf{GL}(V^*)$ defined by

$$\rho_s^* \omega(v) = \omega(\rho_s^{-1} v) \quad \text{for all } s \in G, \omega \in V^* \text{ and } v \in V.$$

When given in matrix form it can be equivalently defined by

$$R_s^* = R_{s^{-1}}^\top = (R_s^{-1})^\top. \tag{7.1}$$

We have seen a basic result for representations of finite groups and we want to extend it to Lie groups. We will see that for compact Lie groups we can arrive to similar results. In order to do this, we have to work with differential forms and integration on manifolds, so that integrals over Lie groups play the role of finite sums indexed over finite groups.

8 Differential Forms and Integration on Manifolds

In this section we use some definitions introduced in the Appendix A such as a covariant k-tensor and the pullback of a smooth map.

8.1 Differential Forms

Definition 8.1. A covariant k-tensor is said to be ***alternating*** if any permutation of the argument causes its value to be multiplied by the sign of the permutation. Alternating covariant k-tensors are called ***exterior forms*** or simply ***k-forms***. The space of all k-forms on V is denoted by $\Lambda^k(V^*)$.

If n is the dimension of V, it can be seen that

$$\dim \Lambda^k(V^*) = \binom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{8.1}$$

Thus, the dimension of the space of n-forms is 1.

8.2 Orientations

Definition 8.2. Let M be a smooth n-manifold. A ***pointwise orientation*** on M is a choice of orientation of each tangent space. If (e_i) is a local frame for TM, we say that (e_i) is ***positively (negatively) oriented*** if $(e_1|_p, \dots, e_n|_p)$ is a positively (negatively) oriented basis for T_pM at each point $p \in U$. A pointwise orientation is said to be ***continuous*** if every point of M is in the domain of an oriented local frame. An ***orientation of*** M is a continuous pointwise orientation. If there exists an orientation for M, we say that M is ***orientable***. Otherwise, we say that it is ***nonorientable***.

Proposition 8.3. *Let M be a smooth n-manifold. Any nonvanishing n-form ω on M determines a unique orientation of M for which ω is positively oriented at each point.*

Proof. Since it is non-vanishing, ω defines a pointwise orientation, so we only need to check that it is continuous. For $n \geq 1$, given $p \in M$, let (e_i) be a local frame on a connected neighbourhood U of p, and let (ε^i) be the dual coframe. The expression for ω in this frame is $\omega = f\varepsilon^1 \wedge \cdots \wedge \varepsilon^n$ for some continuous nonvanishing function f. Therefore,

$$\omega(e_1, \dots, e_n) = f \neq 0$$

at all points of U. The fact that U is connected implies that the previous expression is either always positive or always negative. If it is always positive, the frame is positively oriented and we are done. If it is negative, we can replace e_1 by $-e_1$, to obtain a new frame positively oriented. □

8.3 Integration on Lie Groups

Definition 8.4. Let G be a Lie group. A covariant tensor field A on G is said to be ***left-invariant*** if $L_g^*A = A$ for all $g \in G$.

Proposition 8.5. *Let G be a Lie group endowed with a left-invariant orientation. Then G has a positively oriented left-invariant n-form ω_G unique up to a constant.*

Proof. For $\dim G > 0$, let (e_i) be a left-invariant global frame on G. That is, a basis of $\mathfrak{g}$. We can assume that the frame is positively oriented, replacing e_1 with $-e_1$ if necessary. Let (ε^i) be the dual coframe. Since e_i is left invariant,

$$(L_g^*\varepsilon^i)(e_j) = \varepsilon^i(L_{g*}e_j) = \varepsilon^i(e_j) = \delta^i_j$$

and hence, $L_g^*\varepsilon^i = \varepsilon^i$ and ε^i is left-invariant. Now we define

$$\omega_G = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n.$$

Then,

$$L_g^*(\omega_G) = L_g^*\varepsilon^1 \wedge \cdots \wedge L_g^*\varepsilon^n = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n = \omega_G$$

ω_G is also left invariant. Any other positively oriented left-invariant n-form, $\tilde{\omega}_G$ will be a positive constant multiple of ω_G: we can write $\tilde{\omega}_G|_e = c\omega_G|_e$ and then

$$\tilde{\omega}_G|_g = L^*_{g^{-1}}\tilde{\omega}_G|_e = L^*_{g^{-1}}c\omega_G|_e = c\omega_G|_g.$$

□

This form has a measure associated, $d\mu$, which means that $\int f d\mu = \int f\omega_G$, called the ***Haar measure***. For compact groups, the integral over the group is finite. So we can normalize putting

$$\omega_G = \left(\int_G \tilde{\omega}_G\right)^{-1} \tilde{\omega}_G.$$

Corollary 8.6. *If G is a compact Lie group, then there is a unique left-invariant volume form ω_G such that*

$$\int_G \omega_G = 1.$$

Right and left multiplication commute, hence, the form $R_g^*(\omega_G)$ is left invariant. So, for the previous proposition, $R_g^*(\omega_G) = \lambda(g)\omega_G$. It can be seen that the map λ is a homomorphism of G into $(\mathbb{R}^+, \cdot)$, it is called the ***modular function***. Now, ω_G will be right invariant if and only if $\lambda = 1$ for all $g \in G$. In this case, G is called ***unimodular***. Every compact Lie group G is unimodular, since for each $g \in G$

$$1 = \int_G \omega_G = \lambda(g) \int_G \omega_G = \lambda(g).$$

Thus, every compact Lie group has a unique normalized right and left-invariant volume form.

9 Lie group representations

A representation of a Lie group, G, is a group homomorphism $\rho : G \to \mathsf{GL}(V)$ with V a real or complex vector space, which is also continuous (recall that by Proposition 6.12, then it is automatically smooth). For the case of matrix Lie groups, there is an immediate one,

Definition 9.1. Let G be a matrix Lie group. By definition, it is a subgroup of $\mathsf{GL}(n, K) = \mathsf{GL}(K^n)$. Thus, the inclusion map $i : G \hookrightarrow \mathsf{GL}(K^n)$ is a representation, called the ***standard representation*** of G.

With integrals defined on compact Lie groups thanks to the Haar measure, we can make the representations of G unitary.

Proposition 9.2. *Let G be a compact Lie group and V a complex vector space. Let ρ be a representation of G into $End(V)$. Then there is an inner product on V with respect to which ρ is unitary.*

Proof. Let $\langle v, w\rangle_0$ be an inner product on V and $d\mu$ the Haar volume form on G. We set

$$\langle v, w\rangle = \int_G \langle \rho_g v, \rho_g w\rangle_0 \, d\mu(g) \tag{9.1}$$

where $d\mu(g)$ denotes that the integrand is a function of $g \in G$. It is an inner product and ρ is unitary, since for $h \in G$,

$$\langle \rho_h v, \rho_h w\rangle = \int_G \langle \rho_g \rho_h v, \rho_g \rho_h w\rangle_0 \, d\mu(g) = \int_G \langle \rho_{gh} v, \rho_{gh} w\rangle_0 \, d\mu(g).$$

Using the right invariance of $d\mu(g)$, $\int_G f(gh)d\mu(g) = \int_G f(g)d\mu(g)$,

$$= \int_G \langle \rho_g v, \rho_g w \rangle_0 \, d\mu(g) = \langle v, w \rangle .$$

□

With the unitarity, we can extend the results from finite groups to compact Lie groups,

Theorem 9.3. *Every stable subspace of a representation of a compact Lie group has a stable complement.*

Theorem 9.4. *If G is a compact Lie group, every finite dimensional representation of G is completely reducible.*

Furthermore, one can develop a character theory for compact Lie groups as for finite groups up to some extent, as it is explained in [20].

10 SU(2) and SO(3)

Let us consider the Lie group $G = \mathsf{SU}(2)$ of 2×2 special unitary matrices. Taking into account the restrictions imposed by being unitary and having determinant one, they are the matrices of the form

$$\mathsf{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} ; \;\; \alpha, \beta \in \mathbb{C}, \;\; |\alpha|^2 + |\beta|^2 = 1 \right\}. \tag{10.1}$$

Writing $\alpha = x_1 + ix_2$, $\beta = x_3 + ix_4$, where $x_i \in \mathbb{R}$, the condition is

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \tag{10.2}$$

which is the same condition that defines $\mathbb{S}^3$. Whence, $\mathsf{SU}(2)$ and $\mathbb{S}^3$ are homeomorphic and therefore $\mathsf{SU}(2)$ is simply connected. Even more, as explained in [22], $\mathbb{S}^3$ can be considered as the set of unit quaternions, which is a group under quaternion multiplication.

The Lie algebra of $\mathsf{SU}(2)$, $\mathsf{su}(2)$, is the set of matrices of $M_2(\mathbb{C})$ of the form

$$\left\{ \begin{pmatrix} ia & v \\ -\overline{v} & -ia \end{pmatrix} | \, a \in \mathbb{R}, v \in \mathbb{C} \right\}$$

and we can define a map

$$\begin{array}{rcl} \rho : \mathsf{SU}(2) & \longrightarrow & \mathsf{GL}(\mathsf{su}(2)) \\ A & \longmapsto & \rho(A) : \quad \begin{array}{rcl} \mathsf{su}(2) & \to & \mathsf{su}(2) \\ X & \mapsto & AXA^{-1} \end{array} \end{array} \tag{10.3}$$

From Theorem 4.13, it is well defined and is a representation of $\mathsf{SU}(2)$ in the $\mathbb{R}$-vector space $\mathsf{su}(2)$, which has a basis

$$E_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{10.4}$$

and a norm defined by

$$\|X\| = +\sqrt{\det X}. \tag{10.5}$$

We see then that $\mathsf{su}(2) \cong \mathbb{R}^3$ and $\mathsf{GL}(\mathsf{su}(2)) \cong \mathsf{GL}(\mathbb{R}^3)$, so $\rho : \mathsf{SU}(2) \to \mathsf{GL}(\mathbb{R}^3)$. Moreover, $\rho(A)$ preserves the norm for all $A \in \mathsf{SU}(2)$, so $\rho(A)$ are isometries of $\mathbb{R}^3$, which are the matrices that form the matrix Lie group $\mathsf{O}(3)$. Moreover, $\mathsf{SU}(2)$ is connected and $\rho(I) = I$, so it will map to the connected component of the identity in $\mathsf{O}(3)$. Hence, we have a map

$$\rho : \mathsf{SU}(2) \longrightarrow \mathsf{SO}(3) \tag{10.6}$$

which verifies

i) ρ is surjective and a Lie group homomorphism.

ii) $\ker \rho = \{\pm Id\}$.

iii) ρ is the universal covering map.

