Treball final de grau

GRAU DE MATEMÀTIQUES

Facultat de Matemàtiques i Informàtica
Universitat de Barcelona

The Schur functors and the resolution of determinantal varieties.

Autor: Liena Colarte Gómez.

Director: Dra. Rosa Maria Miró-Roig.
Realitzat a: Departament de Matemàtiques i Informàtica.

Barcelona, January 17, 2017

Contents

Introduction.

Resolutions is one of the most effective methods to obtain information about varieties in Algebraic Geometry. For many years there has been considerable efforts in finding a resolution of determinantal varieties. To put the problem plainly, assume $R = K[x_0, \ldots, x_s]$ is the polynomial ring over an algebraically closed field of characteristic zero and $\mathbb{P}^s$ is the projective space of dimension s over K. Given $(r_{i,j})$ a homogeneous matrix of size $p \times q$ with entries in R, the problem is to find an explicit minimal free resolution of the ideal I_t defined by the $t \times t$ minors of this matrix. Over certain hypothesis on I_t, this is a minimal free resolution of the variety $X = \{z \in \mathbb{P}^s \mid rg((r_{i,j})(z)) < t\}$ of $\mathbb{P}^s$. It provides the Hilbert polynomial of X, the projective dimension and the arithmetically Cohen-Macaulayness of the variety among others characteristics.

In a more general context, this problem was solved by Lascoux in [16] where he gave a minimal free resolution of Schubert's varieties. His construction rests heavily on the theory of Schur functors and the fact that, in characteristic zero, the Schur functors are the irreducible representations of the general linear group. Despite the fact that the techniques developed by Lascoux are out of reach for an undergraduate student of Mathematics, the particular case of determinantal varieties admits a treatment based on developing suitable rudiments of multilinear algebra and combinatorics. In this sense, this paper is an approachment to the work of Lascoux in order to study a minimal free resolution of the ideal associated to the minors of a matrix. Our main goal is to construct a minimal free complex of these ideals whose modules are the modules of the minimal free resolution given by Lascoux, and next describe the resolution. In fact, this minimal free complex is a good candidate to be a resolution, too.

This paper is organized as follows. In Chapter 1 we present a review of background material on Hopf algebras, including exterior, symmetric and divided power algebras, and an exposition of partitions, Tableaux and Young diagrams which are the basic tools used along the main body of the paper. Chapter 2 deals with Schur and CoSchur functors on free R-modules, where R is a commutative ring of arbitrary characteristic. In section 2.1 we define the Schur functor $L_{\lambda/\mu}F$ of a free R-module F with respect to the skew partition λ/μ as the image of a natural transformation $d_{\lambda/\mu} : \Lambda_{\lambda/\mu}F \to S_{\tilde{\lambda}/\tilde{\mu}}F$ of free R-modules. In 2.1.1 the freeness of Schur functors is proved giving an explicit basis of $L_{\lambda/\mu}F$ through a basis of F which is described by means of Young tableaux. We end this subsection giving the rank of $L_\lambda F$ and with a discussion about the functoriality of $L_{\lambda/\mu}(-)$. The CoSchur functor $K_{\lambda/\mu}F$ has a similar treatment in section 2.2, we give a basis of $K_{\lambda/\mu}F$ and we show the duality $(L_{\lambda/\mu}F)^* \cong K_{\tilde{\lambda}/\tilde{\mu}}F^*$, where F^* denote the dual of F.

The remainder of Chapter 2 is devoted to the properties of Schur functors. In section 2.3, we provide a natural filtration of $L_{\lambda/\mu}(F \oplus G)$ with associated graded object $\bigoplus_{\mu\subseteq\gamma\subseteq\lambda} L_{\gamma/\mu}F \otimes L_{\lambda/\gamma}G$, where F and G are free R-modules. In characteristic zero this associated graded object becomes a direct sum decomposition of the Schur functor $L_{\lambda/\mu}(F \oplus G)$. In section 2.4, we address to the Cauchy formula for exterior and symmetric algebra. $S^k(F \otimes G)$ and $\Lambda^k(F \otimes G)$ have natural filtrations with associated graded objects $\bigoplus_{|\lambda|=k} L_\lambda F \otimes L_\lambda G$ and $\bigoplus_{|\lambda|=k} L_\lambda F \otimes K_\lambda G$, respectively. In characteristic zero, this sum becomes a direct decomposition of these functors. The Cauchy formula for exterior algebra is one of the pillars in the Lascoux construction. Finally, section 2.5 is devoted to the

Littlewood-Richardson rule for Schur functors. More precisely, when R contains a field of characteristic zero, the tensor product $L_\lambda F \otimes L_\mu F$ of Schur functors decomposes in a direct sum $\sum_\nu u(\lambda, \mu; \nu) L_\nu F$, where $u(\lambda, \mu; \nu)$ are the multiplicities of the factor $L_\nu F$ given by the Littlewood-Richardson rule. As a consequence, we obtain the Pieri formulas for Schur functors which are essential to describe the boundary maps of the resolution given by Lascoux.

In Chapter 3, we construct a minimal free complex of the ideal generated by the minors of a matrix and a minimal free resolution of determinantal ideals. In section 3.1, we assume that R is a local commutative ring containing a field K of characteristic zero with $\mathfrak{m}$ its maximal ideal, F and G are free R-modules of ranks $m+t-1$ and $n+t-1$ respectively and $(r_{i,j})$ is a matrix of size $(m+t-1) \times (n+t-1)$ with entries in R associated to a R-map $\Phi : F \to G^*$. We denote by I_t the ideal generated by the $t \times t$ minors of $(r_{i,j})$ and we define a complex $(C_\bullet(\Phi, t), d_\bullet)$ of length mn as follows:

(i) For all k, $C_k(\Phi, t)$ is a free R-module, it is a direct sum of tensor product of Schur functors.
(ii) $d_1(C_1(\Phi, t)) = I_t$ and $d_k(C_k(\Phi, t)) \subset \mathfrak{m} C_{k-1}(\Phi, t)$.

We construct the modules $C_k(\Phi, t)$ by means of the action of the group algebra $K(\mathcal{S}_k)$ on the tensor products $T^k F$ and $T^k G$. In subsection 3.1.1 we provide all the results required on $K(\mathcal{S}_k)$ and Young tableaux. In 3.1.2 we construct the modules $C_k(\Phi, t)$ as the images of the morphisms associated to idempotents of $K(\mathcal{S}_k)$. The boundary maps d_k are defined in 3.1.2 and they are essentially the maps induced by the contractions $T^k F \otimes T^k G \to R$ extending Φ. The remainder of the section 3.1 is devoted to prove that $(C_\bullet(\Phi, t), d_\bullet)$ is a complex. This rests on two formulas involving the idempotents defining the modules of the complex.

In section 3.2, assuming notations in 3.1 with $R = K[x_0, \ldots, x_s]$ we define determinantal ideals and determinantal varieties. In 3.2.1 we present the minimal free resolution $(L_{\bullet,t}, d_\bullet)$ of determinantal ideals. The free R-modules $L_{k,t}$ are $\bigoplus_{|I|+n(I)=k, I' \neq 0} L_{\tilde{I}} F \otimes L_{\widetilde{I'}} G$, where I is a partition of weight $m+k-1$, I' and $n(I)$ are described by means of I and its Durfee square. In fact, $L_{k,t} = C_{k,t}(\Phi, t)$. The boundary maps d_k are given by Pieri formulas and the natural contractions $\Lambda^\rho F \otimes \Lambda^\rho G \to R$ induced by Φ. The remains of the Chapter deals with examples of minimal free resolutions and two explicit resolutions computed with the program Macaulay2 [10]. The Eagon-Northcott complex gives a minimal free resolution of determinantal ideals generated by the maximal minors of a homogeneous matrix. We explain this resolution using 3.2.1, but actually the original treatment of this problem due to Eagon-Northcott in [7] does not involve Schur functors and this particular case gives a resolution of determinantal ideals even when R is not of characteristic zero. The Gulliksen-Negard complex is a minimal free resolution of the determinantal ideal generated by the submaximal minors of a square matrix. This complex has only four submodules and boundary maps, we do an accurate description of them.

Acknowledgements.

I would like to express my infinite gratitude to my advisor Rosa Maria Miró-Roig for her dedication and effort week after week to get this work to succeed in. I would like to thank also to P. C. Roberts who kindly sent to us his preprint which was essential to develop this paper. And finally, I would like to thank to my family for their support.

To M.

Chapter 1

Preliminaries.

This chapter is a compilation of all background material on multilinear algebra and combinatorial which is utilized in the main body of the paper. Hopf algebras and discussions of the relevant properties of the exterior, symmetric and divided power algebras are included. We follow [15] and [20] in section 1.1, and [1] in section 1.2.

1.1 Multilinear algebra.

1.1.1 Hopf algebras.

Let R be a commutative ring.

Definition 1.1.1. Let M *and* N be two graded R-modules. We define the *twisting morphism* $T : M \otimes_R N \to N \otimes_R M$ by $T(x \otimes y) = (-1)^{ij} y \otimes x,\ x \in M_i, y \in N_j$.

Definition 1.1.2. A *graded R-Hopf algebra* is a graded R-module $A = \sum_{i \geq 0} A_i$ together with a multiplication $m : A \otimes_R A \to A$, a unit $\eta : R \to A$, a comultiplication or diagonalization $\Delta : A \to A \otimes A$ and a counit $\varepsilon : A \to R$ satisfying:

1. (A, m, η) is a graded R-algebra, (A, Δ, ε) is a graded R-coalgebra, the counit ε is an R-algebra map and the unit η is an R-coalgebra map. That is, the following diagrams are commutative:

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle Id \otimes \Delta} \\ A \otimes A & \xrightarrow{\Delta \otimes Id} & A \otimes A \otimes A \end{array} \qquad \begin{array}{ccccc} & \overset{\cong}{\swarrow} & A & \overset{\cong}{\searrow} & \\ & & \downarrow & & \\ R \otimes A & \xleftarrow{\varepsilon \otimes Id} & A \otimes A & \xrightarrow{Id \otimes \varepsilon} & A \otimes R \end{array} .$$

2. The multiplication m and the comultiplication Δ are compatible in the sense that the following diagram commutes:

$$\begin{array}{ccccc} A\otimes A & \xrightarrow{\quad m \quad} & A & \xrightarrow{\quad \Delta \quad} & A\otimes A \\ {\scriptstyle \Delta\otimes\Delta}\downarrow & & & & \uparrow{\scriptstyle m\otimes m} \\ A\otimes A\otimes A\otimes A & & \xrightarrow{\quad Id\otimes T\otimes Id \quad} & & A\otimes A\otimes A\otimes A \end{array}.$$

In addition, we say that A is a commutative graded R-Hopf algebra if the following diagrams commute,

$$\begin{array}{ccc} A\otimes A & \xrightarrow{\quad T \quad} & A\otimes A \\ & {\scriptstyle m}\searrow \quad \swarrow{\scriptstyle m} & \\ & A & \end{array} \qquad \begin{array}{ccc} & A & \\ & {\scriptstyle \Delta}\swarrow \quad \searrow{\scriptstyle \Delta} & \\ A\otimes A & \xrightarrow[\quad T \quad]{} & A\otimes A \end{array}.$$

We assume that A is connected (i.e. $A_0 = R$) and free (i.e. each $i \geq 0$, A_i is a finitely generated free R-module).

Definition 1.1.3. Let (A, m_A, η_A) *and* (B, m_B, η_B) be two graded R-Hopf algebras. We say that an R-map $\alpha : A \to B$ is a map of graded R-Hopf algebras if $m_B \circ (\alpha \otimes \alpha) = \alpha \otimes m_A$, and $\Delta_B \circ \alpha = (\alpha \otimes \alpha) \circ \Delta_A$.

Definition 1.1.4. Let $A = (A, m_A, \eta_A, \Delta_A, \varepsilon_A)$ and $(B, m_B, \eta_B, \Delta_B, \varepsilon_B)$ be two graded R-Hopf algebras. The tensor product $A \otimes B$ is defined by the graded R-module $A \otimes B = \sum_{i\geq 0} A_i \otimes_R B_i$, a multiplication $m_{A\otimes B} : A \otimes B \otimes A \otimes B \xrightarrow{Id\otimes T\otimes Id} A \otimes A \otimes B \otimes B \xrightarrow{m_A\otimes m_B} A \otimes B$, a comultiplication $\Delta_{A\otimes B} : A \otimes B \xrightarrow{\Delta_A\otimes\Delta_B} A \otimes A \otimes B \otimes B \xrightarrow{Id\otimes T\otimes Id} A \otimes B \otimes A \otimes B$, a unit $\eta_{A\otimes B} : R \xrightarrow{\eta_A\otimes\eta_B} A \otimes B$ and a counit $\varepsilon_{A\otimes B} : A \otimes B \xrightarrow{\varepsilon_A\cdot\varepsilon_B} R$.

1.1.2 The Exterior Algebra.

Let R be a commutative ring and let F be a free R-module of rank n with an ordered basis $\{x_1, \dots, x_n\}$. We denote $F^* = Hom(F, R)$ the dual of F with basis $\{x_1^*, \dots, x_n^*\}$ dual to the basis $\{x_1, \dots, x_n\}$. We denote by $\mathcal{S}_r$ the set of all permutations of $\{1, \dots, r\}$.

Definition 1.1.5. Let $\Lambda^0 F = R$ and $\Lambda^1 F = F$. For all integer $r > 1$ we define the *r-th exterior power* $\Lambda^r F$ of F to be quotient of the r-th tensor power $T^i F := F \otimes \cdots \otimes F$ of F respect to the submodule $\mathfrak{E}$ of $T^r F$ generated by the elements $f_1 \otimes \cdots \otimes f_r - (-1)^{sg(\sigma)} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(r)}$ for all $\sigma \in \mathcal{S}_r$ and $f_1, \dots, f_r \in F$. In other words, $\Lambda^r F$ is the image by the natural surjective map $\pi : T^r F \to T^r F/\mathfrak{E}$ and then, the generators of $\Lambda^r F$ are the elements $\pi(f_1 \otimes \cdots \otimes f_r)$ such that $f_1, \dots, f_r \in F$ which we denote by $f_1 \wedge \cdots \wedge f_r$.

Let $f_1 \wedge \cdots \wedge f_r \in \Lambda^r F$. From the above definition it follows that for each $i \neq j \in \{1, \dots, r\}$, $f_1 \wedge \cdots \wedge f_i \wedge \cdots \wedge f_j \wedge \cdots \wedge f_r = -f_1 \wedge \cdots \wedge f_j \wedge \cdots \wedge f_i \wedge \cdots \wedge f_r$. So, it is clear that if there are two indices such that $f_i = f_j$, then $f_1 \wedge \cdots \wedge f_r = 0$. We assume that in the expression $f_1 \wedge \cdots \wedge f_r$ all elements f_i are different, otherwise it is 0.

Proposition 1.1.6. The set $\{x_{i_1} \wedge \cdots \wedge x_{i_r} \mid 1 \leq i_1 < \cdots < i_r \leq n\}$ form a basis of $\Lambda^r F$. In particular $\Lambda^r F$ is a free R-module of rank $\binom{n}{r}$.

The r-th exterior power $\Lambda^r F$ is a submodule of $T^r F$ through the natural immersion $\imath : \Lambda^r F \to T^r F$ defined by $\imath(f_1 \wedge \cdots \wedge f_r) = \sum_{\sigma \in \mathcal{S}_r} (-1)^{sg(\sigma)} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(r)}$.

Proposition 1.1.7. Let F and E be two free R-modules. If $\phi : E \to F$ is an R-map, then we have a well-defined linear map $\Lambda^r\phi : \Lambda^r E \to \Lambda^r F$ defined by $\Lambda^r\phi(e_1 \wedge \cdots \wedge e_r) = \phi(e_1) \wedge \cdots \wedge \phi(e_r)$. Thus, the r-th exterior power is an endofunctor on the category of R-modules and R-maps.

Definition 1.1.8. We define ΛF to be the graded R-module $\Lambda F := \bigoplus_{r \geq 0} \Lambda^r F$.

For each $s, t \geq 0$ there are natural maps $m_{r,s} : \Lambda^r F \otimes \Lambda^s F \to \Lambda^{r+s} F$ and $\Delta_{r+s} : \Lambda^{r+s} F \to \Lambda^r F \otimes \Lambda^s F$ given by the formulas

$$m_{r,s}(f_1 \wedge \cdots \wedge f_r \otimes g_1 \wedge \cdots \wedge g_s) = f_1 \wedge \cdots \wedge f_r \wedge g_1 \wedge \cdots \wedge g_s$$

$$\Delta_{r+s}(f_1 \wedge \cdots \wedge f_{r+s}) = \sum_{\sigma \in \mathcal{S}_{r+s}} (-1)^{sg(\sigma)} f_{\sigma(1)} \wedge \cdots \wedge f_{\sigma(r)} \otimes f_{\sigma(r+1)} \wedge \cdots \wedge f_{\sigma(r+s)}$$

respectively, where σ is such that $\sigma(1) < \cdots < \sigma(r)$ and $\sigma(r+1) < \cdots < \sigma(r+s)$. These maps induce a natural multiplication $m : \Lambda F \otimes \Lambda F \to \Lambda F$ and comultiplication $\Delta : \Lambda F \to \Lambda F \otimes \Lambda F$ in ΛF given by all components $m_{r,s}$ and Δ_{r+s} respectively. Then, we have

Proposition 1.1.9.

(i) $(\Lambda F, \tilde{m}, \eta, \Delta, \varepsilon)$ is a connected, free and commutative graded R-Hopf algebra where the unit is the natural inclusion $\eta : R \to \Lambda F$ into degree 0 and the counit is the projection $\varepsilon : \Lambda F \to R$ into degree 0.

(ii) The component Δ_{r+s} of the comultiplication map in ΛF^* is the dual map of the component of the multiplication map $m_{r,s}$ in ΛF.

(iii) The component $m_{r,s}$ of the multiplication map in ΛF^* is the dual of the component of the comultiplication map Δ_{r+s} in ΛF.

1.1.3 The Symmetric Algebra.

Definition 1.1.10. Let $S^0F = R$ and $S^1F = F$. For each $r > 1$ we define the *r-th symmetric power* of F to be the quotient of T^rF respect to the submodule $\mathfrak{S}$ of T^rF generated by the elements $f_1 \otimes \cdots \otimes f_r - f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(r)}$ for all $\sigma \in \mathcal{S}_r$ and $f_1, \ldots, f_r \in F$. Then S^rF is the image under the natural projection $\pi : T^rF \to T^rF/\mathfrak{S}$ and it is generated by the elements $\pi(f_1 \otimes \cdots \otimes f_r) := f_1 \cdot \cdots \cdot f_r$, $f_1, \ldots, f_r \in F$.

Observe that the above definition differs from $\Lambda^r F$ only by a sign, $(-1)^{sg(\sigma)}$, however this detail makes important distinctions. Let $f_1 \cdot \cdots \cdot f_r \in S^rF$. For each $i \neq j \in \{1, \ldots, r\}$ we have $f_1 \cdot \cdots \cdot f_i \cdot \cdots \cdot f_j \cdot \cdots \cdot f_r = f_1 \cdot \cdots \cdot f_j \cdot \cdots \cdot f_i \cdot \cdots \cdot f_r$. Differs from $\Lambda^r F$, in the expressions $f_1 \cdot \cdots \cdot f_r$ we can have two equals elements. In this sense, the element $f_1 \cdot \cdots \cdot f_r$ could not have a minimal expression. To emphasize this fact we will use the notation $f_1^{i_r} \cdot \cdots \cdot f_t^{i_r}$ such that $i_1 + \cdots + i_r = r$ where we assume that all elements f_i are different and eventually $i_k = 0$ with $f_i^{i_k} = 1 \in R$.

Proposition 1.1.11. $\{x_1^{i_1} \cdot \cdots \cdot x_n^{i_n} \mid i_1 + \cdots + i_n = r\}$ form a basis of S^rF. In particular, S^rF is a free R-module of rank $\binom{n+r-1}{r}$.

S^rF is a submodule of T^rF through the natural immersion $\jmath : S^rF \to T^rF$ defined by $\jmath(f_1 \cdot \cdots \cdot f_r) = \sum_{\sigma \in \mathcal{S}_r} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(r)}$.

Proposition 1.1.12. Let F and E be two free R-modules and let $\phi : F \to E$ be an R-map. Then, the map $S^r\phi : S^rE \to S^rF$ defined by $S^r(\phi)(e_1 \cdot \cdots \cdot e_r) = \phi(e_1) \cdot \cdots \cdot \phi(e_r)$ is a well-defined R-map. In other words, the rth symmetric power defines an endofunctor on the category of free R-modules and R-maps.

Definition 1.1.13. We define SF to be the graded R-module $SF := \bigoplus_{r \geq 0} S^rF$.

As we have seen in ΛF, there are naturals maps $m_{r,s} : S^rF \otimes S^sF \to S^{r+s}$ and $\Delta_{r+s} : S^{r+s}F \to S^rF \otimes S^sF$ defined by the formulas

$$m_{r,s}(f_1 \cdot \cdots \cdot f_r \otimes g_1 \cdot \cdots \cdot g_r) = f_1 \cdot \cdots \cdot f_r \cdot g_1 \cdot \cdots \cdot g_s$$

$$\Delta_{r+s}(f_1 \cdot \cdots \cdot f_{r+s}) = f_{\sigma(1)} \cdot \cdots \cdot f_{\sigma(r)} \otimes f_{\sigma(r+1)} \cdot \cdots \cdot f_{\sigma(r+s)}$$

respectively where σ runs over the set of all permutations of $\{1, \ldots, r+s\}$ such that $\sigma(1) < \cdots < \sigma(r)$, $\sigma(r+1) < \cdots < \sigma(r+s)$, which induce a multiplication m and Δ comultiplication in SF. In particular, $\Delta_{r+s}(f_1^{i_1} \cdot \cdots \cdot f_{r+s}^{i_{r+s}}) = \sum_{0 \leq j_k \leq i_k} \binom{i_1}{j_1} \cdots \binom{i_{r+s}}{j_{r+s}} f_1^{j_1} \cdot \cdots \cdot f_{r+s}^{j_{r+s}} \otimes f_1^{i_1 - j_1} \cdot \cdots \cdot f_r^{i_{r+s} - j_{r+s}}$, where $j_k = 0$ if $i_k = 0$ and $j_1 + \cdots + j_{r+s} = r$.

Proposition 1.1.14. $(SF, m, \eta, \Delta, \varepsilon)$ is a connected, free and commutative graded R-Hopf algebra with the obvious unit $\eta : R \to SF$ and counit $\varepsilon : SF \to R$.

1.1.4 The Divided Power Algebra.

Definition 1.1.15. Let $D^0F = R$ and $D^1F = F$. For each $r > 1$ we define the *rth divided powerD^rF* of F to be the dual of the rth symmetric power S^rF^*.

Considering the basis $\{(x_1^*)^{i_1} \cdot \cdots \cdot (x_n^*)^{i_n} \mid i_1 + \cdots + i_n = r\}$ of $S^r(F^*)$, we define $x_1^{(i_1)} \cdot \cdots \cdot x_n^{(i_n)}$ to be the dual element of the basis element $(x_1^*)^{i_1} \cdot \cdots \cdot (x_n^*)^{i_n}$. Then,

Proposition 1.1.16. $\{x_1^{(i_1)} \cdot \cdots \cdot x_n^{(i_n)} \mid i_1 + \cdots + i_n = r\}$ form a basis of D^rF. In particular, D^rF is a free R-module of rank $\binom{n+r-1}{r}$.

Proposition 1.1.17. $DF = \bigoplus_{r \geq 0} D^rF$ is a graded dual of the symmetric algebra SF^* with the obvious unit and counit and where

(i) The component $m_{r,s} : D^rF \otimes D^sF \to D^{r+s}F$ of the multiplication on DF is the dual of the component $\Delta_{r+s} : S^{r+s}F^* \to S^rF^* \otimes S^sF^*$ of the comultiplication on SF^*.

$m_{r,s}$ is given by $m_{r,s}(x_1^{(i_1)} \cdot \cdots \cdot x_n^{(i_n)} \otimes x_1^{(j_1)} \cdot \cdots \cdot x_n^{(j_n)}) = \binom{i_1+j_1}{j_1} \cdots \binom{i_n+j_n}{j_n} x_1^{(i_1+j_1)} \cdot \cdots \cdot x_n^{(i_n+j_n)}$.

(ii) The component $\Delta_{r+s} : D^{r+s}F \to D^rF \otimes D^sF$ of the diagonalization on DF is the dual of the component $m_{r,s} : S^rF^* \otimes S^sF^* \to S^{r+s}F^*$ of the multiplication on SF^*.

Δ_{r+s} is given by $\Delta_{r+s}(x_1^{(i_1)} \cdot \dots \cdot x_n^{(i_n)}) = \sum_{0 \le j_k \le i_k} e_1^{(j_1)} \cdot \dots \cdot e_n^{(j_n)} \otimes e_1^{(i_1-j_1)} \cdot \dots \cdot e_n^{(i_n-j_n)}$ where $j_1 + \dots + j_n = r$.

Definition 1.1.18. Let $f \in F$ with $f = \sum_{i=1}^n u_i x_i$. Let $f^{(0)} = 1 \in R$ and $f^{(1)} = f$. For each $r > 1$ we define the r-th divided power $f^{(r)} \in D^rF$ of f to be $\sum_{i_1+\dots+i_n=r} u_1^{i_1} \cdots u_n^{i_n} x_1^{(i_1)} \cdots x_n^{(i_n)}$.

Proposition 1.1.19. Let $f, g \in F$ and let p, q be two non negative integers. The divided powers have the following properties:

1. $f^{(p)}f^{(q)} = \binom{p+q}{q} f^{(p+q)} \in D^{p+q}F$
2. $(f+g)^{(p)} = \sum_{k=0}^p f^{(k)}g^{(p-k)}$.
3. $(fg)^{(p)} = f^{(p)}g^{(p)}$.
4. $(f^{(p)})^{(q)} = \frac{(pq)!}{q!p^{q!}} f^{(pq)}$.

Proposition 1.1.20. Let F and E be two free R-modules and let $\phi : F \to E$ be an R-map. We denote by $\phi^* : E^* \to F^*$ the dual map of ϕ. Then, we have a well defined R-map $D^r\phi : D^rF \to D^rE$ which is the dual map of the map $S^r(\phi^*) : S^rE^* \to S^rF^*$. More precisely, the rth exterior power defines an endofunctor on the category of free R-modules and R-maps.

1.2 Combinatorics.

1.2.1 Partitions and skew partitions.

Let $\mathbb{N}^\infty := \{\lambda := \{\lambda_i\}_{i \ge 1} \mid \lambda_i = 0 \text{ but a finite number of terms}\}$. In other words, for each non negative integer p, let $\mathbb{N}^p := \{(\lambda_1, \dots, \lambda_p) \mid \lambda_i \in \mathbb{N}, i = 1, \dots, p\}$. Hence $\mathbb{N}^\infty = \cup_{p \ge 1} \mathbb{N}^p$.

We will consider $(\lambda_1, \dots, \lambda_p)$ and $\{\lambda_1, \dots, \lambda_p, 0, \dots\}$ the same element.

Definition 1.2.1. A *partition* is an element $\lambda = \{\lambda_i\}_{i \ge 1} \in \mathbb{N}^\infty$ whose components are arranged in decreasing order, that is $\lambda_i \ge \lambda_{i+1}$, $\forall i \ge 0$. The *length* of a partition $\lambda \in N^\infty$ is the number of non zero components in the partition, we often denote it by $l(\lambda)$. The *weight* of a partition $\lambda \in \mathbb{N}^\infty$ is the sum of all its components. We note it by $|\lambda| := \sum_{i \ge 1} \lambda_i$ and we will say that λ is a partition of weight $|\lambda|$.

For example, $(10, 8, 9, 11, 4, 3, 2)$ is not a partition and $(10, 7, 3, 2)$ is a partition of weight 22 and length 4.

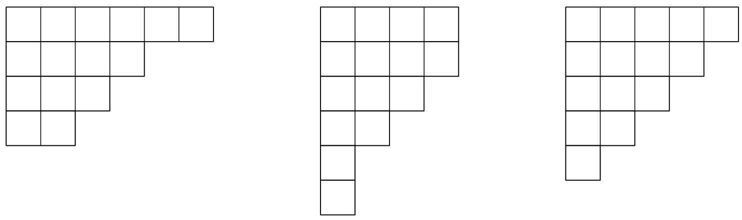
Definition 1.2.2. Let $\lambda = (\lambda_1, \dots, \lambda_p) \in \mathbb{N}^\infty$ be a partition. For each $i \in \{1, \dots, \lambda_1\}$, let $\tilde{\lambda}_i$ be the number of terms of λ which are greater than or equal to i. Clearly $\tilde{\lambda}_i \ge \tilde{\lambda}_{i+1}$, $\forall i \in \{1, \dots, \lambda_1\}$. The *transpose or conjugate* of λ is the partition $\tilde{\lambda} = (\tilde{\lambda}_1, \dots, \tilde{\lambda}_{\lambda_1})$.

For example, the transpose of $(10,7,3,2)$ is the partition $(4,4,3,2,2,2,2,1,1,1)$.

Remark 1.2.3. We consider that all terms of a partition $\lambda = (\lambda_1, \ldots, \lambda_p)$ are non zero. Note that hence, the length of λ is p and equals to $\tilde{\lambda}_1$.

Definition 1.2.4. Let $\lambda = (\lambda_1, \ldots, \lambda_p) \in \mathbb{N}^\infty$ be a partition. The *diagram or shape* of λ is $\Delta_\lambda := \{(i,j) \in \mathbb{N}^2 \mid 1 \le i \le p,\ 1 \le j \le \lambda_i\}$. We will represent the graphic of Δ_λ in $\mathbb{N}^2$, where the points $(i,j) \in \Delta_\lambda$ will be represented by squares.

For example, $\Delta_{(6,4,3,2)}, \Delta_{(4,4,3,2,1,1)}$ and $\Delta_{(5,4,3,2,1)}$ correspond to



respectively.

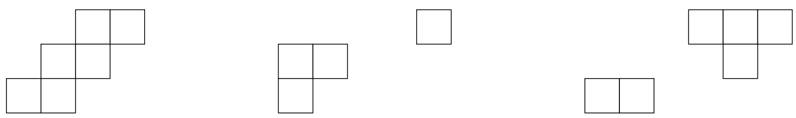
We can consider the shape of a partition $\lambda = (\lambda_1, \ldots, \lambda_p)$ as a kind of matrix with p rows with different lengths and λ_1 columns. The ith row of Δ_λ is of length λ_i. Now, the jth column of Δ_λ is of length the number of terms in λ greater than or equal to j, which equals $\tilde{\lambda}_j$. In fact, $\Delta_{\tilde{\lambda}}$ is the diagram transpose of Δ_λ, and hence this tell us that the conjugate of $\tilde{\lambda}$ is λ. That is, $\tilde{\tilde{\lambda}} = \lambda$ and we can conclude that the conjugation $\lambda \to \tilde{\lambda}$ is an involution on the set of partitions.

The notions of partitions are, in fact, an introduction of a more generalized concept we explain next. In the set of partitions one can define an order. Let $\lambda = (\lambda_1, \ldots, \lambda_p)$ and $\mu = (\mu_1, \ldots, \mu_q)$ be two partitions, we say $\mu \subseteq \lambda$ if $p \ge q$ and $\mu_i \le \lambda_i$, $\forall i \in \{1, \ldots, q\}$. It is equivalent to say that the diagram of μ is contained in the diagram of μ, that is $\Delta_\mu \subseteq \Delta_\lambda$.

Definition 1.2.5. Let $\mu = (\mu_1, \ldots, \mu_q)$ and $\lambda = (\lambda_1, \ldots, \lambda_p)$ be two partitions such that $\mu \subseteq \lambda$. We define the *skew partition* $\lambda/\mu := (\lambda_1 - \mu_1, \ldots, \lambda_q - \mu_q, \lambda_{q+1}, \ldots, \lambda_p)$ with skew shape of diagram $\Delta_{\lambda/\mu} := \Delta_\lambda - \Delta_\mu = \{(i,j) \in \mathbb{N}^2 \mid 1 \le i \le p; \mu_i + 1 \le j \le \lambda_i\}$.

For convenience we will write $\mu = (\mu_1, \ldots, \mu_p)$ where $\mu_{q+1} = \cdots = \mu_p = 0$.

For example, the skew diagrams associated to $(4,3,2)/(2,1)$, $(5,2,1)/(4,0,0)$ and $(7,6,3)/(4,5,1)$ correspond to



Note that, the skew partition λ/μ is a partition if, and only if the length of μ equals the length of λ or μ is the zero partition. Clearly, the set of partitions is a subset of the set of skew partitions, indeed $\lambda/(0) = \lambda$.

1.2.2 Tableaux.

Definition 1.2.6. Let λ/μ a skew partition and let S be a totally ordered set. A *tableau* of shape λ/μ with values in the set S is a function from the skew shape $\Delta_{\lambda/\mu}$ to S. We denote $Tab_{\lambda/\mu}(S)$ the set of such tableaux.

We will represent graphically $T \in Tab_{\lambda/\mu}(S)$ by the skew shape $\Delta_{\lambda/\mu}$ filled with elements of S.

Let us see few simple examples. Consider $S = \{1, \ldots, 5\}$ with the usual order on $\mathbb{N}$ and $\lambda = (3, 2)$. Then, $T : \Delta_\lambda \to S$ such that $T(i,j) = max\{i,j\}$ is a tableau of shape λ with values in S. Two more examples: $R : \Delta_\lambda \to S$ such that $R(i,j) = i + j$ *and* $U : \Delta_\lambda \to S$ such that $U(i,j) = i$.

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 2 & \\ \cline{1-2}\end{array} \qquad R = \begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 3 & 4 & \\ \cline{1-2}\end{array} \qquad U = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 & \\ \cline{1-2}\end{array}$$

Definition 1.2.7. Let $T \in Tab_{\lambda/\mu}(S)$. We say T is *row-standard* if $T(i,j) < T(i,j+1)$ for all $(i,j), (i,j+1) \in \Delta_{\lambda/\mu}$. We say T is *column-standard* if $T(i,j) \leq T(i+1,j)$ for all $(i,j), (i+1,j) \in \Delta_{\lambda/\mu}$. Finally, T is called *standard* if it is row and column standard.

Continuing with the examples above, $T(i,j) = max(i,j)$ and $U(i,j) = i$ are both column-standard but not row-standard while $R(i,j) = i + j$ is standard.

Definition 1.2.8. Let $T \in Tab_{\lambda/\mu}(S)$. We say that T is *co-row-standard* if $T(i,j) \leq T(i,j+1)$ $\forall (i,j), (i,j+1) \in \Delta_{\lambda/\mu}$. We say that T is *co-column-standard* if $T(i,j) < T(i+1,j)$ $\forall (i,j)$, $(i+1,j) \in \Delta_{\lambda/\mu}$. Finally, T is *co-standard* if is co-row and co-column standard.

Observe that T is co-row-standard but not co-column-standard, while R and U are both co-standard.

Chapter 2

Schur Functors and CoSchur Functors.

This chapter is devoted to an introduction of Schur and CoSchur functors theory using only elementary rudiments of multilinear algebra and combinatorics, which we have just presented into the Introductory Material. We follow basically the contents from [1] with a few references from [20] and [9]. In sections 2.1 and 2.2 we define Schur and CoSchur functors and we establish the freeness of both functors. In the remaining sections we give decomposition formulas for Schur functors including Littlewood-Richardson rule and Cauchy formulas.

2.1 Schur Functors.

We start with some notations we will use along this paper. Let R be a commutative ring and let F be a free R-module of rank n. Let $\mu = (\mu_1, \ldots, \mu_p)$ and $(\lambda_1, \ldots, \lambda_p)$ be two partitions such that $\mu \subseteq \lambda$. We define the free R-modules

$$\Lambda_{\lambda/\mu}F := \Lambda^{\lambda_1-\mu_1}F \otimes_R \cdots \otimes_R \Lambda^{\lambda_p-\mu_p}F$$

$$S_{\lambda/\mu}F := S^{\lambda_1-\mu_1}F \otimes \cdots \otimes S^{\lambda_p-\mu_p}F$$

$$D_{\lambda/\mu}F := D^{\lambda_1-\mu_1}F \otimes \cdots \otimes D^{\lambda_p-\mu_p}F$$

$T_{\lambda/\mu}F := F_{(1,\mu_1+1)} \otimes \cdots \otimes F_{(1,\lambda_1)} \otimes \cdots \otimes F_{(p,\mu_p+1)} \otimes \cdots \otimes F_{(p,\lambda_p)} = \bigotimes_{(i,j)\in\Delta_{\lambda/\mu}} F_{(i,j)}$, where $F_{(i,j)}$ denotes a copy of F and the tensor power $\otimes$ is over R. Note that $T_{\lambda/\mu}F$ is an usual way to write the tensor power $T^{|\lambda/\mu|}F$ where $|\lambda/\mu| = |\lambda| - |\mu|$.

Definition 2.1.1. We define the *Schur map* $d_{\lambda/\mu} : \Lambda_{\lambda/\mu}F \to S_{\tilde{\lambda}/\tilde{\mu}}F$ to be the composition

$$\Lambda_{\lambda/\mu}F \xrightarrow{\alpha} T_{\lambda/\mu} \xrightarrow{\beta} S_{\tilde{\lambda}/\tilde{\mu}}F,$$

where α is the tensor product of the natural inclusions or diagonalizations $\iota_i : \Lambda^{\lambda_i-\mu_i}F \to T^{\lambda_i-\mu_i}F = F_{(i,\mu_i+1)} \otimes \cdots \otimes F_{(i,\lambda_i)}, \quad i = 1, \ldots p$, and β is the tensor product of the multiplications $m_j : F_{(\tilde{\mu}_j+1,j)} \otimes_R \cdots \otimes_R F_{(\tilde{\lambda}_j,j)} \to S^{\tilde{\lambda}_j-\tilde{\mu}_j}F, \ j = 1, \ldots, \lambda_1$. Note that ι is a tensor product of appropriated comultiplications on ΛF. We will often denote ι_i simply by Δ.