Indeed,

$$\rho(AB)(X) = ABX(AB)^{-1} = ABXB^{-1}A^{-1} = \rho(A) \circ \rho(B)(X)$$

and $\rho(I) = Id$. So, it is a homomorphism. It is a continuous homomorphism between Lie groups and hence it is a Lie group homomorphism (Theorem 6.12). $-I^{-1} = -I$ and then we see that $-I \in \ker(\rho)$. To see that there is no other element in the kernel except $\pm I$, observe that

$$A = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in \ker \rho \Rightarrow AX = XA^{-1}$$

for all $X \in \mathsf{su}(2)$. Operating a little bit, this leads to the conditions $\alpha = \overline{\alpha}$ and $\beta = 0$. The unique matrices with determinant 1 which fulfill this condition are $\{\pm I\}$.

Now, Lie group homomorphisms have constant rank (Proposition 2.10) and the rank is equal to the codimension of the kernel, which has dimension 0 (as a smooth manifold). So, the map ρ has full rank and so it is a local diffeomorphism around the identities of $\mathsf{SU}(2)$ and $\mathsf{SO}(3)$. Furthermore, $\mathsf{SO}(3)$ is connected and, by Proposition 2.7, it is generated by a neighbourhood of its identity. Thus, ρ is a covering map and since $\mathsf{SU}(2)$ is simply connected, it follows that $\mathsf{SU}(2)$ is the universal cover of $\mathsf{SO}(3)$.

The map ρ is an example of a widely used map related by the exponential map to the adjoint representation of Lie algebras,

Definition 10.1. Let G be a Lie group with Lie algebra $\mathfrak{g}$. We define the ***adjoint representation*** of G as the map

$$\begin{array}{rcl} \mathrm{Ad} : G & \longrightarrow & \mathsf{GL}(\mathfrak{g}) \\ A & \longmapsto & \mathrm{Ad}(A) : \begin{array}{ccc} \mathfrak{g} & \to & \mathfrak{g} \\ X & \mapsto & AXA^{-1} \end{array} \end{array}$$

Recall that by Proposition 6.5, there is an isomorphism between $\mathsf{su}(2)$ and $\mathsf{so}(3)$ given by the induced Lie algebra homomorphism of Ad. Hence, $\mathsf{su}(2)$ and $\mathsf{so}(3)$ are isomorphic Lie algebras, a fact that will be used later.

10.1 A geometric interpretation

The group of rotations of the three dimensional space, $\mathsf{SO}(3)$, can be represented geometrically as the three-dimensional ball of radius π with the antipodal points of the surface identified. To see this, note that every rotation, fixing a convention (left or right hand screw), is determined by the angle $\alpha \in [-\pi, \pi]$ (two rotations that differ a multiple of 2π are equal) and a normal vector $\vec{n} \in \mathbb{R}^3$ pointing in the direction of the axis according to the convention. This information can be compressed into a single vector $\vec{\alpha}$ pointing in the direction of $\pm\vec{n}$ with $\|\vec{\alpha}\| \in [0, \pi]$, the sign being determined by the sign of α.

Fixing an origin, the points pointed by these vectors form the previous solid sphere with the antipodal points of the surface identified, since $\vec{\alpha} \equiv -\vec{\alpha}$ when $\|\vec{\alpha}\| = \pi$.

From $\mathsf{SU}(2)$ being a double cover of $\mathsf{SO}(3)$ and $\mathsf{SU}(2)$ being simply connected, it can be seen that the fundamental group of $\mathsf{SO}(3)$ is $\mathbb{Z}/2\mathbb{Z}$. In fact, it can also be seen that $\mathsf{SO}(3)$ is homeomorphic to $\mathbb{RP}^3$ (c.f. [7]).

Geometrically, this can be seen taking a path of rotations which starts at the identity (the point in the middle of the sphere), goes to the surface and returns to the centre from the antipodal point, making a closed loop representing a rotation of angle 2π. This loop is not contractible. On the other hand, if we take a path which crosses the boundary twice and returns to the centre, we can contract it to the identity point. (There is another way to visualize this phenomenon, called the *Dirac's belt trick*).

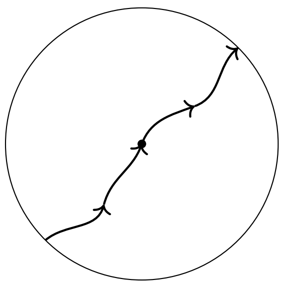

Figure 4: Non-contractible

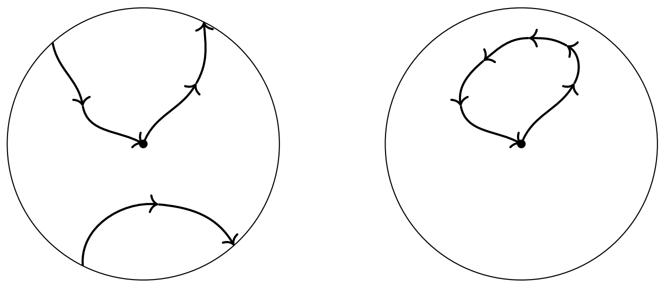

Figure 5: Contractible

11 Lie algebra representations

Analogously to the case of groups, we can define representations of Lie algebras and properties such as faithfulness or reducibility.

Definition 11.1. A (finite-dimensional) ***representation of a Lie algebra***, $\mathfrak{g}$, is a Lie algebra homomorphism of $\mathfrak{g}$ into the set of endomorphisms of a (finite-dimensional) $K-$vector space, V. That is, it is a $K-$linear map

$$\psi : \mathfrak{g} \to \mathrm{End}(V)$$

which preserves the Lie bracket operation.

We can see now that the adjoint map $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ (Definition 4.4) is our first example of a Lie algebra representation. It is the representation of a Lie algebra $\mathfrak{g}$ over $\mathfrak{g}$ (regarded as a $K-$vector space).

As for groups, for finite-dimensional vector spaces over a field K, fixing a basis, to every endomorphism we can associate a square matrix but now including non-invertible matrices. We have,

$$\mathrm{End}(V) \cong \mathsf{gl}(n, K) \text{ and, for } X, Y \in \mathfrak{g},\ \psi([X,Y]) = \psi(X)\psi(Y) - \psi(Y)\psi(X)$$

where n is the dimension of the vector space and, again, it is called the degree of the representation. By Ado's theorem (6.14), we know that every finite-dimensional real Lie algebra will have a faithful finite-dimensional representation on $\mathbb{R}^n$ so, for the study of Lie algebras of Lie groups, which are real and finite-dimensional, we will always have a faithful representation.

Any Lie group representation induces, by Theorem 6.1, a Lie algebra representation. Moreover, Theorem 6.4 tells us that representations of a simply connected Lie group are in one-to-one correspondence with representations of its Lie algebra. We see then that a motivation to study representations of Lie algebras is to find representations of simply connected Lie groups. Complex representations of Lie algebras are generally easier to study than their real ones, so we introduce the next definitions.

11.1 Change of scalars

Definition 11.2. Let K be a field, L a K Lie algebra and $K' \supset K$ another field. A ***K'-representation of*** L is a homomorphism of K-Lie algebras

$$L \longrightarrow \mathrm{End}_{K'}(V) \cong \mathsf{gl}(n, K')$$

where V is a K'-vector space.

If G is a Lie group with Lie algebra $\mathfrak{g}$ and $\rho : G \to \mathsf{GL}(n, \mathbb{C})$ a complex representation of G. Then, ρ induces the Lie algebra homomorphism

$$d\rho : \mathfrak{g} \to \mathsf{gl}(n, \mathbb{C})$$

which is $\mathbb{R}$-linear and a $\mathbb{C}$-representation of the $\mathbb{R}$-Lie algebra $\mathfrak{g}$.

Definition 11.3. Let $\mathfrak{g}_{\mathbb{C}}$ a $\mathbb{C}$-Lie algebra. A ***real form*** of $\mathfrak{g}_{\mathbb{C}}$ is an $\mathbb{R}$-Lie algebra, $\mathfrak{g}_{\mathbb{R}}$, such that

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$$

$\mathfrak{g}_{\mathbb{C}}$ is called the ***complexification*** of $\mathfrak{g}$.

Examples: Let $\mathfrak{g}_{\mathbb{C}} = \mathsf{sl}(2; \mathbb{C}) = \{A \in \mathsf{gl}(2; \mathbb{C}) \,|\, \mathrm{tr} A = 0\}$. Then,

a) $\mathfrak{g}_1 = \mathsf{sl}(2, \mathbb{R}) = \{A \in \mathsf{gl}(2, \mathbb{R}) \,|\, \mathrm{tr} A = 0\}$ is a real form of $\mathfrak{g}_{\mathbb{C}}$.

b) $\mathfrak{g}_2 = \mathsf{su}(2) = \{A \in \mathsf{gl}(2; \mathbb{C}) \,|\, \mathrm{tr} A = 0,\ A^\dagger + A = 0\}$ is another real form of $\mathfrak{g}_{\mathbb{C}}$.

Both statements can be proved using the fact that in both Lie algebras, if $X \in \mathfrak{g}_j$ then $iX \notin \mathfrak{g}_j$, for $j = 1, 2$. The first one is indeed a real form of $\mathsf{sl}(2; \mathbb{C})$ because it has only real matrices and the second one too, since $X^\dagger = -X \Rightarrow (iX)^\dagger = iX \Rightarrow iX \notin \mathfrak{g}_2$. Hence, the complexification can be written as the matrices of the form $X + iY$ for $X, Y \in \mathfrak{g}_i$. Alternatively, the tensor product with $\mathbb{C}$ preserves the property $\mathrm{tr}(A) = 0$ and not the other one, $A^\dagger + A = 0$.