There is another way to define the Schur map which we will often use. It is a simple change of notation using Ferrers matrices which sometimes makes operations easier. We will use both indifferently.

The Ferrers matrix associated to the skew partition λ/μ is a $\lambda_1 \times \lambda_1$ matrix of zeros and ones $\alpha_{\lambda/\mu} = (a_{i,j})$ defined by

$$\begin{cases} a_{i,j} = 1 \ \ if \ \ \mu_i + 1 \leq j \leq \lambda_i \\ a_{i,j} = 0 \ \ if \ \ 1 \leq j \leq \mu_i \text{ or } \lambda_{j+1} \leq j \leq \lambda_1 \end{cases}$$

Keeping in mind that $F_{a_{i,j}} = R \ \ if \ \ a_{i,j} = 0$, $d_{\lambda/\mu}$ is defined as the composition $\Lambda_{\lambda/\mu}F \to \bigotimes_{a_{i,j} \in \alpha_{\lambda/\mu}} F_{a_{i,j}} \to S_{\tilde{\lambda}/\tilde{\mu}}F$, where $F_{a_{i,j}}$ is a copy of F, the first map is the tensor product of the natural injections $\Lambda^{\lambda_i-\mu_i}F \to F_{a_{i,1}} \otimes \cdots \otimes F_{a_{i,\lambda_1}}, i = 1, \ldots, q$, and the second map is the tensor product of the multiplication $F_{a_{1,j}} \otimes \cdots \otimes F_{a_{q,j}} \to S^{a_{1,j}+\cdots+a_{q,j}}F = S^{\tilde{\lambda}_j-\tilde{\mu}_j}F, j = 1, \ldots, \lambda_1$, since $a_{1,j} + \cdots + a_{q,j} = \tilde{\lambda}_j - \tilde{\mu}_j$.

In fact, we can generalize the above map to any matrix of zeros and ones.

Let $\alpha = (\alpha_{i,j})$ be an arbitrary $s \times t$ matrix of zeros and ones. For each $i = 1, \ldots, s$ let $p_i = \sum_{j=1}^{t} a_{i,j}$ and for each $j = 1, \ldots, t$ let $q_j = \sum_{i=1}^{s} a_{i,j}$. Then we can consider free R-modules

$$\Lambda_\alpha F := \Lambda^{p_1}F \otimes \cdots \otimes \Lambda^{p_s}F \ \ and \ \ S_{\tilde{\alpha}}F := S^{q_1}F \otimes \cdots \otimes S^{q_t}F$$

and we can define a natural map $d_\alpha : \Lambda_\alpha F \to S_{\tilde{\alpha}}F$ in the same manner as the Schur map.

Definition 2.1.2. The image of $d_{\lambda/\mu}$ is called the *Schur functor* of F with respect to the skew partition λ/μ, and it is denoted by $L_{\lambda/\mu}F$.

Let us see some examples.

Examples 2.1.3. (1) If $\lambda = (m)$, then $\tilde{\lambda} = (1, \ldots, 1)$ and the Schur map $d_{(m)} : \Lambda_{(m)}F \to S_{(1,\ldots,1)}F$ is just the natural inclusion of $\Lambda^m F$ on $T^m F$. In this case, we see that $L_{(m)} \cong \Lambda^m F$.

(2) If $\lambda = (1, \ldots, 1)$, then $\tilde{\lambda} = (m)$, and the Schur map $d_{(1,\ldots,1)} : \Lambda_{(1,\ldots,1)}F \to S_{(m)}F$ is just the multiplication $F \otimes \cdots \otimes F \to S^m F$. Clearly, $L_{(1,\ldots,1)}F$ is the mth symmetric power $S^m F$.

(3) Let $\lambda = (3, 2)$, first we describe the Schur map associated with λ.

Remember $\bigotimes_{(i,j) \in \Delta_\lambda} F_{(i,j)} = T^{|\lambda|}F$. We have,

$$d_{(3,2)} : \Lambda^3 F \otimes_R \Lambda^2 F \to \bigotimes_{(i,j) \in \Delta_\lambda} F_{(i,j)} = T^5 F \to S^2 F \otimes_R S^2 F \otimes_R F$$

$u \wedge v \wedge w \otimes x \wedge y \to$

$u \otimes v \otimes w \otimes x \otimes y - u \otimes v \otimes w \otimes y \otimes x - u \otimes w \otimes v \otimes x \otimes y + u \otimes w \otimes v \otimes y \otimes x - v \otimes u \otimes w \otimes x \otimes y + v \otimes u \otimes w \otimes y \otimes x + v \otimes w \otimes u \otimes x \otimes y - v \otimes w \otimes u \otimes y \otimes x + w \otimes u \otimes v \otimes x \otimes y - w \otimes u \otimes v \otimes y \otimes x - w \otimes v \otimes u \otimes x \otimes y + w \otimes v \otimes u \otimes y \otimes x$

$\to ux \otimes vy \otimes w - uy \otimes vx \otimes w - ux \otimes wy \otimes v + uy \otimes wx \otimes v - vx \otimes uy \otimes w + vy \otimes ux \otimes w + vx \otimes wy \otimes v - vy \otimes wx \otimes v + wx \otimes uy \otimes v - wy \otimes ux \otimes v - wx \otimes vy \otimes u + wy \otimes vx \otimes u.$

It is clear that $L_{\lambda/\mu}F$ is a submodule of $S_{\tilde{\lambda}/\tilde{\mu}}F$, but it is not trivial that actually $L_{\lambda/\mu}F$ is a free R-module.

2.1.1 The freeness of Schur Functors.

As we have anticipated in the previous subsection, the modules $L_{\lambda/\mu}F$ are free. We will show this fact by finding a basis to $L_{\lambda/\mu}F$ through a basis of F. However, first we need to solve the computational problem of $d_{\lambda/\mu}$.

Let λ/μ be a skew partition with $\tilde{\lambda}_1 = q$ and let $B_F := \{x_1, \ldots, x_n\}$ be a basis of F. We can describe a basis of $\Lambda_{\lambda/\mu}F$ and $S_{\lambda/\mu}F$ as follows.

For each $i \in \{1, \ldots, q\}$ we denote by $I_i = \{\alpha_{(i,\mu_i+1)}, \ldots, \alpha_{(i,\lambda_i)}\}$ a strictly increasing subset of $\{1, \ldots, n\}$ such that $\alpha_1 < \ldots < \alpha_s$. Clearly, the elements $X_{I_1} \otimes \cdots \otimes X_{I_q}$, where each I_i is a such of these subsets, form a basis of $\Lambda_{\lambda/\mu}F$.

Now, if we choose S to be B_F with the order $x_1 < \cdots < x_n$, then for each basis element $X = X_{I_1} \otimes \cdots \otimes X_{I_q}$ we can associate a tableau $T_X \in Tab_{\lambda/\mu}(S)$ defined by $T_X(i,j) = X_{\alpha_{(i,j)}}$. Clearly, T_X is a row-standard tableau. And conversely, if $T \in Tab_{\lambda/\mu}(S)$ is a row-standard tableau, then the element $X_T := X_{I_1} \otimes \cdots \otimes X_{I_q}$ where $X_{I_i} = T(i, \mu_{i+1}) \wedge \cdots \wedge T(i, \lambda_i)$ is a such of basis element we have just described. So, $\{X_T \mid T \in Tab_{\lambda/\mu}(S) \text{ is row-standard}\}$ is a basis of $\Lambda_{\lambda/\mu}F$.

For example, let $\lambda = (4,2,1)$ and $n = 10$. Then, the tableau

$$T = \begin{array}{|c|c|c|c|} \hline x_1 & x_4 & x_5 & x_7 \\ \hline x_2 & x_{10} \\ \cline{1-2} x_3 \\ \cline{1-1} \end{array}$$

define the basis element $X_T = x_1 \wedge x_4 \wedge x_5 \wedge x_7 \otimes x_2 \wedge x_{10} \otimes x_3$ in $\Lambda_{(4,2,1)}F$.

In the same way, we can describe a basis of $S_{\tilde{\lambda}/\tilde{\mu}}F$ through $Tab_{\lambda/\mu}(S)$. If $T \in Tab_{\lambda/\mu}\{x_1, \ldots, x_n\}$ is column-standard, then we can associate to it the element $Z_T = X_{J_1} \otimes \cdots \otimes X_{J_{\lambda_1}}$, where $X_{J_i} = T(\tilde{\mu}_i, i) \cdot \cdots \cdot T(\tilde{\lambda}_i, i)$. Clearly, $\{Z_T \mid T \in Tab_{\lambda/\mu}(S) \text{ is column-standard}\}$ is a basis of $S_{\tilde{\lambda}/\tilde{\mu}}F$.

For example, let $\lambda = (4,3,2,1)$ and $n = 5$. The tableau

$$T = \begin{array}{|c|c|c|c|} \hline x_1 & x_3 & x_4 & x_5 \\ \hline x_1 & x_5 & x_5 \\ \cline{1-3} x_1 & x_5 \\ \cline{1-2} x_2 \\ \cline{1-1} \end{array}$$

define the basis element $Z_T = x_1^3 x_2 \otimes x_3 x_5^2 \otimes x_4 x_5 \otimes x_5$ of $S_{(4,3,2,1)}F$.

From this follows that $\{d_{\lambda/\mu}(X_T) \mid T \in Tab_{\lambda/\mu}(S)$ is row-standard$\}$ generate $L_{\lambda/\mu}F$. We will show that $\{d_{\lambda/\mu}(X_T) \mid T \in Tab_{\lambda/\mu}(S)$ is standard$\}$ is a basis of $L_{\lambda/\mu}F$.

Let $T \in Tab_{\lambda/\mu}(S)$ with $\lambda = (\lambda_1, \dots, \lambda_q)$ and $\mu = (\mu_1, \dots, \mu_q)$. Remember $d_{\lambda/\mu} = (m_1 \otimes \cdots \otimes m_{\lambda_1})(\imath_1 \otimes \cdots \otimes \imath_q)$. Thus, $(\imath_1 \otimes \cdots \otimes \imath_q)(X_T) = \sum_{\sigma=(\sigma_1,\dots,\sigma_q)}(-1)^{sg(\sigma)} X_{T_\sigma}$ where σ_i is a permutation of $1, \dots, \lambda_i - \mu_i$ and T_σ is the tableau defined by $T_\sigma(i,j) = T(i, \sigma_i(j))$. Applying $(m_1 \otimes \cdots \otimes m_{\lambda_1})$ we obtain $d_{\lambda/\mu}(X_T) = \sum_\sigma (-1)^{sg(\sigma)} Z_{T_\sigma}$, where $Z_{T_\sigma} \in S_{\tilde{\lambda}/\tilde{\mu}}F$ similarly as we have seen before.

At this moment we will introduce some facts about tableaux which we will need next. Let $S = B_F$.

Definition 2.1.4. Let $T \in Tab_{\lambda/\mu}(S)$ and let p, q be positive integers. We define $T_{p,q}$ to be the number of times the elements $x_1, \dots, x_q$ appear as entries in the first p rows of T. Formally, $T_{p,q} = |\{(i,j) \in T \mid i \leq p \ \ and \ \ T(i,j) \in \{x_1, \dots, x_q\}\}|$.

For example,

$$T = \begin{array}{|c|c|c|c|} \hline x_1 & x_4 & x_5 & x_7 \\ \hline x_2 & x_{10} \\ \cline{1-2} x_3 \\ \cline{1-1} \end{array}, \qquad T_{5,2} = 4.$$

Definition 2.1.5. Let $T, S \in Tab_{\lambda/\mu}(S)$. We say $S \leq T$ if $S_{p,q} \geq T_{p,q}$ for every positive integers p, q and we say $S < T$ if $S \leq T$ and $S_{p,q} > T_{p,q}$ for at least one pair p, q.

Proposition 2.1.6. $\leq$ is a reflexive and transitive relation on $Tab_{\lambda/\mu}(S)$.
Proof. Obviously $\leq$ is reflexive, $T_{p,q} = T_{p,q}$ for every p, q, so $T \leq T$. Let $T, W, R \in Tab_{\lambda/\mu}(S)$ such that $T \leq W \ \ and \ \ W \leq R$, we want to see $T \leq R$. Let p, q be positive integers, by hypothesis $R_{p,q} > W_{p,q} > T_{p,q}$ and hence $R_{p,q} > T_{p,q}$. □

Remark 2.1.7. If we restrict $\leq$ to the subset of row-standard tableaux, then $\leq$ is consistent with the lexicographic order induced by the correspondence between row-standard tableaux and the basis elements of $\Lambda_{\lambda/\mu}F$.

Lemma 2.1.8. Let $T, R \in Tab_{\lambda/\mu}(S)$ where R is formed by exchanging certain entries from the kth row of T, say $T(k, l_1), \dots, T(k, l_a)$, to certain entries of the $(k+1)$th row of T, say $T(k+1, m_1), \dots, T(k+1, m_a)$, where $T(k+1, m_i) < T(k, l_i)$ for $i = 1, \dots, a$. Then $R < T$.
Proof. First we consider the simple case $a = 1$. We have , $R(k,l) = T(k+1,m)$, $R(k+1,m) = T(k,l)$ and $R(i,j) = T(i,j)$ for all $(i,j) \neq (k,m), (k+1,m)$. By hypothesis $R(k,l) = T(k+1,m) < T(k,l) = R(k+1,m)$.

Let p, q be positive integers, remember $R_{p,q} = |\{(i,j) \in S \mid i \leq p \ \ and \ \ S(i,j) \in \{x_1, \dots, x_q\}|$. Clearly $R_{p,q} = T_{p,q}$ if $p < k$ since R equals to T at the first $k-1$ rows. We fix $p = k$, R differs T only at the entry (k,l), and $R(k,l)$ appears at the first k rows of R one more time than of T, respectively $S(k+1,m)$ one less. Clearly $R_{k,q} = T_{k,q} \ \ if \ \ x_q < R(k,l)$, or $x_q \geq R(k+1,m)$, since $R(k,l) < R(k+1,m)$ and $R_{k,q} = T_{k,q} + 1 - 1$. If $R(k,l) \leq x_q < R(k+1,m)$, then $R_{k,q} = T_{k,q} + 1$. Since $R_{p,q}$ equals $T_{p,q}$ if $p > k$, we conclude $R < T$.

Considering that we can see R as a result of a series of tableau R_i formed by exchanging $R_i(k, l_i) = R_{i-1}(k+1, m_i) \ \ and \ \ R_i(k+1, m_i) = R_{i-1}(k, l_i)$ with $1 \leq i \leq a$ and $R_1 = T$ and hence $R_{i-1} < R_i$ as we have seen at case $a = 1$, the general case follows because of the transitivity of $\leq$. □

Our first goal will be to show that $B = \{d_{\lambda/\mu}(X_T) \mid T \in Tab_{\lambda/\mu}(S) \text{ is standard}\}$ is a system of generators of $L_{\lambda/\mu}F$. We will proceed defining an appropriate map whose image is contained in the kernel of $d_{\lambda/\mu}$ and which will allow us to prove that $d_{\lambda/\mu}(X_T)$, with T row-standard, is a linear combination of elements of B. Moreover, we will establish an isomorphism between the cokernel of this map and $L_{\lambda/\mu}F$. This fact will give us a natural way to describe the modules $L_{\lambda/\mu}F$.

Lemma 2.1.9. For each $i \in \{1, \dots, q-1\}$, the map $d_{\lambda/\mu}$ can be factored as follows:
$\Lambda_{\lambda/\mu}F \xrightarrow{\rho} \Lambda^{\lambda_1-\mu_1}F \otimes \cdots \otimes \Lambda^{\lambda_{i-1}-\mu_{i-1}}F \otimes S_{a_{i,1}+a_{i+1,1}}F \otimes \cdots \otimes S_{a_{i,\lambda_1}+a_{i+1,\lambda_1}}F \otimes \Lambda^{\lambda_{i+2}-\mu_{i+2}}F \otimes \cdots \otimes \Lambda^{\lambda_q-\mu_q}F \xrightarrow{\eta} \bigoplus_{a_{k,j}\in\alpha, k<i} F_{a_{k,j}} \otimes S_{a_{i,1}+a_{i+1,1}}F \otimes \cdots \otimes S_{a_{i,\lambda_1}+a_{i+1,\lambda_1}}F \otimes \bigoplus_{a_{k,j}\in\alpha,k>i+1} F_{(k,j)} \xrightarrow{\nu} S_{\tilde{\lambda}/\tilde{\mu}}F$, where $\rho = Id \otimes \cdots \otimes Id \otimes d_{(\lambda_i,\lambda_{i+1})/(\mu_i,\mu_{i+1})} \otimes Id \otimes \cdots \otimes Id$, $\eta = \imath_1 \otimes \cdots \otimes \imath_{i-1} \otimes Id \otimes \imath_{i+2} \otimes \cdots \otimes \imath_q$ and ν is the tensor product of the multiplication maps $\bar{m}_j : F_{a_{1,j}} \otimes \cdots \otimes F_{a_{i-1,j}} \otimes S_{a_{i,j}+a_{i+1,j}}F \otimes F_{a_{i+2,j}} \otimes \cdots \otimes F_{a_{q,j}} \to S^{\tilde{\lambda}_j-\tilde{\mu}_j}F$, $j = 1, \dots, \lambda_1$ defined over generators by $\bar{m}_j(f_1 \otimes \cdots \otimes f_{i-1} \otimes f_i f_{i+1} \otimes f_{i+2} \otimes \cdots \otimes f_q) = f_1 \cdot \cdots \cdot f_q$, where $f_{a_{i,j}} = 1 \in R \ \ if \ \ a_{i,j} = 0$.

Proof. $d_{(\lambda_i,\lambda_{i+1})/(\mu_i,\mu_{i+1})}$ is the composition of maps $\imath_i \otimes \imath_{i+1} : \Lambda^{\lambda_i-\mu_i}F \otimes \Lambda^{\lambda_{i+1}-\mu_{i+1}}F \to F_{a_{i,1}} \otimes \cdots \otimes F_{a_{i,\lambda_1}} \otimes F_{a_{i+1,1}} \otimes \cdots \otimes F_{a_{i+1,\lambda_1}}$ and the tensor product of the maps $m'_j : F_{a_{i,j}} \otimes F_{a_{i+1,j}} \to S_{a_{i,j}+a_{i+1,j}}F, j = 1, \dots, \lambda_1$. Then, we can write $\nu \circ \eta \circ \nu = (\bar{m}_1 \otimes \cdots \otimes \bar{m}_{\lambda_1})(Id \otimes m'_1 \otimes \cdots \otimes m'_{\lambda_1} \otimes Id)(\imath_1 \otimes \cdots \otimes \imath_q)$, where $(Id \otimes m'_1 \otimes \cdots \otimes m'_{\lambda_1} \otimes Id) : \bigoplus_{a_{i,j}\in\alpha_{\lambda/\mu}} F_{a_{i,j}} \to \bigoplus_{a_{k,j},k<i} F_{a_{k,j}} \otimes S_{a_{i,1}+a_{i+1,1}}F \otimes \cdots \otimes S_{a_{i,\lambda_1}+a_{i+1,\lambda_1}}F \otimes \bigoplus_{a_{k,j},k>i+1} F_{a_{k,j}}$.

Clearly $(\bar{m}_1 \otimes \cdots \otimes \bar{m}_{\lambda_1})(Id \otimes m'_1 \otimes \cdots \otimes m'_{\lambda_1} \otimes Id) = m$. □

As a result of these factorizations we focus our study on $d_{\lambda/\mu}$ when $\mu = (\mu_1, \mu_2)$ and $\lambda = (\lambda_1, \lambda_2)$.

Let $p_1 = \lambda_1 - \mu_1, p_2 = \lambda_2 - \mu_2 \ \ and \ \ k = \lambda_2 - \mu_1$, and consider the partitions $\lambda' = (p_1 + p_2 - k, p_2) \ \ and \ \ \mu' = (p_2 - k)$. Since $\lambda'/\mu' = (p_1, p_2) = \lambda/\mu$, $d_{\lambda/\mu} = d_{\lambda'/\mu'}$.

Definition 2.1.10. Let $p_1, p_2, k \in \mathbb{N}$ such that $p_i \geq k$. We define the map

$$\delta_k^{p_1,p_2} : \Lambda^{p_1}F \otimes \Lambda^{p_2}F \to S^2F \otimes \cdots \otimes S^2F \otimes \Lambda^{p_1-k}F \otimes \Lambda^{p_2-k}F$$

as the composition $\Lambda^{p_1}F \otimes \Lambda^{p_2}F \xrightarrow{\Delta\otimes\Delta} \Lambda^kF \otimes \Lambda^{p_1-k}F \otimes \Lambda^kF \otimes \Lambda^{p_2-k}F \xrightarrow{Id\otimes T\otimes Id} \Lambda^kF \otimes \Lambda^kF \otimes \Lambda^{p_1-k}F \otimes \Lambda^{p_2-k}F \xrightarrow{d_{(k,k)}\otimes Id\otimes Id} S^2F \otimes \cdots \otimes S^2F \otimes \Lambda^{p_1-k}F \otimes \Lambda^{p_2-k}F$, where Δ is the appropriate diagonal map and T is the canonical isomorphism $\Lambda^{p_1-k}F \otimes \Lambda^kF \cong \Lambda^kF \otimes \Lambda^{p_1-k}F$.

Lemma 2.1.11. Let $p_1, p_2, k \in N$ such that $p_i \geq k+1$. Then $\delta_{k+1}^{p_1,p_2} = (Id \otimes \delta_1^{p_1-k,p_2-k}) \circ \delta_k^{p_1,p_2}$.

Proof. Remember $d_{(k+1,k+1)} : \Lambda^{k+1}F \otimes \Lambda^{k+1}F \xrightarrow{\Delta\otimes\Delta} F \otimes \cdots \otimes F \xrightarrow{m_1\otimes\cdots\otimes m_{k+1}} S^2F \otimes \cdots \otimes S^2F$. By coassociativity and cocommutative of diagonal, $\Delta \otimes \Delta$ equals to the map
$\Lambda^{k+1}F \otimes \Lambda^{k+1}F \xrightarrow{\Delta\otimes\Delta} \Lambda^kF \otimes F \otimes \Lambda^kF \otimes F \xrightarrow{Id\otimes T\otimes Id} \Lambda^kF \otimes \Lambda^kF \otimes F \otimes F \xrightarrow{\Delta\otimes\Delta} F \otimes \cdots \otimes F \otimes F \otimes \cdots \otimes F \otimes F \otimes F$ and then $d_{(k+1,k+1)} = d_{(k,k)} \otimes d_{(1,1)}$.

We have, $\delta_{k+1}^{p_1,p_2} = (m_1 \otimes \cdots \otimes m_{k+1} \otimes Id \otimes Id)(\Delta \otimes \Delta \otimes Id \otimes Id)(Id \otimes T \otimes Id)(\Delta \otimes \Delta)$, where $(\Delta \otimes \Delta \otimes Id \otimes Id)(Id \otimes T \otimes Id)(\Delta \otimes \Delta) : \Lambda^{p_1}F \otimes \Lambda^{p_2}F \to F \otimes \cdots \otimes F \otimes F \otimes \cdots \otimes F \otimes \Lambda^{p_1-k-1}F \otimes \Lambda^{p_2-k-1}F$.

Again by coassociativity and cocommutative, the last map equals to $(Id \otimes T \otimes Id)(\Delta \otimes \Delta \otimes \Delta \otimes \Delta)(Id \otimes T \otimes Id)(\Delta \otimes \Delta) : \Lambda^{p_1} F \otimes \Lambda^{p_2} F \to \Lambda^k F \otimes \Lambda^{p_1-k} F \otimes \Lambda^k F \otimes \Lambda^{p_2-k} F \to \Lambda^k F \otimes \Lambda^k F \otimes \Lambda^{p_1-k} F \otimes \Lambda^{p_2-k} F \to F \otimes \cdots \otimes F \otimes F \otimes \cdots \otimes F \otimes F \otimes \Lambda^{p_1-k-1} F \otimes F \otimes \Lambda^{p_2-k-1} F \to F \otimes \cdots \otimes F \otimes F \otimes \cdots \otimes F \otimes F \otimes F \otimes \Lambda^{p_1-k-1} F \otimes \Lambda^{p_2-k-1} F$

And then, applying $m_1 \otimes \cdots \otimes m_k \otimes m_{k+1}$,
$\delta_{k+1}^{p_1,p_2} = (m_1 \otimes \cdots \otimes m_k \otimes m_{k+1} \otimes Id)(Id \otimes T \otimes Id)(\Delta \otimes \Delta \otimes \Delta \otimes \Delta)(Id \otimes T \otimes Id)(\Delta \otimes \Delta)$ which is exactly the map $(Id \otimes \delta_1^{p_1-k,p_2-k})\delta_k^{p_1,p_2}$. □

Lemma 2.1.12. Let $\alpha = (a_{i,j})$ be the $2 \times p_1 + p_2 - k$ matrix of zeros and ones:

$$\begin{matrix} 1\dots1 & 1\dots1 & 0\dots0 \\ 1\dots1 & 0\dots0 & 1\dots1 \\ \underbrace{\qquad}_{k} & \underbrace{\qquad}_{p_1-k} & \underbrace{\qquad}_{p_2-k} \end{matrix}$$

Then the diagram

$$\begin{array}{ccc} \Lambda^{p_1} F \otimes \Lambda^{p_2} F & & \\ \downarrow \delta_k^{p_1,p_2} & \searrow d_\alpha & \\ S^2F \otimes \cdots \otimes S^2F \otimes \Lambda^{p_1-k} F \otimes \Lambda^{p_2-k} F & \xrightarrow[Id\otimes\Delta\otimes\Delta]{} & S_{\tilde{\alpha}} F \end{array}$$

is commutative and $Im\delta_k^{p_1,p_2} \cong Imd_\alpha$. In particular, if $\lambda = (p_1 + p_2 - k, p_2)$ and $\mu = (q_2 - k, 0)$, then $Im\delta_k^{p_1,p_2} \cong L_{\lambda/\mu}$.

Proof. First we describe the map d_α.
$d_\alpha : \Lambda^{p_1} F \otimes \Lambda^{p_2} F \xrightarrow{\iota_1 \otimes \iota_2} \bigoplus_{a_{i,j} \in \alpha} F_{a_{i,j}} \to S^2F \otimes \cdots \otimes S^2F \otimes F \otimes \cdots \otimes F$, since $\sum_{j=1}^{p_1+p_2-k} a_{1,j} = p_1$, $\sum_{j=1}^{p_1+p_2-k} a_{2,j} = p_2$ and $q_1 = 2 = \cdots = 2 = q_k$, $q_{k+1} = 1 = \cdots = 1 = q_{p_1+p_2-k}$. More explicit, we can write $d_\alpha = (m_1 \otimes \cdots \otimes m_k \otimes Id \otimes \cdots \otimes Id)(\Delta \otimes \Delta)$.

Now, $(Id \otimes \Delta \otimes \Delta)\delta_k^{p_1,p_2} = (Id \otimes \cdots \otimes Id \otimes \Delta \otimes \Delta)(m_1 \otimes \cdots \otimes m_k \otimes Id \otimes Id)(\Delta \otimes \Delta \otimes Id \otimes Id)(Id \otimes T \otimes Id)(\Delta \otimes \Delta)$. It is clear that this map equals to $(m_1 \otimes \cdots \otimes m_k \otimes Id \otimes Id)(\Delta \otimes \Delta \otimes \Delta \otimes \Delta)(Id \otimes T \otimes Id)(\Delta \otimes \Delta)$. By coassociativity and commutativity, $(Id \otimes \cdots \otimes Id \otimes \Delta \otimes \Delta)(\Delta \otimes \Delta \otimes Id \otimes Id)(Id \otimes T \otimes Id)(\Delta \otimes \Delta) : \Lambda^{p_1} F \otimes \Lambda^{p_2} F \to \Lambda^k F \otimes \Lambda^{p_1-k} F \otimes \Lambda^k F \otimes \Lambda^{p_2-k} F \to \Lambda^k F \otimes \Lambda^k F \otimes \Lambda^{p_1-k} F \otimes \Lambda^{p_2-k} F \to F \otimes \cdots \otimes F$ equals to the map $\Delta \otimes \Delta : \Lambda^{p_1} F \otimes \Lambda^{p_2} F \to \bigoplus_{a_{i,j} \in \alpha} F_{a_{i,j}}$. Directly, $(Id \otimes \Delta \otimes \Delta)\delta_k^{p_1,p_2} = d_\alpha$.

Observe that the bottom map $Id \otimes \Delta \otimes \Delta$ is an injection, hence $Im(\delta_k^{p_1,p_2}) \cong Im(d_\alpha)$. □

Lemma 2.1.13. Let p_1, p_2, k be positive integers, with $p_i \geq k$. We define the following composite map:

$$\omega_k : \Lambda^{p_1+p_2-k} F \otimes \Lambda^k F \xrightarrow{\Delta \otimes Id} \Lambda^{p_1} F \otimes \Lambda^{p_2-k} F \otimes \Lambda^k F \xrightarrow{Id \otimes \tilde{m}} \Lambda^{p_1} F \otimes \Lambda^{p_2} F$$

where $\tilde{m}$ is the multiplication on the exterior algebra ΛF. Then, the following diagram is commutative

$$\begin{array}{ccc} \Lambda^{p_1+p_2-k} F \otimes \Lambda^k F & \xrightarrow{\quad w_k \quad} & \Lambda^{p_1} F \otimes \Lambda^{p_2} F \\ \downarrow \delta_1^{p_1+p_2-k,k} & & \downarrow \delta_1^{p_1,p_2} \\ S^2F \otimes \Lambda^{p_1+p_2-k-1} F \otimes \Lambda^{k-1} F & \xrightarrow{Id \otimes w_{k-1}} & S^2F \otimes \Lambda^{p_1-1} F \otimes \Lambda^{p_2-1} F \end{array}.$$

Proof. We can write the map $(Id \otimes \omega_{k-1})\delta_1^{p_1+p_2-k,k}$ as the composition $\Lambda^{p_1+p_2-k}F \otimes \Lambda^k F \xrightarrow{\Delta\otimes\Delta} F \otimes \Lambda^{p_1+p_2-k-1}F \otimes F \otimes \Lambda^{k-1}F \xrightarrow{Id\otimes\Delta\otimes Id\otimes Id} F \otimes \Lambda^{p_1-1}F \otimes \Lambda^{p_2-1-(k-1)}F \otimes F \otimes \Lambda^{k-1}F \xrightarrow{(Id\otimes T\otimes T\otimes Id)} F \otimes F \otimes \Lambda^{p_1-1}F \otimes \Lambda^{p_2-1-(k-1)}F \otimes \Lambda^{k-1}F \xrightarrow{m\otimes Id\otimes \tilde{m}} S_2F \otimes \Lambda^{p_1-1}F \otimes \Lambda^{p_2-1}F$.

By coassociativity of diagonal, $\Lambda^{p_1+p_2-k}F \otimes \Lambda^k F \xrightarrow{\Delta\otimes\Delta} F \otimes \Lambda^{p_1+p_2-k-1}F \otimes F \otimes \Lambda^{k-1}F \xrightarrow{Id\otimes\Delta\otimes Id\otimes Id} F \otimes \Lambda^{p_1-1}F \otimes \Lambda^{p_2-1-(k-1)}F \otimes F \otimes \Lambda^{k-1}F$ equals to $\Lambda^{p_1+p_2-k}F \otimes \Lambda^k F \xrightarrow{\Delta\otimes\Delta} \Lambda^{p_1}F \otimes \Lambda^{p_2-1-(k-1)}F \otimes F \otimes \Lambda^{k-1}F \xrightarrow{\Delta\otimes Id\otimes Id\otimes Id} F \otimes \Lambda^{p_1-1}F \otimes \Lambda^{p_2-1-(k-1)}F \otimes F \otimes \Lambda^{k-1}F$.

We denote $f \otimes g := f_1 \wedge \cdots \wedge f_{p_1+p_2-k} \otimes g_{p_1+p_2-k+1} \wedge \cdots \wedge g_{p_1+p_2} \in \Lambda^{p_1+p_2-k}F \otimes \Lambda^k F$. Then, $(Id \otimes \omega_{k-1})\delta_1^{p_1+p_2-k,k}(f \otimes g) = \sum_{i,j,t}(-1)^{sg(i)+sg(j)+sg(t)} f_{ji(1)} \cdot g_{t(1)} \otimes f_{ji(2)} \wedge \cdots \wedge f_{ji(p_1)} \otimes f_{ti(p_1+1)} \wedge \cdots \wedge f_{ti(p_1+p_2-k)} \wedge g_{t(2)} \wedge \cdots \wedge g_{t(k)}$.

Similarly, we can write the map $\delta_1^{p_1,p_2} \circ w_k$ as the composition $\Lambda^{p_1+p_2-k}F \otimes \Lambda^k F \xrightarrow{\Delta\otimes Id} \Lambda^{p_1}F \otimes \Lambda^{p_2-k}F \otimes \Lambda^k F \xrightarrow{Id\otimes\tilde{m}} \Lambda^{p_1}F \otimes \Lambda^{p_2}F \xrightarrow{\Delta\otimes\Delta} F \otimes \Lambda^{p_1-1}F \otimes F \otimes \Lambda^{p_2-1}F \xrightarrow{Id\otimes T\otimes Id} F \otimes F \otimes \Lambda^{p_1-1}F \otimes \Lambda^{p_2-1}F \xrightarrow{m\otimes Id\otimes Id} S^2F \otimes \Lambda^{p_1-1}F \otimes \Lambda^{p_2-1}F$. By the compatibility between comultiplication and multiplication on the exterior algebra ΛF the map $\Lambda^{p_2-k}F \otimes \Lambda^k F \to \Lambda^{p_2}F \to F \otimes \Lambda^{p_2-1}F$ equals to

$\sum_{\alpha=0}^{1} \Lambda^{p_2-k}F \otimes \Lambda^k F \xrightarrow{\Delta\otimes\Delta} \Lambda^{\alpha}F \otimes \Lambda^{p_2-k-\alpha}F \otimes \Lambda^{1-\alpha}F \otimes \Lambda^{k+\alpha-1}F \xrightarrow{Id\otimes T\otimes Id} \Lambda^{\alpha}F \otimes \Lambda^{1-\alpha}F \otimes \Lambda^{p_2-k-\alpha}F \otimes \Lambda^{k+\alpha-1}F \xrightarrow{\tilde{m}\otimes\tilde{m}} F \otimes \Lambda^{p_2-1}F$.

Thus, $(\delta_1^{p_1,p_2} \circ w_k)(f \otimes g) = \sum_{i,j,t}(-1)^{sg(i)+sg(j)+sg(t)} f_{ji(1)} \cdot f_{ti(p_1+1)} \otimes f_{ji(2)} \wedge \cdots \wedge f_{ji(p_1)} \otimes f_{ti(p_1+2)} \wedge \cdots \wedge f_{ti(p_1+p_2-k)} \wedge g + \sum_{i,j,t}(-1)^{sg(i)+sg(j)+sg(t)} f_{ji(1)} \cdot g_{t(1)} \otimes f_{ji(2)} \wedge \cdots \wedge f_{ji(p_1)} \otimes f_{ti(p_1+1)} \wedge \cdots \wedge f_{ti(p_1+p_2-k)} \wedge g_{t(2)} \wedge \cdots \wedge g_{t(k)}$.

To finish the proof it is enough to show that the first addend is zero. In fact, the first term is the image of $f \otimes g$ under the composite map $\Lambda^{p_1+p_2-k}F \otimes \Lambda^k F \xrightarrow{\Delta\otimes Id} \Lambda^{p_1}F \otimes \Lambda^{p_2-k}F \otimes \Lambda^k F \xrightarrow{\Delta\otimes\Delta\otimes Id} F \otimes \Lambda^{p_1-1}F \otimes F \otimes \Lambda^{p_2-k-1}F \otimes \Lambda^k F \xrightarrow{Id\otimes T\otimes Id\otimes Id} F \otimes F \otimes \Lambda^{p_1-1}F \otimes \Lambda^{p_2-k-1}F \otimes \Lambda^k F \xrightarrow{m\otimes\otimes Id\otimes\tilde{m}} S^2F \otimes \Lambda^{p_1-1}F \otimes \Lambda^{p_2-1}$. And again because of coassociativity and cocommutativity of comultiplication, $\Lambda^{p_1+p_2-k}F \otimes \Lambda^k F \xrightarrow{\Delta\otimes Id} \Lambda^{p_1}F \otimes \Lambda^{p_2-k}F \otimes \Lambda^k F \xrightarrow{\Delta\otimes\Delta\otimes Id} F \otimes \Lambda^{p_1-1}F \otimes F \otimes \Lambda^{p_2-k-1}F \otimes \Lambda^k F$ equals to $\Lambda^{p_1+p_2-k}F \otimes \Lambda^k F \xrightarrow{\Delta\otimes Id} \Lambda^{p_1-1}F \otimes \Lambda^{p_2-k+1}F \otimes \Lambda^k F \xrightarrow{Id\otimes\Delta\otimes Id} \Lambda^{p_1-1}F \otimes \Lambda^2F \otimes \Lambda^{p_2-k-1}F \otimes \Lambda^k F \xrightarrow{Id\otimes\Delta\otimes Id\otimes Id} \Lambda^{p_1-1}F \otimes F \otimes F \otimes \Lambda^{p_2-k-1}F \otimes \Lambda^k F$

A simple calculation shows that the image which we are looking for is $\sum_{i,j} f_{i(1)} \wedge \cdots \wedge f_{i(p_1-1)} \otimes (f_{ji(p_1)} \cdot f_{ji(p_1+1)} - f_{ji(p_1+1)} \cdot f_{ji(p_1)})) \otimes f_{ji(p_1+2)} \wedge \cdots \wedge f_{ji(p_1+p_2-k)}$, which is zero since $u \cdot v = v \cdot u \in S_2F$. □

Proposition 2.1.14. Let $p_1, p_2, k \in \mathbb{N}$ *with* $p_i \geq k+1$. Then the composition

$$\Lambda^{p_1+p_2-k}F \otimes \Lambda^k F \xrightarrow{w_k} \Lambda^{p_1}F \otimes \Lambda^{p_2}F \xrightarrow{\delta_{k+1}^{p_1,p_2}} S_2F \otimes \cdots \otimes S_2F \otimes \Lambda^{p_1-k-1}F \otimes \Lambda^{p_2-k-1}$$

is zero.

Proof. We proceed by induction on k. When $k = 0$, we have $\Lambda^{p_1+p_2} \xrightarrow{\Delta} \Lambda^{p_1} \otimes \Lambda^{p_2} \xrightarrow{\Delta\otimes\Delta} F \otimes \Lambda^{p_1-1}F \otimes F \otimes \Lambda^{p_2-1}F \to S_2F \otimes \Lambda^{p_1-1}F \otimes \Lambda^{p_2-1}F$. The same argument we have just seen in the last part of the proof of Lemma 2.1.13 shows that this composition is zero. Suppose $k > 0$. Since $\delta_{k+1}^{p_1,p_2} = (Id \otimes \delta_1^{p_1-k,p_2-k}) \circ \delta_k^{p_1,p_2}$, we can write $\delta_{k+1}^{p_1,p_2} \circ w_k = (Id \otimes$

$\delta_1^{p_1-k,p_2-k}) \circ \delta_k^{p_1,p_2} \circ w_k$. By Lemma 2.1.11 $(Id \otimes \delta_1^{p_1-k,p_2-k}) \circ \delta_k^{p_1,p_2} = (Id \otimes \delta_1^{p_1-k,p_2-k}) \circ (Id \otimes w_{k-1}) \circ \delta_1^{p_1+p_2-k,k}$. The result follows using the induction on $(Id \otimes \delta_1^{p_1-k,p_2-k}) \circ (Id \otimes w_{k-1})$. □

Lemma 2.1.15. Let p_1, p_2, k, u, l be nonnegative integers such that $p_i \geq k$, $0 \leq u \leq l \leq k-1$. Denote by $\bar{\omega}_u$ the composite map

$\Lambda^u F \otimes \Lambda^{p_1+p_2-l} F \otimes \Lambda^{l-u} F \xrightarrow{Id \otimes \Delta \otimes Id} \Lambda^u F \otimes \Lambda^{p_1-u} F \otimes \Lambda^{p_2+u-l} F \otimes \Lambda^{l-u} F \xrightarrow{\tilde{m} \otimes \tilde{m}} \Lambda^{p_1} F \otimes \Lambda^{p_2} F$.

Then $Im(\bar{\omega}_u)$ is contained in the image of the map

$$\omega_k^{p_1,p_2} : \bigoplus_{0 \leq v \leq k-1} \Lambda^{p_1+p_2-v} F \otimes \Lambda^v F \xrightarrow{\sum_{v=0}^{k-1} \omega_v} \Lambda^{p_1} F \otimes \Lambda^{p_2} F.$$

Proof. We proceed by induction on u. When $u = 0$, $\bar{w}_u = w_l$ and the proposition is clear. Assuming $u > 0$, let $x \otimes y \otimes z \in \Lambda^u F \otimes \Lambda^{p_1+p_2-l} F \otimes \Lambda^{l-u} F$, we have $\bar{\omega}_u(x \otimes y \otimes z) = \sum_i (-1)^{sg(i)} x_1 \wedge \cdots \wedge x_u \wedge y_{i(1)} \wedge \cdots \wedge y_{i(p_1-u)} \otimes y_{i(p_1-u+1)} \wedge \cdots \wedge y_{i(p_1+p_2-l)} \wedge z_1 \wedge \cdots \wedge z_{l-u}$. Now, we consider the composition $\omega_{l-u} \circ (m \otimes Id) : \Lambda^u F \otimes \Lambda^{p_1+p_2-l} F \otimes \Lambda^{l-u} \xrightarrow{m \otimes Id} \Lambda^{p_1+p_2-(l-u)} F \otimes \Lambda^{l-u} F \xrightarrow{\Delta \otimes Id} \Lambda^{p_1} F \otimes \Lambda^{p_2-(l-u)} F \otimes \Lambda^{l-u} F \xrightarrow{Id \otimes m} \Lambda^{p_1} F \otimes \Lambda^{p_2} F$. Because of compatibility between the multiplication and comultiplication the composition $\Lambda^u F \otimes \Lambda^{p_1+p_2-l} F \xrightarrow{m} \Lambda^{p_1+p_2-(l-u)} F \xrightarrow{\Delta} \Lambda^{p_1} F \otimes \Lambda^{p_2-(l-u)} F$ equals to

$\sum_{\alpha=0}^{u} \Lambda^u F \otimes \Lambda^{p_1+p_2-l} F \xrightarrow{\Delta \otimes \Delta} \Lambda^{u-\alpha} F \otimes \Lambda^{\alpha} F \otimes \Lambda^{p_1-(u-\alpha)} F \otimes \Lambda^{p_2-(l-u+\alpha)} F \xrightarrow{Id \otimes T \otimes Id} \Lambda^{u-\alpha} F$ $\otimes \Lambda^{p_1-(u-\alpha)} F \otimes \Lambda^{\alpha} F \otimes \Lambda^{p_2-(l-u+\alpha)} F \xrightarrow{m \otimes m} \Lambda^{p_1} F \otimes \Lambda^{p_2-(l-u)} F$.

Thus $\omega_{l-u} \circ (m \otimes Id)(x \otimes y \otimes z) = \bar{\omega}_u(x \otimes y \otimes z) + \sum_{\alpha=1}^{u} \sum_{i,j} (-1)^{sg(i)+sg(j)} x_{i(1)} \wedge \cdots \wedge x_{i(u-\alpha)} \wedge y_{j(p_1)} \wedge \cdots \wedge y_{j(1)} \wedge \cdots \wedge y_{j(p_1-(u-\alpha))} \otimes x_{i(u-\alpha+1)} \wedge \cdots \wedge x_{i(u)} \wedge y_{j(p_1-(u-\alpha)+1} \wedge \cdots \wedge y_{j(p_2+p_1-l)}$. Since $\omega_\alpha(x \otimes y \otimes z) = \sum_i (-1)^{sg(i)} x_{i(1)} \wedge \cdots \wedge x_{i(u-\alpha)} \wedge y \wedge x_{i(u-\alpha+1)} \wedge \cdots \wedge x_{i(u)} \wedge z$, we obtain $\sum_{\alpha=1}^{u} \bar{\omega}_{u-\alpha} \circ \omega_\alpha(x \otimes y \otimes z) = \omega_{l-u} \circ (m \otimes Id)(x \otimes y \otimes z) - \bar{\omega}_u(x \otimes y \otimes z)$. By induction $Im(\bar{\omega}_{u-\alpha} \circ \omega_\alpha) \subset Im(\omega_k^{p_1,p_2})$, $\forall 1 \leq \alpha \leq u$, and hence $\omega_{l-u} \circ (m \otimes Id)(x \otimes y \otimes z) \in Im(\omega_k^{p_1,p_2})$. □

At this moment we define the auxiliary map we have mentioned at the begging of this subsection and next we state our first main result.

Definition 2.1.16. Let $\lambda = (\lambda_1, \ldots, \lambda_q)$ and $\mu = (\mu_1, \ldots, \mu_q)$ be two partitions such that $\mu \subseteq \lambda$. For each $i = 1, \ldots, q-1$ consider partitions $\lambda^i = (\lambda_i, \lambda_{i+1})$ and $\mu^i = (\mu_i, \mu_{i+1})$. One defines a map $\omega_{\lambda/\mu}$ to be the sum of maps $Id_1 \otimes \cdots \otimes Id_{i-1} \otimes \omega_{\lambda^i/\mu^i} \otimes Id_{i+2} \otimes \ldots \otimes Id_q$, where $i = 1, \ldots, q-1$ and $\omega_{\lambda^i/\mu^i} = \omega_{k_i}^{p_i,p_{i+1}}$ with $p_i = \lambda_i - \mu_i$, $k_i = \lambda_{i+1} - \mu_i$. We define $\bar{L}_{\lambda/\mu}(F)$ to be the cokernel of map $\omega_{\lambda/\mu}$.

Theorem 2.1.17. The image of $\omega_{\lambda/\mu}$ is contained in the kernel of $d_{\lambda/\mu}$.

Proof. Keeping in mind the factorization of $d_{\lambda/\mu}$, Lemma 2.1.9 it is enough to consider $\lambda = (\lambda_1, \lambda_2)$, $\mu = (\mu_1, \mu_2)$ and show that $Im(\omega_k^{p_1,p_2}) \subseteq ker(d_{\lambda/\mu})$.

Now, since the composition $\delta_{k+1}^{p_1,p_2} \circ \omega_k = 0$ as we have seen before, $\omega_k \subset ker(\delta_{k+1}^{p_1,p_2})$ and hence $Im(\sum_{v=0}^{k-1} \omega_v) \subseteq ker(\sum_{v=0}^{k-1} \delta_{v+1}^{p_1,p_2})$. We have already finished considering that for each $k \in \mathbb{N} \mid p_i \geq k$, $(Id \otimes \Delta \otimes \Delta) \circ \delta_k^{p_1,p_2} = d_{\lambda/\mu} \Rightarrow ker(\delta_k^{p_1,p_2}) \subseteq ker(d_{\lambda/\mu})$ and then $Im(\sum_{v=0}^{k-1} \omega_v) \subset ker(\sum_{v=0}^{k-1} \delta_{v+1}^{p_1,p_2}) \subset ker(d_{\lambda/\mu})$. □

As a direct corollary of Theorem 2.1.17 the map $d_{\lambda/\mu}$ induces a surjective map $\bar{d}_{\lambda/\mu} : \bar{L}_{\lambda/\mu}(F) \to L_{\lambda/\mu}(F)$ defined by $\bar{d}_{\lambda/\mu}(\bar{x}) = d_{\lambda/\mu}(x)$, where $\bar{x}$ is the image under the natural map $\pi' : \Lambda_{\lambda/\mu}F \to \bar{L}_{\lambda/\mu}F$. Note that $\bar{x} = \bar{y} \Rightarrow \bar{d}_{\lambda/\mu}(\bar{x}) = \bar{d}_{\lambda/\mu}(\bar{y}) \Leftrightarrow d_{\lambda/\mu}(x) = d_{\lambda/\mu}(y)$ which implies $d_{\lambda/\mu} = \bar{d}_{\lambda/\mu} \circ \pi'$.

Lemma 2.1.18. Let $T \in Tab_{\lambda/\mu}(S)$ be a row-standard tableau which is not standard. Then there exist standard tableaux T_i with $T_i < T$ such that $X_T - \sum \pm X_{T_i} \in Im(w_{\lambda/\mu})$.

Proof. Let $T^j = (T(j, \mu_j + 1), \dots, T(j, \lambda_j))$ and $T^{j+1} = (T(j+1, \mu_{j+1}+1), \dots, T(j+1, \lambda_{j+1})$ be the first two rows of T in which column-standardness is violated. Let $\bar{T} = (T^j, T^{j+1})$ be a tableau of shape λ^j/μ^j where $\lambda^j = (\lambda_j, \lambda_{j+1}), \mu^j = (\mu_j, \mu_{j+1})$. First, we will proof the lemma in this particular case.

Let $p_1 = \lambda_j - \mu_j$, $p_2 = \lambda_{j+1} - \mu_{j+1}$ and $k = \lambda_{j+1} - \mu_j$. For convenience, we will denote $T(j,i) = a_i, \quad i = 1, \dots, p_1 \quad and \quad T(j+1, l) = b_l, \ l = 1, \dots, p_2$. Let $a_{u+1} > b_{p_2-k+u+1}$ be the first entries in which the violation takes place. Let $\bar{X} = a_1 \wedge \dots \wedge a_u \otimes b_1 \wedge \dots \wedge b_{p_2-k+u+1} \wedge a_{u+1} \wedge \dots \wedge a_{p_1} \otimes b_{p_2-k+u+2} \wedge \dots \wedge b_{p_2} \in \Lambda^u F \otimes \Lambda^{p_1+p_2-l} F \otimes \Lambda^{l-1-u} F$, with $l = k - 1$. Graphically:

			a_1	$\dots$	a_{u+1}	$\dots$	
b_1	$\dots$	b_{p_2-k}	b_{p_2-k+1}	$\dots$	$b_{p_2-k+u+1}$	$\dots$	

We have $\bar{w}_u(\bar{X}) = \sum (-1)^{sg(i)} a_1 \wedge \dots \wedge a_u \wedge c_I \otimes c_{I'} \wedge b_{p_2-k+u+2} \wedge \dots \wedge b_{p_2} =: \sum \pm \bar{X}_I$ where c_I is the corresponding exterior product of $p_1 - u$ terms of $b_1 \wedge \dots \wedge b_{p_2-k+u+1} \wedge a_{u+1} \wedge \dots \wedge a_{p_1}$ and $c_{I'}$ of the complementary terms after applying $Id \otimes \Delta \otimes \Delta$. We denote $c_I = c_{i_1} \wedge \dots \wedge c_{i_{p_1-u}}$ and $c_{I'} = c_{i_{p_1-u+1}} \wedge \dots \wedge c_{p_1+p_2-k+u+1}$. By Lemma 2.1.15, $\sum \pm \bar{X}_I \in Im(w_k^{p_1,p_2}) = Im(\omega_{\lambda^j/\mu^j})$.

For each I let $\bar{T}_I$ be the tableau of shape λ^j/μ^j associated to $\bar{X}_I$. There is only one term I, say I_0, such that $c_I = a_{u+1} \wedge \dots \wedge a_{p_1}$, and hence the tableau associated to I_0 is the tableau associated to $\bar{X}$. The rest of tableaux $\bar{T}_I$ are a result of exchanging some entries $a_{u+1}, \dots, a_{p_1}$ by some entries $b_1, \dots, b_{p_2-k+u+1}$, with necessarily $c_{i_1} = b_t$ for some $t \in \{1, \dots, p_2 - k + u + 1\}$, by a previous lemma $\bar{T}_I < \bar{T}$.

Observe that each $a_1 \wedge \dots \wedge a_u \wedge c_I \otimes c_{I'} \wedge b_{p_2-k+u+2} \wedge \dots \wedge b_{p_2}$ as an element of $\Lambda^{p_1} F \otimes \Lambda^{p_2} F$ can be arranged such that equals $x_{I_1} \wedge \dots \wedge x_{I_{p_1}} \otimes x_{I'_1} \wedge \dots \wedge x_{I'_{p_2}}$ with $x_{I_1} < \dots < x_{I_{p_1}}$ and $x_{I'_1} < \dots < x_{I'_{p_2}}$. Because of this we assume that $\bar{T}_I$, $I \neq I_0$ is row-standard. In this situation for each $\bar{T}_I$ we have $c_{i_1} \leq c_{p_2-k+u+1}$, since $a_{u+1} > b_{p_2-k+u+1} \Rightarrow a_{u+1} > b_t$, $t = 1, \dots, p_2 - k + u + 1$. So, if $\bar{T}_I$ is not standard, then the first index where the column standardness violation in $\bar{T}_I$ takes place is greater than $u + 1$.

Since there is a finite number of tableaux in $Tab_{\lambda^j/\mu^j}(S)$, repeating the same procedure for each non standard tableau $\bar{T}_I$ and continue as many times we need, finally we will obtain $\sum \bar{X}_{T_I} = \bar{X} - \sum_{I \neq I_0} \pm \bar{X}_{T_I} \in Im(\omega_{\lambda^j/\mu^j})$.

Now the general case is clear, we only have to exchange $\bar{X}$ by $T(1, \mu_1 + 1) \wedge \dots \wedge T(1, \lambda_1) \otimes \dots \otimes \bar{X} \otimes T(j+2, \mu_{j+2} + 1) \wedge \dots \wedge T(j+2, \lambda_{j+2}) \otimes \cdots$, $\bar{w}_u$ by the natural generalization $Id_1 \otimes \dots \otimes Id_{i-1} \otimes \bar{w}_{u_i} \otimes Id_{i+2} \otimes \dots \otimes Id_q$, w_{λ^j/μ^j} by $w_{\lambda/\mu}$ and repeat this procedure for each couple of rows which column-standardness is violated. □

Let us to apply the above lemma to the element associated to the tableau $T = \begin{array}{|c|c|} \hline x_1 & x_6 \\ \hline x_2 & x_5 \\ \hline \end{array}$.

We have, $p_1 = 2, p_2 = 2, k = 2, u = 1$ and $l = 1$. We want to compute $\bar{\omega}_2(x_1 \otimes x_2 \wedge x_5 \wedge x_6)$, where $\bar{\omega}_2$ is the map $F \otimes \Lambda^3 F \xrightarrow{Id\otimes\Delta} F \otimes F \otimes \Lambda^2 F \xrightarrow{m\otimes Id} \Lambda^2 F \otimes \Lambda^2 F$. We obtain,

$$\begin{array}{|c|c|}\hline x_1 & x_2 \\ \hline x_5 & x_6 \\ \hline \end{array} - \begin{array}{|c|c|}\hline x_1 & x_5 \\ \hline x_2 & x_6 \\ \hline \end{array} + \begin{array}{|c|c|}\hline x_1 & x_6 \\ \hline x_2 & x_5 \\ \hline \end{array}$$

Directly from Lemma 2.1.18

Corollary 2.1.19. $\{d_{\lambda/\mu}(X_T) \mid T \in Tab_{\lambda/\mu}(S)$ is standard$\}$ is a system of generators in $L_{\lambda/\mu}F$.

Corollary 2.1.20. $\{\bar{X}_T \mid T \in Tab_{\lambda/\mu}(S)$ is standard$\}$ spans $\bar{L}_{\lambda/\mu}F$.

Proof. Since $\{X_T \mid T \in Tab_{\lambda/\mu}(S)$ is row-standard$\}$ generate $\Lambda_{\lambda/\mu}(F)$, it is enough to see that each $\bar{X}_T$, with T row-standard can be expressed as a combination of elements of that such.

Let $T \in Tab_{\lambda/\mu}(S)$ be a row-standard tableau. By Lemma 2.1.18 we know that there exist $T_I \in Tab_{\lambda/\mu}(S)$ standard tableaux such that $X_T - \sum_I X_{T_I} \in Im(\omega_{\lambda/\mu})$. Clearly, $\bar{X}_T = \sum_I \bar{X}_{T_I}$. □

Theorem 2.1.21. $\{d_{\lambda/\mu}(X_T) \mid T \in Tab_{\lambda/\mu}\{x_1, \dots, x_n\}$ is standard$\}$ is a basis of $L_{\lambda/\mu}F$ called *the standard basis* of $L_{\lambda/\mu}F$. So, $L_{\lambda/\mu}F$ is a free R-module.

Proof. We denote $\lambda = (\lambda_1, \dots, \lambda_q), \mu = (\mu_1, \dots, \mu_q)$. Since $\{d_{\lambda/\mu}(X_T) \mid T \in Tab_{\lambda/\mu}\{x_1, \dots, x_n\}$ is standard$\}$ is a system of generators of $L_{\lambda/\mu}$ as we have just seen, we only must proof that these elements are independent. We proceed as follows:

Let $\Lambda'_{\lambda/\mu}F$ and $S'_{\tilde{\lambda}/\tilde{\mu}}F$ be the free R-submodules of $\Lambda_{\lambda/\mu}$ and $S_{\tilde{\lambda}/\tilde{\mu}}F$ respectively generated by $\{X_T \mid T \in Tab_{\lambda/\mu}(S)$ is standard$\}$ and $\{Z_T \mid T \in Tab_{\lambda/\mu}(S)$ respectively. Consider the natural injection $i : \Lambda'_{\lambda/\mu}F \hookrightarrow \Lambda_{\lambda/\mu}F$ and the natural projection $\pi : S_{\tilde{\lambda}/\tilde{\mu}}F \to S'_{\tilde{\lambda}/\tilde{\mu}}F$. Keeping in mind that $Tab_{\lambda/\mu}(S)$ is totally ordered lexicographically and $d_{\lambda/\mu}(\Lambda'_{\lambda/\mu}F) = L_{\lambda/\mu}F$, if we prove that the map associated to the composition $\pi \circ d_{\lambda/\mu} \circ i$ is triangular with ones on the diagonal respect these basis in this order, then $d_{\lambda/\mu} \circ i$ is an injection and hence $L_{\lambda/\mu}F \cong L'_{\lambda/\mu}F$ which implies that $\{d_{\lambda/\mu}(X_T) \mid T \in Tab_{\lambda/\mu}\{x_1, \dots, x_n\}$ is standard$\}$ is a basis of $L_{\lambda/\mu}F$.

Let $T \in Tab_{\lambda/\mu}(S)$ be a standard tableau. As we have seen at the beginning of this section, $d_{\lambda/\mu}(X_T) = \sum_\sigma (-1)^{sg(\sigma)} Z_{T_\sigma}$ where $\sigma = (\sigma_1, \dots, \sigma_q)$, σ_i runs through all permutations of $\{\mu_i + 1, \dots, \lambda_i\}$ and $T_\sigma \in Tab_{\lambda/\mu}\{x_1, \dots, x_n\}$ is the tableau of entries $T_\sigma(i, j) = T(i, \sigma_i(j))$. Let T'_σ be the column standardization of T_σ, since T'_σ and T_σ define the same element Z_T in $S_{\tilde{\lambda}/\tilde{\mu}}F$ because the properties of the symmetric power, we can write $d_{\lambda/\mu}(X_T) = \sum_\sigma Z_{T_{\sigma'}}$ where σ' runs over the set of permutations such that $T_{\sigma'}$ is column-standard. Then $\pi \circ d_{\lambda/\mu}(X_T) = \sum_\sigma Z_{T'_\sigma}$ which σ runs over the set of permutations such that T'_σ is standard.

We observe that Z_T occurs in the sum above, then it is enough to prove that $T'_\sigma < T$ for all $\sigma \neq Id$. First we observe that T_σ cannot be row-standard since the standardness of T, and hence $T_\sigma \neq T'_\sigma$ if $\sigma \neq Id$. For $\sigma \neq Id$, T'_σ is obtained from T_σ as a result of iterating the process described in Lemma 2.1.8, so $T'_\sigma < T_\sigma$, and by the same reason $T_\sigma < T$. Clearly this implies $T'_\sigma < T$ for all $\sigma \neq Id$. □

The above theorem give us an explicit basis of Schur functors by means of combinatoric tools. As a particular case we have the following theorem (see [15], Theorem 6.3):

Theorem 2.1.22. Let $\lambda = (\lambda_1, \ldots, \lambda_q)$ be a partition. If $\lambda_1 \leq n$, then

$$dim(L_\lambda F) = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

otherwise, $L_\lambda F$ is zero.

Remark 2.1.23. Consider $L_{\lambda/\mu}F$. In general, if $\lambda_1 - \mu_1$ is greater than the rank of F, then $\Lambda^{\lambda_1 - \mu_1}F$ is zero. Thus, $L_{\lambda/\mu}F$ is trivially null for skew partitions such that the length of λ/μ is greater than the rank of F.

Theorem 2.1.24. $L_{\lambda/\mu}F$ is isomorphic to $\bar{L}_{\lambda/\mu}F$.
Proof. Since $d_{\lambda/\mu} = \bar{d}_{\lambda/\mu} \circ \pi'$, $\pi \circ \bar{d}_{\lambda/\mu} \circ \pi' \circ i$ is an isomorphism, and hence $\bar{L}_{\lambda/\mu}F$ is isomorphic to $\Lambda'_{\lambda/\mu}F$ as we have argued before. Then $\bar{L}_{\lambda/\mu}F$ is a free R-module with a basis $\{\bar{X}_T \mid T \in Tab_{\lambda/\mu}\{x_1, \ldots, x_n\}$ is standard$\}$. Clearly $\bar{d}_{\lambda/\mu}$ is an isomorphism. □

The isomorphism $L_{\lambda/\mu}F \cong \bar{L}_{\lambda/\mu}F$ and the natural definition of $\omega_{\lambda/\mu}$ gives us a natural way to describe Schur functors as the following examples illustrate.

Examples 2.1.25. (1) $L_{(2,1)}F$. We have $p_1 = 2, p_2 = 1$ and $k = 1$. Then, $\omega_{(2,1)}$ is just the map $\omega_0^{2,1} : \Lambda^3 F \to \Lambda^2 F \to \Lambda^1 F$. That is, $L_{(2,1)}F$ is the cokernel of the diagonal map $\Delta_{2+1} : \Lambda^3 F \to \Lambda^2 F \otimes F$.

(2) $L_{(3,2)}F$. Here, $p_1 = 3, p_2 = 2$ and $k = 2$. In that case $\omega_{(3,2)}$ is the sum of the two maps $\omega_0^{3,2} : \Lambda^5 F \to \Lambda^3 F \otimes \Lambda^2 F$ and $\omega_1^{3,2} : \Lambda^4 F \otimes F \to \Lambda^3 F \otimes \Lambda^2 F$. The first map is the diagonal map Δ_{3+2} and the second map is the composition $\Lambda^4 F \otimes F \xrightarrow{\Delta_{4+1} \otimes Id} \Lambda^3 F \otimes F \otimes F \xrightarrow{Id \otimes \tilde{m}_{1,1}} \Lambda^3 F \otimes \Lambda^2$. So, $L_{(3,2)}F$ is the cokernel of the sum of these two maps.

We want to end this section talking about the functoriality of $L_{\lambda/\mu}(-)$.

Let E and F be two free R-modules, let $\phi : E \to F$ be an R-map and let λ/μ a skew partition. From Proposition 1.1.7 we have a well defined map $\Lambda_{\lambda/\mu}(\phi) : \Lambda_{\lambda/\mu}F \to \Lambda_{\lambda/\mu}E$ given by $\Lambda_{\lambda/\mu}(\phi)(X_1 \otimes \cdots \otimes X_q) = (\Lambda^{\lambda_1 - \mu_1}(\phi)(X_1)) \otimes \cdots \otimes (\Lambda^{\lambda_q - \mu_q}(\phi)(X_q))$ where X_i represent an element of $\Lambda^{\lambda_i - \mu_i}F$. Denoting $d^F_{\lambda/\mu}$ and $d^E_{\lambda/\mu}$ the Schur maps associated to F and E respectively, we can define a natural R-map $L_{\lambda/\mu}(\phi) : L_{\lambda/\mu}F \to L_{\lambda/\mu}E$ by taking $L_{\lambda/\mu}(\phi)(d^F_{\lambda/\mu}(X)) = d^E_{\lambda/\mu}(\Lambda_{\lambda/\mu}(X))$. Then, $L_{\lambda/\mu}(-)$ define an endofunctor on the category of free R-modules and R-map.

2.2 CoSchur Functors and the freeness of the CoSchur functors.

CoSchur functors are defined in a similar way as Schur functor using divided and exterior powers.

Definition 2.2.1. Let $\lambda = (\lambda_1, \ldots, \lambda_p)$ and $\mu = (\lambda_1, \ldots, \lambda_p)$ be two partitions with $\mu \subseteq \lambda$. We define the *coSchur map* $d'_{\lambda/\mu} : D_{\lambda/\mu}F \to \Lambda_{\lambda/\mu}F$ associated to the skew partition λ/μ to be the composite map $D^{\lambda_1-\mu_1}F \otimes \cdots \otimes D^{\lambda_p-\mu_p}F \xrightarrow{\alpha'} \bigotimes_{(i,j)\in\Delta_{\lambda/\mu}} F_{(i,j)} \xrightarrow{\beta'} \Lambda^{\tilde{\lambda}_1-\tilde{\mu}_1}F \otimes \cdots \otimes \Lambda^{\tilde{\lambda}_{\lambda_1}-\tilde{\mu}_{\lambda_1}}$, where α' is the tensor product of the inclusions $\iota'_i : D^{\lambda_i-\mu_i}F \to F_{(i,\mu_i+1)} \otimes \cdots \otimes F_{(i,\lambda_i)}, i = 1, \ldots, p$ and β is the tensor product of the multiplications $\tilde{m}_j : F_{(\tilde{\mu}_1+1,j)} \otimes \cdots \otimes F_{(\tilde{\lambda}_i,j)} \to \Lambda^{\tilde{\lambda}_j-\tilde{\mu}_j}F, j = 1, \ldots, \lambda_1$.

Definition 2.2.2. We define the *coSchur functor* $K_{\lambda/\mu}F$ associated to F and the skew partition λ/μ to be the image of $d'_{\lambda/\mu}$.

Clearly, $K_{\lambda/\mu}F$ can be defined through Ferrers matrices in analogous way as $L_{\lambda/\mu}$ and the coSchur map can be generalized as a natural map d'_α where α is an arbitrary matrix of zeros as we have seen in previous sections.

Examples 2.2.3. (1) $K_{(t)}F = D^tF$.

(2) $K_{(1,\ldots,1)}F = \Lambda^{(t)}F, t = 1 + \cdots + 1$.

The Standard Basis Theorem of Schur functors has its dual version to CoSchur functors. Indeed, we easily check that $\{d'_{\lambda/\mu}(X_T) \mid T \in Tab_{\lambda/\mu}$ is co-column-standard$\}$ is a basis of $D_{\lambda/\mu}$ and arguing as in section 2.1.1 we prove:

Theorem 2.2.4. $\{d'_{\lambda/\mu}(X_T) \mid T \in Tab_{\lambda/\mu}\{x_1, \ldots, x_n\}$ is co-standard$\}$ is a basis of $K_{\lambda/\mu}F$ called *the co-standard basis* of $K_{\lambda/\mu}F$. So, $K_{\lambda/\mu}F$ is a free R-module.

The functoriality of $K_{\lambda/\mu}(-)$ follows from the functoriality of $D^r(-)$ in the same manner we have argued from Schur functors in section 2.1.1. We will finish our general discussion of Schur and CoSchur functors by stating the duality $(L_{\lambda/\mu}F)^* \cong K_{\tilde{\lambda}/\tilde{\mu}}(F^*)$. It follows directly from the dual nature of the Schur and CoSchur maps.

Theorem 2.2.5. $(L_{\lambda/\mu}F)^* \cong K_{\tilde{\lambda}/\tilde{\mu}}(F^*)$.

Proof. We denote $d'_{\tilde{\lambda}/\tilde{\lambda}} : D^{\tilde{\lambda}/\tilde{\mu}}F \to \Lambda_{\lambda/\mu}F$ and $d_{\lambda/\mu} : \Lambda_{\lambda/\mu}F \to S_{\tilde{\lambda}/\tilde{\mu}}F$. The coSchur map is the composite map $\beta' \circ \alpha'$, where β' is the tensor product of diagonal maps in DF and α' the tensor product of multiplications in ΛF. Similarly we have $d_{\lambda/\mu} = \alpha \circ \beta$. From the Introductory Material, β' is the dual map of α and α' is the dual map β. □

Moreover, if R contains a field of characteristic zero, then $L_{\lambda/\mu}F \cong K_{\tilde{\lambda}/\tilde{\mu}}F$. See [2], Corollary (2.3.3). From Theorem 2.2.5, it follows that $K_{\lambda/\mu}F \cong (L_{\tilde{\lambda}/\tilde{\mu}}F^*)^*$. This isomorphism give the following formula of rank of CoSchur functor.

Theorem 2.2.6. Let $\lambda = (\lambda_1, \ldots, \lambda_q)$ be a partition. If $\tilde{\lambda}_1 \leq n$, then

$$dim(K_\lambda F) = \prod_{1\leq i<j\leq n} \frac{\tilde{\lambda}_i - \tilde{\lambda}_j + j - i}{j - i}$$

otherwise, $K_\lambda F$ is zero.

For example let $\lambda = (4, 2, 1)$ and let $n = 4$. Since $\tilde{\lambda} = (3, 2, 1, 1)$ and $3 < 4$, then from the above theorem $K_{(4,2,1)}F$ is a free R-module of rank 10,i.e $K_{(4,2,1)}F \cong R^{10}$.

2.3 Decomposition of Schur Functors.

The aim of this section will be to define a filtration on $L_{\lambda/\mu}(F \oplus G)$ with associated graded module $\bigoplus_{\mu\subseteq\gamma\subseteq\lambda} L_{\gamma/\mu}F \otimes L_{\lambda/\gamma}G$.

Let F and G be free R-modules of rank n and m with ordered basis $\{x_1, \ldots, x_n\}$ and $\{x_{n+1}, \ldots, x_{n+m}\}$ respectively. We will consider the ordered basis $\{x_1, \ldots, x_{n+m}\}$ in $F \oplus G$.

Definition 2.3.1. Let F be a module over a ring R. A *finite filtration* of F is a sequence of submodules of F

$$0 =: F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n := F.$$

One defines the associated graded object of the filtration $\sum_{i=0}^{n-1} F_{i+1}/F_i$.

Once we have defined the modules of the filtration, we will need to describe basis of these modules. We do it through a basis of $F \oplus G$ and $Tab_{\lambda/\mu}(F \oplus G)$. We start introducing some facts about partitions and tableaux we will use in this description.

Definition 2.3.2. Let $\lambda = (\lambda_1, \ldots, \lambda_q), \mu = (\mu_1, \ldots, \mu_q)$ be two partitions. We say $\lambda \geq \mu$ if $\lambda_1 = \mu_1, \ldots, \lambda_i = \mu_i$ *and* $\lambda_{i+1} > \mu_{i+1}$ for some i.

For example, $(1, 1, 1, 1, 3, 5) \geq (1, 1, 1, 1, 2, 5)$.

Proposition 2.3.3. $\geq$ is a total order on $\mathbb{N}^\infty$.

Proof. First we will see that $\geq$ is an order. Clearly, $\lambda \geq \lambda$ for all partition $\lambda \in \mathbb{N}^\infty$. Let λ, μ be two partitions such that $\lambda \geq \mu$ *and* $\mu \geq \lambda$ and let i, j be the indexes such that $\lambda_1 = \mu_1, \ldots, \lambda_i = \mu_i, \lambda_{i+1} > \mu_{i+1}$ *and* $\mu_1 = \lambda_1, \ldots, \lambda_j = \mu_j, \mu_{j+1} > \lambda_{j+1}$. Clearly we must have $i = j = q$. Finally, let λ, μ and σ be partitions such that $\lambda \geq \mu$ and $\mu \geq \sigma$. As before, we say $\lambda_i > \mu_i$ and $\mu_j > \sigma_j$. Then, $\lambda_1 = \mu_1, \ldots, \lambda_{i-1} = \mu_{i-1}$, $\lambda_i > \mu_i$, and $\mu_1 = \sigma_1, \ldots, \mu_{j-1} = \sigma_{j-1}$, $\mu_j > \sigma_j$. It is enough to take $k = min\{i, j\}$.

And second, we will show that $\geq$ is a total order. Given λ, μ two partition we say $i = max\{i \mid \lambda_1 = \mu_1, \ldots, \lambda_i = \mu_i\}$. We will see that $\lambda \leq \mu$ or $\mu \leq \lambda$. The case $i = q$ is clearly, so we suppose $i < q$. Since we have $\lambda_{i+1} \geq \mu_{i+1}$ or $\mu_{i+1} \geq \lambda_{i+1}$, the result follows. □

Definition 2.3.4. Let E, F be two free modules with basis $\{x_1, \ldots, x_m\}$ and $\{x_{m+1}, \ldots, x_{m+n}\}$ respectively and let $T \in Tab_{\lambda/\mu}(\{x_1, \ldots, x_{m+n}\})$. For each $i \in \{1, \ldots, q\}$ let η_i to be μ_i plus the number of basis elements of F in the ith row of T. We define the sequence $\eta(T) = (\eta_1, \ldots, \eta_q) \in \mathbb{N}^\infty$.

Observe that when $T \in Tab_{\lambda/\mu}(S)$ is standard, then $\eta(T)$ is a partition and $\mu \subseteq \eta(T) \subseteq \lambda$. Indeed, let T^i, T^{i+1} be two rows of T. We say $\eta_i(T) = \mu_i + k_i$ where k_i is the number of basis elements of F in the ith row. If $T(i, l)$ is a basis element of E, then $T(i+1, l)$ must be a basis element of E too since T is column-standard and thus $T(i, l) \leq T(i+1, l)$ (remember the ordered basis $\{x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}\}$). Necessarily, $k_i \geq k_{i+1}$ and then $\mu_i + k_i \geq \mu_{i+1} + k_{i+1}$.

Lemma 2.3.5. Let $S, T \in Tab_{\lambda/\mu}\{x_1, \ldots, x_{n+m}\}$ with $S \leq T$. Then $\eta(S) \geq \eta(T)$.

Proof. We assume $\eta(S) \neq \eta(T)$, otherwise there is nothing to prove. Let k the first integer such that $\eta_k(S) \neq \eta_k(T)$, we will see $\eta_k(S) > \eta_k(T)$. Remember that $S \leq T$ if $S_{p,q} \geq T_{p,q}$ for all p, q, where $S_{p,q}$ and $T_{p,q}$ are the number of times the first q elements of the basis appear in the first p rows of S and T respectively. From the definition of $\eta(S)$ and $\eta(T)$ it is clear that $S_{p,m} = \sum_{j=1}^{p} \eta_j(S) - \mu_j \;\; and \;\; T_{p,m} = \sum_{j=1}^{p} \eta_j(T) - \mu_j$, and hence $\eta_i(S) = \eta_i(T), \forall i < k$ implies $S_{p,m} = T_{p,m}$, $\forall p < k$.