We have seen that the $\mathbb{R}$-Lie algebras $\mathsf{sl}(2,\mathbb{R})$ and $\mathsf{su}(2)$ are real forms of the same $\mathbb{C}$-Lie algebra $\mathsf{sl}(2;\mathbb{C})$ and both have dimension three over $\mathbb{R}$. However, they are not isomorphic because $\mathsf{sl}(2,\mathbb{R})$ has a 2-dimensional subalgebra and $\mathsf{su}(2)$ does not have it.

We introduce the notation $\mathrm{Rep}_K(\mathfrak{g}_{K'})$ that denotes the set of $K-$representations of the $K'-$Lie algebra $\mathfrak{g}$.

Proposition 11.4. *Let $\mathfrak{g}_\mathbb{R}$ be a real form of $\tilde{\mathfrak{g}}_\mathbb{C}$. Then the natural map*

$$\begin{array}{ccc} \mathrm{Rep}_\mathbb{C}(\tilde{\mathfrak{g}}_\mathbb{C}) & \to & \mathrm{Rep}_\mathbb{C}(\mathfrak{g}_\mathbb{R}) \\ \rho & \mapsto & \rho|_{\mathfrak{g}_\mathbb{R}} \end{array}$$

is bijective.

Proof. We omit the subscripts $\mathbb{R}$ and $\mathbb{C}$, keeping in mind that $\mathfrak{g}$ is an $\mathbb{R}$-Lie algebra and $\tilde{\mathfrak{g}}$ a $\mathbb{C}$-Lie algebra. Let ρ_1, ρ_2 be two $\mathbb{C}$-linear representations of $\tilde{\mathfrak{g}}$. We have

$$\mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}} \xrightarrow{\rho_i} \mathsf{gl}(n_i,\mathbb{C}).$$

If $\rho_1|_\mathfrak{g} = \rho_2|_\mathfrak{g}$, since $\mathfrak{g}$ generates $\tilde{\mathfrak{g}}$ as a $\mathbb{C}$-Lie algebra and ρ_1 and ρ_2 are linear, this implies that $\rho_1 = \rho_2$.
To prove exhaustivity it is enough to see that for a given $\rho : \mathfrak{g} \to \mathsf{gl}(n,\mathbb{R})$, we can take $\rho\ \otimes \mathrm{id}_\mathbb{C}$ as an antiimage. $\square$

In particular, since $\mathsf{SU}(2)$ is simply connected, we have

$$\mathrm{Rep}_\mathbb{C}(\mathsf{SU}(2)) \longleftrightarrow \mathrm{Rep}_\mathbb{C}(\mathsf{su}(2)_\mathbb{R}) \longleftrightarrow \mathrm{Rep}_\mathbb{C}(\mathsf{sl}(2;\mathbb{C})_\mathbb{C}) \longleftrightarrow \mathrm{Rep}_\mathbb{C}(\mathsf{sl}(2,\mathbb{R})_\mathbb{R}).$$

In general, if G is a simply connected Lie group, we can study its complex representations through the representations of its Lie algebra $\mathfrak{g}$ or through the representations of its complexification $\mathfrak{g}_\mathbb{C}$.

11.2 Representations of $\mathsf{sl}(2;\mathbb{C})$

Recall that the matrix Lie algebra $\mathsf{sl}(2;\mathbb{C})$ is the Lie algebra of 2×2 traceless complex matrices. That is, matrices of the form

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \qquad \text{where } a,b,c \in \mathbb{C}.$$

We choose a basis for this algebra

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

which has the commutation relations

$$[H,X] = 2X, \quad [H,Y] = -2Y, \quad [X,Y] = H. \tag{11.1}$$

Now, consider an irreducible representation of the Lie algebra $\mathsf{sl}(2;\mathbb{C})$ over a complex vector space V of dimension m,

$$\psi : \mathsf{sl}(2;\mathbb{C}) \to \mathsf{gl}(V).$$

The images of the basis vectors will satisfy the same commutation relations that H, X and Y. Let $u \in V$ be an eigenvector of $\psi(H)$ (it exists since V is a complex vector space) with eigenvalue α,

$$\psi(H)(u) = \alpha u.$$

From now on, to simplify the notation we write H, X, Y in the place of $\psi(H), \psi(X)$ and $\psi(Y)$. Using the commutations relations we can see that $X(u)$ is also an eigenvector of H with eigenvalue $\alpha + 2$,

$$H(X(u)) = X(H(u)) + 2X(u) = (\alpha + 2)X(u)$$

and similarly for Y, $H(Y(u)) = (\alpha - 2)Y(u)$. Moreover, applying it repeatedly n times,

$$H(X^n(u)) = (\alpha + 2n)X^n(u), \quad H(Y^n(u)) = (\alpha - 2n)Y^n(u).$$

Since an operator on a finite-dimensional space has a finite number of eigenvalues, there is some $N \geq 0$ such that

$$X^N(u) \neq 0 \quad \text{and} \quad X^{N+1}(u) = 0$$

Defining $\lambda = \alpha + 2N$ and $u_0 = X^N(u)$, we have

$$H(u_0) = \lambda u_0, \qquad X(u_0) = 0.$$

Furthermore, if we set $u_k = Y^k(u_0)$ for $k \geq 0$, then

$$H(u_k) = (\lambda - 2k)u_k \tag{11.2}$$

and the operator Y will act on u_k like

$$Y(u_k) = u_{k+1}. \tag{11.3}$$

Besides, X acts on u_k in the following way,

$$\begin{aligned} X(u_k) &= X(Y^k(u_0)) = H(Y^{k-1}(u_0)) + YX(Y^{k-1}(u_0)) \\ &= (\lambda - 2(k-1))u_{k-1} + YX(u_{k-1}) \end{aligned}$$

Now, using

$$\sum_{j=1}^{k} \lambda - 2(k-j) = k[\lambda - (k-1)]$$

we get

$$X(u_k) = k[\lambda - (k-1)]u_{k-1}. \tag{11.4}$$

Again, H cannot have an infinite number of eigenvalues, then there is a positive integer m such that

$$u_k \neq 0 \;\text{ for }\; k \leq m \;\text{ and }\; u_{m+1} = Y^{m+1}u_0 = 0.$$

This last equality, in particular, leads to

$$0 = X(u_{m+1}) = (m+1)(\lambda - m) = 0 \Rightarrow m = \lambda \in \mathbb{Z}.$$

That is, we have proven that for every irreducible representation of $\mathsf{sl}(2;\mathbb{C})$, $\psi : \mathsf{sl}(2;\mathbb{C}) \to \mathsf{gl}(V)$, there exists a positive integer m and $m+1$ vectors $u_0, \ldots, u_m \in V$ such that the images of the basis H, X, Y will act on the vectors as in (11.2), (11.4) and (11.3), respectively. That is,

$$\begin{cases} \psi(H)(u_k) & = & (m-2k)u_k \\ \psi(X)(u_k) & = & k[m-(k-1)]u_{k-1} \quad \text{if } k>0 \quad \text{and} \quad \psi(X)(u_0)=0 \\ \psi(Y)(u_k) & = & u_{k+1} \quad \text{if } k<m \quad \text{and } \psi(Y)(u_m)=0 \end{cases} \tag{11.5}$$

The vectors u_k are linearly independent, since they have different eigenvalues for $\psi(H)$. The vector space they generate is invariant under these three operators (and hence under any element of the representation). Since ψ is an irreducible representation, $V = \langle u_0, \ldots, u_m \rangle$.

$$\begin{tikzcd} \cdots & \underset{U_{k-2}}{\bullet} \arrow[r, "\psi(Y)"'] & \underset{U_{k-1}}{\bullet} \arrow[l, bend right, "\psi(X)"'] \arrow[r, "\psi(Y)"'] & \underset{U_k}{\bullet} \arrow[l, bend right, "\psi(X)"'] \arrow[r, "\psi(Y)"'] & \underset{U_{k+1}}{\bullet} \arrow[l, bend right, "\psi(X)"'] \arrow[r, "\psi(Y)"'] & \underset{U_{k+2}}{\bullet} \arrow[l, bend right, "\psi(X)"'] & \cdots \end{tikzcd}$$

Figure 6: Action of $\psi(X)$ and $\psi(Y)$ over the eigenspaces of $\psi(H)$, $U_k = \langle u_k \rangle$.

Thus, we have seen that every irreducible representation of dimension $m+1$ will have a basis of eigenvectors of the image of H with eigenvalues $-m, -m+2, \ldots, m-2, m$. So, two irreducible representations with the same dimension are isomorphic and we can label every irreducible representation by its dimension. From now on, ψ_m will mean the irreducible representation of dimension $m+1$.

We have seen the rules that have to obey the representations of $\mathsf{sl}(2;\mathbb{C})$. Moreover, we can define on every complex vector space V of dimension $m+1$ the operators $\psi(H), \psi(X)$ and $\psi(Y)$ acting on a basis by (11.5). It can be seen that they will obey the commutation relations (11.1) and they will generate an irreducible representation of $\mathsf{sl}(2;\mathbb{C})$. Thus, every irreducible representation of $\mathsf{sl}(2;\mathbb{C})$ is realizable.

Furthermore, as we have already pointed out, the representations of $\mathsf{SU}(2)$ are in one-to-one correspondence with the representations of $\mathsf{sl}(2;\mathbb{C})$. Now the usefulness goes in the other direction: since $\mathsf{SU}(2)$ is a compact Lie group, by Theorem 9.4 it is completely reducible and hence every representation of $\mathsf{sl}(2;\mathbb{C})$ will also be completely reducible. Then, every representation of $\mathsf{sl}(2;\mathbb{C})$ is a direct sum of representations which satisfy (11.5). We have classified the representations of $\mathsf{sl}(2;\mathbb{C})$.

As a consequence, for any (finite dimensional) complex representation $\psi : \mathsf{sl}(2;\mathbb{C}) \to \mathsf{gl}(V)$, working separately on each one of the irreducible representations it can be seen that the operators $\psi(X)$ and $\psi(Y)$ are nilpotent and that if k is an eigenvalue of $\psi(H)$, then it is an integer and the integers

$$-|k|, -|k|+2, \ldots, |k|-2, |k|$$

are also eigenvalues of $\psi(H)$.

11.3 Representations of $\mathsf{SO}(3)$

We know the representations of $\mathsf{sl}(2;\mathbb{C})$ and are in one-to-one correspondence with the ones of $\mathsf{su}(2)$. These ones in turn are in one-to-one correspondence with the representations of $\mathsf{SU}(2)$ due to the fact that it is simply connected.