Writing $S_{k,m} = \sum_{j=1}^{k-1} \eta_j(S) - \mu_j + \eta_k(S) - \mu_k$ and $T_{k,m} = \sum_{j=1}^{k-1} \eta_j(T) - \mu_j + \eta_k(T) - \mu_k$, since we have assumed that $\eta_i(S) = \eta_i(T)$, $i = 1 \ldots, k-1$, then $S_{k,m} > T_{k,m} \Rightarrow \eta_k(S) > \eta_k(T)$. □

Proposition 2.3.6. Let $T \in Tab_{\lambda/\mu}(F \oplus G)$. Then there exist unique standard tableaux T_i, and unique integers $c_i \neq 0$, such that $d_{\lambda/\mu}(X_T) = \sum c_i d_{\lambda/\mu}(X_{T_i})$ and $\eta(T_i) \geq \eta(t)$.
Proof. Directly from Lemma 2.3.5 and the standard basis theorem Theorem 2.1.21. □

Definition 2.3.7. Let μ, γ and λ be partitions such that $\mu \subseteq \gamma$ and $\gamma \subseteq \lambda$. We define submodules

$$M_\gamma(\Lambda_{\lambda/\mu}(F \oplus G)) := Im(\varphi : \bigoplus_{\mu \subseteq \sigma \subseteq \lambda, \sigma \geq \gamma} \Lambda_{\sigma/\mu}(F) \otimes \Lambda_{\lambda/\sigma}(G) \to \Lambda_{\lambda/\mu}(F \oplus G)),$$

$$\dot{M}_\gamma(\Lambda_{\lambda/\mu}(F \oplus G)) := Im(\varphi' : \bigoplus_{\mu \subseteq \sigma \subseteq \lambda, \sigma > \gamma} \Lambda_{\sigma/\mu}(F) \otimes \Lambda_{\lambda/\sigma}(G) \to \Lambda_{\lambda/\mu}(F \oplus G)).$$

where φ and φ' are the sum of maps obtained by tensoring the maps $\Lambda_{\sigma_i - \mu_i} F \otimes \Lambda_{\lambda_i - \sigma_i} G \to \Lambda_{\lambda_i - \mu_i}(F \oplus G)$ defined by $x_1 \wedge \cdots \wedge x_{\sigma_i - \mu_i} \otimes y_1 \wedge \cdots \wedge y_{\lambda_i - \sigma_i} \to x_1 \wedge \cdots \wedge x_{\sigma_i - \mu_i} \wedge y_1 \wedge \cdots \wedge y_{\lambda_i - \sigma_i}$. We define submodules:

$$M_\gamma(L_{\lambda/\mu}(F \oplus G)) := d_{\lambda/\mu}(M_\gamma(\Lambda_{\lambda/\mu}(F \oplus G))),$$

$$\dot{M}_\gamma(L_{\lambda/\mu}(F \oplus G)) := d_{\lambda/\mu}(\dot{M}_\gamma(\Lambda_{\lambda/\mu}(F \oplus G))).$$

The modules $M_\gamma(L_{\lambda/\mu}(F \oplus G))$ are the submodules of the filtration and the quotients $M_\gamma(L_{\lambda/\mu}(F \oplus G))/\dot{M}_\gamma(L_{\lambda/\mu}(F \oplus G))$ define the graded object. Clearly $\{X_{T'} \otimes X_{T''} \mid T' \in Tab_{\sigma/\mu}\{x_1, \ldots, x_n\}, T'' \in Tab_{\lambda/\sigma}\{x_{n+1}, \ldots, x_{n+m}\}$ are row- standard, $\sigma \geq \gamma\}$ and $\{X_{T'} \otimes X_{T''} \mid T' \in Tab_{\sigma/\mu}\{x_1, \ldots, x_n\}, T'' \in Tab_{\lambda/\sigma}\{x_{n+1}, \ldots, x_{n+m}\}$ are row-standard, $\sigma > \gamma\}$ represent basis of the modules $\bigoplus_{\mu \subseteq \sigma \geq \lambda, \sigma \geq \gamma} \Lambda_{\sigma/\mu}(F) \otimes \Lambda_{\lambda/\sigma}(G)$ and $\bigoplus_{\mu \subseteq \sigma \geq \lambda, \sigma > \gamma} \Lambda_{\sigma/\mu}(F) \otimes \Lambda_{\lambda/\sigma}(G)$ respectively.

Note that, $\varphi(X_{T'} \otimes X_{T''}) = X_{T''}$ where $T'' \in Tab_{\lambda/\mu}(S'')$ such that $T''(i,j) = T(i,j) \;\; if \;\; j \leq \sigma_i \;\; and \;\; T''(i, j + \sigma_i) = T'(i,j) \;\; if \;\; j \leq \lambda_i$ which clearly is a row-standard tableau with $\eta(T'') = \sigma$ and then $\eta(T'') \geq \gamma$ or $\eta(T'') > \gamma$ depending on the case. Now, if $T \in Tab_{\lambda/\mu}(\{x_1, \ldots, x_{n+m}\})$ is a standard tableau such that $\eta(T) \geq \gamma$ and hence $\mu \subseteq \eta(T'') \subseteq \lambda$, $T'(i,j) = T(i,j) \;\; if \;\; j \leq \eta(T)_i$ defines a standard tableau of shape σ/μ and $T''(i,j) = T(i, j + \eta(T'')_i) \;\; if \;\; j \leq \lambda_i$ defines a standard tableau of shape λ/σ. Obviously, $X_{T''}$ is the image of $X_T \otimes X_{T'}$ by φ. It follows that $\{d_{\lambda/\mu}(X_T) \mid T \in Tab_{\lambda/\mu}$ is standard, $\eta(T) \geq \gamma\}$ form an R-basis of $M_\gamma(L_{\lambda/\mu}(F \oplus G))$ and $\{d_{\lambda/\mu}(X_T) \mid T \in Tab_{\lambda/\mu}$ is standard, $\eta(T) > \gamma\}$ form an R-basis of $M_\gamma(L_{\lambda/\mu}(F \oplus G))$.

Proposition 2.3.8. The map $\Lambda_{\gamma/\mu}(F) \otimes \Lambda_{\lambda/\gamma}(G) \xrightarrow{\psi} M_\gamma(\Lambda_{\lambda/\mu}(F \oplus G))$ induces a map

$$\tilde{\psi}_\gamma : L_{\gamma/\mu}(F) \otimes L_{\lambda/\gamma}(G) \to M_\gamma(L_{\lambda/\mu}(F \oplus G))/\dot{M}_\gamma(L_{\lambda/\mu}(F \oplus G)).$$

Proof. The map $\tilde{\psi}_\gamma$ is defined by sending $d_{\gamma/\mu}(x) \otimes d_{\lambda/\gamma}(y)$ to the class of $d_{\lambda/\mu}(\psi(x \otimes y))$. We only need to verify that $\tilde{\psi}_\gamma$ sends 0 to $\bar{0}$, since $ker(d_{\lambda/\mu}) = Im(w_{\lambda/\mu})$, it is sufficient to prove that $\psi(Im(w_{\gamma/\mu}) \otimes \Lambda_{\lambda/\gamma}G)$ and $\psi(\Lambda_{\gamma/\mu}(F) \otimes Im(w_{\lambda/\gamma}))$ are both contained in $N := Im(w_{\lambda/\mu}) + \dot{M}_\gamma(L_{\lambda/\mu}(F \oplus G))$.

Remember $w_{\gamma/\mu} = \sum_{i=1}^{q-1} Id_1 \otimes \ldots \otimes Id_{i-1} \otimes w_{\gamma^i/\mu^i} \otimes Id_{i+2} \otimes \ldots \otimes Id_q$ where $w_{\gamma^i/\mu^i} = \sum_{t=0}^{\gamma_{i+1}-\mu_i-1} \Lambda^{\gamma_i-\mu_i+\gamma_{i+1}-\mu_{i+1}-t}F \otimes \Lambda^t F \to \Lambda^{\gamma_i-\mu_i}F \otimes \Lambda^{\gamma_{i+1}-\mu_{i+1}}F$. By the change $l = \gamma_i - \mu_{i+1} - t$, we can write the last map as $w_{\gamma^i/\mu^i} = \sum_{l=\mu_i-\mu_{i+1}+1}^{\gamma^{i+1}-\mu_{i+1}} \Lambda^{\gamma_i-\mu_i+l}F \otimes \Lambda^{\gamma_{i+1}-\mu_{i+1}-l}F \to \Lambda^{\gamma_i-\mu_i}F \otimes \Lambda^{\gamma_{i+1}-\mu_{i+1}}F$. And thus, $w_{\gamma/\mu} = \sum_{i=1}^{q-1} \sum_{l=\mu_i-\mu_{i+1}+1}^{\gamma_{i+1}-\mu_{i+1}} \Lambda^{\gamma_1-\mu_1}F \otimes \cdots \otimes \Lambda^{\gamma_i-\mu_i+l}F \otimes \Lambda^{\gamma_{i+1}-\mu_{i+1}-l}F \otimes \cdots \otimes \Lambda^{\gamma_q-\mu_q}F \to \Lambda_{\gamma/\mu}F$. Fixing i and l, if we see that $\psi(Im(\Lambda^{\gamma_1-\mu_1}F \otimes \cdots \otimes \Lambda^{\gamma_i-\mu_i+l}F \otimes \Lambda^{\gamma_{i+1}-\mu_{i+1}-l}F \otimes \cdots \otimes \Lambda^{\gamma_q-\mu_q}F \otimes \Lambda_{\lambda/\gamma}G) \subseteq N$, clearly we will have show that $\psi(Im(w_{\gamma/\mu}) \otimes \Lambda_{\lambda/\gamma}G) \subseteq N$.

Let $X = x_{I_1} \otimes \cdots \otimes x_{I_{i-1}} \otimes x_{I_i} \otimes x_{I_{i+1}} \otimes x_{I_{i+2}} \otimes \cdots \otimes X_{I_q}$ and $Y = y_{J_1} \otimes \cdots \otimes y_{J_q}$ be basis elements of $\Lambda^{\gamma_1-\mu_1}F \otimes \cdots \otimes \Lambda^{\gamma_i-\mu_i+l}F \otimes \Lambda^{\gamma_{i+1}-\mu_{i+1}-l}F \otimes \cdots \otimes \Lambda^{\gamma_q-\mu_q}F$ and $\Lambda_{\lambda/\gamma}G$ respectively. Using the coassociativity and commutativity of the comultiplication map, we can write $w_l(x_{I_i} \otimes x_{I_{i+1}}) = \sum_U \pm x_U \otimes x_{I_{i+1}} \wedge x_{U'}$, where U runs over all subsets of order $\gamma_i - \mu_i$ of I_i and U' is the complement of U in I_i. Keeping this in mind,

$\psi(X \otimes Y) = \sum_U \pm x_{I_1} \wedge y_{I_1} \otimes \cdots \otimes x_U \wedge y_{J_i} \otimes x_{I_{i+1}} \wedge x_{U'} \wedge y_{J_{i+1}} \otimes \cdots \otimes x_{I_q} \wedge y_{I_q} =: \sum_U \pm Z_{T_U}$
where each $T_U \in Tab_{\lambda/\mu}\{x_1, \ldots, x_{n+m}\}$ and clearly $\eta(T_U) = \gamma$.

Applying the same decomposition to $\omega_{\lambda/\mu}$ and fixing the same i and l, if we consider the basis element $W := x_{I_1} \wedge y_{I_1} \otimes \cdots \otimes x_{I_q} \wedge y_{I_q}$ in $\Lambda^{\lambda_1-\mu_1}(F \oplus G) \otimes \cdots \otimes \Lambda^{\lambda_i-\mu_i+l}(F \oplus G) \otimes \Lambda^{\lambda_{i+1}-\mu_{i+1}-l}(F \oplus G) \otimes \cdots \otimes \Lambda^{\lambda_q-\mu_q}(F \oplus G)$ we obtain that the corresponding factor of $w_{\lambda/\mu}$ send that basis element to $\sum_{w_1,w_2} \pm x_{I_1} \wedge y_{I_1} \otimes \cdots \otimes x_{w_1} \wedge y_{w_2} \otimes x_{w_1'} \wedge x_{I_{i+1}} \wedge y_{w_2'} \wedge y_{J_{i+1}} \otimes \cdots \otimes x_{I_q} \wedge y_{I_q}$ where w_1, w_2 are subsets of I_i and J_i whose orders add up to $\lambda_i - \mu_i$ and w_1', w_2' are the complements of w_1, w_2 in I_i, I_{i+1} respectively.

Since J_i is of order $\lambda_i - \gamma_i$, then w_1 must be of order $\geq \lambda_i - \mu_i - \lambda_i + \gamma_i = \gamma_i - \mu_i$, then we can write the image above as:

$$\sum_U \pm X_{T_U} + \sum_{w_1,w_2} \pm x_{I_1} \wedge y_{I_1} \otimes \cdots \otimes x_{w_1} \wedge y_{w_2} \otimes x_{w_1'} \wedge x_{I_{i+1}} \wedge y_{w_2'} \wedge y_{J_{i+1}} \otimes \cdots \otimes x_{I_q} \wedge y_{I_q}$$

where now $w_1 + w_2 > \lambda_i - \mu_i$. Each summand of the second addend corresponds to an element $W_{T_{w_1,w_2}}$ where T_{w_1,w_2} is a tableau satisfying $\eta(T_{w_1,w_2}) > \gamma$, since $w_1 > \gamma_i - \mu_i$. Then, the second summand is contained in $\dot{M}(\Lambda_{\lambda/\mu}(F \oplus G))$ and $\psi(X \otimes Y) = \omega_{\lambda/\mu}(W) - \sum_{w_1,w_2} W_{T_{w_1,w_2}}$ which clearly is an element of N.

The proof that $\psi(\Lambda_{\gamma/\mu}F \otimes Im(w_{\lambda/\gamma})) \subseteq N$ proceeds formally in the same way that $\psi(Im(w_{\gamma/\mu}) \otimes \Lambda_{\lambda/\gamma}G) \subseteq N$. However we need to define appropriate maps instead of $\omega_{\lambda/\gamma}$ and $\omega_{\lambda/\mu}$. Observe that considering the maps $\omega_{\lambda^i/\gamma^i} = \sum_{l=\gamma_i-\gamma_{i+1}+1}^{\lambda_{i+1}-\gamma_{i+1}} \Lambda^{\lambda_i-\gamma_i+l}F \otimes \Lambda^{\lambda_{i+1}-\gamma_{i+1}-l}F \to \Lambda^{\lambda_i-\gamma_i}F \otimes \Lambda^{\lambda_{i+1}-\gamma_{i+1}}F$ and $w_{\lambda^i/\mu^i} = \sum_{l=\mu_i-\mu_{i+1}+1}^{\lambda_{i+1}-\mu_{i+1}} \Lambda^{\lambda_i-\mu_i+l}(F \oplus G) \otimes$

$\Lambda^{\lambda_{i+1}-\mu_{i+1}-l}(F\oplus G)\to\Lambda^{\lambda_i-\mu_i}(F\oplus G)\otimes\Lambda^{\lambda_{i+1}-\mu_{i+1}}(F\oplus G)$, there can be some $l>\gamma_i-\gamma_{i+1}$ which not occurs at w_{λ^i/μ^i}.

Let $\lambda_i-\lambda_{i+1}+1\leq t\leq\lambda_i-\gamma_i$ and $u:=\lambda_i-\gamma_i-t=:l\leq\lambda_{i+1}-\gamma_i-1$. We define the $\tilde{\omega}_{\lambda/\gamma}=\sum_t\tilde{w}_{u,t}$ where $\tilde{w}_{u,t}:\Lambda^{\lambda_i-\gamma_i-t}G\otimes\Lambda^{\lambda_{i+1}-\gamma_{i+1}+t}G\to\Lambda^{\lambda_i-\gamma_i}G\otimes\Lambda^{\lambda_{i+1}-\gamma_{i+1}}G$ whose image is contained in the image of the map w_{λ^i/γ^i}, for all t. In the same way we define $\tilde{\omega}_{\lambda/\mu}$ considering $\mu_i-\mu_{i+1}+1\leq t\leq\lambda_i-\mu_i$, $u=\lambda_i-\mu_i-t=t\leq\lambda_{i+1}-\mu_i-1$. The remainder of the proof is exactly the same as in the first part but replacing $\omega_{\lambda/\gamma}$ and $\omega_{\lambda/\mu}$ by $\tilde{\omega}_{\lambda/\gamma}$ and $\tilde{\omega}_{\lambda/\mu}$ respectively. □

Theorem 2.3.9. The map $\tilde{\psi}_\gamma$ is an isomorphism.

Proof. $\tilde{\psi}_\gamma$ is clearly surjective since $\{d_{\lambda/\mu}(X_T)\mid T \text{ is standard}, \eta(T)=\gamma\}$ represent an R-basis of $M_\gamma(L_{\lambda/\mu}(F\oplus G))/\dot{M}_\gamma(L_{\lambda/\mu}(F\oplus G))$. Indeed, if T is a standard basis of $Tab_{\lambda/\mu}\{x_1,\ldots,x_{n+m}\}$ with $\eta(T)=\gamma$, then $T'(i,j)=T(i,j)\ \ if\ \ j\in\{\mu_i+1,\ldots,\gamma_i\}$ and $T''(i,j)=T(i,j)\ \ if\ \ j\in\{\gamma_i+1,\ldots,\lambda_i\}$ defines standard tableaux of $Tab_{\gamma/\mu}\{x_1,\ldots,x_n\}$ and $Tab_{\lambda/\gamma}\{x_{n+1},\ldots,x_{n+m}\}$ respectively. Then, $\tilde{\psi}_\gamma(d_{\gamma/\mu}(X_{T'})\otimes d_{\lambda/\gamma}(X_{T''}))=d_{\lambda/\mu}(\psi(X_{T'}\otimes X_{T''}))=d_{\lambda/\mu}(X_T)$.

Moreover, $\tilde{\psi}$ carries a basis element of $L_{\gamma/\mu}F\otimes L_{\lambda/\mu}G$ to a basis element of $M_\gamma(L_{\lambda/\mu}(F\oplus G))/\dot{M}_\gamma(L_{\lambda/\mu}(F\oplus G))$. If $T'\in Tab_{\gamma/\mu}\{x_1,\ldots,x_n\}$ and $T''\in Tab_{\lambda/\gamma}\{x_{n+1},\ldots,x_{n+m}\}$ are standard tableaux, $\psi(X_{T'}\otimes X_{T''})=X_T$ where T a standard tableau with $\eta(T)=\gamma$, as we have seen before. □

Corollary 2.3.10. The submodules $\{M_\gamma(L_{\lambda/\mu}(F\oplus G))\mid\mu\subseteq\gamma\subseteq\lambda\}$ give a filtration of $L_{\lambda/\mu}(F\oplus G)$, whose associated graded module is isomorphic to $\bigoplus_{\mu\subseteq\gamma\subseteq\lambda}L_{\gamma/\mu}F\otimes L_{\lambda/\gamma}G$.

Proof. Clearly, $\{M_\gamma(L_{\lambda/\mu}(F\oplus G))\mid\mu\subseteq\gamma\subseteq\lambda\}$ is a filtration of $L_{\lambda/\mu}(F\oplus G)$ with the total order on partitions which we defined at the beginning of this section . If $M_\gamma(L_{\lambda/\mu}(F\oplus G))\subset M_{\gamma'}(L_{\lambda/\mu}(F\oplus G))$ is a piece of this filtration, then $\gamma=max\{\sigma\in N\mid\gamma<\gamma'\}$, and hence $M_\gamma(L_{\lambda/\mu}(F\oplus G))$ equals to $\dot{M}_\gamma(L_{\lambda/\mu}(F\oplus G))$. □

Let us see a couple of examples.

Examples 2.3.11. (1) We first start computing the decomposition of $L_{(4,2)/(2,1)}(F\oplus G)$.

From the above discussion we only need to determinate all the partitions γ such that $\mu\subseteq\gamma\subseteq\lambda$. These partitions are $(2,1),(2,2),(3,1),(3,2),(4,1)$ and $(4,2)$. Then, the graded object of the filtration is $L_{(4,2)/(2,1)}G\oplus L_{(2,2)/(2,1)}F\otimes L_{(4,2)/(2,2)}G, L_{(3,1)/(2,1)}F\otimes L_{(4,2)/(3,1)}G, L_{(3,2)/(2,1)}F\otimes L_{(4,2)/(3,2)}G, L_{(4,1)/(2,1)}F\otimes L_{(4,2)/(4,1)}G$ and $L_{(4,2)/(2,1)}F$.

(2) The decomposition of $L_{(1,1,1)}(F\oplus G)$ corresponds to $S^3G, F\otimes L_{(1,1,1)/(1,0,0)}G, S^2(F)\otimes L_{(1,1,1)/(1,1,0)}G$ and S^3F.

In fact, when R contains a field of characteristic zero we have an isomorphism

$$L_{\lambda/\mu}(F\oplus G)\cong\bigoplus_{\mu\subseteq\gamma\subseteq\lambda}L_{\gamma/\mu}F\otimes L_{\lambda/\gamma}F$$

(See [2], Proposition (2.3.1).)

2.4 Cauchy Decomposition Formulas for Schur Functors.

In this section we explain the relation between Schur and CoSchur functors and the symmetric and exterior algebra. More precisely, we will construct filtrations of $S^k(F \otimes G)$ and $\Lambda^k(F \otimes G)$ with associated graded objects $\bigoplus_{|\lambda|=k} L_\lambda F \otimes L_\lambda G$ and $\bigoplus_{|\lambda|=k} L_\lambda F \otimes K_\lambda G$ respectively. Moreover, if R is a ring of characteristic zero, these filtrations are direct sum decompositions of $S^k(F \otimes G)$ and $\Lambda^k(F \otimes G)$. The Cauchy formula for the exterior algebra will be essential in Chapter 4 to understand the construction of general resolutions of determinantal varieties.

2.4.1 The decomposition of the Symmetric Algebra.

Definition 2.4.1. Let p be a positive integer and let $\lambda = (\lambda_1, \dots, \lambda_t)$ be a partition of weight k. We define a natural paring $\langle , \rangle_p : \Lambda^p F \otimes \Lambda^p G \to S^p(F \otimes G)$ by sending $f_1 \wedge \cdots \wedge f_p \otimes g_1 \wedge \cdots \wedge g_p$ to the $p \times p$ determinant $(-1)^{p(p-1)/2} \sum (-1)^{sg(\sigma)} (f_{\sigma(1)} \otimes g_1) \cdot \cdots \cdot (f_{\sigma(p)} \otimes g_p) =: \langle f_1 \wedge \cdots \wedge f_p, g_1 \wedge \cdots \wedge g_p \rangle$. Formally,

$$\langle f_1 \wedge \cdots \wedge f_p, g_1 \wedge \cdots \wedge g_p \rangle_p = \begin{vmatrix} f_1 \otimes g_1 & \cdots & f_1 \otimes g_p \\ \vdots & & \vdots \\ f_p \otimes g_1 & \cdots & f_p \otimes g_p \end{vmatrix}.$$

We extend the above to a pairing $\langle , \rangle : \Lambda_\lambda F \otimes \Lambda_\lambda G \to S^k(F \otimes G)$ by $\langle f_1 \wedge \cdots \wedge f_t, g_1 \wedge \cdots \wedge g_t \rangle = \langle f_1, g_1 \rangle_{\lambda_1} \cdot \cdots \cdot \langle f_t, g_t \rangle$, where $f_i \in \Lambda^{\lambda_i} F$ and $g_i \in \Lambda^{\lambda_i} G$ for all $i \in \{1, \dots, t\}$.

For example, $\langle , \rangle_2 : \Lambda^2 F \otimes \Lambda^2 G \to S^2(F \otimes G)$ is given by $\langle f_1 \wedge f_2, g_1 \wedge g_2 \rangle_2 = (f_1 \otimes g_1)(f_2 \otimes g_2) - (f_1 \otimes g_2)(f_2 \otimes g_1)$, $f_1, f_2 \in F$, $g_1, g_2 \in G$.

Definition 2.4.2. Let $\lambda = (\lambda_1, \dots, \lambda_t)$ be a partition of weight k. We define submodules of $S_k(F \otimes G)$,

$$M_\lambda(S^k(F \otimes G)) := \sum_{\gamma \geq \lambda, |\gamma|=k} \langle \Lambda_\gamma F, \Lambda_\gamma G \rangle,$$

$$\dot{M}_\lambda(S_k(F \otimes G)) := \sum_{\gamma > \lambda, |\gamma|=k} \langle \Lambda_\gamma F, \Lambda_\gamma G \rangle.$$

where $\langle \Lambda_\gamma F, \Lambda_\gamma F \rangle$ denotes the image of the pairing $\langle , \rangle : \Lambda_\gamma F \otimes \Lambda_\gamma G \to S^k(F \otimes G)$.

Note that $\langle F \otimes \cdots \otimes F, G \otimes \cdots \otimes G \rangle = M_{(1,\dots,1)}(S^k(F \otimes G)) = S^k(F \otimes G)$, $1 + \cdots + 1 = k$. We have,

Definition 2.4.3. $\{M_\lambda(S^k(F \otimes G) \mid |\lambda| = k\}$ define a natural filtration

$$0 \subseteq M_{(k)}(S^k(F \otimes G)) \subseteq M_{(k-1,1)}(S^k(F \otimes G)) \subseteq \cdots \subseteq M_{(1,\dots,1)}(S^k(F \otimes G))$$

induced by the lexicographic order $\geq$ on partitions of weight k.

Directly from the definition of both submodules, $\{\langle X_S, Y_T \rangle \mid S \in Tab_\gamma\{x_1, \dots, x_n\},\ T \in Tab_\gamma\{x_{n+1}, \dots, x_{n+m}\}, \gamma \geq \lambda,\ |\gamma| = k\}$ form a system of generators of $M_\lambda(S^k(F \otimes G))$, and $\{\langle X_S, Y_T \rangle \mid S \in Tab_\gamma\{x_1, \dots, x_n\},\ T \in Tab_\gamma\{x_{n+1}, \dots, x_{n+m}\}, \gamma > \lambda,\ |\gamma| = k\}$ spans $\dot{M}_\lambda(S^k(F \otimes G))$.

For convenience we denote $M_\lambda(S^k(F\otimes G))$ by M_λ and $\dot{M}_\lambda(S^k(F\otimes G))$ by $\dot{M}_\lambda$. Our goal is to show that $M_\lambda/\dot{M}_\lambda$ is isomorphic to $L_\lambda F\otimes L_\lambda G$. So first we need to define a morphism β_γ from $L_\lambda F\otimes L_\lambda G$ to $M_\lambda/\dot{M}_\lambda$. In [1] Corollary $[III.1.2]$, $[I.5]$ and Proposition $[III.1.1]$, using that the paring $\langle,\rangle$ is in fact a restriction of a natural map between extended R-Hopf algebras, one sees that $\langle\omega_\lambda(\Lambda_\lambda F),\Lambda_\lambda F\rangle+\langle(\Lambda_\lambda F),\omega_\lambda(\Lambda_\lambda F)\rangle\subseteq\dot{M}_\lambda$. We have,

Proposition 2.4.4. The natural map $\langle,\rangle:\Lambda_\lambda F\otimes\Lambda_\lambda F\to M_\lambda$ induces a surjective map $\beta_\lambda:L_\lambda F\otimes L_\lambda G\to M_\lambda/\dot{M}_\lambda$ given by $\beta_\lambda(d_\lambda(X)\otimes d_\lambda(Y))=\overline{\langle X,Y\rangle}$, where $\overline{\langle X,Y\rangle}$ denotes the class of $\langle X,Y\rangle$ in the quotient $M_\lambda/\dot{M}_\lambda$.

Proof. Since $L_\lambda F\cong\Lambda_\lambda F/Im(\omega_\lambda)$, see Definition 2.1.16 and Theorem 2.1.24, it follows from $\langle\omega_\lambda(\Lambda_\lambda F),\Lambda_\lambda F\rangle+\langle(\Lambda_\lambda F),\omega_\lambda(\Lambda_\lambda F)\rangle\subseteq\dot{M}_\lambda$ that β_λ is well defined. Since $M_\lambda/\dot{M}_\lambda$ is generated by $\{\overline{\langle X_S,Y_T\rangle}\mid S\in Tab_\lambda\{x_1,\ldots,x_n\},T\in Tab_\lambda\{x_{n+1},\ldots,x_{n+m}\}\}$, it is enough to see that for each $\overline{\langle X_S,Y_T\rangle}$ of that such there exists $X\in L_\lambda F$ and $Y\in L_\lambda G$ such that $\beta_\gamma(d_\lambda(X)\otimes d_\lambda(Y))=\overline{\langle X_S,Y_T\rangle}$. Obviously, X_S and Y_T are the elements we are looking for. □

Remark 2.4.5. As a consequence of the above proposition we obtain $\{\overline{\langle X_S,Y_T\rangle}\mid S\in Tab_\lambda\{x_1,\ldots,x_n\},T\in Tab_\lambda\{x_{n+1},\ldots,x_{n+m}\}$ are standard$\}$ generates $M_\lambda/\dot{M}_\lambda$.

Corollary 2.4.6. $\{\langle X_S,Y_T\rangle\mid S\in Tab_\gamma\{x_1,\ldots,x_n\},T\in Tab_\gamma\{x_{n+1},\ldots,x_{n+m}\}$ are standard, $\gamma\geq\lambda,|\gamma|=k\}:=B_\lambda$ generate M_λ.

Proof. Since $\{M_\lambda,|\lambda|=k\}$ is ordered lexicographically, we proceed by induction on λ. We can see easily that the first piece of the decomposition is $\Lambda^kF\otimes\Lambda^kG=L_{(k)}F\otimes L_{(k)}G$. Indeed, $M_{(k)}$ is isomorphic to $\Lambda^kF\otimes\Lambda^kG$, since the pairing $\langle,\rangle:\Lambda^kF\otimes\Lambda^kG\to S^k(F\otimes G)$ is the natural embedding of $\Lambda^kF\otimes\Lambda^kG$ in $S^k(F\otimes G)$. So, the initial case is clear. Consider $\lambda>(k)$, we want to prove that B_λ generate M_λ. Observe that the corresponding piece of the filtration is $\dot{M}_\lambda\subseteq M_\lambda$. From Proposition 2.4.4, each element of M_λ can be written as a sum of an element of $\dot{M}_\lambda$ and an element generated by $\{\langle X_S,Y_T\rangle\mid S\in Tab_\lambda\{x_1,\ldots,x_n\},T\in Tab_\lambda\{x_{n+1},\ldots,x_{n+m}\}$ are standard$\}$. The result follows by induction on $\dot{M}_\lambda$. □

Theorem 2.4.7. The maps $\beta_\lambda:L_\lambda F\otimes L_\lambda G\to M_\lambda/\dot{M}_\lambda$ are isomorphisms and therefore the associated graded object of the filtration $\{M_\lambda(S_k(F\otimes G)\mid|\lambda|=k\}$ is $\bigoplus_{|\lambda|=k}L_\lambda F\otimes L_\lambda G$.

Proof. Let $B_k:=\{\langle X_S,Y_T\rangle\mid S\in Tab_\lambda\{x_1,\ldots,x_n\}$ is standard, $T\in Tab_\lambda\{x_{n+1},\ldots,x_{n+m}\}$ is standard, $|\lambda|=k\}$. By Corollary 2.4.6, B_k generates $S^k(F\otimes G)$. Then directly from the equality ranks

$$rank(S^k(F\otimes G))=\sum_{|\lambda|=k}rank(L_\lambda F\otimes L_\lambda G)$$

(see [1] Theorem $[III.1.4]$), B_k must be an R-basis of $S^k(F\otimes G)$. This, in turn, implies that B_λ is an R-basis of M_λ and $\{\langle X_S,Y_T\rangle\mid S\in Tab_\gamma\{x_1,\ldots,x_n\},T\in Tab_\gamma\{x_{n+1},\ldots,x_{n+m}\}$ are standard, $\gamma>\lambda,|\gamma|=k\}:=\dot{B}_\lambda$ is an R-basis of $\dot{M}_\lambda$. Consequently, $\{\overline{\langle X_S,Y_T\rangle}\mid S\in Tab_\lambda\{x_1,\ldots,x_n\},T\in Tab_\lambda\{x_{n+1},\ldots,x_{n+m}\}$ are standard$\}$ is an R-basis of $M_\lambda/\dot{M}_\lambda$. It follows that β_λ send an element basis of $L_\lambda F\otimes L_\lambda G$ to an element basis of $M_\lambda/\dot{M}_\lambda$. □

Examples 2.4.8. (1) The decomposition of $S^2(F\otimes G)$ corresponds to $\Lambda^2F\otimes\Lambda^2F$ and $S^2F\otimes S^2G$.

(2) The three partitions of weight 3 are $(3), (2,1)$ and $(1,1,1)$, then the respective terms of the decomposition of $S^3(F \otimes G)$ are $\Lambda^3 F \otimes \Lambda^3 F, L_{(2,1)}F \otimes L_{(2,1)}G$ and $S^3 F \otimes S^3 G$.

The Cauchy formula for $S^k(F \otimes G)$ has an important consequence. When R is a ring of characteristic zero the decomposition becomes an isomorphism, more precisely (see [20] Corollary (2.3.3))

Theorem 2.4.9. If R is a ring of characteristic zero, then $S^k(F \otimes G) \cong \bigoplus_{|\lambda|=k} L_\lambda F \otimes L_\lambda G$.

2.4.2 The decomposition of the exterior algebra.

Definition 2.4.10. Let p be a positive integer and let $\lambda = (\lambda_1, \ldots, \lambda_t)$ be a partition of weight k. We define a natural pairing $\langle , \rangle_p : \Lambda^p F \otimes D^p G \to \Lambda^p(F \otimes G)$ by induction on $p \geq 1$. For $p = 1$ we define $\langle f, g \rangle_p = f \otimes g$. For $p > 1$ we define $\langle f_1 \wedge \cdots \wedge f_p, g_1^{(\alpha_1)} \cdot \cdots \cdot g_t^{(\alpha_t)} \rangle_p$ by

$$\sum_{i=1}^{p} \langle f_1, g_i \rangle \wedge \langle f_1 \wedge \cdots \wedge f_p, g_1^{(\alpha_1)} \cdot \cdots \cdot g_i^{(\alpha_i - 1)} \cdot \cdots \cdot g_t^{(\alpha_t)} \rangle$$

where $\sum_{i=1}^{t} = p$ and $\alpha_i \geq 1$ for all i. We define a pairing $\langle , \rangle : \Lambda_\lambda F \otimes D_\lambda G \to \Lambda^k(F \otimes G)$ by the natural extension $\langle f_1 \otimes \cdots \otimes f_t, g_1 \otimes \cdots \otimes g_t \rangle = \langle f_1, g_1 \rangle_{\lambda_1} \wedge \cdots \wedge \langle f_t, g_t \rangle_{\lambda_t}$, where $f_i \in \Lambda^{\lambda_i} F$ and $g_i \in D^{\lambda_i} G$ for all $i \in \{1, \ldots, t\}$.

Definition 2.4.11. Let $\lambda = (\lambda_1, \ldots, \lambda_t)$ be a partition of weight k. We define submodules of $\Lambda^k(F \otimes G)$

$$M_\lambda(\Lambda^k(F \otimes G)) := \sum_{\gamma \geq \lambda, |\gamma| = k} \langle \Lambda_\gamma F, D_\gamma G \rangle,$$

$$\dot{M}_\lambda(\Lambda^k(F \otimes G)) := \sum_{\gamma > \lambda, |\gamma| = k} \langle \Lambda_\gamma F, D_\gamma G \rangle.$$

Clearly $M_{(1,\ldots,1)} = \Lambda^k(F \otimes G)$, $1 + \cdots + 1 = k$. We have,

Definition 2.4.12. $\{M_\lambda(\Lambda^k(F \otimes G) \mid |\lambda| = k\}$ define a natural filtration

$$0 \subseteq M_{(k)}(\Lambda^k(F \otimes G)) \subseteq M_{(k-1,1)}(\Lambda^k(F \otimes G)) \subseteq \cdots \subseteq M_{(1,\ldots,1)}(\Lambda^k(F \otimes G))$$

induced by the lexicographic order $\geq$ on partitions of weight k.

From the above definitions, $\{\langle X_S, Y_T \rangle \mid S \in Tab_\gamma\{x_1, \ldots, x_n\},\ T \in Tab_\gamma\{x_{n+1}, \ldots, x_{n+m}\},\ \gamma \geq \lambda,\ |\gamma| = k\}$ and $\{\langle X_S, Y_T \rangle \mid S \in Tab_\gamma\{x_1, \ldots, x_n\},\ T \in Tab_\gamma\{x_{n+1}, \ldots, x_{n+m}\}, \gamma > \lambda,\ |\gamma| = k\}$ spans $M_\lambda(\Lambda^k(F \otimes G))$ and $\dot{M}_\lambda(\Lambda^k(F \otimes G))$, respectively. Arguing as in Subsection 2.4.1 we get

Proposition 2.4.13. The natural map $\langle , \rangle : \Lambda_\lambda F \otimes \Lambda_\lambda G \to M_\lambda$ induces a surjective R-map $\beta_\lambda : L_\lambda F \otimes K_\lambda G \to M_\lambda / \dot{M}_\lambda$.

Lemma 2.4.14. $\{\langle X_S, Y_T \rangle \mid S \in Tab_\gamma\{x_1, \ldots, x_n\}$ is standard, $T \in Tab_\gamma\{x_{n+1}, \ldots, x_{n+m}\}$ is standard, $\gamma \geq \lambda, |\gamma| = k\}$ spans M_λ.

Theorem 2.4.15. The maps $\beta_\lambda : L_\lambda F \otimes K_\lambda G \to M_\lambda / \dot{M}_\lambda$ are isomorphisms and therefore the associated graded object of the filtration $\{M_\lambda(\Lambda^k(F \otimes K), |\lambda| = k\}$ is $\bigoplus_{|\lambda|=k} L_\lambda F \otimes K_\lambda G$.

Proof. It follows as Theorem 2.4.7 using the equality rank

$$rank(\Lambda^k(F \otimes G)) = \sum_{|\lambda|=k} rank(L_\lambda F \otimes K_\lambda G)$$

which we can find in [1] Theorem $[III.2.2]$. □

Theorem 2.4.16. If R is a ring of characteristic zero, then $\Lambda^k(F \otimes G) \cong \bigoplus_{|\lambda|=k} L_\lambda F \otimes K_\lambda F$. Moreover, since $K_\lambda G \cong L_{\tilde{\lambda}} G$, $\Lambda^k(F \otimes G) \cong \bigoplus_{|\lambda|=k} L_\lambda F \otimes L_{\tilde{\lambda}} G$.

2.5 The Littlewood-Richardson rule for Schur functors.

The tensor product of Schur functors decomposes into a direct sum of Schur functors, provided that R is a ring of characteristic zero. We finish this exposition about Schur and CoSchur functors presenting a combinatoric algorithm, the Littlewood-Richardson rule, which computes the multiplicities of the factors in the decomposition of $L_\lambda F \otimes L_\mu G$. As a corollary of the Littlewood-Richardson rule we will obtain the Pieri formulas for Schur functors.

An accurate exposition of the following result, based on representation theory, is found in [20] Sections [2.2] and [2.3].

Theorem 2.5.1. Let R be a ring of characteristic zero, let F be a free R-module of rank n and let λ and μ be two partitions. Then,

$$L_\lambda F \otimes L_\mu F \cong \sum_{|\nu|=|\lambda|+|\mu|} u(\lambda, \mu; \nu) L_\nu F$$

where $u(\lambda, \mu; \nu)$ denotes the multiplicity of the factor $L_\nu F$.

The Littlewood-Richardson rule provides an algorithm which computes the multiplicities $u(\lambda, \mu; \nu)$. The statement and proof of Littlewood-Richardson rule require two combinatorial notions we present immediately.

2.5.1 The Schensted Process and Words of Yamanouchi.

The following procedure is known as the Schensted process.

Definition 2.5.2. Let λ be a partition, S a totally ordered set, $U \in Tab_\lambda(S)$ a standard tableau and $p \in S$. We define $p \to U$ to be the tableau obtained in the following recursive manner.

(1). If $p_1 = min\{U(1, j) \mid U(1, j) \geq p\}$ exists, let $p \to U$ be the tableau of shape λ obtained by replacing of p_1 to p. We say this step *bumping of p into U.*

(2). If p_1 does not exist we define $p \to U$ to be the tableau of shape $\lambda_1 = (\lambda_1 + 1, \ldots, \lambda_q)$ obtained from U by adjoining a new box to the first row of U with entry p and finish.

If (1) takes place repeat the process with U_1 and $p = p_1$, where U_1 is the tableau obtained from U by removing the first row, and continue. Finally, $p \to U$ is the tableau obtained from U by adjoining a new box to a non empty row or by adjoining a new bottom row. In the first case $p \to U$ is a tableau of shape $\lambda^1 = (\lambda_1, \dots, \lambda_i + 1, \dots, \lambda_n)$ for some i, and in the second case, $\lambda^1 = (\lambda_1, \dots, \lambda_n, 1)$.

Remark 2.5.3. Note that λ^1 is a partition. When $\lambda^1 = (\lambda_1, \dots, \lambda_n, 1)$ the result is clear. Assume that $\lambda^1 = (\lambda_1, \dots, \lambda_i + 1, \dots, \lambda_n)$, then in the ith step of the above procedure p does not bump any element in the ith row of U. The standardness of U implies $U(i-1, j) \leq U(i,j)$, for every (i,j) of its diagram. If $p = U(i-1,j)$ and $j \leq \lambda_i$, then $p_{i-1} \leq U(i,j) \Rightarrow \exists min\{U(i,j) \mid U(i,j) \geq p_{i-1}\}$, which is a contradiction. Necessarily, $j > \lambda_i \Rightarrow \lambda_{i-1} > \lambda_i$ and hence, $\lambda_{i-1} \geq \lambda_i + 1$.

Let us to compute $p \to U$ for $p = 2$ and $U = \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 3 & \\ 4 & & \end{array}$.

(1). $p_1 = U(1,2) = 3$, $p \to U = \begin{array}{ccc} 1 & 2 & 4 \\ 2 & 3 & \\ 4 & & \end{array}$, $\lambda^1 = \lambda$.

Let $p = 3$ and $U_1 = \begin{array}{cc} 2 & 3 \\ 4 & \end{array}$.

(2). $p_1 = U(2,2) = 3$, $p \to U$.

Let $p = 3$ and $U_2 = \begin{array}{c} 4 \end{array}$.

(3). $p_1 = U(3,1) = 4$, $p \to U = \begin{array}{ccc} 1 & 2 & 4 \\ 2 & 3 & \\ 3 & & \end{array}$, $\lambda^1 = \lambda$.

Finish, $U \to p = \begin{array}{ccc} 1 & 2 & 4 \\ 2 & 3 & \\ 3 & & \\ 4 & & \end{array}$ and $\lambda^1 = (3, 2, 1, 1)$.

Lemma 2.5.4. The tableau $p \to U$ in $Tab_{\lambda^1}(S)$ is standard.

Proof. Row-standardness is clear, we only have to prove column-standardness. It is enough to consider two adjacent rows, i and $i+1$, of U in which the step *bumping of p into U* took place, otherwise there is nothing to prove since λ^1 is a partition. For convenience we denote by $u_i, i = 1, \dots, \lambda_i$ the entries in the ith row and by $v_i, i = 1, \dots, \lambda_{i+1}$ the entries in the $(i+1)$th row.

(i) Suppose p does bump u_t in the ith row and u_t does bump v_s in the $(i+1)$th row. Since $u_{t-1} \leq p \leq u_t$ and $v_{s-1} \leq u_t \leq v_s$, we must have $s \leq t$ and we only have to check that $p \leq v_t$. Clearly $s \leq t$ and the standardness of U implies $u_t \leq v_s \leq v_t$ which, in turn, implies $p \leq v_t$.

(ii) Suppose p does bump u_t, but u_t does not bump any element in the $i+1$th row of U. In this case we adjoin the box $(i+1, \lambda_i + 1)$ with the entry u_t. Then $p \leq u_t$ and $u_t > v_j$ for all $j = 1, \dots, \lambda_{i+1}$ and we must have $t > \lambda_{i+1}$ since t is standard, and hence the result is clear. □

Definition 2.5.5. Let U^1 be an standard tableau in $Tab_{\lambda^1}(S)$. Let (i,j) be an extremal box of λ^1, that is $j = \lambda^1_i$ *and* $\lambda^1_i > \lambda^1_{i+1}$, and let $r = U^1(i,j)$. We define $U^1(i,\lambda_i) \leftarrow U^1$ to be the tableau in the following recursive manner.
(1) Let $r_1 = \max\{U^1(i-1,k) \mid U^1(i-1,k) \leq r\}$ and replace r_1 by r.
(2) Repeat step (1) with $r = r_1$ and the row $i-2$, and continue. The result of this algorithm is a tableau $U^1(i,\lambda_i) \leftarrow U^1$ of shape $\lambda = (\lambda^1_1, \ldots, \lambda^1_i - 1, \ldots, \lambda^1_t)$ obtained from U^1 by removing the extremal box (i,j). We call *the bumped out element of* U^1 to the element in the first row of U^1 which was bumped out in the final step of the recursive process above.

Lemma 2.5.6. Let U be a standard tableau in $Tab_\lambda(S)$, $p \in S$ and $U^1 = p \to U$. If r is the element adjoined through *the bumping of p into U*, then $r \leftarrow U^1$ is U and *the bumped out element of U^1* is p.
Proof. We proceed by induction on the number of rows of U. If U only has one row, then U^1 is one of the following cases:
(a) If p does bump an element u_t of U, then $\lambda^1 = (\lambda, 1)$, $U^1(1,j) = (1,j)$ if $j \neq t$, $U^1(1,t) = p$ and $U^1(2,1) = u_t$. Since $(2,1)$ is an extremal box, let $r = u_t$. Now $r_1 = \max\{U^1(1,k) \mid U^1(1,k) \leq u_t\}$, since U is standard we know $u_{t-1} < u_t < u_{t+1}$, so r_1 only could be u_{t-1} or p. By definition, $u_t = \min\{U1,k \mid U(1,k) \geq p\}$, so $u_{t-1} < p \leq u_t$, it follows $r_1 = p$ and p is the bumped out element of U^1. Clearly $r \leftarrow U^1$ is U.
(b) If p does not bump any element of U, then $\lambda^1 = (\lambda+1)$ and $U^1(1,j) = U(i,j)$ if $j \leq \lambda$ and $U(1,\lambda+1) = p$. Clearly $(1,\lambda+1)$ is the extremal box of U and p is the bumped out element.

Suppose U has more than one row and the lemma is true for standard tableau of shape λ' of length less than the length of λ. If p does not bump any element of the first row of U, the lemma is clear. Assume that p bump u_t in the first row of U, let $\tilde{U}$ be the tableau U with the top row removed and consider $u_t \to \tilde{U}$. Clearly U^1 is the tableau $u_t \to \tilde{U}$ with the top row of U^1. Note that if r is an extremal box of $u_t \to \tilde{U}$, then r is an extremal box of U^1 too. By induction, $r \leftarrow (u_t \to \tilde{U}) = \tilde{U}$ and u_t is the bumped out element of $\tilde{U}$. The last step of the reverse process to the Schensted process must bump out p and return us U. □

Definition 2.5.7. Let $a = (a_1, \ldots, a_n)$ be a sequence of positive integers and let $S = \{x_1 \leq \cdots \leq x_m\}$ an ordered set. We define the *content* of a to be the sequence $\lambda = (\lambda_1, \ldots)$, where λ_i is the number of times i appears in a, for all $i \in \mathbb{N}$. We say that a is a *word of Yamanouchi or a Y-word* if for each $k = 1, \ldots, n$ the number of times i appears in the sequence $(a_1, \ldots, a_k)$ is not smaller than the number of times $i+1$ appears, for all $i \in \mathbb{N}$. We say that $(x_{i_1}, \ldots, x_{i_n})$ is a Y-word if $(i_1, \ldots, i_n)$ is a Y-word.

Remark 2.5.8. If a is a Y-word, then clearly the content of $(a_1, \ldots, a_k)$ is a partition for $k = 1, \ldots, n$. Reciprocally, if the content of $(a_1, \ldots, a_k)$ is a partition for $k = 1, \ldots, n$, then a is a Y-word.

For example, the sequence $a = (1,1,2,2,1,3,2,1,2,1,2,3)$ has content $(5,5,2)$ and it is a word of Yamanouchi as the following tabular of subsequences and contents of a shows us.

$k=1$	(1)	(1)
$k=2$	$(1,1)$	(2)
$k=3$	$(1,1,2)$	$(2,1)$
$k=4$	$(1,1,2,2)$	$(2,2)$
$k=5$	$(1,1,2,2,1)$	$(3,2)$
$k=6$	$(1,1,2,2,1,3)$	$(3,2,1)$
$k=7$	$(1,1,2,2,1,3,2)$	$(3,3,1)$
$k=8$	$(1,1,2,2,1,3,2,1$	$(4,3,1)$
$k=9$	$(1,1,2,2,1,3,2,1,2)$	$(4,4,1)$
$k=10$	$(1,1,2,2,1,3,2,1,2,1)$	$(5,4,1)$
$k=11$	$(1,1,2,2,1,3,2,1,2,1,2)$	$(5,5,1)$
$k=12$	$(1,1,2,2,1,3,2,1,2,1,2,3)$	$(5,5,2)$

Proposition 2.5.9. There exists a bijection between the set of words of Yamanouchi of content λ and the set of standard tableau of shape λ with distinct entries from the set $\{1,\ldots,n\}$, where n is the weight of λ.

Proof. Let $\mathcal{T}$ be a set of standard tableau of shape λ with distinct entries from the set $\{1,\ldots,n\}$ and Y the set of Y-words of content λ.

Let $T \in Tab_\lambda(\{1,\ldots,n\})$ be a standard tableau. By hypothesis, each $i \in \{1,\ldots,n\}$ appears as an entry $T(k,j)$ only one time. Let $a_i = k$, that is, a_i is the number of the row of T where i is an entry, and consider the sequence $a = (a_1,\ldots,a_n)$. First we compute the content of a. The number of times i appears in a is exactly the number of elements in the ith row of T, that is λ_k, then a is a sequence of content λ. And second, we prove that a is an Y-word. Let $k \in \{1,\ldots,n\}$ and consider the subsequence $a' = (a_1,\ldots,a_k)$. Fixed i let j_i and j_{i+1} be the number of times i and $i+1$ appear in the subsequence respectively, that is j_i is the number of elements of ith row of T smaller than k. If we have j_{i+1} elements in the $i+1$th row of T smaller than k, then since T is standard, $(T(i,l) \leq T(i+1,l), \quad l = 1,\ldots,\lambda_{i+1})$, we have at least j_{i+1} elements in the ith row of T smaller than k too, so it is clear that $j_i \geq j_{i+1}$. We define $\alpha : \mathcal{T} \to Y$ to be the map which sends T to a as above.

Now let $a = (a_1,\ldots,a_n)$ be a Y-word of content λ. For each $i \in \{1,\ldots,n\}$ we consider the subsequence of a with entries equal to i and we denote it by $(a_{k_1},\ldots,a_{k_{\lambda_i}})$. Then $T(i,j) = k_j$ define a row-standard tableau of shape λ with distinct entries in $\{1,\ldots,n\}$. We want to prove that T is standard. Let $(a_{k_1},\ldots,a_{k_{\lambda_i}})$ be the subsequence of a with entries equal to i and $(a_{j_1},\ldots,a_{j_{\lambda_{i+1}}})$ the subsequence of a with entries equal to $i+1$. First we see that $k_1 \leq j_1$. We know that k_1 is the first subindex such that $a_l = i$ and j_1 is the first subindex such that $a_l = i+1$. If $j_1 < k_1$, then for $k = j_1$ the number of times i appears in the sequence $(a_1,\ldots,a_{j_1})$ is smaller than the number of $i+1$, which is a contradiction since a is an Y-word. With a symmetric argue we see that $k_i \leq j_i, \quad i = 1,\ldots,\lambda_{i+1}$. We define $\beta : Y \to \mathcal{T}$ to be the map which sends a to T as above.

Clearly α and β are inverses. □

Definition 2.5.10. Let $a = (a_1,\ldots,a_n)$ a Y-word of content λ. If $T \in \mathcal{T}$, then $\tilde{T}(i,j) = T(j,i)$ is a standard tableau of shape $\tilde{\lambda}$ with distinct entries from $\{1,\ldots,n\}$. We define $\tilde{a}$ to be the Y-word $\alpha(\widetilde{\beta(a)})$ of content $\tilde{\lambda}$, we call $\tilde{a}$ the transpose of a.

Remark 2.5.11. $\tilde{a}_i$ is the number of a_k such that $a_k = a_i$ and $k \leq i$.

Definition 2.5.12. Let λ, μ be two partitions. We define A to be the set of pairs (U_1, U_2) where U_1 is a standard tableau of shape λ and U_2 is a standard tableau of shape μ. We define B to be the set of triples (ν, U, V) where ν is a partition containing λ, U is a standard tableau of shape ν and if the diagram of ν/λ is $\{(i,j) \mid i = 1, \ldots, q, \lambda_i + 1 \leq j \leq \nu_i$, then V is a standard tableau of shape ν/λ of content $\tilde{\mu}$ such that the sequence $(V(q, \lambda_q + 1), V(q-1, \lambda_q + 1), \ldots, V(q, \nu_q), V(q-1, \nu_q), \ldots, V(1, \lambda_1)$ is a Y-word.

Remark 2.5.13. Observe that the last sequence is the sequence obtained by reading each column of V from the bottom up, starting with the left-most column and moving to the right column by column.

For example, if $\lambda = (3,2,1)$, $\mu = (3,3,2,2,2)$ and $\nu = (6,4,3,3,2)$, then

$$V = \begin{array}{|c|c|c|c|c|c|} \hline & & & 1 & 2 & 3 \\ \hline & & 1 & 2 & & \\ \hline & 1 & 2 & & & \\ \hline 1 & 2 & 3 & & & \\ \hline 1 & 2 & & & & \\ \hline \end{array}$$

is a standard tableau of shape ν/λ, content $(5,5,2) = \tilde{\mu}$ and with associated sequence $a = (1,1,2,2,1,3,2,1,2,1,2,3)$, which is a Y-word.

The main goal of this subsection is to construct an injection from A to B which will play an important role in the next subsection. Defining a map $\phi : A \to B$ will take us a while, first we will obtain from a pair (U_1, U_2) of A a triple (ν, U, V) where ν and U satisfying B-conditions, the difficult step will be to verify V is of that such.

Let $(U_1, U_2) \in A$ and let $p_1 \geq p_2 \geq \cdots \geq p_{\tilde{\mu}_1}$ the entries of the first column of U_2 from the bottom up. We define $U_1^{(1)} = (p_{\tilde{\mu}_1} \to (\cdots \to (p_2 \to (p_1 \to U_1))\cdots))$ to be the tableau obtained from U_1 by bumping in the column of U_2 into U_1. We repeat the procedure with $U_1^{(1)}$ and the second column of U_2. Continuing in this manner we obtain a standard tableau U of some shape ν containing λ and, since we have bumped all elements of U_2, $|\nu| = |\lambda| + |\mu|$.

Each box of the diagram of λ/μ, $\{(i,j) \mid i = 1, \ldots, q, \lambda_i + 1 \leq j \leq \nu_i\}$ was a a result of having bumped some entry $U_2(l,k)$ into U_1, with the assignment $V(i,j) = k$ we define a tableau V of shape ν/λ. The number of times i appears as an entry of V is clearly the number of elements in the i column of U_2, therefore the content of V is $\tilde{\mu}$. We say that (ν, U, V) is the triple associated to (U_1, U_2). We want to show that V is a standard tableau and the sequence associated to V is a word of Yamanouchi.

Lemma 2.5.14. Let U' be a standard tableau of shape λ' and let $p_1 \geq \cdots \geq p_r$ be elements of S. Let U'' be the standard tableau $p_r \to (\cdots \to (p_2 \to (p_1 \to U''))\cdots)$ of some shape λ''. Then, the diagram of λ''/λ contains at most one box in each row. Moreover, the box adjoined by bumping in p_{i+1} is below the box adjoined by bumping in p_i, for $i = 1, \ldots, r-1$.

Proof. From the construction of U'' it is enough to consider $p_2 \to (p_1 \to U'')$. If p_1 does not bump any entry in the first row of U', then $p_1 \to U'$ has p_1 in the first row. Since $p_2 \leq p_1$, p_2 must bump some element p_2' in the first row of $p_1 \to U'$, and hence the box adjoined by bumping p_2 into $p_1 \to U'$ is below the box adjoined by bumping p_1 verifying the lemma.

Otherwise, if p_1 does bump some entry p_1' in the first row of U', then $p_2 \leq p_1$ must also bump some entry p_2' with $p_2' \leq p_1'$ in the first row of $p_1 \to U'$. Let $p_1 = p_1'$ and $p_2 = p_2'$, we repeat the argument with U' and $p_1 \to U'$ without the first rows of U' and $p_1 \to U'$. Continuing in the same manner we must stop when p_1 does not bump any entry in the ith row of U'. □

Lemma 2.5.15. Let U' be a standard tableau of shape λ, let $l_s > k_s \geq \cdots \geq k_1$ and let U be the tableau $l_s \to (k_1 \to (\cdots \to (k_s \to U')\cdots))$ of some shape λ'. Then, $U = k_1 \to (\cdots \to (k_{s-1} \to (l_s \to (k_s \to U')))\cdots)$

Proof. It is sufficient to prove it for $s = 2$. We proceed by induction on the number of rows of U'. Suppose U' has only one row. If l_2 does not bump any element U', the result is clear. Otherwise, if l_2 bumps an element l_2' of U' U', then k_2 must have bump and element k_2' and k_1 an element k_1', with $k_1' \leq k_2' < l_2'$. We consider $k_1' < k_2' < l_2'$, the other case is similar. We have,

$$k_2 \to U' = \begin{array}{|c|c|c|} \hline \quad\dots\quad & k_2 & \quad\dots\quad \\ \hline k_2' \\ \cline{1-1} \end{array},$$

$$k_1 \to (k_2 \to U') = \begin{array}{|c|c|c|c|c|} \hline \quad\dots\quad & k_1 & \quad\dots\quad & k_2 & \quad\dots\quad \\ \hline k_1' \\ \cline{1-1} k_2' \\ \cline{1-1} \end{array}$$

$$\text{and } l_2 \to (k_1 \to (k_2 \to U')) = \begin{array}{|c|c|c|c|c|c|c|} \hline \quad\dots\quad & k_1 & \quad\dots\quad & k_2 & \quad\dots\quad & l_2 & \quad\dots\quad \\ \hline k_1' & l_2' \\ \cline{1-2} k_2' \\ \cline{1-1} \end{array}$$

$$l_2 \to (k_2 \to U') = \begin{array}{|c|c|c|c|c|} \hline \quad\dots\quad & k_2 & \quad\dots\quad & l_2 & \quad\dots\quad \\ \hline k_2' & l_2' \\ \cline{1-2} \end{array}$$

$$\text{and } l_2 \to (k_1 \to (k_2 \to U')) = \begin{array}{|c|c|c|c|c|c|c|} \hline \quad\dots\quad & k_1 & \quad\dots\quad & k_2 & \quad\dots\quad & l_2 & \quad\dots\quad \\ \hline k_1' & l_2' \\ \cline{1-2} k_2' \\ \cline{1-1} \end{array}$$

Assume that U' has more than one row. If l_2 does not bump any element in the first row of U', the result is clear. Suppose that l_2 does bump an element l_2' in the first row of U', as before k_2 and k_1 do bump elements $l_2' > k_2' \geq k_1'$ in the first row of U'. Let U_1' be the tableau U' without the first row, clearly U is the tableau obtained from $l_2' \to (k_1' \to (k_2' \to U_1')$ by adjoining the first row of U. The result follows by induction on U_1'. □

Lemma 2.5.16. Let U' be a standard tableau of shape λ, let $k_t \geq \cdots \geq k_1,\ l_s \geq \cdots \geq l_1,\quad t \geq s$ be elements of S such that $k_i < l_i,\quad i = 1, \dots s$ and let U be the tableau $l_1 \to (\cdots l_s \to (\cdots k_1 \to (\cdots \to (k_t \to U')\cdots)))\cdots)$ of some shape λ'. Then, the box of U' adjoined to U by bumping in l_α is strictly to the right of the box of U' adjoined to U by bumping in k_α, for $\alpha = 1, \dots, s$.

Proof. We can assume $s = t$, since it is the same to prove that the box of U' adjoined to $k_{s+1} \to (\cdots \to (k_t \to U')\cdots)$ by bumping in l_α is strictly to the right of the box of U adjoined to $k_{s+1} \to (\cdots \to (k_t \to U')\cdots)$ by bumping in k_α, for $\alpha = 1, \dots, s$. We proceed by induction on s.

When $s = 1$, we have $k_1 > l_1$. By lemma 2.5.15, the box $(i, \lambda_i + 1)$ adjoined by bumping in k_1 is below that the box $(j, \lambda_j + 1)$ adjoined by bumping in l_1: So $j > i \Rightarrow \lambda_i + 1 < \lambda_j + 1$, and hence $(j, \lambda_j + 1)$ is strictly to the right of $(i, \lambda_i + 1)$. Now, suppose that is true for $k = s - 1$. By lemma 2.5.15, if we let $U'_1 = l_s \to (k_s \to U')$, then $U = l_1 \to (\cdots \to (l_{s-1} \to (k_1 \to (\cdots \to (k_{s-1} \to U'_1) \cdots))) \cdots)$. The result follows by induction. □

Proposition 2.5.17. Let $(U_1, U_2) \in A$ and let (ν, U, V) be its triple associated. Then $(\nu, U, V) \in B$.
Proof. We only have to prove that V is a standard tableau whose sequence associated is a Y-word. Remember V is a tableau of shape ν/λ such that $V(i, j) = k$, where (i, j) is the box adjoined by bumping in the element $U_2(l, k)$.

We start showing the standardness of V. Let $(i, j), (i, j + 1)$ *and* $(i + 1, j)$ be adjacent boxes of V. By Lemma 2.5.16, the tableau obtained by bumping in a standard column $p_1 \geq p_2 \geq \cdots \geq p_{\mu_i}$ of U^2 never has two boxes in the same row, moreover the box adjoined by bumping in p_{i+1} is below to the box adjoined by bumping in p_i. Therefore, if (i, j) is the box adjoined by bumping in the element $U^2(l, k)$, then $(i, j + 1)$ is the box adjoined by bumping in an element $U_2(l', k')$ such that $k' > k$, and hence $V(i, j + 1) > V(i, j)$. The row-standardness of V is clear. Let $(i + 1, j)$ be the box adjoined by bumping in the element $U^2(l', k')$ and let U' be the tableau of some shape ν' obtained by bumping in the first k' columns of U_2 into U_1. Since $\nu'_i \geq \nu'_{i+1}$, necessarily the box $(i, j) = (i, \nu'_i)$ was adjoined first, thus $k \leq k'$. So $V(i, j) = k \leq k' = V(i + 1, j)$ and the column-standardness of V follows.

It remains to prove that the sequence associate to V is a Y-word. For convenience we name this sequence by $a = (p^1_1, \ldots, p^1_{\tilde{\mu}_1}, \ldots, p^{\mu_1}_1, \ldots, p^{\mu_1}_{\tilde{\mu}_q})$. For each $k = 1, \ldots, |\lambda/\nu|$ we consider the subsequence $a^k = (p^1_1, \ldots, p^1_{\tilde{\mu}_1}, \ldots, p^i_1, \ldots, p^i_k)$ of a, we want to see that the content of a^k is a partition. Let $1 \leq j < k$, the number of time $j + 1$ appears in a^k is the number of boxes adjoined by bumping in the elements of the $(j + 1)$th column of U_2 into U_1. If $j + 1$ appears l_{j+1} times, then these boxes correspond to the element $U^2(j + 1, 1), \ldots, U^2(j + 1, l)$. By Lemma 2.5.16 we know that the boxes adjoined by bumping in $U^2(j + 1, 1), \ldots, U^2(j + 1, l)$ are strictly to the right of the boxes associated to $U^2(j, 1), \ldots, U^2(j, l)$. Moreover, if the box associated to $U^2(j + 1, h)$ appears in the subsequence a^k, then the box associated to $U^2(j, h)$ must appear too. Clearly $l_j \geq l_{j+1}$, and hence the content of a^k is a partition. □

Definition 2.5.18. We define a map $\Phi : A \to B$ by mapping a pair (U_1, U_2) of A to its triple associated $(\nu, U, V) \in B$.

Proposition 2.5.19. The map $\Phi : A \to B$ is an injection.
Proof. Let $\bar{A}$ be the set of pairs (U_1, U_2) such that $U_1 \in Tab_\lambda(S)$ *and* $U_2 \in Tab_\mu(S)$, we want to define a map $\Psi : B \to \bar{A}$ such that $(\Psi \circ \Phi)(U_1, U_2) = (U_1, U_2)$ for all pairs (U_1, U_2) in A. Clearly, this implies the proposition.

Let $(\nu, U, V) \in B$. Remember V is a standard tableau of shape ν/λ over $\{1, \ldots, \mu_1\}$ and content $\tilde{\mu} = (\tilde{\mu}_1, \ldots, \tilde{\mu}_t)$. If μ_1 is an entry in the k-th row of V, it is clear that the box associated to it is (k, μ_k). Since V is a standard tableau, if k is the last row in which μ_1 appears, or equivalent, the first time μ_1 appears in the Y-word associated to V, this entry must be an extremal box of V, and thus of U. Therefore we can consider the standard tableau $U(k, \nu_k) \leftarrow U$ of some shape ν_1. Let p_1 be the element bumped out of U, since the second appearance of μ_1 is an extremal box of $U(k, \nu_k) \leftarrow U$ too, we can repeat the argument above. After $\tilde{\mu}_t$ iterations we obtain a standard tableau U' of some shape ν', which is a partition containing λ, and $p_1, \dots, p_{\tilde{\mu}_t}$ elements bumpered out of U, we put them into a column γ_t. Let V' be the standard tableau of shape ν'/λ obtained from V by removing all boxes of V with the entry μ_1. Clearly, the sequence associated to V' is the sequence of V but deleting the elements μ_1, and thus is an Y-word too. Repeat the above argument with the triple (ν', U', V') and μ_2, and continue. Finally, we will obtain a standard tableau U_1 of shape λ from U by bumping out the elements of V, and $\gamma_t, \dots, \gamma_1$ columns. We define a map $\Psi : B \to \bar{A}$ sending (ν, U, V) to the pair (U_1, U_2), where U_2 be the tableau of shape μ and columns $\gamma_1, \dots, \gamma_t$.

Let $(U_1, U_2) \in A$, we want to prove that $\Psi(\Phi(U_1, U_2))) = (U_1, U_2)$. We proceed by induction on the number of column of U_2. Suppose that U_2 is one column tableau, we denote it by $p_1 \geq \cdots \geq p_r$. Consider $(\nu, U, V) := \Phi(U_1, U_2)$ and $(U_1', U_2') := \Psi(\nu, U, V)$. Remember $U = p_r \to (\cdots \to (p_1 \to U_1) \cdots)$ is a standard tableau of shape ν and V is a standard tableau of shape ν/λ with all entries 1. If $r = 1$, then $U = p_1 \to U_1$ and V is a single box tableau with entry 1. Let $U(i, \lambda_i + 1)$ be the box adjoined by bumping in p_1, since the bumped out element of $U_1' = U(i, \lambda_i + 1) \leftarrow U$ is p_1, we must have $U_1' = U_1$, and thus $U_2' = U_2$. Otherwise, if $r > 0$, the lowest extremal box of V must be the box adjoined by bumping in p_r and it must be an extremal box of U too. Therefore, the first step of Ψ give us back the tableau $U' = p_{r-1} \to (\cdots \to (p_1 \to U_1) \cdots)$ with p_r the bumped out element of U. By induction on r $\Psi(\nu', U', V')$, with V' to be the tableau V but removing p_r, is (U_1, U_2''), where U_2'' is the tableau U_2 but removing p_r. Clearly $\Psi(\nu, U, V) = (U_1, U_2)$.

Assuming that U_2 has more than one column, let U_2' be the tableau U_2 but removing its last column $p_1 \geq \cdots \geq p_r$. Let (ν', U'', V'') be the triple obtained from the first step of Ψ applied to (ν, U, V), as we have seen before, U'' is the standard tableau obtained by bumping out the boxes with entries μ_1 in V and V'' is obtained from V by removing all boxes with entry μ_1. From the definition of Ψ, clearly $\Psi(\Phi(U_1, U_2))$ is obtained from $\Psi(\nu', U'', V'') = (U_1, U_2'')$ by adjoining to U_2'' the bumped out column γ_t in the first step of Ψ. By induction, $\Psi(\Phi(U_1, U_2')) = \Psi(\nu', U', V') = (U_1, U_2')$. So, in order to finish the proof we have to show that $(\nu', U'', V'') = (\nu', U', V')$ and the last column of U_2 is the column γ_t of bumped out elements. In this case, $(U_1, U_2'') = (U_1, U_2')$ and the result follows. It is clear that $V' = V''$, since V' is the tableau V but removing all boxes with entry μ_1, which is the definition of V''. From definition of Φ, the tableau U' is the tableau obtained from U_1 by bumping in since the last column of U_2, and thus $U = p_r \to (\cdots \to (p_1 \to U') \cdots)$. By the initial case, $\Psi(\Phi(U', (p_1, \dots, p_r)) = \Psi(\nu, U, V_1) = (U', (p_1, \dots, p_r))$, where V_1 is obtained from V but deleting all boxes with entries $1, \dots, \mu_1 - 1$. Because of that U' is the tableau obtained from U by bumping out the elements with entries μ_1, that is U'', and the bumping out column is the last column of U_2. □

2.5.2 The Littlewood-Richardson Rule.

This subsection is devoted to the proof of the Littlewood-Richardson rule:

Theorem 2.5.20. Let R be a ring of characteristic zero, let F be a free R-module and let λ

and μ two partitions. Then, there is a natural isomorphism

$$L_\lambda F \otimes L_\mu F \cong \sum_\nu u(\lambda, \mu; \nu) L_\nu F$$

where $u(\lambda, \mu; \nu)$ is the number of standard tableau $V \in Tab_{\nu/\lambda}(\{1, \ldots, |\mu|\})$ of content $\tilde{\mu}$ such that the sequence associated to V is a word of Yamanouchi.

Let $S = \{x_1, \ldots, x_n\}$ be an ordered basis of F. Since A is the set of all pairs (U_1, U_2) where U_1 and U_2 are standard basis in $Tab_\lambda(S)$ and $Tab_\mu(S)$ respectively, from the standard basis theorem for Schur functors, (Theorem 2.1.21), $\{d_\lambda(X_{U_1}) \otimes d_\mu(X_{U_2}) \mid (U_1, U_2) \in A\}$ is an R-basis of $L_\lambda F \otimes L_\mu F$. Now, if $u(\lambda, \mu; \nu)$ is the number of standard tableau $V \in Tab_{\nu/\lambda}(\{1, \ldots, |\mu|\})$ of content $\tilde{\mu}$ such that the sequence associated to V is a word of Yamanouchi, then $\{d_{\nu/\lambda}(X_{(\nu,U,V)}) \mid (\nu, U, V) \in B\}$ must be an R-basis of $\sum_\nu u(\lambda, \mu; \nu) L_\nu F$. Since Proposition 2.5.19 tell us that the cardinality of A is less than the cardinality of B, to prove the Littlewood-Richardson rule we only have to construct an injection $\sum_\nu u(\lambda,\mu;\nu) L_\nu F \to L_\lambda F \otimes L_\mu F$.

We will proceed as follows. First we will define for each standard tableau V satisfying conditions in Theorem 2.5.20 a morphism $\Phi_V : L_\nu F \to L_\lambda F \otimes L_\mu F$ which gives $u(\lambda, \mu; \nu)$ independent copies of $L_\nu F$ inside $L_\lambda F \otimes L_\mu F$. That is $\Phi_V(L_\nu F) \cong u(\lambda, \mu; \nu) L_\nu F$, and hence $u(\lambda, \mu; \nu) L_\nu F$ is a submodule of $L_\lambda F \otimes L_\mu F$, which give us the injection we are looking for.

In order to define morphisms $\Phi_V : L_\nu F \to L_\lambda F$ we need to introduce two more combinatorial definitions and one Lemma.

Definition 2.5.21. Let V be a standard tableau satisfying conditions in Theorem 2.5.20. We denote by $a = (a_1, \ldots, a_m)$ the sequence associated to V and let $\tilde{a} = (\tilde{a}_1, \ldots, \tilde{a}_m)$ the transpose of a. We define $\tilde{V}$ to be the tableau of shape ν/λ and content μ with associated sequence $\tilde{a}$.

As an example, $V = \begin{array}{cc} & 1 \\ 1 & 2 \\ 1 & \end{array}$ and $\tilde{V} = \begin{array}{cc} & 3 \\ 2 & 1 \\ 1 & \end{array}$.

Lemma 2.5.22. $\tilde{V}$ is decreasing in rows and strictly decreasing in columns.

Proof. We start with the row condition. Let (k, l) and $(k, l+1)$ to adjacent boxes with entries $a_i < a_j$ in V respectively. We want to see that $\tilde{a}_i \geq \tilde{a}_j$. Remember $\tilde{a}_i$ is the number of a_h such that $a_h = a_i$ and $h \leq i$, from this if we remove from V all boxes containing entries $< a_i$ we still have a standard tableau whose associated sequence is a Y-word and the picture $\boxed{\tilde{a}_i}\boxed{\tilde{a}_j}$ not change in $\tilde{V}$. Therefore we may assume that $a_i = 1$ and $l = \lambda_k + 1$. We proceed by induction on the number of elements above the box $(k, \lambda_k + 1)$. If there is nothing above the box of a_i, then a_i is the last appearance of 1 in a. So, $\tilde{a}_i$ and $\tilde{a}_j$ are the number of time a_i and a_j appear in the subsequence $(a_1, \ldots, a_j)$. Since a is a Y-word and $a_i < a_j$, it follows that $\tilde{a}_i > \tilde{a}_j$. Assuming that there are n boxes above a_i, consider the boxes $(k-1, l)$ and $(k-1, l+1)$ with entries a_{i+1} and a_{j+1} respectively. It is clear that $\tilde{a}_i = \tilde{a}_{i+1} + 1$ and $\tilde{a}_j = \tilde{a}_{j+1} + 1$. By induction, $\tilde{a}_{i+1} \geq \tilde{a}_{j+1}$ and hence, $\tilde{a}_i \geq \tilde{a}_j$.

In order to finish the proof we check the column condition. Let (k, l) and $(k+1, l)$ to adjacent boxes with entries $a_{i+1} \leq a_i$ of V respectively. If $a_{i+1} = a_i$, then clearly

$\tilde{a}_{i+1} = \tilde{a}_i + 1 > \tilde{a}_i$. Suppose $a_{i+1} < a_i$, $\tilde{a}_i$ is the number of $a_k = \tilde{a}_i, \quad k \leq i$, indeed $\tilde{a}_i$ is the number of time that a_i appears in the subsequence $(a_1, \ldots, a_i)$. Since a is a Y-word and $a_{i+1} \leq a_i$, if t_{i+1} is the number of time a_{i+1} appears in the above subsequence, then $t_{i+1} \geq \tilde{a}_i$. We obtain the desire result since $\tilde{a}_{i+1} = t_{i+1} + 1$. □

Definition 2.5.23. Let V be a tableau satisfying conditions in Theorem 2.5.20, let $\Delta_{\nu/\lambda}$ be the diagram of V. We denote $\mu = (\mu_1, \ldots, \mu_q)$ and $a = (a_1, \ldots, a_m)$ the associated sequence of V. From definition of content, $\tilde{V}$ has entries in $\{1, \ldots, \tilde{\mu}_1\}$ and V has entries in $\{1, \ldots, \mu_1\}$. Now, for each $k \in \{1, \ldots, \tilde{\mu}_1\}$ there exist μ_k boxes in $\Delta_{\nu/\lambda}$ such that $\tilde{V}(i,j) = k$. For a fixed k, consider the sequence $(i_1, j_1), \ldots, (i_{\mu_k}, j_{\mu_k})$ of these boxes formed by listing them from bottom to top in each column, starting from the left-most column. In fact, the entries of these boxes in this order is the subsequence a^k of a. Since $\tilde{\tilde{a}} = a$, $V(i_l, j_l)$ is the position of the entry in the box (i_l, j_l) in the subsequence a^k. By Lemma 2.5.22 $j_1 < j_2 < \cdots < j_{\mu_k}$, and hence $V(i_l, j_l) = l, \quad l = 1, \ldots, \mu_k$. Consequently, for each $k \in \{1, \ldots, \tilde{\mu}_1\}$ and $l \in \{1, \ldots, \mu_k\}$ there exists a unique box $(i,j) \in \Delta_{\nu/\lambda}$ such that $\tilde{V}(i,j) = k$ and $V(i,j) = l$. We define a bijection $\sigma_V : \Delta_{\nu/\lambda} \to \Delta_\mu$ by $\sigma_V(i,j) = (\tilde{V}(i,j), V(i,j))$.