We have seen that $\mathsf{su}(2)$ is isomorphic to $\mathsf{so}(3)$ and now we want to study the relation between the representations of $\mathsf{SO}(3)$, which is not simply connected, and those of $\mathsf{so}(3)$. Take the basis

$$F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad F_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{11.6}$$

of $\mathsf{so}(3)$. It has the same commutation relations that the basis E_j of $\mathsf{su}(2)$ given previously in (10.4) and both Lie algebras are of dimension 3. Hence, the linear map $\phi : \mathsf{su}(2) \to \mathsf{so}(3)$ which sends $E_j \to F_j$, for $j = 1, 2, 3$, will be a Lie algebra isomorphism. Note that the matrix $E_1 = iH/2$.

Whence, the irreducible representations of $\mathsf{so}(3)$ will be of the form $\sigma_m = \psi_m \circ \phi^{-1}$: $\mathsf{so}(3) \to \mathsf{gl}(V)$ and we want to see which ones come from a representation Σ_m of the Lie group $\mathsf{SO}(3)$. That is, the ones which satisfy

$$\Sigma_m(\exp X) = \exp(\sigma_m(X)) \quad \forall X \in \mathsf{so}(3). \tag{11.7}$$

First, we assume that m is an odd integer and that this representation exists. Computing

$$e^{2\pi F_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\pi & -\sin 2\pi \\ 0 & \sin 2\pi & \cos 2\pi \end{pmatrix} = I.$$

We will have

$$\Sigma_m(e^{2\pi F_1}) = I \tag{11.8}$$

whereas

$$\sigma_m(F_1) = \psi_m(\phi^{-1}(F_1)) = \psi_m(E_1) = \frac{i}{2}\psi_m(H)$$

the eigenvectors of $\psi_m(H)$, u_k, will be then eigenvectors of $\sigma_m(F_1)$ with eigenvalue $i(m - 2k)/2$. Thus, in the basis formed by the u_k, $\sigma_m(F_1)$ will be a diagonal matrix with the eigenvalues on the diagonal. The eigenvalues of $\exp(2\pi\sigma_m(F_1))$ will be $e^{2\pi i(m-2k)/2}$ and since m is odd, so is $m - 2k$. Hence,

$$e^{2\pi\sigma_m(F_1)} = -I. \tag{11.9}$$

Looking at (11.8) and (11.9), we have obtained

$$\Sigma_m(\exp(2\pi F_1)) = I \neq \exp(\sigma_m(2\pi F_1)) = -I.$$

Thus, there is no irreducible representation of $\mathsf{SO}(3)$ of even dimension (m odd implies $m + 1$ even).

11.3.1 A representation of $\mathsf{SU}(2)$

We make a pause here to introduce a representation of $\mathsf{SU}(2)$ which will come in handy for the construction of the representation Σ_m.

Let V_m be the vector space of homogeneous polynomials of degree m in two complex variables. That is, the space of polynomials of the form

$$f(z_1, z_2) = \sum_{k=0}^{m} a_k z_1^{m-k} z_2^k, \quad a_k \in \mathbb{C}.$$

We define a linear transformation on the $(m+1)$-dimensional space V_m

$$\begin{array}{rcl} \Pi_m(U) : V_m & \to & V_m \\ f(z) & \to & [\Pi_m(U)f](z) = f(U^{-1}z) \end{array} \tag{11.10}$$

where $U \in \mathsf{SU}(2), z \in \mathbb{C}^2$ and $f \in V_m$. We check that it is a representation,

$$\Pi_m(U_1)[\Pi_m(U_2)f](z) = [\Pi_m(U_2)f](U_1^{-1}z) = f(U_2^{-1}U_1^{-1}z) = [\Pi_m(U_1U_2)f](z).$$

We can find the associated representation of $\mathsf{su}(2)$, which we will denote by π_m, with the formula (6.4),

$$[\pi_m(X)f](z) = \frac{d}{dt}f(e^{-tX}z)\bigg|_{t=0}.$$

Computing it,

$$\frac{d}{dt}f(e^{-tX}z)\bigg|_{t=0} = D_z f(-X\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}) = \frac{\partial f}{\partial z_1}(-X_{11}z_1 - X_{12}z_2) + \frac{\partial f}{\partial z_2}(-X_{21}z_1 - X_{22}z_2)$$

and applying it to the basis of $\mathsf{su}(2)$, H, X, Y, we get

$$\pi_m(H) = -z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}, \quad \pi_m(X) = -z_2\frac{\partial}{\partial z_1}, \quad \pi_m(Y) = -z_1\frac{\partial}{\partial z_2}$$

and they satisfy the commutation relations (11.1). In fact, it can be seen that $\pi_m \cong \psi_m$ (cf. [7]). Hence, we have found the irreducible representations of $\mathsf{SU}(2)$ associated to the ones of $\mathsf{sl}(2;\mathbb{C})$.

11.3.2 Irreducible representations of $\mathsf{SO}(3)$

Now, returning to the case of $\mathsf{SO}(3)$, assume that m is an even integer and consider the representation Π_m.

$$e^{2\pi E_1} = -I \implies \Pi_m(-I) = \Pi_m(e^{2\pi E_1}) = e^{\pi_m(2\pi E_1)}.$$

Recall the universal cover ρ that we found in the previous section. It has kernel $\{I, -I\}$ and hence for two elements U and $-U$

$$\rho(-U) = \rho(-I)\rho(U) = \rho(U)$$

so every element $R \in \mathsf{SO}(3)$ will have two antiimages. Now, $\pi_m(2\pi E_1)$ has eigenvalues $2\pi i(m-2j)/2$ for $j = 0, \ldots, m$. Then its exponential will be diagonal on the basis $u_0, \ldots, u_m$ and, since m is even, the eigenvalues will be $e^{2\pi i(m-2j)/2} = 1$. Thus,

$$\Pi_m(-I) = I.$$

So we see that $\Pi_m(-U) = \Pi_m(U)$ too, for any element $U \in \mathsf{SU}(2)$. We can define then

$$\Sigma_m = \Pi_m \circ \rho^{-1}$$

because every two related antiimages of ρ, U and $-U$, will go to the same element $\Pi_m(U)$. Finally, the map

$$\sigma_m = \pi_m \circ \phi^{-1}$$

is the one related by the exponentials with $\Sigma_m = \Pi_m \circ \rho^{-1}$ and $\pi_m \cong \psi_m$.

11.4 Representations of $\mathsf{sl}(3;\mathbb{C})$

Before studying the representations of $\mathsf{sl}(3;\mathbb{C})$, we introduce a definition that will be used. Recall the tensor product of representations for Lie groups, its associated Lie algebra representation comes from applying formula (6.4) to the tensor product representation:

Definition 11.5. Let $\mathfrak{g}$ be a Lie algebra and let $\pi_1 : \mathfrak{g} \to \mathsf{gl}(U)$ and $\pi_2 : \mathfrak{g} \to \mathsf{gl}(V)$ be representations acting on vector spaces U and V. The tensor product of the representations is defined for all $X \in \mathfrak{g}$ by

$$\begin{array}{rccl} \pi_1 \otimes \pi_2 : & \mathfrak{g} & \to & \mathsf{gl}(U \otimes V) \\ & X & \to & \pi_1(X) \otimes I + I \otimes \pi_2(X) \end{array}$$

Now, for the case of $\mathsf{sl}(3;\mathbb{C})$ and $\mathsf{SU}(3)$ we have the same equivalence between the complex representations as for $\mathsf{sl}(2;\mathbb{C})$ and $\mathsf{SU}(2)$:

$$\mathrm{Rep}_{\mathbb{C}}(\mathsf{SU}(3)) \quad \longleftrightarrow \quad \mathrm{Rep}_{\mathbb{C}}(\mathsf{su}(3)_{\mathbb{R}}) \quad \longleftrightarrow \quad \mathrm{Rep}_{\mathbb{C}}(\mathsf{sl}(3;\mathbb{C})_{\mathbb{C}}).$$

Thus, from now on, we will only consider complex representations. We can take the following basis for the traceless matrices forming $\mathsf{sl}(3;\mathbb{C})$.

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad Y_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Note that, discarding the last row and last column of the matrices H_1, X_1, Y_1, and discarding the first row and first column of the matrices H_2, X_2, Y_2; we recover the Lie algebra $\mathsf{sl}(2;\mathbb{C})$. Thus, $\langle H_1, X_1, Y_1\rangle \cong \langle H_2, X_2, Y_2\rangle \cong \mathsf{sl}(2;\mathbb{C})$ and they satisfy the same commutation relations between them. The other commutation relations are:

$$\begin{array}{llll} & [H_1,H_2] = 0 & [X_3,Y_3] = H_1 + H_2 & \\ [H_1,X_2] = -X_2, & [H_1,Y_2] = Y_2, & [H_1,X_3] = X_3, & [H_1,Y_3] = -Y_3; \\ [H_2,X_1] = -X_1, & [H_2,Y_1] = Y_1, & [H_2,X_3] = X_3, & [H_2,Y_3] = -Y_3; \\ [X_1,X_2] = X_3, & [Y_1,Y_2] = -Y_3, & [X_1,Y_2] = 0, & [X_2,Y_1] = 0; \\ [X_1,X_3] = 0, & [Y_1,Y_3] = 0, & [X_2,X_3] = 0, & [Y_2,Y_3] = 0; \\ [X_2,Y_3] = Y_1, & [X_3,Y_2] = X_1, & [X_1,Y_3] = -Y_2, & [X_3,Y_1] = -X_2. \end{array}$$

Now, we would like to apply the same reasoning as for $\mathsf{sl}(2;\mathbb{C})$ and have the representations determined by the eigenspaces of the image of $H \in \mathsf{sl}(2;\mathbb{C})$. However, now any traceless diagonal matrix will be expressed as a linear combination of H_1 and H_2 instead of being expressed by a multiple of H. We have the following elementary proposition (c.f. [9]),

Proposition 11.6. *Commuting diagonalizable linear operators on a complex vector space are simultaneously diagonalizable.*

Since $[H_1, H_2] = 0$, we see that we should study the *eigenvectors* of the subalgebra spanned by H_1 and H_2, namely $\mathfrak{h} = \langle H_1, H_2\rangle$.