Definition 2.5.24. Let V be a satisfying conditions in Theorem 2.5.20. We denote $a = (a_1, \ldots, a_m)$ its associated sequence. We define a map ϕ_V to be the composition:

$$L_\nu F \cong K_{\tilde{\nu}} F \xrightarrow{\imath} \Lambda_\nu F \xrightarrow{\Delta} \Lambda_\lambda F \otimes \Lambda_{\nu/\lambda} F \xrightarrow{Id \otimes \eta_V} \Lambda_\lambda F \otimes \Lambda_\mu F \xrightarrow{d_\lambda \otimes d_\mu} L_\lambda F \otimes L_\mu F$$

where the first map is the isomorphism between $L_\nu F$ and $K_{\tilde{\nu}} F$, $\imath$ is the natural inclusion of $K_{\tilde{\lambda}} F$ in $\Lambda_\lambda F$, Δ is the tensor product of comultiplications $\Lambda_{\nu_i} F \to \Lambda_{\lambda_i} F \otimes \Lambda_{\nu_i - \lambda_i} F$, and finally η_ν is the composition $\Lambda_{\nu/\lambda} F \xrightarrow{\imath'} T_{\nu/\lambda} F \xrightarrow{\jmath} \otimes_\mu F \xrightarrow{m} \Lambda_\mu F$, where $\imath'$ is the tensor of the inclusions of $\Lambda^{\nu_i - \lambda_i} F$ in $T^{\nu_i - \lambda_i} F$, $\jmath : \otimes_{\nu/\lambda} F \to \otimes_\mu F$ sends $(\cdots \otimes x_{(i,j)} \otimes \cdots)$ to $(\cdots \otimes x_{\sigma_V(i,j)} \otimes \cdots)$ and m is the multiplication.

Lemma 2.5.25. Let V be a satisfying conditions in Theorem 2.5.20. Then, the morphism $\Phi_\nu : L_\nu F \to L_\lambda F \otimes L_\mu F$ give $u(\lambda, \mu; \nu)$ independent copies of $L_\nu F$ inside $L_\lambda F \otimes L_\mu F$.
Proof. The proof of this lemma is found in [1], Lemma $IV.2.5$. It follows from arguments which imply knowledges about Representation Theory. □

As we have discuss at the begging of this subsection, the Littlewood-Richardson rule follows from the above Lemma. We present here two consequence of Theorem 2.5.20.

Corollary 2.5.26. The map $\Phi : A \to B$ is a bijection.

Corollary 2.5.27. Let R be a ring of characteristic zero, let F be a free R-module, let λ be a partition and p a positive integer. Then, there are natural isomorphisms:

(1) $L_\lambda F \otimes S^p F \cong \sum_\nu L_\nu F$, where ν runs over all partitions containing λ such that $|\nu| = |\lambda| + p$ and $\nu_i \leq \lambda_i + 1$ for all i.

(2) $L_\lambda F \otimes \Lambda^p F \cong \sum_\nu L_\nu F$, where ν runs over all partitions containing λ such that $|\nu| = |\lambda| + p$ and $\tilde{\nu}_i \leq \tilde{\lambda}_i + 1$ for all i.

Proof. Observe that $S^p F = L_{(1,\ldots,1)} F$, $1 + \cdots + 1 = p$ and $\Lambda^p F = L_{(p)} F$.

In the first case we have that if V is a standard tableau of shape ν/λ, content (p) with associated sequence a word of Yamanouchi. then, V must have in $\{1\}$. The standardness of V force that the diagram of V only has one box in each row. Then, ν_i is at most $\lambda_i + 1$. Clearly, $|\nu| = |\lambda| + p$.

In the second case, V has content $(1,\ldots,1)$, $1 + \cdots + 1 = p$. Then, the tableau $\tilde{V}$ is of shape ν/λ, has content (p) and entries in $\{1\}$. Since $\tilde{V}$ is strictly decreasing in columns, for each column of $\tilde{V}$ there is only one box. Then, $\tilde{\nu}_i \leq \tilde{\lambda}_i + 1$. Clearly, $|\nu| = |\lambda| + p$. □

We want to finish with some examples.

Examples 2.5.28. (1) We start computing the Pieri formulas when $\lambda = (2,1)$ and $p = 2$. First, the partitions ν containing $(2,1)$ such that $|\nu| = 5$ are $(2,1,1,1), (2,2,1), (3,1,1)$, $(3,2)$ and $(4,1)$. The condition $\nu_i \leq \lambda_i + 1$ remove the partition $(4,1)$, and the condition $\tilde{\nu}_i \leq \tilde{\lambda}_i + 1$ eliminate the partition $(2,1,1,1)$. We obtain the followings isomorphism.

$$L_{(2,1)}F \otimes S^2F \cong L_{(2,1,1,1)}F \otimes L_{(2,2,1)}F \oplus L_{(3,1,1)}F \oplus L_{(3,2)}F$$

$$L_{(2,1)}F \otimes \Lambda^2F \cong L_{(2,2,1)}F \oplus L_{(3,1,1)}F \oplus L_{(3,2)}F \oplus L_{(4,1)}F$$

(2) Let us see the direct decomposition of $L_{(2,1)}F \otimes L_{(2,1)}F$. The partitions ν containing $(2,1)$ such that $|\nu| = 6$ are $(2,1,1,1,1), (2,2,1,1), (2,2,2), (3,1,1,1), (3,2,1), (3,3), (4,1,1)$, $(4,2)$ and $(5,1)$. We must study the set of standard tableau $V \in Tab_{\nu/\lambda}\{1,2\}$ of content $(2,1)$ which sequence associated is a Y-word for each ν in the above list.

$\nu = (2,1,1,1,1)$, $V = \begin{array}{cc} & \boxed{2} \\ \boxed{1} & \\ \boxed{1} & \end{array}$ but the sequence associated to V is $(2,1,1)$ which is not a Y-word. $u(\lambda,\mu,(2,1,1,1,1)) = 0$.

$\nu = (2,2,1,1)$, $V = \begin{array}{cc} & \boxed{2} \\ \boxed{1} & \\ \boxed{1} & \end{array}$ or $\begin{array}{cc} & \boxed{1} \\ \boxed{1} & \\ \boxed{2} & \end{array}$, $(1,1,2)$ is a Y-word. $u(\lambda,\mu,(2,2,1,1)) = 1$.

$\nu = (2,2,2)$, $V = \begin{array}{cc} & \boxed{1} \\ \boxed{1} & \boxed{2} \end{array}$, $(1,2,1)$ is a Y-word. $u(\lambda,\mu,(2,2,2)) = 1$.

$\nu = (3,1,1,1)$, $V = \begin{array}{cc} & \boxed{1} \\ & \\ \boxed{1} & \\ \boxed{2} & \end{array}$ or $\begin{array}{cc} & \boxed{2} \\ & \\ \boxed{1} & \\ \boxed{1} & \end{array}$, $(1,1,2)$ is a Y-word.

$u(\lambda,\mu,(3,1,1,1)) = 1$.

$\nu = (3,2,1)$, $V = \begin{array}{cc} & \boxed{1} \\ & \\ \boxed{1} & \\ \boxed{2} & \end{array}$ or $\begin{array}{cc} & \boxed{1} \\ & \\ \boxed{2} & \\ \boxed{1} & \end{array}$ or $\begin{array}{cc} & \boxed{2} \\ & \\ \boxed{1} & \\ \boxed{1} & \end{array}$, $(1,1,2)$ and $(1,2,1)$ are Y-words. $u(\lambda,\mu,(3,2,1)) = 2$.

$\nu = (3,3)$, $V = \begin{array}{cc} & \boxed{1} \\ \boxed{1} & \boxed{2} \end{array}$, $(1,2,1)$ is a Y-word. $u(\lambda,\mu,(3,3)) = 1$.

$\nu = (4,1,1)$, $V = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}$ $\begin{array}{|c|} \hline 1 \\ \hline \end{array}$, $(1,1,2)$ is a Y-word. $u(\lambda, \mu, (4,1,1)) = 1$.

$\nu = (4,2)$, $V = \begin{array}{cc} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \\ \begin{array}{|c|} \hline 1 \\ \hline \end{array} & \end{array}$, $(1,1,2)$ is a Y-word. $u(\lambda, \mu, (4,2)) = 1$.

$\nu = (5,1)$, there is not standard tableau V with entries in $\{1,2\}$ and content $(2,1)$. $u(\lambda, \mu, (5,1)) = 0$.

We have, $L_{(2,1)}F \otimes L_{(2,1)}F \cong L_{(2,2,1,1)}F \oplus L_{(2,2,2)}F \oplus L_{(3,1,1,1)}F \oplus 2L_{(3,2,1)}F \oplus L_{(3,3)}F \oplus L_{(4,1,1)}F \oplus L_{(4,2)}F$.

Chapter 3

A minimal free complex associated to the minors of a matrix.

Finding a minimal free resolution of the ideal associated to the minors of a matrix is a classical subject in Algebra and Algebraic Geometry. This problem was solved by Lascoux in [16]. In this chapter, we construct a minimal free complex associated to the minors of a matrix. Mainly, we follow [18] and [3], and the last section is based on [17], [4], [12], [16] and [13].

3.1 A minimal free complex.

Let R be a local commutative ring contained a field K of characteristic zero and $\mathfrak{m}$ its maximal ideal. Let n, m and t be positives integers such that $n \leq m$ and let F and G be free R-modules of ranks $m+t-1$ and $n+t-1$, with basis $\{f_1, \ldots, f_{m+t-1}\}$ and $\{g_1, \ldots, g_{n+t-1}\}$, respectively. We denote $G^* = Hom(G, R)$ the dual of G with basis $\{g_1^*, \ldots, g_{n+t-1}^*\}$ dual to the basis $\{g_1, \ldots, g_{n+t-1}\}$. Let $(r_{i,j})$ be a $(n+t-1) \times (m+t-1)$ matrix with entries in R associated to a map $\Phi : F \to G^*$. We denote by I_t to be the ideal of R generated by the $t \times t$ minors of the matrix $(r_{i,j})$.

Definition 3.1.1. Let M be a free R-module. We say that a complex

$$M_\bullet : 0 \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \to \cdots \to M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} 0$$

of free R-modules is a *minimal free complex of* M if $d_1(M_1) = M$ and for each $i = 1, \ldots, n$, we have $d_i(M_i) \subset \mathfrak{m}M_{i-1}$. The last condition is equivalent to the following: after choosing basis in M_i, for each morphism d_i the matrix associated to d_i has entries in $\mathfrak{m}$. We say that $M_\bullet$ is a *minimal free resolution of* M if $M_\bullet$ is a minimal free exact complex of M. Equivalently, if $M_\bullet$ is a minimal free complex of M and for each $i = 1, \ldots, n+1$ the homology $H_i(M_\bullet) = 0$.

Our aim is to construct a minimal free complex

$$C_\bullet(\Phi, t) : 0 \to C_{mn}(\Phi, t) \xrightarrow{d_{mn}} C_{mn-1}(\Phi, t) \to \cdots \to C_1(\Phi, t) \xrightarrow{d_1} R \to 0$$

of I_t.

3.1.1 The group algebra and combinatorics.

Definition 3.1.2. The *group algebra* $K(\mathcal{S}_n)$ is formally the set of all finite linear combinations

$$\sum_{\sigma} \alpha(\sigma)\sigma, \ \ \sigma \in \mathcal{S}_n, \ \alpha(\sigma) \in K$$

with a sum $(\sum_\sigma \alpha(\sigma)\sigma) + (\sum_\sigma \beta(\sigma)\sigma) := \sum_\sigma(\alpha(\sigma) + \beta(\sigma))\sigma$ and a product $(\sum_\sigma \alpha(\sigma)\sigma) \cdot (\sum_\rho \beta(\rho)\rho) := \sum_{\sigma,\rho} \alpha(\sigma)\beta(\rho)\sigma\rho$. The elements of $\mathcal{S}_n$ are called *group elements*, which are linearly independents and constitute a basis of $K(\mathcal{S}_n)$.

Definition 3.1.3. Let $e \in K(\mathcal{S}_n)$. We say that e is *essentially idempotent* if there is a nonzero element $\kappa \in K$ such that $e^2 = \kappa e$. We say that e is *idempotent* if $e^2 = e$.

Remark 3.1.4. If e is essentially idempotent with $e^2 = \kappa e$, then $(\kappa^{-1}e)(\kappa^{-1}e) = \kappa^{-1}e$ and hence $\kappa^{-1}e$ is idempotent.

Definition 3.1.5. Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a partition of weight n. A *Young tableau* is any tableau of shape λ with distinct entries on $\{1, \dots, n\}$.

Examples 3.1.6. For example, if $\lambda = (4, 3, 2, 1)$ two possible Young tableau are

$T =$

<table><tr><td>3</td><td>6</td><td>4</td><td>5</td></tr><tr><td>7</td><td>1</td><td>10</td><td></td></tr><tr><td>2</td><td>8</td><td></td><td></td></tr><tr><td>9</td><td></td><td></td><td></td></tr></table>

and $R =$

<table><tr><td>1</td><td>2</td><td>3</td><td>4</td></tr><tr><td>5</td><td>6</td><td>7</td><td></td></tr><tr><td>8</td><td>9</td><td></td><td></td></tr><tr><td>10</td><td></td><td></td><td></td></tr></table>

Remark 3.1.7. Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a partition of weight n. Clearly there are $n!$ possible Young tableaux of shape λ. In fact, the $n!$ possible ways of filling Δ_λ with the elements $\{1, \dots, n\}$. We can define each Young tableau as follows. We define the Young tableau λ_{id} by $\lambda_{id}(i, j) = \sum_{k=1}^{i-1} \lambda_k + j$. Each $\alpha \in K(\mathcal{S}_n)$ define a unique Young tableau λ_α by $\lambda_\alpha(i, j) = \alpha(\lambda_{id}(i, j))$, and hence any Young tableau is of the form λ_α for some $\alpha \in K(\mathcal{S}_n)$.

Observe that the above tableau R is λ_{id} for $\lambda = (4, 3, 2, 1)$. Each Young tableau λ_α defines two subgroups P_{λ_α} and Q_{λ_α} of $\mathcal{S}_n$. P_{λ_α} is the set of permutations that map each element to an element in the same row in λ_α. Q_{λ_α} is the set of permutations that map each element to an element in the same column in λ_α.

Definition 3.1.8. Let λ be a partition of weight n and let λ_α be a Young tableau. We define $P_\alpha = \sum_{p \in P_{\lambda_\alpha}} p \in K(\mathcal{S}_n)$, $\quad Q_\alpha = \sum_{q \in Q_{\lambda_\alpha}} (-1)^{sg(q)} q \in K(\mathcal{S}_n)$ and $E_{\lambda_\alpha} = P_\alpha Q_\alpha$.

Theorem 3.1.9. Let λ be a partition of weight n and let λ_α be a Young tableau. Then, E_{λ_α} is essentially idempotent. If $E_{\lambda_\alpha}^2 = \kappa_{\lambda_\alpha} E_{\lambda_\alpha}$, we denote the idempotent element $e(\lambda_\alpha) = \kappa_{\lambda_\alpha}^{-1} E_{\lambda_\alpha}$.

Definition 3.1.10. Let λ be a partition of weight n and let λ_α and λ_β be two Young tableaux. We define $s_{\alpha,\beta}$ to be the permutation that takes λ_β to λ_α. More precisely, $\lambda_\alpha(i, j) = s_{\alpha,\beta}(\lambda_\beta(i, j))$, for all $(i, j) \in \Delta_\lambda$. We denote $\lambda_\alpha = s_{\alpha,\beta}\lambda_\beta$.

Theorem 3.1.11. Let λ be a partition of weight n and let λ_α and λ_β be two Young tableaux. Then, $P_{\lambda_\alpha} = s_{\alpha,\beta}P_{\lambda_\beta}s_{\alpha,\beta}^{-1}$, $Q_{\lambda_\alpha} = s_{\alpha,\beta}Q_{\lambda_\beta}s_{\alpha,\beta}^{-1}$ and $E_{\lambda_\alpha} = s_{\alpha,\beta}E_{\lambda_\beta}s_{\alpha,\beta}^{-1}$.

Definition 3.1.12. Let λ be a partition of weight n and let λ_α and λ_β be two Young tableaux. We say that $\lambda_\alpha \ll \lambda_\beta$ if not two elements in the same column of λ_α are in the same row of λ_β.

Theorem 3.1.13. Let λ be a partition of weight n and let λ_α and λ_β be two Young tableaux. $\lambda_\alpha \ll \lambda_\beta$ if, and only if $s_{\alpha,\beta} = pq$ with $p \in P_{\lambda_\beta}$ and $q \in Q_{\lambda_\beta}$. We say that $s_{\alpha\beta}$ is a pq.

Proposition 3.1.14. Let λ be a partition of weight n and let λ_α be a Young tableau. For all $p \in P_{\lambda_\alpha}$ and $q \in Q_{\lambda_\alpha}$, $pP = Pp = P$ and $qQ = Qq = (-1)^{sg(q)}Q$.

Theorem 3.1.15. Let λ be a partition of weight n and let λ_α and λ_β be two Young tableaux. If $\lambda_\alpha \not\ll \lambda_\beta$, then $e(\lambda_\alpha)e(\lambda_\beta) = 0$.

Remark 3.1.16. $\lambda_\alpha \not\ll \lambda_\beta$ if, and only if there is two entries x and y in the same row of λ_β and in the same column of λ_α. Let (xy) be the permutation which transposes x and y. Since $(xy) \in P_{\lambda_\beta}$ and $(xy) \in Q_{\lambda_\alpha}$, (xy) has the same coefficient in $e(\lambda_\beta)$ and in $(xy)e(\lambda_\alpha)$. Then, $(xy)e(\lambda_\beta) = e(\lambda_\beta)$ and $e(\lambda_\alpha)(xy) = -e(\lambda_\alpha)$ imply that $e(\lambda_\beta)$ is divisible on the right by $id - (xy)$ and $e(\lambda_\alpha)$ is divisible on the left by $id + (xy)$.

Theorem 3.1.17. Let λ be a partition of weight n and let λ_α and λ_β be two standard Young tableaux. If $\lambda_\alpha \ll \lambda_\beta$, then $\lambda_\alpha < \lambda_\beta$, where $<$ is the usual order in tableaux induced lexicographically.

Theorem 3.1.18. Let λ be a partition of weight n and let λ_α and λ_β be two Young tableaux. The elements $s_{\alpha\beta}e(\lambda_\beta) = e(\lambda_\alpha)s_{\alpha\beta}$, where λ_α ranges over all the standard tableaux, form a basis for $K(\mathcal{S}_k)e(\lambda_\beta)$.

3.1.2 The complex.

We keep the notation introduced in section 3.1.1. In addition, we note $(mn) = (n, \ldots, n)$, such that $n + \cdots + n = nm$, and we denote $\lambda_{(mn)} = \lambda_{id}$, in the same manner we consider $(m + t - 1, n)$ and $\lambda_{((m+t-1)n)}$.

We proceed to construct the complex $C_\bullet(\Phi, t)$. First we define the modules $C_k(\Phi, t)$ by means of idempotents and we prove that they are a sum of tensor product of Schur functors. This implies that $C_k(\Phi, k)$ is a free R-module and we will describe a basis of $C_k(\Phi, k)$. In second place we determinate the morphisms of the complex, we often call them the boundary maps. And finally, we will prove that these elements define a minimal free complex of I_t.

Definition 3.1.19. Let λ be a partition. A diagonal box of Δ_λ is a box of the form (i, i). The *Durfee square* of λ is the number of diagonal boxes in Δ_λ. In other words, we say that λ has Durfee square d if $\lambda_d \geq d$ but $\lambda_{d+1} \leq d$. We arbitrarily assign (0) Durfee square 0. It is clear that λ and $\tilde{\lambda}$ have the same Durfee square.

Definition 3.1.20. Let k be an integer such that $1 \leq k \leq mn$ and let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition of weight k such that $\lambda \subset (mn)$. For $t = 1$, we define $\lambda_{F,1}$ to be the subtableau of

$\lambda_{(mn)}$ of shape λ, and we define $\lambda_{G,1}$ to be the tableau transpose of $\lambda_{F,1}$, i.e. the subtableau of $\tilde{\lambda}_{(mn)}$ of shape $\tilde{\lambda}$. Assume $t > 1$.

To the square $\lambda_{(mn)}(i,j)$ we associate:

(i) The square $\lambda_{((m+t-1)n)}(i,j)$ if $j > i$.

(ii) The vertical string of t squares $\lambda_{((m+t-1)n)}(i,j), \dots, \lambda_{((m+t-1)n)}(i+t-1,j)$ if $j = i$.

(iii) The square $\lambda_{((m+t-1)n)}(i+t-1,j)$ if $j < i$.

We define $\lambda_{F,t}$ to be the tableau obtained from $\lambda_{F,1}$ by replacing each square of $\lambda_{F,1}$ by its associate square or string of t squares. If d is the Durfee square of $\lambda_{F,1}$, then the diagram of $\lambda_{F,t}$ is obtained from the diagram of $\lambda_{F,1}$ by adjoining $t-1$ boxes to the first d columns. So, $\lambda_{F,t}$ is a tableau of shape $\lambda(F,t) = (\lambda_1, \dots, \lambda_d, d, \dots, d, \lambda_{d+1}, \dots, \lambda_m)$, with $d + \cdots + d = t-1$ and with transpose $\tilde{\lambda}(F,t) = (\tilde{\lambda}_1 + t - 1, \dots, \tilde{\lambda}_d + t - 1, \tilde{\lambda}_{d+1}, \dots, \tilde{\lambda}_{\lambda_1})$.

To the square $\tilde{\lambda}_{(mn)}(i,j)$ we associate:

(i) The square $\lambda_{((m+t-1)n)}(i,j)$ if $j < i$.

(ii) The vertical string of t squares $\lambda_{((m+t-1)n)}(i,j), \dots, \lambda_{((m+t-1)n)}(i+t-1,j)$ if $i = j$.

(iii) The square $\lambda_{((m+t-1)n)}(i+t-1,j)$ if $j > i$.

We define the tableau $\lambda_{G,t}$ in the same manner of $\lambda_{F,t}$ with the above association. The shape of $\lambda_{G,t}$ is $\lambda(G,t) := (\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_d, d, \dots, d, \widetilde{\lambda}_{d+1}, \dots, \widetilde{\lambda}_{\lambda_1})$ with $d + \cdots + d = t-1$ and with transpose $\tilde{\lambda}(G,t)$ = $(\lambda_1 + t - 1, \dots, \lambda_d + t - 1, \lambda_{d+1}, \dots, \lambda_m)$.

Remark 3.1.21. Note that if $t > 1$, in general $\lambda_{G,t}$ is not the transpose of $\lambda_{F,t}$.

For example, consider $m = 4, n = 3, t = 3, k = 5$ and $\lambda = (2,2,1)$. We have,

$$\lambda_{((m+t-1)n)} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline 10 & 11 & 12 \\ \hline 13 & 14 & 15 \\ \hline 16 & 17 & 18 \\ \hline \end{array} \qquad \lambda_{(mn)} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline 10 & 11 & 12 \\ \hline \end{array} \qquad \lambda_{F,1} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 5 \\ \hline 7 \\ \cline{1-1} \end{array} \qquad \lambda_{G,1} = \begin{array}{|c|c|c|}\hline 1 & 4 & 7 \\ \hline 2 & 5 \\ \cline{1-2} \end{array}$$

$$\lambda_{F,t} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 5 \\ \hline 7 & 8 \\ \hline 10 & 11 \\ \hline 13 \\ \cline{1-1} \end{array} \qquad \lambda_{G,t} = \begin{array}{|c|c|c|}\hline 1 & 10 & 13 \\ \hline 4 & 5 \\ \cline{1-2} 7 & 8 \\ \cline{1-2} 2 & 11 \\ \cline{1-2} \end{array}$$

Observe that $\lambda_{((m+t-1)n)}$ and $\lambda_{(mn)}$ have Durfee square 3, while the others tableaux have Durfee square 2.

Consider $\mathcal{S}_r$. If M is a free R-module, $\mathcal{S}_r$ acts on $T^r M$ as follows. If $\sigma \in \mathcal{S}_r$, we define $\sigma(m_1 \otimes \cdots \otimes m_r) = m_{\sigma^{-1}(1)} \otimes \cdots \otimes m_{\sigma^{-1}(n)}$, $m_i \in M$ for all i. So, for each $\sigma, \rho \in \mathcal{S}_r$, $(\sigma\rho)(m_1 \otimes \cdots \otimes m_r) = \sigma(\rho((m_1 \otimes \cdots \otimes m_r)))$. Since R contains K, the group algebra $K(\mathcal{S}_r)$ acts on $T^r M$ and each element of $K(\mathcal{S}_r)$ acts as an R-map.

Keeping this in mind, we consider $\lambda_{F,t}$ as in Definition 3.1.20 and $\mathcal{S}_{k+d(t-1)}$ to be the symmetric group on $\{\lambda_F(t)(1,1),\ldots,\lambda_F(t)(1,\bar{\lambda}_1),\ldots,\lambda_F(t)(m+t-1,\bar{\lambda}_{m+t-1})\}$ with d the Durfee square of $\lambda_{F,1}$. The idempotent element $e(\lambda_{F,t})$ acts on $T^{k+d(t-1)}F$ as an R-map, thus $e(\lambda_{F,t})(T^{k+d(t-1)}F)$ is a R-module of $T^{(k+d(t-1)}F$. Apply the same argue with $\lambda_{G,t}$ and G. For convenience we denote $e(\lambda_{F,t})(T^{k+d(t-1)}F)$ and $e(\lambda_{G,t})(T^{k+d(t-1)}G)$ by $e(\lambda_{F,t})F$ and $e(\lambda_{G,t})F$, respectively. We define the modules $C_k(\Phi,t)$:

Definition 3.1.22. We define the R-module $C_k(\Phi,t) := \bigoplus_{|\lambda|=k} e(\lambda_F(t))(F)\otimes e(\lambda_G(t))(G)$ for each $1\leq k\leq mn$. For $k=0$, let $C_0(\Phi,t)\cong R$.

Proposition 3.1.23. For each $1\leq k\leq mn$, $C_k(\Phi,t)\cong \bigoplus_{|\lambda|=k} L_{\tilde{\lambda}(F,t)}F\otimes L_{\tilde{\lambda}(G,t)}G$. Hence, $C_k(\Phi,t)$ is a free R-module.

Proof. We will show that $e(\lambda_{F,t})F\cong L_{\tilde{\lambda}(F,t)}F$, the proof of $e(\lambda_{G,t})G\cong L_{\tilde{\lambda}(G,t)}G$ is analogous. Remember $e(\lambda_{F,t}) = \frac{1}{\kappa_{\lambda_{F,t}}}(\sum_{\sigma\in P_{\lambda_{F,t}}}\sigma)(\sum_{\tau\in Q_{\lambda_{F,t}}}(-1)^{sg(\tau)}\tau)$ where $P_{\lambda_{F,t}}$ is the set of permutations that map each element to an element in the same row in $\lambda_{F,t}$ and $Q_{\lambda_{F,t}}$ is the set of permutations that map each element to an element in the same column in $\lambda_{F,t}$. Then, $\sigma\in P_{\lambda_{F,t}}$ has the form $(\sigma_1,\ldots,\sigma_{m+t-1})$ where σ_i is a permutation on the i-th of $\lambda_{F,t}$ and $\tau\in Q_{\lambda_{F,t}}$ has the form $(\tau_1,\ldots,\tau_{\lambda(F,t)_1})$ where τ_j is a permutation on the j-column of $\lambda_{F,t}$. For convenience we introduce the following notation to describe the factors of the module $e(\lambda_F(t))$. Since $\lambda_F(t)$ is a tableau with $k+d(t-1)$ distinct entries in $\{1,\ldots,(m+t-1)n\}$, we can establish the one-one correspondence $\lambda_F(t)(i,j)\to(i,j)$ for each $(i,j)\in\Delta_{\lambda(F,t)}$. Therefore, each permutation of the entries of $\lambda_{F,t}$ induces a permutation of the boxes of $\Delta_{\lambda(F,t)}$ and, clearly on $\Delta_{\tilde{\lambda}(F,t)}$. Then, we can consider the tensor product $T^{k+d(t-1)}F$ indexed by $\Delta_{\lambda(F,t)}$, more precisely, $T^{k+d(t-1)}F = \bigotimes_{(i,j)\in\Delta_{\lambda(F,t)}}F_{i,j} = T_{\lambda(F,t)}F$, where $F_{i,j}=F$.

Let $f = f_{(1,1)}\otimes\cdots\otimes f_{(m+t-1,\lambda(F,t)_{m+t-1})}\in T^{k+d(t-1)}F$, we have $(\sum_{\tau\in Q_{\lambda_{F,t}}}\tau)(f) = \sum_{\tau\in Q_{\lambda_{F,t}}}(-1)^{sg(\tau_1)}\cdots(-1)^{sg(\tau_q)}f_{\tau_1(1,1)}\otimes\cdots\otimes f_{\tau_{\lambda(F,t)_1}(1,\lambda(F,t)_1)}\otimes\cdots\otimes f_{\tau_1(m+t-1,1)}\otimes\cdots\otimes f_{\tau_{\lambda(F,t)_{m+t-1}}(m+t-1,\lambda(F,t)_{m+t-1})}$ and $(\sum_{\sigma\in P_{\lambda_{F,t}}}\sigma)(f) = \sum_{\tau\in P_{\lambda_{F(t)}}}f_{\sigma_1((1,1))}\otimes\cdots\otimes f_{\sigma_1((1,\lambda(F,t)_1))}\otimes\cdots\otimes f_{\sigma_{m+t-1}((m+t-1,1))}\otimes\cdots\otimes f_{\sigma_{m+t-1}((m+t-1,\lambda(F,t)_{m+t-1})}$. Let $\Upsilon : T_{\lambda(F,t)}F\to T_{\tilde{\lambda}(F,t)}$ be the canonical isomorphism permuting factors, from the above $Im(\Upsilon\circ(\sum_{\tau\in Q_{\lambda_{F,t}}}\tau))$ is the immersion of $\Lambda_{\tilde{\lambda}(F,t)}F$ in $T_{\lambda(F,t)}F$ and $Im(\sum_{\tau\in Q_{\lambda_{F,t}}}\tau)$ is the immersion of $S_{\lambda(F,t)}F$ in $T_{\lambda(F,t)}F$. More precisely, $(\Upsilon\circ(\sum_{\tau\in Q_{\lambda_{F,t}}}\tau))(f)$ corresponds bijectively to $f_{(1,1)}\wedge\cdots\wedge f_{(1,\tilde{\lambda}(F,t)_1)}\wedge\cdots\wedge f_{(\lambda(F,t)_1,1}\otimes\cdots\otimes f_{(\lambda(F,t)_1,\tilde{\lambda}(F,t)_{\lambda(F,t)_1})}$ and $(\sum_{\sigma\in P_{\lambda_{F,t}}}\sigma)(f)$ corresponds bijectively to $f_{(1,1)}\cdot\cdots\cdot f_{(1,\lambda(F,t)_1)}\otimes\cdots\otimes f_{(m+t-1,1)}\cdot\cdots\cdot f_{(m+t-1,\lambda(F,t)_{m+t-1})}$.

Since the element $e(\lambda_{F,t})(f) = \kappa^{-1}_{\lambda(F,t)}\sum_{\sigma\in P_{\lambda(F,t)}}\sum_{\tau\in Q_{\lambda_{F(t)}}}(-1)^{sg(\tau_1)}\cdots(-1)^{sg(\tau_q)}f_{\sigma_1\tau_1(1,1)}\otimes\cdots\otimes f_{\sigma_1\tau_{\lambda(F,t)_1}(1,\lambda(F,t)_1)}\otimes\cdots\otimes f_{\sigma_{m+t-1}\tau_1(m+t-1,1)}\otimes\cdots\otimes f_{\sigma_{m+t-1}\tau_{\lambda(F,t)_{m+t-1}}(m+t-1,\lambda(F,t)_{m+t-1})}$, clearly $e(\lambda_{F,t})(f)$ corresponds bijectively to
$d_{\tilde{\lambda}(F,t)}(f_{(1,1)}\wedge\cdots\wedge f_{(1,\tilde{\lambda}(F,t)_1)}\otimes\cdots\otimes f_{(\lambda(F,t)_1,1}\wedge\cdots\wedge f_{(\lambda(F,t)_1,\tilde{\lambda}(F,t)_{\lambda(F,t)_1})})$. □

We can describe a basis of $C_k(\Phi,t)$ through a basis of F and G. Let $T\in Tab_{\lambda(F,t)}\{f_1,\ldots,f_{m+t-1}\}$, we define $\hat{T}$ to be tableau of shape $\tilde{\lambda}(F,t)$ transpose to T. From the above proposition, $\{e(\lambda_{F,t})(X_T)\ \mid\ T\in Tab_{\lambda(F,t)}\{f_1,\ldots,f_{m+t-1}\}$ *and* $\hat{T}$ is standard$\}$ form a basis of $e(\lambda_{F,t})F$. Similarly, $\{e(\lambda_{G,t})(Y_T)\ \mid\ S\in Tab_{\lambda(F,t)}\{g_1,\ldots,g_{n+t-1}\}$ *and* $\hat{S}$ is standard$\}$ form a basis of $e(\lambda_{G,t})G$. Finally,

Corollary 3.1.24. $\{e(\lambda_{F,t})(X_T) \otimes e(\lambda_{G,t})(Y_S) \mid T \in Tab_{\lambda(F,t)}\{f_1, \dots, f_{m+t-1}\}$ *and* $\hat{T}$ is standard, $S \in Tab_{\lambda(F,t)}\{g_1, \dots, g_{n+t-1}\}$ *and* $\hat{S}$ is standard, $|\lambda| = k\}$ form a basis of $C_k(\Phi, t)$.

Now, we proceed to construct the boundary maps $d_k : C_k(\Phi, t) \to C_{k-1}(\Phi, t)$. As we have anticipated, the morphism d_k are induced by Φ in a natural way. The canonical isomorphisms $Hom(F, G^*) \cong F^* \otimes G^* \cong Hom(F \otimes G, R)$ allow us define

Definition 3.1.25. If $T^n F$ and $T^n G$ are indexed by sets in one-one correspondence I and $\tilde{I}$ of cardinality n, we define a map $\Phi_* : T^n F \otimes T^n G \to R$ by $((\bigotimes_{i \in I} f_i) \otimes (\bigotimes_{j \in \tilde{I}} g_j)) \to \prod_{i \in I} \Phi_{f_i}(g_{\tilde{i}})$. We say that Φ_* is the natural contraction extending Φ. More general, for any corresponding subsets of I and $\tilde{I}$ of k elements in one-one correspondence we have a map $\Phi_* : T^n F \otimes T^n G \to T^{n-k} F \otimes T^{n-k} G$ contracting these k elements.

Consider now the involution $i^{-1} : K(\mathcal{S}_n) \to K(\mathcal{S}_n)$ induced by the map of $\mathcal{S}_n$ sending $\sigma \to \sigma^{-1}$. That is, $i^{-1}(r = \sum k_s s) = \sum k_s s^{-1} := r^*$. The following proposition will be essential to proof that $C_\bullet(\Phi, t)$ is a complex.

Proposition 3.1.26. Let $\Phi_* : T^n F \otimes T^n G \to R$, where the two factors are indexed by sets I and $\tilde{I}$ in one-one correspondence $i \to \tilde{i}$. Let x and y be elements of $T^n F$ and $T^n G$ respectively, and let s and t be elements of $K(\mathcal{S}_n)$, where $\mathcal{S}_n$ denotes the symmetric group on I. Each $\sigma \in \mathcal{S}_n$ induces a permutation $\tilde{\sigma}$ on $\tilde{I}$ such that $\sigma(i) = j$ if, and only if $\tilde{\sigma}(\tilde{i}) = \tilde{\sigma}(\tilde{j})$. Then $\Phi_*((sx) \otimes (\tilde{t}y)) = \Phi_*((t^* s)x \otimes y)$.
Proof. By linearity, it suffices to prove it when σ and τ are elements of S_n. We consider the permutation on $\tilde{I}$ induced by τ^{-1}. We denote $\sigma x = \otimes_{i \in I} x_{\sigma^{-1}(i)}$ and $\tilde{\tau} y = \otimes_{j \in I'} y_{\tilde{\tau}^{-1}(j)}$. Considering the induced one-one correspondence $k' = \sigma^{-1}(i) \to \tilde{k} = \tilde{\tau}^{-1}(\tilde{i})$, we have $\Phi_*(\sigma x \otimes \tilde{\tau} y) = \prod_{i \in I} \Phi_{x_{\sigma^{-1}(i)}} y_{\tilde{\sigma}^{-1}(\tilde{i})}$. If $k' = \tilde{\tau}^{-1}(\tilde{i})$, then $\tilde{i} = \tilde{\tau}(k')$. This implies that $k' = \tilde{l}$ such that $\tau(l) = i$ and hence $k = \sigma^{-1}\tau(l)$. The above expression becomes $\prod_{l \in I} \Phi_{x_{\sigma^{-1}\tau(l)}} y_{\tilde{l}}$ which equals to $\Phi_*((\tau^{-1}\sigma)x \otimes y)$ with the induced one-one correspondence $\sigma^{-1}\tau(l) \to \tilde{l}$.
□

We consider $C_k(\Phi, t)$ as in Definition 3.1.22, where λ is a partition of weight k, $\lambda_{F,t}$ and $\lambda_{G,t}$ are the Young tableaux constructed in 3.1.20. Observe that $\lambda_{F,t}$ and $\lambda_{G,t}$ have the same entries. If a is an entry of $\lambda_{F,t}$ we denote by $\tilde{a}$ the corresponding entry in $\lambda_{G,t}$. Clearly, $a \to \tilde{a}$ defines a one-one correspondence $\sim$, and thus we can consider the tensor products $T^{(k+d(t-1)}F$ and $T^{(k+d(t-1)}G$ indexed by the entries of the tableaux $\lambda_{F,t}$ and $\lambda_{G,t}$ in one-one correspondence $\sim$.

For convenience we denote $C_\lambda(t)$ the summand $e(\lambda_{F,t})F \otimes e(\lambda_{G,t})G$ of $C_k(\phi, t)$. Let λ and μ be partitions of weight k and $k-1$, respectively. First we define a map $d_{\mu\lambda} : C_\lambda(t) \to C_\mu(t)$, we distinguish two cases:
(i) If there is an entry of $\mu_{F,1}$ that is not in $\lambda_{F,1}$, let $d_{\mu\lambda} = 0$.
(ii) Assume $\mu_{F,1} \subseteq \lambda_{F,1}$. Therefore $\mu_{F,1}$ is obtained by removing one corner a of $\lambda_{F,1}$. We often denote $\mu_{F,1}$ by $\lambda_{F,1} - a$. If d_μ is the Durfee square of μ, $C_\mu(t)$ is a submodule of $T^{k-1+d_\mu(t-1)}F \otimes T^{k-1+d_\mu(t-1)}G$ where the factors of $T^{k-1+d_\mu(t-1)}F$ and $T^{k-1+d_\mu(t-1)}G$ are indexed over the same sets as $T^{k+d(t-1)}F$ and $T^{k+d(t-1)}G$ except the entries associated to a

and $\tilde{a}$ respectively, we call them the entries of a and $\tilde{a}$ respectively. In this situation we consider the map $\Phi_* : T^{k+d(t-1)}F \otimes T^{k+d(t-1)}G \to T^{k-1+d_\mu(t-1)}F \otimes T^{k-1+d_\mu(t-1)}G$ contracting the entries of a and $\tilde{a}$ with the one-one correspondence $\sim$.

Definition 3.1.27. Let a be a corner of $\lambda_{F,1}$. If a is in the i-column of $\lambda_{F,1}$, we define $s(\lambda, a)$ to be the number of squares in the i-column above a plus the number of squares in the columns $i+1, \ldots, \lambda(F,1)_1$. Formally, $s(\lambda_{F,1}, a) := (\sum_{j=i}^{\lambda(F,t)_1} \tilde{\lambda}_j) - 1$.

Definition 3.1.28. We define the map $d_{\mu\lambda} : C_\lambda(t) \to C_\mu(t)$ to be $(-1)^{s(\lambda_{F,1},a)}(e(\mu_F(t)) \otimes e(\mu_G(t))) \circ \phi_*$, where ϕ_* denotes the restriction of Φ_* to $C_\lambda(t)$.

With this ingredients we can finally define the boundary maps of the complex $C_\bullet(\Phi, t)$.

Definition 3.1.29. For each $1 \leq k \leq mn$ we define the morphism $d_k := \bigoplus_{|\lambda|=k} d_\lambda = \bigoplus_{|\lambda|=k} \bigoplus_{|\mu|=k-1} d_{\mu\lambda}$ from $C_k(\Phi, t)$ to $C_{k-1}(\Phi, t)$.

Observe that ϕ_* is defined by a matrix with coefficients in $\mathfrak{m}$, and hence $[e(\mu_{F,t}) \otimes e(\mu_{G,t}]\phi_*$ sends C_λ into $\mathfrak{m}C_\mu$. Therefore, $d_k(C_k(\Phi, t)) \subset \mathfrak{m}C_{k-1}(\Phi, t))$.

Proposition 3.1.30. $d_1(C_1(\Phi, t)) = I_t$.
Proof. There is only one partition $\lambda = (1)$ satisfying $|\lambda| = 1$. In this case, $\lambda(F, t) = (1, \ldots, 1) = \lambda(G, t)$ and

$$\lambda_{F,t} \quad = \quad \begin{array}{|c|} \hline 1 \\ \hline \vdots \\ \hline t \\ \hline \end{array} \quad = \quad \lambda_{G,t}$$

Therefore, $C_1(\Phi, t) = e(\lambda_{F,t})F \otimes e(\lambda_{G,t})G$ and $\{e(\lambda_{F,t})(f_{i_1} \otimes \cdots \otimes f_{i_t}) \otimes e(\lambda_{G,t})(g_{j_1} \otimes \cdots \otimes g_{j_t}) \mid 1 \leq i_1 < \cdots < i_t \leq m+t-1, 1 \leq j_1 < \cdots < j_t < n+t-1\}$ form a basis of $C_1(\Phi, t)$. Since $\sum_{\sigma \in P_{\lambda_{F,t}}} \sigma = id$, we have from Proposition 3.1.26 that $\phi_*(e(\lambda_{F,t})(f_{i_1} \otimes \cdots \otimes f_{i_t}) \otimes e(\lambda_{G,t})(g_{j_1} \otimes \cdots \otimes g_{j_t})) = \phi_*(\sum_{\sigma \in \mathcal{S}_t} (-1)^{sg(\tau)} f_{\sigma(i_1)} \otimes \cdots \otimes f_{\sigma(i_t)} \otimes g_{j_1} \otimes \cdots \otimes g_{j_t}) = \sum_{\sigma \in \mathcal{S}_t} (-1)^{sg(\tau)} \Phi_{f_{\sigma(i_1)}} g_{j_1} \cdots \Phi_{f_{\sigma(i_t)}} g_{j_t} = \sum_{\sigma \in \mathcal{S}_t} (-1)^{sg(\tau)} r_{\sigma(i_1)j_1} \cdots r_{\sigma(i_t)j_t}$, which is the $t \times t$ minor of $(r_{i,j})$ corresponding to the rows $j_1, \ldots, j_t$ and the columns $i_1, \ldots, i_t$ of $(r_{i,j})$. $\square$

It remains to prove that $(C_\bullet(\Phi, t), d_\bullet)$ is a complex. More specifically, we must show that

$$\bigoplus_{|\nu|=k-2, |\mu|=k-1, |\lambda|=k-1} d_{\nu\mu} d_{\mu\lambda} = 0.$$

The partitions ν involved in this sum must be such that $\nu_{F,1} = \lambda_{F,1} - a - b = \lambda_{F,1} - b - a$, where a is a corner of $\lambda_{F,1}$ and b becomes a corner of $\lambda_{F,t} - a$ or vice versa. There are two possible configurations of two entries a and b in $\lambda_{F,1}$ which can be removed.
(1) a or b is not a corner of $\lambda_{F,1}$ and there is only one order to remove them. If a is corner of $\lambda_{F,1}$, then $\lambda_{F,1}$ is of the form:

$$\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & \\ \cline{1-3} & b & a \\ \cline{1-3} \\ \cline{1-1} \end{array} \qquad \text{or} \qquad \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & b \\ \cline{1-3} & & a \\ \cline{1-3} \\ \cline{1-1} \end{array}$$

In this case, there is only one partition μ such that $d_{\nu\mu}$ and $d_{\mu\lambda}$ are nonzero.
(2) a and b are both corners of $\lambda_{F,1}$, and hence there are two possible ways of removing them. In this second case $\lambda_{F,1}$ is of the form:

			a	
	b			

or

			b	
	a			

Then, there are two partitions μ such that $d_{\nu\mu}$ and $d_{\mu\lambda}$ are non zero. We may assume that b is bellow a, the other case is analogous. Observe that $s(\lambda - a, b) = s(\lambda, b) - 1$ and $s(\lambda - b, a) = s(\lambda, a)$. Therefore, $(-1)^{s(\lambda-a,b)}(-1)^{s(\lambda,a)} = -(-1)^{s(\lambda-b,a)}(-1)^{s(\lambda,b)}$.

From those arguments to prove that $C_\bullet(\Phi, t)$ is a complex follows from showing that given two entries a and b which can be removed, with b a corner:
(i) If a is not a corner of $\lambda_{F,1}$, then $d_{(\lambda_{F,1}-b-a)(\lambda_{F,1}-b)}d_{(\lambda_{F,1}-b)\lambda_{F,1}} = 0$.
(ii) If a is a corner of $\lambda_{F,1}$ above b, then

$$d_{(\lambda_{F,1}-b-a)(\lambda_{F,1}-b)}d_{(\lambda_{F,1}-b)\lambda_{F,1}} + d_{(\lambda_{F,1}-b-a)(\lambda_{F,1}-a)}d_{(\lambda_{F,1}-a)\lambda_{F,1}} = 0.$$

Now, $d_{\nu\mu}d_{\mu\lambda}$ module sign correspond to $[e(\nu_{F,t}) \otimes e(\nu_{G,t})]\phi^1_*[e(\mu_{F,t}) \otimes e(\mu_{G,t}]\phi^2_*$. Since $e(\lambda_{F,t}) \otimes e(\lambda_{G,t})$ as an idempotent is the identity to C_λ, we can write $d_{\nu\mu}d_{\mu\lambda}$ as $[e(\nu_{F,t}) \otimes e(\nu_{G,t})]\phi^1_*[e(\mu_{F,t}) \otimes e(\mu_{G,t}]\phi^2_*[e(\lambda_{F,t}) \otimes e(\lambda_{G,t})]$. If $\mu_{F,1} = \lambda_{F,1} - b$ for some corner b of $\lambda_{F,1}$, then ϕ^2_* contracts b and $\tilde{b}$ and not acts on $e(\mu_{F,t})F$ and $e(\mu_{G,t})G$. So, we can apply ϕ^2_* after the idempotent $[e(\mu_{F,t}) \otimes e(\mu_{G,t}]$. The composition $\phi^1_*\phi^2_*$ defines a contraction on the union of the entries of a and b. Applying the above argument, $d_{\nu\mu}d_{\mu\lambda}$ can be written as the composition

$$\phi_*[e(\nu_{F,t}) \otimes e(\nu_{G,t})][e(\mu_{F,t}) \otimes e(\mu_{G,t})][e(\lambda_{F,t}) \otimes e(\lambda_{G,t})]. \quad (*)$$

For the remainder of this section we fix k, λ a partition of weight k, b and a be squares which can be removed. We write $\lambda_{F,t} - b = \mu_{F,t}$ and $\lambda_{F,t} - b - a = \nu_{F,t}$. For convenience we denote $\hat{e}(\lambda) = [e(\lambda_{F,t}) \otimes e(\lambda_{G,t})], e(\lambda) = e(\lambda_{F,t})$ and $e(\tilde{\lambda}) = e(\lambda_{G,t})$. The purpose of expressing $d_{\nu\mu}d_{\mu\lambda}$ as $(*)$ is that identities involving $\hat{e}(\nu)\hat{e}(\mu)\hat{e}(\lambda)$ become identities involving $d_{\nu\mu}d_{\mu\lambda}$ just by adding ϕ_* on the left. More precisely, conditions (i) and (ii) are equivalent to:

Lemma 3.1.31. Let b a corner of λ_F and let a be a square of λ_F directly above or to the left of b in such a way that it becomes a corner of $\lambda_F - b$. Then,

$$\Phi_*\hat{e}(\lambda - a - b)\hat{e}(\lambda - b)\hat{e}(\lambda) = 0.$$

Lemma 3.1.32. Let a and b be corners of λ_F. Then,

$$\Phi_*\hat{e}(\lambda - a - b)\hat{e}(\lambda - b)\hat{e}(\lambda) = \Phi_*\hat{e}(\lambda - a - b)\hat{e}(\lambda - a)\hat{e}(\lambda).$$

Clearly, we can write $\hat{e}(\lambda - a - b)\hat{e}(\lambda - b)\hat{e}(\lambda) = [e(\lambda - a - b)e(\lambda - b)e(\lambda)][e(\tilde{\lambda} - a - b)\hat{e}(\tilde{\lambda} - b)\hat{e}(\tilde{\lambda})]$. We dedicate the following subsections to the proof of both Lemmas, which implies that $C_\bullet(\Phi, t)$ is a complex. First we consider the following.

Remember Theorem 3.1.18 and notations in subsection 3.1.1, let $\mathcal{S}_{k+d(t-1)}$. In particular, noting $s_{\alpha,1}$ permutations from $\lambda_{F,t}$ to λ_α. The following propositions are found in [18] and [3].

Proposition 3.1.33. We can write $e(\lambda - b)e(\lambda) = \sum_* k_\alpha s_{\alpha,1} e(\lambda)$, where $*$ denotes that the sum runs over the standard tableaux $\lambda_\alpha = s_{\alpha,1}\lambda_{F,t}$. We denote it simply by $\sum k_\alpha s_{\alpha,1} e(\lambda)$.

Remark 3.1.34. There is not one way of write $s_{\alpha 1}$ in the form $s_{\sigma 1}pq$. Therefore, in general the coefficient of $s_{\alpha 1}$ in $e(\lambda - b)e(\lambda)$ is not k_α. In fact, it is the sum of all the terms $\frac{(-1)^{sg(q)}k_\sigma}{\kappa_{\lambda_{F,t}}}$ such that $s_{\alpha 1} = s_{\sigma 1}pq$.

Definition 3.1.35. Let $\lambda_\alpha = s_{\alpha 1}\lambda_{F,t}$ be a standard tableaux. We define j_α to be $\kappa_{\lambda_{F,t}}$ times the coefficient of $s_{\alpha 1}$ in $e(\lambda - b)e(\lambda)$.

Proposition 3.1.36. Let $\lambda_\alpha = s_{\alpha,1}\lambda_{F,t}$. We can write $s_{\alpha 1} = s_{\sigma 1}pq$ where $p \in P_{\lambda_{F,t}}$ and $q \in Q_{\lambda_{F,t}}$ if, and only if $\lambda_\alpha \ll \lambda_\sigma$.

Proposition 3.1.37. Consider $e(\lambda - b)e(\lambda) = \sum k_\alpha s_{\alpha 1} e(\lambda)$. Suppose there are two elements in the same column in λ_α and in the same row in $\lambda_{F,t}$, and neither of these is an entry of b. Then the coefficient of $s_{\alpha 1}$ in $e(\lambda - b)e(\lambda)$ is zero.

Proposition 3.1.38. Let a be an entry of $\lambda_{F,t}$ which becomes a corner of $\lambda_{F,t} - b$. Consider $e(\lambda - b)e(\lambda) = \sum k_\alpha s_{\alpha 1} e(\lambda)$. Suppose that

1. All entries of $\lambda_{F,t} - a - b$ precede the entries of a and b. That is, the entries in a and b are the highest entries in $\lambda_{F,t}$.
2. The entry in a given row of $\lambda_{F,t} - a - b$ precedes any element in any lower row. That is, given an entry $(\lambda_{F,t} - a - b)(i,j)$, then $(\lambda_{F,t} - a - b)(k,l) > (\lambda_{F,t} - a - b)(i,j), \quad \forall k > i, \forall l$.

Then, if $s_{\alpha 1}$ moves any entry of $\lambda_{F,t} - a - b$, we have either

(i) $k_\alpha = 0$, or

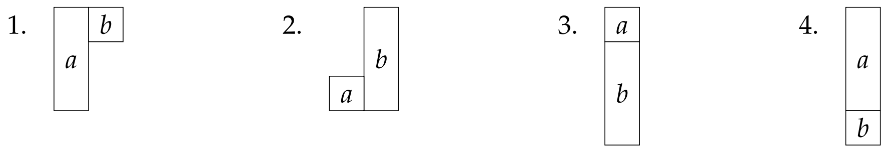
(ii) $e(\lambda - a - b)s_{\alpha 1} e(\lambda) = 0$.

In addition, if $s_{\alpha 1}$ moves only elements of a and b, we have $j_\alpha = k_\alpha$.

3.1.3 The Proof of Lemma 3.1.31.

We consider $\mathcal{S}_{k+d(t-1)}$ on the entries of $\lambda_{F,t}$ ordered first row by row in $\lambda_{F,t} - a - b$ and then the entries of a and b. In other words, $\lambda_{F,t}$ is a tableau as in Proposition 3.1.38. The proof of both Lemmas are based on effectives calculations of the product of the idempotents $e(\lambda - a - b), e(\lambda - a), e(\lambda - b)$ and $e(\lambda)$. Our goal is to develop techniques which simplify this expressions.

There are these possible configurations for b and a, depending on whether b and a are in the same row or the same column in $\lambda_{F,1}$, and depending on whether a or b corresponds to a single square or a string of t squares. The four possible arrangements in $\lambda_{F,t}$ are



where we have considered the case where both a and b are single squares as a special case of either of these. First we study cases 1. and 2.

Proposition 3.1.39. Let b a corner of $\lambda_{F,1}$ and let a be a square directly to the left of b. Then, the product $e(\lambda - a - b)e(\lambda - b)e(\lambda)$ is a sum of terms each of which is divisible on the left by $id + (xy)$, for some distinct elements x and y in the entries of a and b.

Proof. We let $e(\lambda - b)e(\lambda) = \sum k_\alpha s_{\alpha 1} e(\lambda)$. By Proposition 3.1.38, in the above expression we only have to consider terms for which $s_{\alpha 1}$ only moves the entries of a and b. Let $s_{\alpha 1}$ be of that such, then there are two elements x and y which end up in a row of λ_α.

Denoting (xy) the permutation which transposes x and y, hence (xy) is a p in P_α. Since $e(\lambda_\alpha) = (xy)e(\lambda_\alpha)$, $e(\lambda_\alpha)$ is divisible on the left by $id + (xy)$. Therefore, $e(\lambda - a - b)e(\lambda - b)e(\lambda)$ is divisible on the left by $id + (xy)$, because $s_{\alpha 1} e(\lambda) = e(\lambda_\alpha) s_{\alpha 1}$ and $e(\lambda - a - b)$ does not act over the entries of a and b. □

Computation of cases 3. and 4. require a previous result.

Proposition 3.1.40. Let λ be any tableau, let c be a column of consecutive squares of λ all of which are at the right ends of their rows and such that the bottom square of c is a corner. Then, $e(\lambda - c)$ is defined and for any permutation q of the entries in c, we have $qe(\lambda - c)e(\lambda) = (-1)^{sg(q)} e(\lambda - c - d)e(\lambda)$. Similarly, if d is another corner square of λ, we have $qe(\lambda - c - d)e(\lambda) = (-1)^{sg(q)} e(\lambda - c - d)e(\lambda)$.

Proof. Suppose c is in the nth column of λ. For $i = 1, 2, \ldots, n-1$, let q_i be the permutation on the ith column such that if q moves the entry in the box (k, n) to (k', n), then q_i moves the entry in the box (k, i) to (k', i). We define $\bar{q}$ to be $\bar{q} = q_1 q_2 \cdots q_{n-1}$ and consider $q\bar{q}$, clearly $q\bar{q} \in Q_\lambda$, $\bar{q} \in Q_{\lambda - c}$ and $\bar{q} \in Q_\lambda$. Since $qe(\lambda - c) = e(\lambda - c)q$ and $q\bar{q} = \bar{q}q$, we have $qe(\lambda - c)e(\lambda) = (-1)^{sg(q)} e(\lambda - c)e(\lambda)$. Let d be another corner squares of λ, since d must be below c and the left of c, $\bar{q} \in Q_{\lambda - c - d}$ and the results follows. □

Proposition 3.1.41. Let b a corner of $\lambda_{F,1}$ and let a be a square directly above of b. Then $e(\lambda - a - b)e(\lambda - b)e(\lambda)$ is divisible on the left by $id - (xy)$ for all pairs x and y of entries in a and b.

Proof. We let $e(\lambda - b)e(\lambda) = \sum k_\alpha s_{\alpha 1} e(\lambda)$. By Proposition 3.1.38, we only need to consider factors $s_{\alpha 1}$ moving elements in the entries of a and b. Since these entries are consecutive, they are in the same column in strictly increasing order. If λ_α is a standard tableau, $s_{\alpha 1}$ must be the identity, and hence $e(\lambda - b)e(\lambda) = k_{id} e(\lambda)$. Observe that for all pairs x and y of entries in a and b, the transposition (xy) is a q in a. From the above proposition $e(\lambda - a - b)e(\lambda) = -qe(\lambda - a - b)e(\lambda)$. It follows that $e(\lambda - a - b)e(\lambda)$ is divisible on the left by $id - (xy)$ for all pairs x and y in the entries of a and b. □

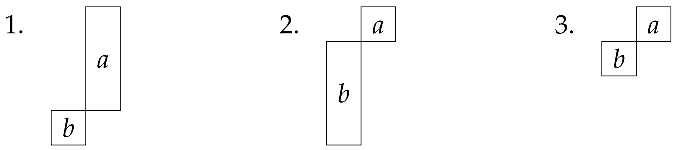
Observe that b is a corner of $\lambda_{F,1}$ and a is directly on the left of b if, and only if $\tilde{b}$ is a corner of $\lambda_{G,1}$ and $\tilde{a}$ is directly above of $\tilde{b}$, and vice versa. Assume that 1. or 2. holds for $\lambda_{F,t}$, from Proposition 3.1.40 we can write $e(\lambda - a - b)e(\lambda - b)e(\lambda) = \sum_{x,y}(1 + (xy))A_{x,y}$, where $A_{x,y}$ are elements of $K(S_{k+d(t-1)})$ indexed over pairs x and y of entries of a and b. By Proposition 3.1.41, for each pair $\tilde{x}$ and $\tilde{y}$ of entries in $\tilde{a}$ and $\tilde{b}$ we can find $B_{x,y}$ such that $e(\tilde{\lambda} - \tilde{a} - \tilde{b})e(\tilde{\lambda} - \tilde{b})e(\tilde{\lambda}) = (1 - (\tilde{x}\tilde{y})B_{x,y}$. At this moment we are in disposition to prove Lemma 1.

With the above expressions we have $\hat{e}(\lambda - a - b)\hat{e}(\lambda - b)\hat{e}(\lambda) = e(\lambda - a - b)e(\lambda - b)e(\lambda)e(\tilde{\lambda} - \tilde{a} - \hat{b})e(\tilde{\lambda} - \tilde{b})e(\tilde{\lambda}) = \sum_{x,y}(id + (xy))A_{x,y}e(\tilde{\lambda} - \tilde{a} - \tilde{b})e(\tilde{\lambda} - \tilde{b})e(\tilde{\lambda}) = \sum_{x,y}(id + (xy))A_{x,y}(id - (\tilde{x}\tilde{y}))B_{x,y}$. Applying ϕ_*, it follows from Proposition 3.1.26 that

$\phi_*(\hat{e}(\lambda - a - b)\hat{e}(\lambda - b)\hat{e}(\lambda)) = \phi_*(\sum_{x,y}(1 + (xy))A_{x,y}(1 - (\tilde{x}\tilde{y}))B_{x,y}) = \phi_*(\sum_{x,y}(1 + (xy))(1 - (xy))A_{x,y}B_{x,y})) = 0$,
where we have used that $(id - (xy))^* = id - (xy)$. □

3.1.4 The proof of Lemma 3.1.32.

We may assume that b is below a, the other case is similar. The proof of Lemma 3.1.32 is basically the same as Lemma 3.1.31, however the expressions involving the idempotents are significantly more complicated as before. The three possible arrangements for a and b in $\lambda_{F,t}$ are



We start estimating the coefficient of id.

Proposition 3.1.42. Let $e(\lambda - a)e(\lambda) = \sum k_\alpha s_{\alpha 1} e(\lambda)$. Then, $k_{id} = 1$.
Proof. Since id does not move any element of $\lambda - b - a$, by Proposition 3.1.38 we have $j_{id} = k_{id}$. Remember j_{id} is $k_{\lambda_{F,1}}$ times the coefficient of id in $e(\lambda - a)e(\lambda) = \frac{1}{\kappa_\lambda \kappa_{\lambda - a}} E_{\lambda - a}E_\lambda$. In fact, j_{id} is $\frac{1}{\kappa_{\lambda - a}}$ times the coefficient of id in $E_{\lambda - a}E_\lambda$. If $Id = pqp'q'$ where $p \in P_{\lambda - a}, q \in Q_{\lambda - a}, p' \in P_\lambda$ and $q' \in Q_\lambda$, then since pq does not act over the entries of a, necessarily p' and q' cannot permute the entries of a either. So, $Id = pqp'q'$ if, and only if $Id = pq\rho\sigma$, where $\rho \in P_{\lambda - a}$ and $\sigma \in P_{\lambda - a}$. This means that the coefficient of id in $E_{\lambda - a}E_\lambda$ equals to the coefficient of id in $E_{\lambda - a}E_{\lambda - a} = \kappa_{\lambda - a}E_{\lambda - a}$, which is $\kappa_{\lambda - a}$. We obtain that $j_{id} = \frac{\kappa_{\lambda - a}}{\kappa_{\lambda - a}} = 1$.
□

Proposition 3.1.43. $e(\lambda - a - b)e(\lambda - b)e(\lambda) = e(\lambda - a - b)e(\lambda)$
Proof. Let $e(\lambda - b)e(\lambda) = \sum k_\alpha s_{\alpha 1} e(\lambda)$, by Proposition 3.1.38 we only have to consider permutations $s_{\alpha 1}$ which move only the entries in a and b, and we know that $j_\alpha = k_\alpha$. Let $s_{\alpha 1}$ of that such and λ_α. Since λ_α is standard and the entries in a and b are in increasing order, if $s_{\alpha 1}$ is not the identity, then at least one entry from a must go to a position in b in λ_α. Let j and j' be the columns of a and b respectively. If $s_{\alpha 1}$ moves the entry of a in the box (i, j) to the box (i', j') in b, then the entries in the boxes $(i, j))$ and (i, j') are in the same row in $\lambda_{F,t}$ and in the same column of λ_α. Considering that b is below a and to the left of a, the entry in the box (i, j') is not in b. Therefore, there are two elements in the

same column in λ_α and in the same row in $\lambda_{F,t}$, and neither of these is an entry of b. By Proposition 3.1.37 $j_\alpha = 0 = k_\alpha$ and hence, the only term which contributes to the product $\sum k_\alpha s_{\alpha 1} e$ is id. We have, $e(\lambda - a - b)e(\lambda - b)e(\lambda) = e(\lambda - a - b)k_{id}e(\lambda) = e(\lambda - a - b)e(\lambda)$, since as we have just seen before $k_{id} = 1$. □

However, the calculation of $e(\lambda - a - b)e(\lambda - a)e(\lambda)$ is considerably more complicated. In this case one has $e(\lambda - a)e(\lambda) = \sum k_\alpha s_{\alpha 1} e(\lambda)$ and again by Proposition 3.1.38 we only have to consider permutations $s_{\alpha 1}$ which only move the entries of a and b. Nevertheless, we cannot apply the last proposition in that case. In fact, it is true that there are two elements in the same column in λ_α and in the same row in $\lambda_{F,t}$ but, one of this elements is an entry of a and hence we cannot apply Proposition 3.1.37. At the beginning, there are nontrivial permutations $s_{\alpha 1}$ with $k_\alpha \neq 0$ and we will have to find these values. For convenience we denote the entries of a and b by $1, \ldots, t+1$, where 1 denotes the entry in whichever a or b is a single square.

Proposition 3.1.44. Let s and s' be permutations of $1, \ldots, t+1$ such that $s(1) = s'(1)$. We have:

(i) $e(\lambda - a - b)se(\lambda) = (-1)^{s+s'} e(\lambda - a - b)s'e(\lambda)$.

(ii) If j is the coefficient of s in $e(\lambda - a)e(\lambda)$ and j' of s', then $j = (-1)^{s+s'} j'$.

Proof. (i) Let $q = s^{-1}s'$, since $s(1) = s'(1)$ we are in the hypothesis of Proposition 3.1.40. Then, $qe(\lambda - a - b)e(\lambda) = (-1)^s g(q)e(\lambda) = (-1)^{sg(s^{-1})+sg(s')} e(\lambda - a - b)e(\lambda)$ and the result follows.

(ii) We have $s' = sq$. Since $q \in Q_\lambda$ we know that $e(\lambda)q = (-1)^{sg(q)} e(\lambda)$. Then, the coefficient of s in $e(\lambda - a)e(\lambda)$ is the coefficient of s' in $(-1)^{sg(q)} e(\lambda - a)e(\lambda)$. □

Let $e(\lambda - a)e(\lambda) = \sum k_\alpha s_{\alpha 1} e(\lambda)$. If $s_{\alpha,1}$ only moves $1, 2, \ldots, t+1$, then $s_{\alpha 1}$ is characterized by the image of 1 and we know that $j_\alpha = k_\alpha$. Let $s_{\alpha 1}(1) = r \neq 1$, from the above proposition considering $(1r)$, $e(\lambda - a - b)s_{\alpha 1} e(\lambda) = (-1)^{sg(s_{\alpha 1})-1} e(\lambda - a - b)(1r)e(\lambda)$ and the coefficient of $s_{\alpha 1}$ equals to $(-1)^{sg(s_{\alpha 1})-1}$ times the coefficient of $(1r)$ in $e(\lambda - a)e(\lambda)$. We have $e(\lambda - a - b)e(\lambda - a)e(\lambda) = \sum k_\alpha e(\lambda - a - b)s_{\alpha 1} e(\lambda) = \sum k_\alpha (-1)^{sg(s_{\alpha 1})-1} e(\lambda - a - b)(1r)e(\lambda) \sum k_r e(\lambda - a - b)(1r)e(\lambda) = e(\lambda - a - b)(1 + \sum_{r=2}^{t+1} k_r (1r))e(\lambda)$.

Proposition 3.1.45. If r and r' are the entries in the t string of boxes, then $k_r = k_{r'}$.
Proof. We can write $(1r') = (rr')(1r)(rr')$. If r and r' are in the nth column in the positions (j, n) and (j', n), for $i = 1, \ldots, n-1$ we define q_i to be the transposition on the ith column which transposes the elements in the positions (j, i) and (j', i). Let $\bar{q} = q_1 \cdots q_{n-1}$, we note that $\bar{q}^2 = Id$. Since $(1r') = (rr')\bar{q}\bar{q}(1r)(rr') = \bar{q}(rr')(1r)\bar{q}(rr')$ and $\bar{q}(rr')$ commutes with $e(\lambda - a)$ and $e(\lambda)$, we have $\bar{q}(rr')e(\lambda - a)e(\lambda)\bar{q}(rr') = e(\lambda - a)e(\lambda)[\bar{q}(rr')]^2 = e(\lambda - a)e(\lambda)$. The coefficient of $(1r')$ in $e(\lambda - a)e(\lambda)$ is the coefficient of $(1r)$. □

Proposition 3.1.46. Suppose a is a column of t squares. Let λ' be the tableau obtained from λ by removing the last $t-1$ entries of a. Then, $j_{(12)}$ is $\kappa_{\lambda'}$ times the coefficient of (12) in $e(\lambda' - a')e(\lambda')$.
Proof. Since $\lambda' - a' = \lambda - a$, $\frac{1}{\kappa_{\lambda - a}} = \frac{1}{\kappa_{\lambda' - a'}}$ and the coefficient of (12) in $E_{\lambda - a}E_\lambda$ is the coefficient in $E_{\lambda' - a'}E_\lambda$. It is sufficient to see that the coefficient of (12) in $E_{\lambda - a}E_\lambda$ is the

coefficient of (12) in $E_{\lambda'-a'}E_{\lambda'}$. If $(12) = p'q'pq \in E_{\lambda'-a'}E_{\lambda}$, since $p'q'$ and (12) do not act on the last entries of a, then pq does also and clearly $pq \in E_{\lambda'}$. Hence, the coefficient of (12) in $E(\lambda - a)E(\lambda)$ is the coefficient of (12) in $E(\lambda' - a')E(\lambda')$. □

From the above propositions if a is a column of t squares, we can write $e(\lambda - a - b)e(\lambda - a)e(\lambda) = e(\lambda - a - b)(1 + k_2 \sum_{r=2}^{t+1}(1r))e(\lambda)$ and when b is a column of t squares, $e(\lambda - a - b)e(\lambda - a)e(\lambda) = e(\lambda - a - b)(1 + k_{t+1} \sum_{r=2}^{t+1}(1r))e(\lambda)$.

First we prove Lemma 3.1.32 in the case 3. when a and b are both single squares. Next, we will see that we can reduce the other cases to this. We need one more result which we can find in [18] Proposition 13, it is based on Young's Semiformals representation of $K(\mathcal{S}_n)$. First a simple definition.

Definition 3.1.47. The *axial distance* between two boxes x and y of a standard Young tableau T is the number of steps to get from x to y, where steps contribute with $+1$ when going down or to the left or -1 when going up or to the right. The axial distance does not depend on the path from x to y.

Proposition 3.1.48. Let a and b be two corners of λ, with b below a. Then, the coefficient of (ab) in $e(\lambda - a)e(\lambda)$ is $\frac{1}{(k_{\lambda\eta})}$, where η is the axial distance from a to b in λ. Hence we have $k_{(ab)} = j_{(ab)} = \frac{1}{\eta}$.

Let η be the axial distance from a to b, we have $e(\lambda - a - b)e(\lambda - a)e(\lambda) = e(\lambda - a - b)[1 + \frac{1}{\eta}(ab)]e(\lambda)$ and $e(\lambda - a - b)e(\lambda - b)e(\lambda) = e(\lambda - a - b)e(\lambda)$. In the same manner, $e(\tilde{\lambda} - \tilde{a} - \tilde{b})e(\tilde{\lambda} - \tilde{a})e(\tilde{\lambda}) = e(\tilde{\lambda} - \tilde{a} - \tilde{b})$ and $e(\tilde{\lambda} - \tilde{a} - \tilde{b})e(\tilde{\lambda} - \tilde{b})e(\tilde{\lambda}) = e(\tilde{\lambda} - \tilde{a} - \tilde{b})[1 + \frac{1}{\eta}(\tilde{a}\tilde{b})]e(\tilde{\lambda})$.

Since permutation (ab) commutes with $e(\lambda - a - b)$ and $(\tilde{a}\tilde{b})$ with $e(\mu - \tilde{a} - \tilde{b})$, we can write $\hat{e}(\lambda - a - b)\hat{e}(\lambda - a)\hat{e}(\lambda) = [1 + \frac{1}{\eta}(ab)]e(\lambda - a - b)e(\lambda)e(\tilde{\lambda} - \tilde{a} - \tilde{b})e(\tilde{\lambda})$ and $\hat{e}(\lambda - a - b)\hat{e}(\lambda - b)\hat{e}(\lambda) = e(\lambda - a - b)e(\lambda)[1 + \frac{1}{\eta}(\tilde{a}\tilde{b})]e(\tilde{\lambda} - \tilde{a} - \tilde{b})e(\tilde{\lambda})$. Applying ϕ_*, $\phi_*(\hat{e}(\lambda - a - b)\hat{e}(\lambda - b)\hat{e}(\lambda)) = \phi_*(e(\lambda - a - b)e(\lambda)[1 + \frac{1}{\eta}(\tilde{a}\tilde{b})]e(\mu - \tilde{a} - \tilde{b})e(\tilde{\lambda}) = \phi_*([1 + \frac{1}{\eta}(ab)]e(\lambda - a - b)e(\lambda)e(\tilde{\lambda} - \tilde{a} - \tilde{b})e(\tilde{\lambda}) = \phi_*(\hat{e}(\lambda - a - b)\hat{e}(\lambda - a)\hat{e}(\lambda))$. This completes the proof of Lemma 3.1.32 in case 3.

Suppose now a is a column of t squares, as we have seen $e(\lambda - a - b)e(\lambda - a)e(\lambda) = e(\lambda - a - b)(1 + k_2 \sum(1r))e(\lambda)$. From Proposition 3.1.46 k_2 is $\kappa_{\lambda'}$ times the coefficient of (12) in $e(\lambda' - a)e(\lambda')$, where λ' is the tableau obtained from $\lambda_{f,t}$ by removing the squares $3, \ldots, t+1$. Considering λ', a' and b, we are in the hypothesis of Proposition 3.1.48, and hence k_2 is $\frac{1}{\eta}$ where η is the axial distance from a' to b which is, in fact, $\eta_0 + t - 1$, where η_0 is the axial distance from a to b in $\lambda_{F,1}$. Therefore, $\hat{e}(\lambda - a - b)\hat{e}(\lambda - a)\hat{e}(\lambda) = [1 + \frac{1}{\eta} \sum_{r=2}^{t+1}(1r)]e(\lambda - a - b)e(\lambda)e(\tilde{\lambda} - \tilde{a} - \tilde{b})e(\tilde{\lambda})$.

In $\lambda_{G,t}$ $\tilde{a}$ is below $\tilde{b}$, with $\tilde{a}$ a column of t squares and b a single square. As we have seen before, $e(\tilde{\lambda} - \tilde{a} - \tilde{b})e(\tilde{\lambda} - \tilde{b})e(\tilde{\lambda}) = [1 + k_{\tilde{2}} \sum_{\tilde{r}=2}^{\widetilde{t+1}}(1\tilde{r})]e(\tilde{\lambda} - \tilde{a} - \tilde{b})e(\tilde{\lambda})$. If we consider $\tilde{\lambda}'$ to be the tableau obtained from $\lambda_{G,t}$ by removing the last $t-1$ squares from $\tilde{a}$, then $k_{\tilde{2}}$ is $\kappa_{\tilde{\lambda}'}$ times the coefficient of (12) in $e(\tilde{\lambda}' - \tilde{a}')e(\tilde{\lambda}')$, which is $\frac{1}{\tilde{\eta}}$ and where $\tilde{\eta}$ is the axial distance from $\tilde{b}$ to $\tilde{a}'$, that is, the axial distance from $\tilde{b}$ to the top square of $\tilde{a}$. Clearly $\tilde{\eta} = \eta_0 + t - 1 =$

η, since the axial distance distance from $\tilde{b}$ to $\tilde{a}$ in $\lambda_{G,1}$ is the axial distance from a to b in $\lambda_{F,1}$. Therefore, $\hat{e}(\lambda-a-b)\hat{e}(\lambda-b)\hat{e}(\lambda) = e(\lambda-a-b)e(\lambda)[1+\frac{1}{\eta}\sum(1\tilde{r})]e(\tilde{\lambda}-\tilde{a}-\tilde{b})e(\tilde{\lambda})$. Since $[1+\frac{1}{\eta}\sum(1\tilde{r})]^* = [1+\frac{1}{\eta}\sum(1\tilde{r})]$, we have $\phi_*(\hat{e}(\lambda-a-b)\hat{e}(\lambda-b)\hat{e}(\lambda)) = \phi_*(e(\lambda-a-b)e(\lambda)[1+\frac{1}{\eta}\sum(1\tilde{r})]e(\mu-\tilde{a}-\tilde{b})e(\mu)) = \phi_*([1+\frac{1}{\eta}\sum(1r)]e(\lambda-a-b)e(\lambda)e(\tilde{\lambda}-\tilde{a}-\tilde{b})e(\tilde{\lambda}) = \phi_*(\hat{e}(\lambda-a-b)\hat{e}(\lambda-b)\hat{e}(\lambda))$.