Definition 11.7. Given V a finite dimensional complex vector space and $\mathcal{A} \subset \mathrm{End}(V)$ a subspace. A ***weight*** is an element $\lambda \in \mathcal{A}^*$ such that there exists $v \in V$, $v \neq 0$ satisfying

$$f(v) = \lambda(f)v, \text{ for all } f \in \mathcal{A}.$$

Given a weight $\lambda \in \mathcal{A}^*$, a ***weight vector*** for λ is a $v \in V$ such that

$$f(v) = \lambda(f)v, \text{ for all } f \in \mathcal{A}.$$

The dimension of the subspace

$$V_\lambda = \{v \in V \mid f(v) = \lambda(f)v \ \ \forall f \in \mathcal{A}\ \}$$

is called the ***multiplicity*** of λ.

In our case, given $\psi : \mathsf{sl}(3;\mathbb{C}) \to \mathsf{gl}(V)$ a Lie algebra representation on a complex vector space V, our $\mathcal{A}$ will be $\psi(\mathfrak{h})$.

The following proposition is easily proved (see, for instance, [9]),

Proposition 11.8. *Commuting linear operators over a complex vector space have a common eigenvector.*

Thus, we see that every representation of $\mathsf{sl}(3;\mathbb{C})$ has at least one weight. Furthermore, if $\lambda \in \mathfrak{h}^*$ is a weight, then $\lambda(\psi(H_1))$ and $\lambda(\psi(H_2))$ are integers, as can be seen by restricting the representation to $\langle H_i, X_i, Y_i\rangle$, for $i = 1, 2$. Recall the procedure followed for $\mathsf{sl}(2;\mathbb{C})$: we studied the eigenvalues of $H(X(u))$ and $H(Y(u))$ for an eigenvector u of H, which in fact is equivalent to study the eigenvalues of ad_H.

Definition 11.9. In the special case $\psi = \mathrm{ad} : \mathsf{sl}(3;\mathbb{C}) \to \mathrm{End}(\mathsf{sl}(3;\mathbb{C}))$, nonzero weights are called ***roots*** and weight vectors are called ***root vectors***.

Since H_1 and H_2 form a basis of $\mathfrak{h}$, every weight (and in particular every root) is determined by its value on these two vectors. To denote this, we write a weight $\lambda \in \psi(\mathfrak{h})^*$ as the ordered pair $\lambda = (\lambda(\psi(H_1)), \lambda(\psi(H_2)))$. Thus, $\alpha \in \mathrm{ad}(\mathfrak{h})^*$ is a root if and only if $\alpha \neq (0, 0)$ and there exists $Z \in \mathsf{sl}(3;\mathbb{C})$ such that $\mathrm{ad}_{H_i}(Z) = a_i Z$ for $i = 1, 2$. For example, $X_1, X_2, X_3, Y_1, Y_2, Y_3$ are root vectors of roots,

$$\begin{array}{lll} X_1 : (2,-1), & X_2 : (-1,2), & X_3 : (1,1) \\ Y_1 : (-2,1), & Y_2 : (1,-2), & Y_3 : (-1,-1). \end{array} \tag{11.11}$$

In fact, these are the only roots of $\mathsf{sl}(3;\mathbb{C})$ and it is said that these six roots form the ***root system*** of $\mathsf{sl}(3;\mathbb{C})$.

Remark: For $\mathsf{sl}(2;\mathbb{C})$, the root system would be the roots of X, Y which are, respectively, 2 and -2.

Now, let $\lambda = (m_1, m_2)$ be a weight and $v \neq 0$ a corresponding weight vector. Let $\alpha = (a_1, a_2)$ be a root and $Z_\alpha \in \mathsf{sl}(3;\mathbb{C})$ a corresponding root vector. Then, for $j = 1, 2$, using that the Lie bracket of a root vector is, by definition, $[H_j, Z_\alpha] = a_j Z_\alpha$, we have

$$H_j(Z_\alpha)v = (m_j + a_j)Z_\alpha v. \tag{11.12}$$

This implies that $Z_\alpha v$ is a new weight vector with weight $(m_1 + a_1, m_2 + a_2)$ or that $Z_\alpha v = 0$. Therefore, the root vectors now play the role of $X, Y \in \mathsf{sl}(2;\mathbb{C})$. We introduce a result from [7],

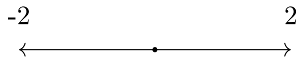

Figure 7: Root system of $\mathsf{sl}(2;\mathbb{C})$

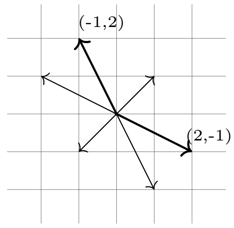

Figure 8: Root system of $\mathsf{sl}(3;\mathbb{C})$

Proposition 11.10. *Weight vectors with different weights are linearly independent.*

That will help us to prove the following statement.

Proposition 11.11. *Let $\psi : \mathsf{sl}(3;\mathbb{C}) \to \mathsf{gl}(V)$ be an irreducible representation. Then, $\psi(H_1)$ and $\psi(H_2)$ can be simultaneously diagonalized (i.e., every irreducible representation is the direct sum of its weight spaces).*

Proof. We have seen that every representation has at least one weight. We denote by λ this weight and by E_λ its weight space ($\dim E_\lambda \geq 1$). Now, let Z_α be a root vector with root α, $\psi(Z_\alpha)$ maps E_λ into the weight space $E_{\lambda+\alpha}$. Thus, the space $W = \bigoplus_\lambda E_\lambda$ (by Proposition 11.10, it is indeed the direct sum) is invariant under the action of the elements of the basis and hence is invariant under $\mathsf{sl}(3;\mathbb{C})$. Since ψ is irreducible, it follows that $W = V$. $\square$

Recall again the construction of the representations of $\mathsf{sl}(2;\mathbb{C})$: we found the greatest eigenvalue of H which was associated to the number N which held $X^N(u) \neq 0$ and $X^{N+1}(u) = 0$; and then we iterated Y over $X^N(u)$ to find all the vectors in V. Looking at (11.12), we fix a set of roots:

Definition 11.12. Let Φ be the previous root system of $\mathsf{sl}(3;\mathbb{C})$, a subset Φ^+ is a set of ***positive roots*** if for each $\alpha \in \Phi$ exactly one of the roots $\alpha, -\alpha$ is contained in Φ^+ and for any two distinct roots $\alpha, \beta \in \Phi^+$ such that $\alpha + \beta \in \Phi$ then $\alpha + \beta \in \Phi^+$. The ***positive simple roots*** are the elements of Φ^+ which cannot be written as the sum of two elements of Φ^+.

We see that the roots $\alpha_1 = (2, -1)$ and $\alpha_2 = (-1, 2)$ are a set of positive simple roots of the root system (11.11). Now, in order to find the analogous to the *greatest eigenvalue* we introduce a partial ordering,

Definition 11.13. Let $\alpha_1 = (2, -1)$ and $\alpha_2 = (-1, 2)$ be the positive simple roots. Let λ_1 and λ_2 be two weights. λ_1 is ***higher*** than λ_2, denoted $\lambda_1 \succeq \lambda_2$, if $\lambda_1 - \lambda_2$ is a nonnegative linear combination of the positive simple roots. That is, if it can be written in the form

$$\lambda_1 - \lambda_2 = a\alpha_1 + b\alpha_2, \quad \text{with } a \geq 0 \text{ and } b \geq 0.$$

A weight λ_0 is called a ***highest weight*** if for all weights λ of the representation, $\lambda_0 \succeq \lambda$. It is an order relation (it is reflexive, antisymmetric and transitive) but it is not total.

The finite-dimensionality of H_j, $j = 1, 2$, together with (11.12), implies that there is a weight vector in V that is killed by X_i, for $i = 1, 2$ and 3. So, consider an irreducible representation $\psi : \mathsf{sl}(3;\mathbb{C}) \to \mathsf{gl}(V)$ and a weight vector $v \in V$ with weight λ such that $\psi(X_j)(v) = 0$ for $j = 1, 2, 3$. Consider the subspace W of V spanned by vectors of the form

$$w = \psi(Y_{j_1})\psi(Y_{j_2})\ldots\psi(Y_{j_N})v \tag{11.13}$$

with $j_k = 1, 2$ or 3 and $N \geq 0$. For $N = 0$ we have $w = v$. Taking the ordered basis of $\mathsf{sl}(3;\mathbb{C})$, $\{X_1, X_2, X_3, H_1, H_2, Y_1, Y_2, Y_3\}$, we apply the Reordering Lemma[8] to the action of an element of the basis over w and rewrite the expression as

$$\psi(Y_3)^{i_3}\psi(Y_2)^{i_2}\psi(Y_1)^{i_1}\psi(H_2)^{j_2}\psi(H_1)^{j_1}\psi(X_3)^{k_3}\psi(X_2)^{k_2}\psi(X_1)^{k_1}v.$$

If $k_n \neq 0$ for $n = 1, 2, 3$, then the vector is zero because v gets killed by $\psi(X_{k_n})$. Otherwise, v is an eigenvector of both $\psi(H_1)$ and $\psi(H_2)$ so we get a linear combination of elements of the form

$$\psi(Y_3)^{i_3}\psi(Y_2)^{i_2}\psi(Y_1)^{i_1}v$$

which is an element of the subspace W spanned by the elements of the form (11.13). This procedure works for the action of any element of the basis so W is invariant and $W = V$.

The elements Y_1, Y_2 and Y_3 are root vectors with roots $-\alpha_1, -\alpha_2$ and $-\alpha_1 - \alpha_2$, respectively, and, by the formula (11.12), every element of the form (11.13) with $N > 0$ will be a weight vector with weight lower than λ. Thus, the only vectors in V with weight λ are the multiples of v and we have have proven the result:

Proposition 11.14. *Every irreducible representation of* $\mathsf{sl}(3;\mathbb{C})$ *has a unique highest weight and its highest weight space has multiplicity 1.*

We introduce a Lemma,

Lemma 11.15. *Let* $\psi : \mathsf{sl}(3;\mathbb{C}) \to V$ *be a (complex) representation with a vector* v *with weight* λ *killed by* $\psi(X_j)$ *for* $j = 1, 2, 3$ *such that the smallest invariant subspace of* V *containing* v *is* V*. Then* v *is the unique (up to scalar multiplication) highest weight vector of the representation.*

Proof. The proof is the same as for Proposition 11.14 changing the hypothesis of irreducibility for the hypothesis of V being the smallest invariant space such that $v \in V$. (Note that both hypotheses can be used to show that $W = V$). ☐

Now we can prove that the highest weight determines the irreducible representation.

Proposition 11.16. *Two irreducible representation of* $\mathsf{sl}(3;\mathbb{C})$ *with the same highest weight are isomorphic.*

Proof. Let $\psi : \mathsf{sl}(3;\mathbb{C}) \to \mathsf{gl}(V)$ and $\phi : \mathsf{sl}(3;\mathbb{C}) \to \mathsf{gl}(W)$ be two irreducible representations with highest weight λ with respective highest weight vectors $v \in V$ and $w \in W$. Consider the direct sum of this representations, $\psi \oplus \phi : \mathsf{sl}(3;\mathbb{C}) \to \mathsf{gl}(V \oplus W)$, and let U be the smallest invariant subspace of $V \oplus W$ which contains the vector (v, w). This invariant subspace has an associated subrepresentation with weight vector $(v, w) \in U$ killed by the images of X_i, $i = 1, 2, 3$. Then, by Lemma 11.15, (v, w) is the highest weight vector of the subrepresentation. Besides, the subrepresentation is completely reducible (this comes from the complete reducibility of $\mathsf{SU}(3)$) and we can write it as a direct sum of irreducible representations,

$$U = \bigoplus_j U_j.$$

[8]See A.14.

Every irreducible representation is the direct sum of its weight spaces (Proposition 11.11), so $(v, w) \in U_k$ for some k. Since U is the smallest invariant space that contains (v, w), it follows that $U_k = U$ and U is irreducible.

The projections of U on every component of $V \oplus W$,

$$\begin{array}{rccc} \pi_V : & U & \to & V \\ & (u_1, u_2) & \mapsto & u_1 \end{array} \quad \text{and} \quad \begin{array}{rccc} \pi_W : & U & \to & W \\ & (u_1, u_2) & \mapsto & u_2 \end{array}$$

for $u_1 \in V$, $u_2 \in W$ such that $(u_1, u_2) \in U$, hold

$$\pi_V \circ (\psi \oplus \phi)|_U = \psi \circ \pi_V \text{ and } \pi_W \circ (\psi \oplus \phi)|_U = \phi \circ \pi_W.$$

Besides, U, V and W are all irreducible and we can apply Schur's Lemma[9]. Since π_V and π_W are nonzero because $(v, w) \in U$, it follows that $V \cong U \cong W$. □

We have seen that the weights of $\mathsf{sl}(3; \mathbb{C})$ are pairs of integers. For highest weights we can restrict even more their possible value: let $\lambda = (m_1, m_2)$ the highest weight of an irreducible representation and $v \neq 0$ its weight vector. Then, $X_1 v = 0$ and $X_2 v = 0$ or it would not be the highest weight vector. The study of the representations of $\mathsf{sl}(2; \mathbb{C})$ makes the values m_1 and m_2 automatically non-negative. So,

Proposition 11.17. *The highest weight of an irreducible representation is a pair of non-negative integers.*

Conversely,

Proposition 11.18. *For every pair (m_1, m_2) of non-negative integers there exists an irreducible representation of $\mathsf{sl}(3; \mathbb{C})$ with highest weight $\lambda = (m_1, m_2)$.*

Proof. The trivial representation is an irreducible representation with highest weight $(0, 0)$. The standard representation, the inclusion of $\mathsf{sl}(3; \mathbb{C})$ into $\mathsf{gl}(\mathbb{C}^3)$, has the canonical basis vectors as weight vectors $e_1, e_2, e_3 \in \mathbb{C}^3$ and corresponding weights $(1, 0)$, $(-1, 1)$, $(0, -1)$. It is irreducible and has highest weight $(1, 0)$. Analogously to groups, for Lie algebras one can define the dual of a given representation. In particular, the dual representation of the standard representation is given by

$$\psi(Z) = -Z^\top$$

for every $Z \in \mathsf{sl}(3; \mathbb{C})$. It is also irreducible and has weight vectors e_1, e_2, e_3 with corresponding weights $(-1, 0)$, $(1, -1)$ and $(0, 1)$. The highest weight being $(0, 1)$.

Now we can build all the other irreducible representations from the tensor product of these two, called ***fundamental representations***. So, let $V = \mathbb{C}^3$ and V^* its dual. Let $\psi_1 : \mathsf{sl}(3; \mathbb{C}) \to \mathsf{gl}(V)$ be the standard representation and $\psi_2 : \mathsf{sl}(3; \mathbb{C}) \to \mathsf{gl}(V^*)$ its dual representation and let $v_1 = e_1$ and $v_2 = e_3$ be the respective highest weight vectors. Consider the tensor product of the representations,

$$\phi = (\psi_1 \otimes \psi_2) : \mathsf{sl}(3; \mathbb{C}) \to \mathsf{gl}(V \otimes V^*)$$

and consider the vector $v_1 \otimes v_2 \in V \otimes V^*$. It is killed by $\phi(X_j)$ for $j = 1, 2, 3$ and has weight $(1, 1)$, as it can be seen by the action of $\phi(H_j)$, $j = 1, 2$ over it:

$$(\psi_1 \otimes \psi_2)(H_1)(v_1 \otimes v_2) = \psi_1(H_1)(v_1) \otimes I(v_2) + I(v_1) \otimes \psi_2(H_1)(v_2) = v_1 \otimes v_2 + v_1 \otimes 0 = v_1 \otimes v_2$$

[9] See A.15

and similarly for H_2. Thus, applying the same reasoning of the proof of Proposition 11.16 to the smallest invariant subspace W which contains $v_1 \otimes v_2$, we get that W is irreducible and $\phi_W : \mathsf{sl}(3;\mathbb{C}) \to \mathsf{gl}(W)$ is the irreducible representation with highest weight $(1,1)$.

Moreover, the same reasoning can be applied to the m_1-times tensor product of V tensor product with the m_2-times tensor product of V^* to get the desired irreducible representation with highest weight (m_1, m_2). □

Remark: One could go further and ask what can be said about two given isomorphic representations which are the direct sum of irreducible representations. As we have said, the character theory for finite groups can be extended to compact Lie groups and hence, it is possible to show that the two representations are the same up to the order in which appear the irreducible representations.

11.4.1 Weight diagrams of $\mathsf{sl}(3;\mathbb{C})$

In this section we show a way to *visualize* the representations by means of weight diagrams. Defining an inner product in $\mathfrak{h}$ by $\langle H, H' \rangle = \mathrm{trace}(H^\dagger H')$, we can identify $\mathfrak{h}^*$ with $\mathfrak{h}$. Thus, the matrices associated to the weights $\lambda_1 = (1,0)$ and $\lambda_2 = (0,1)$ are,

$$\lambda_1 = \mathrm{diag}(2/3, -1/3, -1/3), \quad \lambda_2 = \mathrm{diag}(1/3, 1/3, -2/3). \tag{11.14}$$

Therefore, the roots $\alpha_1 = 2\lambda_1 - \lambda_2$ and $\alpha_2 = -\lambda_1 + 2\lambda_2$ are the matrices,

$$\alpha_1 = \mathrm{diag}(1, -1, 0), \quad \alpha_2 = \mathrm{diag}(0, 1, -1).$$

Now, as vectors, λ_1 and λ_2 span a two dimensional real space. With the defined inner product they have norm $\sqrt{2}$ and the angle between them is 60^{o}. Then, the weight diagram for the standard representation will be the weights $(1,0), (-1,1), (0,-1)$ in the basis (λ_1, λ_2).

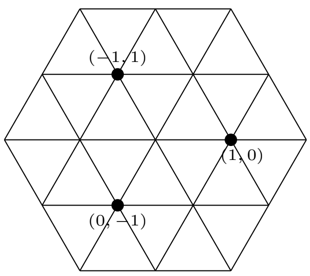

Figure 9: Weight diagram of the standard representation, highest weight $(1,0)$.

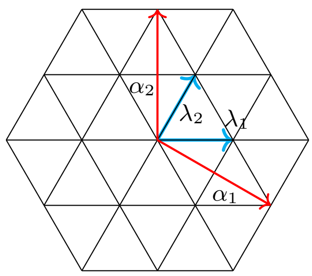

Figure 10: The vectors corresponding to λ_1, λ_2 and α_1, α_2.

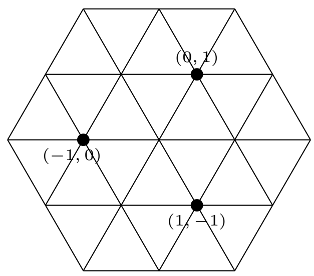

Figure 11: Weight diagram of the dual representation, highest weight $(0,1)$.

Note that all the weights, if not killed, are connected by the root vectors α_1, α_2. Moreover, we can see that a given weight connected by α_1, α_2 or $\alpha_1 + \alpha_2$ to other weights spans a representation of $\mathsf{sl}(2;\mathbb{C})$. We can construct the irreducible representation corresponding to a given highest weight with the repeated translation by $-\alpha_1$ and $-\alpha_2$, keeping in mind that the new weights cannot be higher than the highest weight.

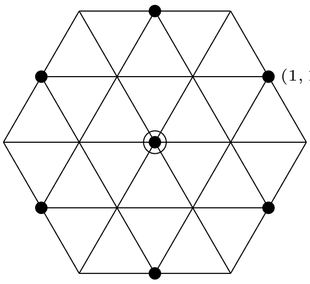

Figure 12: Weight diagram of the representation with highest weight $(1,1)$. The multiplicity of $(0,0)$ is 2.

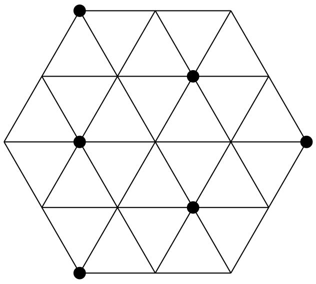

Figure 13: Weight diagram of the representation with highest weight $(2,0)$.

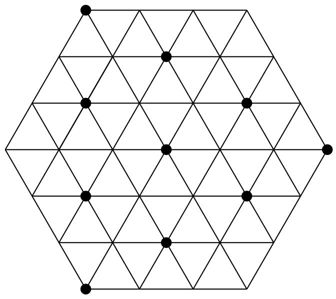

Figure 14: Weight diagram of the representation with highest weight $(3,0)$.

12 The Eightfold way

This section is devoted to the study of a classification scheme which, as we will see, is related to the representations of the Lie algebra $\mathsf{sl}(3;\mathbb{C})$. This geometric approach is known as the Eightfold way, a term coined by Gell-Mann in reference to the buddhist Noble Eightfold Path. It led to the realization of the existence of a symmetry between elementary particles and this, in turn, to the postulation of the existence of quarks.

In 1947, the known particles (the nucleons p and n, the leptons e, μ, their neutrinos ν_e, ν_μ and the mesons π^+, π^-) were *almost*[10] understood and classified until a new particle entered the game: a neutral particle, denoted K^0, which decayed into two oppositely charged pions,

$$K^0 \longrightarrow \pi^+ + \pi^-.$$

In the following years, new decays associated to new particles were observed:

$$\begin{gathered} K^+ \longrightarrow \pi^+ + \pi^+ + \pi^- \\ \Lambda \longrightarrow p + \pi^-. \end{gathered}$$

The decaying particles were assumed to be different from the first particle, K^0, by charge conservation for K^+ and by baryon number conservation for Λ. The differences between the time of production of the particles and their time of decay led physicists to assume that the processes involved were different (now it is said that the particles are produced by the strong interaction and decay, way more slowly, by the weak interaction). Gell-Mann and Nishijima, independently, assigned to the new particles a new quantum number, called *strangeness*, with integer value. In particular, the proton and neutron were assigned zero strangeness. From the observed decays, they derived the Gell-Mann-Nishijima formula

$$Q = I_3 + \frac{1}{2}(B + S) \tag{12.1}$$

where Q is the charge, I_3 the third component of isospin, B the baryon number (1 for baryons and 0 for mesons) and S the strangeness. This last property was assumed to be conserved in the creation of the particles but not conserved in their decay. That is, strangeness is conserved by the strong force but not conserved by the weak force. This simple idea solved the apparent contradictions between possible and observed decays. More particles were discovered and each one was assigned a strangeness value coherent

[10]The muon and its neutrino were still puzzling.

with their production and decay. However, there seemed to be no logic behind the *zoo* of particles.

In 1961, Gell-Mann and Ne'eman independently arranged particles with the same spin by strangeness and charge, obtaining the following geometric figures:

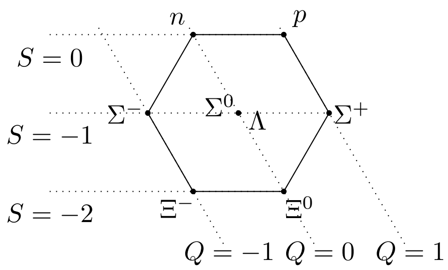

Figure 15: The Baryon octet (spin $= \frac{1}{2}$)

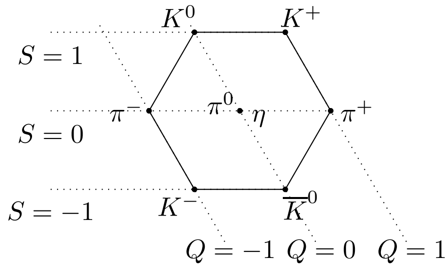

Figure 16: The Meson octet (spin $= 0$)

They are clearly reminiscent of the weight diagram of the $\mathfrak{sl}(3;\mathbb{C})$ representation with highest weight $(1,1)$. Furthermore, by that time, the following diagram was not complete:

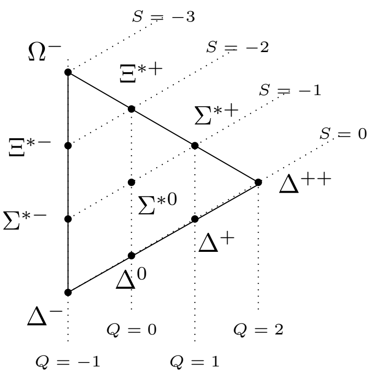

Figure 17: The baryon decuplet (spin $= \frac{3}{2}$)

There was no known baryon with $Q = -1$ and $S = -3$. Gell-Mann predicted its existence (he even predicted its mass with another empirical formula) and in 1964 the Ω^- was discovered.

The Eightfold way was, hence, the equivalent to Mendeleev's Periodic Table for particle physics: a geometric representation of the known particles provided empirical formulas and led to predictions that were later verified. It was a matter of time for somebody to find the analogue to the electron shells in nuclear physics.

12.1 Lie Groups and Lie Algebras in Physics

Symmetries in physics were already studied through Lie groups and Lie algebras before the Eightfold way. A symmetry of a physical system is a set of transformations acting on the system such that the physical observables are invariant. Moreover, symmetry transformations correspond to elements of a group and one symmetry transformation followed by another corresponds to group multiplication. If the transformation is continuous, then the group is a Lie group (physicists work on locally euclidean spaces).

Noether's theorem asserts, in an informal way, that every continuous symmetry has a corresponding quantity constant in time. Thus, studying the symmetries of a system one can obtain its conservation laws.

In quantum mechanics, the group associated to a symmetry must be unitary in order to preserve the probability of a system to be in a given state. In the case of the wave functions describing electrons, the group associated to the rotational invariance is $\mathsf{SU}(2)$. Recall that $\mathrm{Rep}_{\mathbb{C}}(\mathsf{SU}(2)) \leftrightarrow \mathrm{Rep}_{\mathbb{C}}(\mathsf{sl}(2;\mathbb{C})_{\mathbb{C}})$ and the irreducible representations of $\mathsf{sl}(2;\mathbb{C})$ are the ψ_m. The representation acting on a single electron is ψ_1 (dimension 2) and the ladder operators (X and Y) change its spin from down to up and vice versa. The representation of the coupling of two electrons is $\psi_1 \otimes \psi_1$, which is isomorphic to $\psi_2 \oplus \psi_0$.

Comment: It is usually said that rotating an electron 360^{o} adds a minus sign to its wave function. The reason for this is the fact that electrons are fermions and hence have semi-integer spin. That is, the irreducible representations of $\mathsf{SU}(2)$, ψ_m, acting on them have odd m. Thus, there is no irreducible representation of $\mathsf{SO}(3)$ associated to ψ_m and the exponential of an element of $\mathsf{so}(3)$ times 2π is $-I$ (c.f. (11.9)). On the other hand, if the factor is 4π, the value of the exponential is I.

The proton and the neutron have very similar masses and this led Heisenberg to consider them as two states of the same particle, the hypothetical nucleon. That is, like the two states of an electron (spin up/down). Thus, it was introduced the approximate[11] symmetry of isospin, which is therefore associated to the group $\mathsf{SU}(2)$. Isospin is short for "isotopic spin", in reference to different isotopes having different isospin.

12.2 The quark model

In 1964, Gell-Mann (again) and Zweig, inspired by the concept of symmetry of isospin and seeing that the group underlying the symmetry of the diagrams was the group $\mathsf{SU}(3)$, proposed that all hadrons are composed of a more elementary particle, which Gell-Mann called quark. The symmetry was even more approximate than the isospin symmetry, since masses in the same octet or decuplet vary considerably. The quark appeared in three different states, called *flavours*: up, down and strange. Each one with spin $= 1/2$ and carrying baryon number $1/3$. Moreover, the strange quark carried strangeness $S = -1$ and they had charge $2/3$, $-1/3$ and $-1/3$, respectively (cf. (11.14)). They were represented by the vectors

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

These are the basis vectors of the irreducible representation of $\mathsf{sl}(3;\mathbb{C})$ of highest weight $(1,0)$, which we denote now by $\mathbf{3}$. The antiparticles of these quarks, $\overline{u}, \overline{d}, \overline{s}$, are the basis of the dual representation, denoted $\overline{\mathbf{3}}$. The previous quantum numbers will be represented by the operators $I_3 = \frac{1}{2}H_1 = \mathrm{diag}(1/2, -1/2, 0)$, $B = \mathrm{diag}(1/3, 1/3, 1/3)$ and $S = \mathrm{diag}(0, 0, -1)$. The Gell-Mann-Nishijima formula (12.1) holds with $Q = \lambda_1 = \mathrm{diag}(2/3, -1/3, -1/3)$.

They postulated that mesons are formed by pairs of quarks and antiquarks, hence, they will be in the representation of the tensor product of both representations, $\mathbf{3} \otimes \overline{\mathbf{3}}$. As we have seen, this representation decomposes as the direct sum of an irreducible representation of

[11]Since their mass (energy) is not the same (but close), the symmetry is approximate.

highest weight $(1,1)$ with dimension 8 and a representation of dimension 1 (highest weight $(0,0)$). Thus, the irreducible representation forms the Meson octet, which together with the weight $(0,0)$ (which represents the particle $\eta' = \frac{u\overline{u}+d\overline{d}+s\overline{s}}{\sqrt{3}}$, invariant for the action of $\mathsf{SU}(3)$), forms the Meson nonet.

In the following diagrams we have represented the fundamental representations (quarks and antiquarks) and their tensor product, the Meson nonet. The multiplicity of the weight $(0,0)$ in the Meson nonet is 3, one corresponds to η' and the other two to $\pi^0 = \frac{1}{\sqrt{2}}(u\overline{u}-d\overline{d})$ and $\eta = \frac{1}{\sqrt{6}}(u\overline{u}+d\overline{d}-2s\overline{s})$. In the diagram, we show the quark content. Being specific, the highest weight vector is $u \otimes \overline{s}$ and from it we can compute the quark states with the images of the Y_i, $i = 1,2,3$ acting iteratively on $u \otimes \overline{s}$.

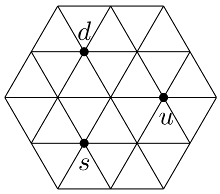

Figure 18: Quarks u, d, s.

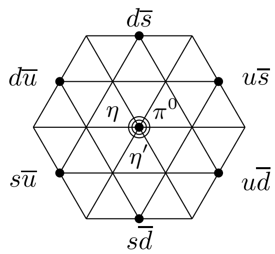

Figure 19: The Meson nonet.

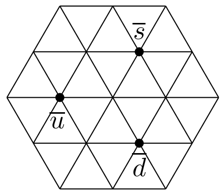

Figure 20: Antiquarks $\overline{u}, \overline{d}, \overline{s}$.

Baryons are considered to be composites of three quarks. Thus, they arise in the representation $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$. This representation has an irreducible representation of highest weight $(3,0)$ and dimension 10, the baryon decuplet, two of dimension 8 and one of dimension 1. That is, $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3} = \mathbf{10}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{1}$. This can be seen with the help of weight diagrams and using that $\mathbf{3}\otimes\mathbf{3} = \mathbf{6}\oplus\overline{\mathbf{3}}$, with highest weights $(2,0)$ and $(0,1)$. Then, $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3} = (\mathbf{6}\oplus\overline{\mathbf{3}})\otimes\mathbf{3}$ and the highest weights will be $(3,0)$, $(1,1)$, $(1,1)$ and $(0,0)$.

However, there was no experimental evidence of the existence of quarks and the beauty of this construction is not compelling enough. Besides, there were flaws in the theory. For instance, the baryon Δ^{++} ought to be composed of three quarks u and have spin $= \frac{3}{2}$. Thus, since quarks have spin $= \frac{1}{2}$, the three quarks should be in the same state, in contradiction with the Pauli exclusion principle.

In 1964, Greenberg introduced a new quantum property of quarks, called *color*, which could be red, green or blue. The associated symmetry was again the one of $\mathsf{SU}(3)$ but this time it was an exact symmetry: quarks with different color are indistinguishable except for their color. This solved the problem with Pauli exclusion principle (the three quarks in Δ^{++} had different color) and proposed that naturally occurring particles are colorless, thus, explaining the non direct observation of quarks. This initiated Quantum Chromodynamics, a theory of exact symmetry called color $\mathsf{SU}(3)$, in contrast with the approximate symmetry of flavour $\mathsf{SU}(3)$. Since the symmetry is exact, the implications are deeper and it is associated to an elementary interaction, the strong interaction.

Even though the theory was flawless, it seemed artificial. It was not widely accepted until the discovery of a new particle that was only explainable through the quark theory, adding a new flavour, charm.

Nowadays, the known flavours are up, down, strange, charm, top (or truth) and bottom (or beauty).

Conclusions and further work

Following a differential geometry approach along with a matrix group approach we have studied Lie groups and Lie algebras, first separately and next, with the exponential map, in relationship to each other. That way we have been able to prove the Closed subgroup theorem which has allowed us to prove that matrix Lie groups are Lie groups.

Studying the universal covering space of a connected Lie group, which can be made into a Lie group, the condition of simple connectedness arises to ensure that Lie algebra homomorphisms induce Lie group homomorphisms. Thus, we have seen the equivalence between the category of simply connected Lie groups and the category of real Lie algebras, and hence the equivalence between the representations of a simply connected Lie group and the representations of its associated Lie algebra.

Moreover, the existence of left invariant forms on Lie groups allow to describe in concrete terms the Haar measure on compact Lie groups. Then, the left invariant normalized volume form is also right invariant and thanks to this fact every representation of a compact Lie group can be shown to be completely reducible.

Finally, the study of complex representations of Lie algebras simplifies the study of their associated Lie groups, especially when they are compact and simply connected. This relation between representations of Lie algebras and Lie groups has been key to the field of elementary particle physics, leading to the concept of quarks.

Although we left Lie algebra representations at $\mathfrak{sl}(3;\mathbb{C})$, the generalization of our results to include any semisimple Lie algebra should be simple since the basic concepts in relation to weights and roots are the same. In other words, a further extension of the present study would deal with the theory of representations of semisimple Lie algebras. This, in turn, could be completed with the classification of semisimple Lie algebras via Dynkin diagrams.

Another natural extension of our study would be to explore the Lie group - Lie algebra relationship in depth along with the Baker-Campbell-Hausdorff formula. Furthermore, the characters and orthogonality relations for compact Lie groups might also be explored since, as mentioned before, we left them out of the current work. Examples of representations of non-compact Lie groups such as the Lorentz group would be another interesting topic as well.

We have seen the irruption of Lie groups and Lie algebras in physics. After that, they were implemented in all gauge theories. Furthermore, it would be really interesting to study how string theory deals with the exceptional Lie algebra E_8.

A Basic definitions and results

Definition A.1. A ***differentiable*** or ***smooth manifold*** of dimension m and class $\mathcal{C}^\infty$ is a topological space M, Hausdorff, satisfying the second axiom of countability and an equivalence class of atlas of dimension m and class $\mathcal{C}^\infty$.

Definition A.2. A continuous mapping $f : X \to Y$ is a ***local homeomorphism*** if every $x \in X$ has an open neighbourhood V such that $f(V)$ is open and f is a homeomorphism of V over $f(V)$.

Definition A.3. The ***diameter*** of a non-empty bounded subset S of a metric space X is defined to be the least upper bound (or supremum) of the set $\{d(x, y) \,|\, x, y \in S\} \subset \mathbb{R}$.

Definition A.4. For any $X \in M_n(\mathbb{C})$, we define the ***Hilbert-Schmidt*** norm as

$$\|X\| = \left(\sum_{j,k=1}^{n} |X_{jk}|^2 \right)^{1/2}$$

Definition A.5. If M and N are smooth manifolds and $F : M \to N$ is a smooth map, for each $p \in M$ the map

$$dF_p : T_pM \to T_{F(p)}N$$

is the ***differential of*** F ***at*** p. Let $v \in T_pM$ and $f \in \mathcal{C}^\infty$, the map is defined by

$$dF_p(v)(f) = v(f \circ F)$$

Definition A.6. Let M be a smooth manifold and TM its tangent bundle, a ***vector field*** is a continuous map

$$\begin{array}{rccc} X : & M & \to & TM \\ & p & \to & X_p \end{array}$$

with the property that $X_p \in T_pM$ for all $p \in M$. When TM is given its natural smooth structure, the vector field is said to be a ***smooth vector field*** if it is a smooth map. The set of all smooth vector fields in M is denoted by $\mathfrak{X}(M)$.

Definition A.7. Let $F : M \to N$ be a smooth map between smooth manifolds, $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$. We say that X and Y are $F-$***related*** if and only if for all $p \in M$

$$dF_p(X_p) = Y_{F(p)}$$

Definition A.8. Let $F : M \to N$ be a smooth map between smooth manifolds, let $p \in M$. The differential $d_pF : T_pM \to T_{F(p)}N$ yields the ***pullback by F at p***, the dual linear map

$$dF_p^* : T_{F(p)}^*N \to T_p^*M$$

Definition A.9. Let M and N be smooth manifolds and $F : M \to N$ a diffeomorphism. Let X be a smooth vector field on M, the unique smooth vector field on N that is $F-$related to X is called the ***pushforward of*** X ***by*** F. For $p \in M$, it is defined by

$$(F_*X)_p = dF_{F^{-1}(p)}(X_{F^{-1}(p)})$$

Definition A.10. Let V be a finite-dimensional vector space. If k is a positive integer, a ***covariant k-tensor on*** V is an element of the tensor product of the dual space of V, V^*, k times, $V^* \otimes \cdots \otimes V^*$, typically thought as a real-valued multilinear function of k elements of V:

$$\alpha : \underbrace{V \times \cdots \times V}_{k} \to \mathbb{R} \tag{A.1}$$

k is called the ***rank***. The space of all covariant k-tensors on V is denoted

$$T^k(V^*) = \underbrace{V^* \otimes \cdots \otimes V^*}_{k} \tag{A.2}$$

A ***contravariant tensor of rank*** k is an element of

$$T^k(V) = \underbrace{V \otimes \cdots \otimes V}_{k} \tag{A.3}$$

The space of ***mixed tensors on*** V ***of type (k,l)*** is defined as

$$T^{(k,l)}(V) = \underbrace{V \otimes \cdots \otimes V}_{k} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{l} \tag{A.4}$$

It is called the space of k times contravariant and l times covariant tensors. An element of $T^{(1,0)}$ is a vector and an element of $T^{(0,1)}$ is a form.

Lemma A.11 (Lebesgue's number Lemma). *For every open cover $\{U_i\}$ of a compact metric space X, there is a positive real number ϵ, called Lebesgue's number, such that every subset of X of diameter less than ϵ is contained in some element of $\{U_i\}$.*

Lemma A.12 (Uniform Time Lemma (from [11])). *Let V be a smooth vector field on a smooth manifold M, and let ϕ be its flow. Suppose there is a positive number ε such that for every $p \in M$, the domain of $\phi^{(p)}$ contains $(-\varepsilon, \varepsilon)$. Then V is complete.*

Lemma A.13 (Naturality of integral curves (from [11])). *Let M and N be smooth manifolds and $F : M \to N$ a smooth map. Then $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are $F-$related if and only if for each integral curve γ of X, $F \circ \gamma$ is an integral curve of Y.*

Lemma A.14 (Reordering Lemma (from [7])). *Suppose that $\mathfrak{g}$ is any Lie algebra and that π is a representation of $\mathfrak{g}$. Suppose that $X_1, \dots, X_m$ is an ordered basis for $\mathfrak{g}$ as a vector space. Then any expression of the form*

$$\pi(X_{j_1})\pi(X_{j_2}) \dots \pi(X_{j_N})$$

can be expressed as a linear combination of terms of the form

$$\pi(X_m)^{k_m}\pi(X_{m-1})^{k_{m-1}} \dots \pi(X_1)^{k_1}$$

where each k_l is a non-negative integer and where $k_1 + k_2 \dots k_m \leq N$.

Lemma A.15 (Schur's Lemma (from [20])). *Let $\rho^1 : G \to \mathsf{GL}(V_1)$ and $\rho^2 : G \to \mathsf{GL}(V_2)$ be two irreducible representations of G over complex vector spaces and let $f : V_1 \to V_2$ be a linear mapping such that $\rho^2_s \circ f = f \circ \rho^1_s$ for all $s \in G$. Then,*

1. *If ρ^1 and ρ^2 are not isomorphic, we have $f = 0$.*
2. *If $V_1 = V_2$ and $\rho^1 = \rho^2$, f is a homothety.*

Remark: The Lemma is valid for Lie algebras (changing GL for gl).

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