When a is a single square and b is a column of t squares, the proof is exactly the same as in the previous case considering a simple modification of Proposition 3.1.46.

Proposition 3.1.49. Suppose b is a column of t squares. Let λ' be the tableau obtained from λ by removing the first $t-1$ entries of b. Then, $j_{(1[t+1])}$ is $\kappa_{\lambda'}$ times the coefficient of $(1[t+1])$ in $e(\lambda'-b')e(\lambda')$.

3.2 Determinantal ideals and determinantal varieties.

Let K be an algebraically closed field of characteristic zero, let $\mathbb{P}^s$ be the s-dimensional projective space over K, $R = K[x_0,\ldots,x_s]$ and $\mathfrak{m} = (x_0,\ldots,x_s)$ its homogeneous maximal ideal. Let $m \geq n$ be positives integers.

Definition 3.2.1. Let A be a commutative ring, let $P \subset A$ be a prime ideal and let $I \subset A$ be an ideal. The *codimension* of P is the maximum of the lengths of chains of prime ideals descending from P. We define the *codimension or the height* of I to be the minimum of the codimension of primes ideals containing I. We denote it by $ht(I)$.

Definition 3.2.2. Let A be a commutative ring, let $r_1,\ldots,r_n \in A$ and let $I \subset A$ be a proper ideal. We say that $r_1,\ldots,r_n$ is a *regular sequence or A-sequence* if $(r_1,\ldots,r_n) \neq A$ and for each $i \in \{1,\ldots,n\}$, $r_i \neq Ann(A/(r_1,\ldots,r_{i-1}))$. We define the *grade or depth* of I to be the maximal length of a A-sequences contained in I. We denote it by $depth(I)$.

In the particular case $A = R$, then if $I \subset R$ is a proper ideal, $ht(I) = depht(I)$.

Definition 3.2.3. Let $(r_{i,j})$ be a $m \times n$ matrix with entries in R and let $1 \leq t \leq n$. We say that $(r_{i,j})$ is a *t-homogeneous* matrix if the $k \times k$ minors of $(r_{i,j})$ are homogeneous polynomials for all $k \leq t$. We say that $(r_{i,j})$ is *homogeneous matrix* if all the minors of $(r_{i,j})$ are homogeneous polynomials. If $(r_{i,j})$ is t-homogeneous, then for each $k \leq t$ we define I_k to be the ideal generated by the $k \times k$ minors of $(r_{i,j})$.

A classical result from Eagon-Northcott [7] gives us an estimation of the height of I_t.

Theorem 3.2.4. Let A be a commutative ring and let $(r_{i,j})$ be a t-homogeneous matrix of size $m \times n$ with entries in A. Then, for any $r \leq t$ the ideal I_r has $ht(I_r) \leq (m-t+1)(n-t+1)$. Moreover, if $A = R$ for any $r \leq t$, $depth(I_r) \leq (m-t+1)(n-t+1)$.

Definition 3.2.5. Let $(r_{i,j})$ be a t-homogeneous matrix of size $m \times n$ with entries in R and let $k \leq t$. We say that I_k is *determinantal* if it has maximal height, that is, $ht(I_k) = (m-k+1)(n-k+1)$.

Definition 3.2.6. Let $J \subset R$ be a homogeneous ideal of R and let $X \subset \mathbb{P}^s$ be a projective variety. We say that J is a *determinantal ideal* if there is a $t-$homogeneous $m \times n$ matrix

$(r_{i,j})$ with entries in R such that $J = I_t$. If J is determinantal, we say that J is a *standard determinantal* ideal if $t = n$ and we say that J is a *linear standard determinantal* if J is standard determinantal and all entries of $(r_{i,j})$ are linear forms. Finally, we say that X is a *determinantal variety* if its homogeneous ideal is determinantal.

Let us see a significant example of determinantal ideals. Let m and n be positive integers and let $R = K[X_{i,j}, 1 \leq i \leq m, 1 \leq j \leq n]$. We call generic matrix to the $m \times n$ matrix of indeterminates $(X_{i,j})$, note that $(X_{i,j})$ is a homogeneous matrix of size $m \times n$. We have,

Theorem 3.2.7. For each $t \leq min\{m,n\}$, $ht(I_t) = (m-t+1)(n-t+1)$.

The above theorem is a particular case of a more general result found in [17] Theorem 2.5. Directly from Theorem 3.2.7, I_t is a determinantal ideal for all $t \leq min\{m,n\}$.

In fact, $\{I_t, t \leq min\{m,n\} - 1\}$ are the homogeneous ideal defining a historically important class of determinantal varieties. Let M be the set of all $m \times n$ matrices over K and let $\mathbb{P}^{mn-1}$ be the projective space associated to M. For each $t \leq min\{m,n\} - 1$ we let $M_t \subset M$ be the subset of matrices of rank t or less. Observe that $M_t = \{Z = [Z_{i,j}, 1 \leq i \leq m, 1 \leq j \leq n] \in \mathbb{P}^{mn-1} \mid rg((X_{i,j})(Z)) \leq t\}$, where $(X_{i,j})(Z)$ denotes the matrix $(Z_{i,j})$. Thus, one easily can see that M_t is just the projective variety defined by I_{t+1}. We call M_t generic determinantal variety. Moreover, as we can see in [12] Lecture 9, M_1 is the Segre variety, M_2 is the chordal variety of the Segre variety and more generally, M_t is the union of the secant $(t-1)$-planes to the Segre variety for $t > 2$.

We end this section with another large family of determinantal ideals. Let $R = K[x_1, \ldots, x_s]$ and let $(r_{i,j})$ be a matrix of linear forms in R.

Definition 3.2.8. A *generalized row* of $(r_{i,j})$ is a nonzero linear combination of the rows of $(r_{i,j})$. A *generalized column* of $(r_{i,j})$ is a nonzero linear combination of the columns of $(r_{i,j})$. A *generalized entry* of $(r_{i,j})$ is a nonzero linear combination of the entries of some generalized row or a nonzero linear combination of some generalized column. We say that $(r_{i,j})$ is 1-*generic* if every generalized entry of $(r_{i,j})$ is nonzero.

For example, the generic matrices are 1-generic. The homogeneous ideal defined by the maximal minors of a 1-generic matrix is determinantal. More precisely,

Proposition 3.2.9. Let $m \geq n$. If $(r_{i,j})$ is a 1-generic matrix of size $m \times n$ in R, then $ht(I_n) = m - n + 1$.

The proof of the above proposition is found in [8] Appendix 6B, Theorem 6.4. However the definition of 1-generic matrix seems sophisticated, there are well known examples of determinantal varieties defined by the maximal minor of a 1-generic matrix. One of this examples is the rational normal curve. As we can see in [12] Example 9.3 the rational normal curve $X \subset \mathbb{P}^s$ is the variety defined by the maximal minors of the 1-generic matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{s-1} \\ x_1 & x_2 & \cdots & x_s \end{pmatrix}$$

and hence X is a determinantal variety.

3.2.1 A minimal free resolution of determinantal ideals.

Let K be an algebraically closed field of characteristic zero, let $\mathbb{P}^s$ be the s-dimensional projective space over K, $R = K[x_0, \ldots, x_s]$ and $\mathfrak{m} = (x_0, \ldots, x_s)$ its homogeneous maximal ideal Let n, m and t be positives integers such that $n \leq m$ and let F and G be free R-modules of ranks $m+t-1$ and $n+t-1$ with basis $\{f_1, \ldots, f_{m+t-1}\}$ and $\{g_1, \ldots, g_{n+t-1}\}$ respectively. We denote $G^* = Hom(G, R)$ the dual of G with basis $\{g_1^*, \ldots, g_{n+t-1}^*\}$ dual to the basis $\{g_1, \ldots, g_{n+t-1}\}$. Let $(r_{i,j})$ be a $(n+t-1) \times (m+t-1)$ homogeneous matrix with entries in R, we consider the ideal I_t. $(r_{i,j})$ defines a R-map $\Phi : F \to G^*$ sending f_i to $\sum_{j=1}^{n+t-1} r_{i,j} g_j^*$, $i = 1, \ldots, m+t-1$. We note $(mn) = (n, \ldots, n)$, such that $n + \cdots + n = nm$, and we denote $\lambda_{(mn)} = \lambda_{id}$, in the same manner we consider $(m+t-1, n)$ and $\lambda_{((m+t-1)n)}$.

We present an outline explanation of the work by Lascoux in [16]. We have based partially on notations used in [19] and [13] and our main purpose is connected it with the complex $C_\bullet(\Phi, t)$ and the Schur theory which we have developed. We start with a proposition which allows us to describe the modules of the resolution plainly.

Proposition 3.2.10. Let $I = (i_1, \ldots, i_{m+t-1})$ be a partition of weight $k + d(t-1)$ such that $I \subset (m+t-1, n)$ and d is the Durfee square of I. Then,

(i) If $\tilde{i}_d \leq d+t-1$, we let $I' = 0$.

(ii) If $\tilde{i}_d \geq d+t-1$, we let $I' = (\tilde{i}_1 - t + 1, \ldots, \tilde{i}_d - t + 1, d, \ldots, d, \tilde{i}_{d+1}, \ldots, \tilde{i}_{i_{m+t-1}})$, $d + \cdots + d = (t-1)d$.

Definition 3.2.11. For each $1 \leq k \leq mn$ we define $L_{k,t} = \bigoplus_{|I|-n(I)=k, I' \neq 0} L_{\tilde{I}} F \otimes L_{\widetilde{I'}} G$, where $I = (i_1, \ldots, i_m) \subset (m+t-1, n)$, $n(I) = d(t-1)$, d is the Durfee square of I and I' is the partition describe in Proposition 3.2.10. Let $L_{0,t} = R$.

Let I be a partition with Durfee square d such that $|I| \subset (m+t-1,)$, $|I| = k + n(I) = k + d(t-1)$ and $I' \neq 0$. Since $\tilde{i}_d \geq d+t-1$, let $\tilde{\lambda}$ be the partition of terms $\tilde{\lambda}_1 = \tilde{i}_1 - t + 1, \ldots, \tilde{\lambda}_d = \tilde{i}_d - t + 1, \tilde{\lambda}_{d+1} = \tilde{i}_{d+1}, \ldots, \tilde{\lambda}_{i_{m+t-1}} = \tilde{i}_{i_{m+t-1}}$. Considering Definition 3.1.20 we see that λ is a partition of weight $k + d(t-1)$ and Durfee square d such that $I = \lambda(F, t)$, and hence $\tilde{I} = \tilde{\lambda}(F, t)$ and $\widetilde{I'} = \tilde{\lambda}(G, t)$. Therefore, $L_{k,t} = C_k(\Phi, t)$.

Definition 3.2.12. For any pairs of partitions (I, H) such that $|I| - n(I) = k$ and $|H| - n(H) = k-1$ with I' and H' nonzero we define a morphism $\psi_{I,H} : L_{\tilde{I}} F \otimes L_{\widetilde{I'}} G \to L_{\tilde{H}} F \otimes L_{\widetilde{H'}} G$ as follows,

(i) If $H \not\subset I$, let $\psi_{I,H} = 0$.

(ii) Assume $H \subset I$, as we have seen $\tilde{I}$ is of the form $\tilde{\lambda}_{F,t} = (\tilde{\lambda}_1 + t - 1, \ldots, \tilde{\lambda}_d + t - 1, \tilde{\lambda}_{d+1}, \ldots, \tilde{\lambda}_{\lambda_1})$ for a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ of weight k with Durfee square d. $\tilde{H}$ is the shape of a tableau of the form $\lambda_{F,t} - a$ where a is a corner of $\lambda_{F,1}$. Then, $\lambda_{F,t}$ and $\lambda_{F,t} - a$ coincide except in one square or a string of t squares in which case, $\tilde{I}$ and $\tilde{H}$ coincide except in one term i_j and h_j such that $i_j - h_j = 1$ or $i_j - h_j = t$. The same is true for $\widetilde{I'}$ and $\widetilde{H'}$. Let $q = i_m$, we denote $\tilde{I} = (i_1, \ldots, i_l, i, i_{l+1}, \ldots, i_q)$, $\tilde{H} = (i_1, \ldots, i_l, h, i_{l+1}, \ldots, i_q)$, $\widetilde{I'} = (j_1, \ldots, j_r, i', j_{r+1}, \ldots, j_s)$ and $\widetilde{H'} = (j_1, \ldots, j_r, h', j_{r+1}, \ldots, j_s)$. Denoting $\rho = i - h = i' - h'$, by the Pieri formulas (Corollary 2.5.27) we have injections

$$\delta_{I,\rho} : L_{\tilde{I}} F \to L_{\tilde{H}} F \otimes \Lambda^\rho F \;\; and \;\; \delta_{I',\rho} : L_{\widetilde{I'}} G \to L_{\widetilde{H'}} G \otimes \Lambda^\rho G$$

which induce, together with the contraction $\Lambda^\rho F \otimes \Lambda^\rho G \to R$ induced by Φ, a morphism $\psi_{I,H} : L_{\tilde{I}}F \otimes L_{\widetilde{I'}}G \to L_{\tilde{H}}F \otimes L_{\widetilde{H'}}G$.

Definition 3.2.13. For each $1 \leq k \leq mn$ we define a morphism $d_k : L_{k,t} \to L_{k-1,t}$ to be $\bigoplus \psi_{I,H}$.

Proposition 3.2.14. $d_1(L_{1,t}) = I_t$.
Proof. We are looking for partitions $\tilde{I}$ of Durfee square d such that $|\tilde{I}| = 1 + d(t-1)$ and $\tilde{i}_1 \geq d + t - 1$. This last condition implies that $|\tilde{I}| \geq d(d+t-1) = d^2 + d(t-1)$, then necessarily $d = 1$. Therefore, $|\tilde{I}| = t$ and $\tilde{i}_1 \geq t$, and hence $\tilde{I}$ must be the line partition $\tilde{I} = (t)$. We obtain $L_{1,t} = \Lambda^t F \otimes \Lambda^t G$ and d_1 must be the natural contraction map $\Lambda^t F \otimes \Lambda^t G \to R$ sending $f_{i_1} \wedge \cdots \wedge f_{i_t} \otimes g_{j_1} \wedge \cdots \wedge g_{j_t}$ to the minors of $(r_{i,j})$ corresponding to the $i_1, \ldots, i_t$ rows and $j_1, \ldots, j_t$ columns, where $1 \leq i_1 < \cdots < i_t \leq m+t-1$ and $1 \leq j_1 < \cdots < j_t < n+t-1$. □

Theorem 3.2.15. If $ht(I_t) = mn$, then $L_{\bullet,t}$ is a minimal free resolution of I_t.

We finish this chapter describing classical examples of minimal free resolutions of determinantal varieties. We start with the minimal resolution of the ideal generated by the maximal minors.

The Eagon-Northcott Complex.

We analyze the case $n = 1, m \geq 1$ and $t \geq 1$. Now $(r_{i,j})$ is a matrix of size $(m+t-1, t)$ and I_t is the ideal of R defined by the maximal minors of $(r_{i,j})$. Assuming $ht(I_t) = m$, $(L_{\bullet,t}, d_\bullet)$ is a minimal free resolution of I_t. First we describe the modules $L_{k,t}$. Since $n = 1$, each partition $I \subset (m+t-1, 1)$ has Durfee Square 1. Therefore, for each $1 \leq k \leq m$ there is only one partition $I \subseteq (m+t-1, 1)$ such that $|I| = t - 1 + k$, we have $I = (1, \ldots, 1), 1 + \cdots + 1 = t - 1 + k$ and $\tilde{I} = (k+t-1)$. Given that $k + t - 1 \geq t$, from Proposition 3.2.10 $I' = (k, 1, \ldots, 1), 1 + \cdots + 1 = t - 1$ and then $\widetilde{I'} = (t, 1, \ldots, 1), 1 + \cdots + 1 = k - 1$. We have, for $1 \leq k \leq m$

$$L_{k,t} = L_{(k+t-1)}F \otimes L_{(t,1,\ldots,1)}G = \Lambda^{k+t-1}F \otimes L_{(t,1,\ldots,1)}G.$$

Let us to examine the boundary maps $d_k : L_{k,t} \to L_{k-1,t}$, for convenience we denote $1, \ldots, 1$ such that $1 + \cdots + 1 = s$ by 1^s. Let $2 \leq k \leq m$, we have partitions $\tilde{I} = (k+t-1), \tilde{H} = (k+t-2), \widetilde{I'} = (t, 1^{k-1})$ and $\widetilde{H'} = (t, 1^{k-2})$. By the Pieri formula we have injections $\Lambda^{k+t-1}F \xrightarrow{\Delta} \Lambda^{k+t-2}F \otimes F$ and $L_{(t,1^{k-1})}G \to L_{(t,1^{k-2})}G \otimes G$, where Δ is the appropriated component of the diagonal map. The second injection is the composition $L_{(t,1^{k-1})}G \cong K_{(k,1^{t-1})}G \hookrightarrow \Lambda_{(t,1^{k-1})}G \to \Lambda_{(t,1^{k-2})}G \otimes G \to L_{(t,1^{k-2})}G \otimes G$ sending $d_{\widetilde{I'}}(g_{i_1} \wedge g_2 \wedge \cdots \wedge g_t \otimes g_{i_2} \otimes \cdots \otimes g_{i_k}$ such that $1 = i_1 < i_2 \leq \cdots \leq i_{k-2} < t$ to $\sum_{u=1}^{k-1} g_{i_1} \cdot \cdots \cdot \hat{g_{i_u}} \cdot \cdots \cdot g_{i_{k-1}} \otimes g_2 \otimes \cdots \otimes g_t \otimes g_{i_u}$. And the first injection sends $f_{j_1} \wedge \cdots \wedge f_{j_{k+t-1}}$ such that $1 \leq j_1 < \cdots < j_{k+t-1} \leq m$ to $\sum_{s=1}^{k+t-1} (-1)^{s+1} f_{j_1} \wedge \cdots \wedge \hat{f_{j_s}} \wedge \cdots \wedge f_{k+t-1} \otimes f_{j_s}$, where $\hat{f_{i_s}}$ means that this element is not in the exterior product. Hence, d_k is defined by $d_k(d_{\widetilde{I'}}(f_{j_1} \wedge \cdots \wedge f_{j_{k+t-1}} \otimes g_{i_1} \wedge g_2 \wedge \cdots \wedge g_t \otimes g_{i_2} \otimes \cdots \otimes g_{i_k}) =$

$$\sum_{u=1}^{k-1} \sum_{s=1}^{k+t-1} r_{u,i_k} (-1)^{s+1} f_{j_1} \wedge \cdots \wedge \hat{f_{j_s}} \wedge \cdots \wedge f_{k+t-1} \otimes g_{i_1} \cdot \cdots \cdot \hat{g_{i_u}} \cdot \cdots \cdot g_{\sigma(i_{k-1})} \otimes g_2 \otimes \cdots \otimes g_t.$$

This is, in fact, the Eagon-Northcott complex (see [7]). Indeed, the Pieri formula gives us an isomorphism $L_{(t,1^{k-1})}G \cong S^{k-1}G \otimes \Lambda^k G$. Since G has rank k, it follows that $\Lambda^k G \cong R$ and then, $L_{k,t} \cong \Lambda^{k+t-1}F \otimes S^{k-1}G$. Under this isomorphism we can describe more easily the boundary maps d_k. In this case, d_k is given by
$d_k((f_{i_1} \wedge \cdots \wedge f_{i_{k+t-1}}) \otimes (g_1^{j_1} \cdot \cdots \cdot g_t^{j_t}) =$

$$\sum_{k^*}(\sum_{p=1}^{k+t-1} (-1)^{p+1} r_{k,i_p} f_{i_1} \wedge \cdots \wedge \hat{f}_{i_p} \wedge \cdots \wedge f_{i_{k+t-1}}) \otimes g_1^{j_1} \cdot \cdots \cdot g_k^{j_k-1} \cdot \cdots \cdot g_t^{j_t}$$

where the sum runs over all values k such that $j_k > 0$. This is the original form of the minimal resolution gives by Eagon-Northcott. The original construction of the resolution does not involve Schur functors and it is made over a commutative ring R of any characteristic.

Theorem 3.2.16. If $ht(I_t) = m$, then a minimal free resolution of R/I_t is given by

$$(L_{\bullet,t}, d_\bullet) : 0 \to \Lambda^{m+t-1}F \otimes S^{m-1}G \xrightarrow{d_m} \Lambda^{m+t-2}F \otimes S^{m-2}G \to \cdots \to \Lambda^t F \xrightarrow{d_1} R \to R/I_t \to 0.$$

The minimal free resolution of the rational normal curve is given by the Eagon-Northcott complex. For example, consider $K = [x_0, \ldots, x_7]$ and the projective space $\mathbb{P}^7$. The rational normal curve X of $\mathbb{P}^7$ is the variety is given by the ideal the maximal minors of the $2 \times (6+2-1)$ matrix:

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_6 \\ x_1 & x_2 & \cdots & x_7 \end{pmatrix}.$$

So, $F = R^7(-1)$ and $G = R^2$. We have, for $1 \le k \le 6$, $\quad L_{k,2} = \Lambda^{k+1}F \otimes S^{k-1}G$. Since $rank(\Lambda^{k+1}F) = \binom{7}{k+1}$ and $rank(S^{k-1}G) = \binom{2+k-2}{k-1} = k$, explicitly:
$L_{6,2} = R(-7) \otimes R^6 \cong R^6(-7)$, $L_{5,2} = R^7(-6) \otimes R^5 \cong R^{35}(-6)$, $L_{4,2} = R^{21}(-5) \otimes R^4 \cong R^{84}(-5)$, $L_{3,2} = R^{35}(-4) \otimes R^3 \cong R^{105}(-4)$, $L_{2,2} = R^{35}(-3) \otimes R^2 = R^{70}(-3)$, $L_{1,2} = R^{21}(-2) \otimes R \cong R^{21}(-2)$.

The resolution of the rational normal curve of $\mathbb{P}^7$ looks like:

$0 \to R^6(-7) \to R^{35}(-6) \to R^{84}(-5) \to R^{105}(-4) \to R^{70}(-3) \to R^{21}(-2) \to R \to R/I_t \to 0$

The Gulliksen-Negard complex.

Now we will construct the minimal free resolution of determinantal ideals given by the submaximal minors of a square matrix, in particular the case $m = n = 2$ and $t \ge 1$. Let $(r_{i,j})$ be a matrix of size $t \times t$. There are 4 free R-modules involved in the complex, to find them we have to describe partitions $\tilde{I} = (\tilde{i}_1, \tilde{i}_2)$ of Durfee square $d = 1$ or $d = 2$ such that $\tilde{i}_d \ge d + t - 2$ and $\tilde{i}_1 + \tilde{i}_2 = k + d(t-2)$. We distinguish two cases according to the Durfee square:

(1) When $d = 1$, we obtain conditions $\tilde{i}_1 \ge t-1, \tilde{i}_1 + \tilde{i}_2 = k + t - 2$ and $i_2 \le 1$

(2) When $d = 2$, we obtain conditions $\tilde{i}_2 \ge t$ and $\tilde{i}_1 + \tilde{i}_2 = k + 2t - 4$ and $i_2 \ge 2$.

For $k = 2$ and $d = 2$, since $\tilde{i}_1 = 2t - 2 - \tilde{i}_2 \leq t - 2 \leq i_1$, there are no partitions $\tilde{I}$. If $d = 1$, since $\tilde{i}_2 = t - i_1$ and $i_1 \geq t - 1$, there are two partitions $\tilde{I}$ satisfying the above conditions. We have $\tilde{I} = (t-1, 1)$ and $\tilde{I} = (t, 0)$. In this case $L_{2,t-1} = L_{(t-1,1)}F \otimes \Lambda^t G \oplus \Lambda^t F \otimes L_{(t-1,1)}G$.

For $k = 3$ and $d = 2$, $\tilde{i}_1 = 2t - 1 - \tilde{i}_2$ and $\tilde{i}_2 \geq t$ implies $\tilde{i}_1 \leq t - 1 < \tilde{i}_2$, and hence $(\tilde{i}_1, \tilde{i}_2)$ is not a partition. Otherwise, if $d = 1$ it results $\tilde{i}_1 \geq t - 1$ and $\tilde{i}_2 = t + 1 - \tilde{i}_1$ which implies $i_2 = 1$ and $i_2 = 2$. Given that $i_2 \leq 1$, it follows that $\tilde{I} = (t, 1)$ and $\widetilde{I'} = (t, 1)$. Therefore $L_{3,t-1} = L_{(t,1)}F \otimes L_{(t,1)}G$.

For $k = 4$ and $d = 1$, $\tilde{i}_2 = t + 2 - \tilde{i}_1$ and $i_1 \geq t - 1$. Since $\tilde{i}_2 \leq 1$, we obtain the following possibilities $(t+2, 0)$ and $(t+1, 1)$, but $\tilde{i}_1$ is the length of $I \subset (t, 2)$. So, in this cases there are no partitions $\tilde{I}$. When $d = 2$, conditions says $\tilde{i}_1 = 2t - \tilde{i}_2$ and $\tilde{i}_2 \geq t$. There is only one partition $\tilde{I} = (t, t)$, and hence $\tilde{I}' = (t, t)$. Thus, $L_{4,t-1} = L_{(t,t)}F \otimes L_{(t,t)}G$.

The complex looks like

$$0 \to L_{(t,t)}F \otimes L_{(t,t)}G \to L_{(t,1)}F \otimes L_{(t,1)}G \to L_{(t-1,1)}F \otimes \Lambda^t G \oplus \Lambda^t F \otimes L_{(t-1,1)}G \to$$

$$\to \Lambda^{t-1}F \otimes \Lambda^{t-1}G \to R \to 0.$$

It remains to describe the boundary maps $d_\bullet$. Let $\varphi : \Lambda^{t-1}F \otimes \Lambda^{t-1}G \to R$ be the contraction map, $d_4 : L_{(t,t)}F \otimes L_{(t,t)}G \to L_{(t,1)}F \otimes L_{(t,1)}G$ is the composition of the injections $L_{(t,t)}F \to L_{(t,1)}F \otimes \Lambda^{t-1}F$ and $L_{(t,t)}G \to L_{(t,1)}G \otimes \Lambda^{t-1}G$ followed by φ. One can easily see that d_k is defined by sending $d_{(t,t)}(f_1 \wedge \cdots \wedge f_t \otimes f_1 \wedge \cdots \wedge f_t)$ to $\sum_{i,j}(-1)^{i+j}\varphi(f_1 \wedge \cdots \wedge \hat{f}_i \wedge \cdots \wedge f_t \otimes g_1 \wedge \cdots \wedge \hat{g}_j \wedge \cdots \wedge g_t) f_1 \wedge \cdots \wedge f_t \otimes f_i \otimes g_1 \wedge \cdots \wedge g_t \otimes g_j$. Since $\Lambda^t F \cong R \cong \Lambda^t G$ which are generated by $f_1 \wedge \cdots \wedge f_t$ and $g_1 \wedge \cdots \wedge g_t$ respectively and $\varphi(f_1 \wedge \cdots \wedge \hat{f}_i \wedge \cdots \wedge f_t \otimes g_1 \wedge \cdots \wedge \hat{g}_j \wedge \cdots \wedge g_t)$ is the determinant of $(r_{i,j})$ with the ith row and the jth column deleted, denoting them by 1 and $det_{i,j}$ respectively, $d_4(d_{(t,t)}(1,1)) = \sum_{i,j}(-1)^{i+j} det_{i,j} f_i \otimes g_j \otimes 1$.

d_3 is the sum of the maps $\psi_{(2,1^{t}-1),(2,1^{t-2})}$ and $\psi_{(2,1^{t}-1),(1^t)}$ as in Definition 3.2.12. The injection $L_{(t,1)}F \to L_{(t-1,1)}F \otimes F$ maps $d_{(t,1)}(f_1 \wedge \cdots \wedge f_t \otimes f_i)$ to $\sum_{j=1}^{t}(-1)^{j+1} f_1 \wedge \cdots \wedge \hat{f}_j \wedge \cdots \wedge f_t \otimes f_i \otimes f_j$ if $i \neq 1$ and to $2\sum_{j=1}^{t}(-1)^{j+1} f_1 \wedge \cdots \wedge \hat{f}_j \wedge \cdots \wedge f_t \otimes f_1 \otimes f_j$ if $i = 1$. While the injection $L_{(t,1)}G \to \Lambda^t G \otimes G$ sends $d_{(t,1)}(g_1 \wedge \cdots \wedge g_t \otimes g_s)$ to $g_1 \wedge \cdots \wedge g_t \otimes g_s$ if $s \neq 1$ and to $2g_1 \wedge \cdots \wedge g_t \otimes g_1$ if $s = 1$. Therefore, the respective images of $\psi((t,1),(t-1,1))$ of this elements are in each case:

$\sum_{j=1}^{t} r_{j,s}(-1)^{j+1} f_1 \wedge \cdots \wedge \hat{f}_j \wedge \cdots \wedge f_t \otimes f_i \otimes g_1 \wedge \cdots \wedge g_t = \sum_{j=1}^{t} r_{j,s}(-1)^{j+1} f_1 \wedge \cdots \wedge \hat{f}_j \wedge \cdots \wedge f_t \otimes f_i \otimes 1$ if $i, s \neq 1$.

$2\sum_{j=1}^{t}(-1)^{j+1} r_{j,1} f_1 \wedge \cdots \wedge \hat{f}_j \wedge \cdots \wedge f_t \otimes f_i \otimes 1$ if $i \neq 1$ and $s = 1$.

$2\sum_{j=1}^{t}(-1)^{j+1} f_1 r_{j,s} \wedge \cdots \wedge \hat{f}_j \wedge \cdots \wedge f_t \otimes f_1 \otimes f_1 \otimes 1$ if $i = 1$ and $s \neq 1$.

$4\sum_{j=1}^{t}(-1)^{j+1} f_1 r_{j,1} \wedge \cdots \wedge \hat{f}_j \wedge \cdots \wedge f_t \otimes f_1 \otimes f_1 \otimes 1$ if $i = 1 = s$.

Symmetrically, the map $\psi_{(t-1,1),(t,1)}$ is the same as the above map changing f by g. Finally, d_2 is the sum of maps $\psi_{(2,1^{t-2}),(1^{t-1})}$ and $\psi_{(1^t),(1^{t-1})}$. Observe that these two maps coincide with the boundary maps of the Eagon- Northcott complex. The complex $(L_{\bullet,t-1}, d_\bullet)$ is called the Gulliksen-Negard complex originally presented in [11].

Two concrete examples computed with Macaulay2.

Let $R = K[x_0, \dots, x_5]$ and consider the following 2×5 matrix:

$$M = \begin{pmatrix} x_0 + x_1 & x_1 + x_2 & x_2 + x_3 & x_3 + x_4 & x_4 + x_5 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 \end{pmatrix}.$$

The ideal I_2 generated by the maximal minors of M is prime and $ht(I_2) = 4$. Therefore, I_2 is determinantal and the homogeneous ideal of the variety $X = \{z \in \mathbb{P}^5 \mid rg(M(z)) = 1\}$ is I_2. Hence X is determinantal and a minimal free resolution of I_2 is given by the Eagon-Northcott complex:

$$0 \to R(-6) \oplus R(-7) \oplus R(-8) \oplus R(-9) \xrightarrow{d_4} R^5(-5) \oplus R^5(-6) \oplus R^5(-7) \xrightarrow{d_3}$$

$$R^{10}(-4) \oplus R^{10}(-5) \xrightarrow{d_2} R^{10}(-3) \xrightarrow{d_1} R \to R/I_2 \to 0.$$

In the next pages we give the matrix associated to the boundary maps d_k for $k = 1, \dots, 4$.

The matrix associated to $d_1 : R^{10}(-3) \to R$ is the transpose of the matrix:

$$\begin{pmatrix}
x_1^3 + x_1^2x_2 - x_0x_2^2 - x_1x_2^2 \\
x_1^2x_2 + x_1^2x_3 - x_0x_3^2 - x_1x_3^2 \\
x_2^3 + x_2^2x_3 - x_1x_3^2 - x_2x_3^2 \\
x_1^2x_3 + x_1^2x_4 - x_0x_4^2 - x_1x_4^2 \\
x_2^2x_3 + x_2^2x_4 - x_1x_4^2 - x_2x_4^2 \\
x_3^3 + x_3^2x_4 - x_2x_4^2 - x_3x_4^2 \\
x_1^2x_4 + x_1^2x_5 - x_0x_5^2 - x_1x_5^2 \\
x_2^2x_4 + x_2^2x_5 - x_1x_5^2 - x_2x_5^2 \\
x_3^2x_4 + x_3^2x_5 - x_2x_5^2 - x_3x_5^2 \\
x_4^3 + x_4^2x_5 - x_3x_5^2 - x_4x_5^2
\end{pmatrix}$$

$d_2 : R^{10}(-4) \oplus R^{10}(-5) \to R^{10}(-3)$

$$\left(\begin{matrix}
-x_2-x_3 & -x_3-x_4 & 0 & 0 & -x_4-x_5 & 0 & 0 & 0 \\
x_1+x_2 & x_3+x_4 & -x_3-x_4 & 0 & x_4+x_5 & -x_4-x_5 & 0 & 0 \\
-x_0-x_1 & 0 & 0 & -x_3-x_4 & 0 & 0 & -x_4-x_5 & 0 \\
0 & x_1-x_3 & x_2+x_3 & 0 & -x_4-x_5 & x_4+x_5 & 0 & -x_4-x_5 \\
0 & -x_0-x_1 & 0 & x_2+x_3 & 0 & 0 & x_4+x_5 & 0 \\
0 & x_0+x_1 & -x_0-x_1 & -x_1-x_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & x_1+x_4 & x_2-x_4 & 0 & x_3+x_4 \\
0 & 0 & 0 & 0 & -x_0-x_1 & 0 & x_2-x_4 & 0 \\
0 & 0 & 0 & 0 & x_0+x_1 & -x_0-x_1 & -x_1-x_2 & 0 \\
0 & 0 & 0 & 0 & -x_0-x_1 & x_0+x_1 & x_1+x_2 & -x_0-x_1
\end{matrix}\right.$$

$$\left.\begin{matrix}
0 & 0 & x_3^2 & x_4^2 & 0 & 0 & x_5^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -x_2^2 & 0 & x_4^2 & 0 & 0 & x_5^2 & 0 & 0 & 0 & 0 \\
0 & 0 & x_1^2 & 0 & 0 & x_4^2 & 0 & 0 & x_5^2 & 0 & 0 & 0 \\
0 & 0 & 0 & -x_2^2 & -x_3^2 & 0 & 0 & 0 & 0 & x_5^2 & 0 & 0 \\
-x_4-x_5 & 0 & 0 & x_1^2 & 0 & -x_3^2 & 0 & 0 & 0 & 0 & x_5^2 & 0 \\
0 & -x_4-x_5 & 0 & 0 & x_1^2 & x_2^2 & 0 & 0 & 0 & 0 & 0 & x_5^2 \\
0 & 0 & 0 & 0 & 0 & 0 & -x_2^2 & -x_3^2 & 0 & -x_4^2 & 0 & 0 \\
x_3+x_4 & 0 & 0 & 0 & 0 & 0 & x_1^2 & 0 & -x_3^2 & 0 & -x_4^2 & 0 \\
0 & x_3+x_4 & 0 & 0 & 0 & 0 & 0 & x_1^2 & x_2^2 & 0 & 0 & -x_4^2 \\
-x_1-x_2 & -x_2-x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1^2 & x_2^2 & x_3^2
\end{matrix}\right)$$

$d_3 : R^5(-5) \oplus R^5(-6) \oplus R^5(-7) \to R^{10}(-4) \oplus R^{10}(-5)$

$$
\left(\begin{array}{ccccccc}
x_3+x_4 & x_4+x_5 & 0 & 0 & 0 & x_4^2 & x_5^2 \\
-x_2-x_3 & -x_4-x_5 & x_4+x_5 & 0 & 0 & -x_3^2 & -x_5^2 \\
x_1-x_3 & -x_4-x_5 & 0 & x_4+x_5 & 0 & x_2^2-x_3^2 & -x_5^2 \\
-x_0-x_1 & 0 & 0 & 0 & x_4+x_5 & -x_1^2 & 0 \\
0 & -x_2+x_4 & -x_3-x_4 & 0 & 0 & 0 & -x_3^2+x_4^2 \\
0 & x_1+x_4 & 0 & -x_3-x_4 & 0 & 0 & x_2^2-x_3^2+x_4^2 \\
0 & -x_0-x_1 & 0 & 0 & -x_3-x_4 & 0 & -x_1^2 \\
0 & 0 & x_1+x_4 & x_2-x_4 & 0 & 0 & 0 \\
0 & 0 & -x_0-x_1 & 0 & x_2-x_4 & 0 & 0 \\
0 & 0 & x_0+x_1 & -x_0-x_1 & -x_1-x_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -x_3-x_4 & -x_4-x_5 \\
0 & 0 & 0 & 0 & 0 & x_2+x_3 & x_4+x_5 \\
0 & 0 & 0 & 0 & 0 & -x_1-x_2 & 0 \\
0 & 0 & 0 & 0 & 0 & x_0+x_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x_2-x_4 \\
0 & 0 & 0 & 0 & 0 & 0 & -x_1-x_2 \\
0 & 0 & 0 & 0 & 0 & 0 & x_0+x_1 \\
0 & 0 & 0 & 0 & 0 & 0 & x_1+x_2 \\
0 & 0 & 0 & 0 & 0 & 0 & -x_0-x_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right.
$$

$$
\left.\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_5^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-x_2x_4-x_4^2-x_2x_5-x_4x_5+x_5^2 & x_5^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x_5^2 & 0 & 0 & 0 & 0 & 0 \\
-x_4^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_2x_3+x_2x_4+x_3x_4 & -x_4^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -x_4^2 & 0 & 0 & 0 & 0 & 0 \\
x_4^2 & x_3^2-x_4^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-x_1^2 & 0 & x_3^2-x_4^2 & 0 & 0 & 0 & 0 & 0 \\
x_0x_2+x_1x_2+x_0x_4+x_1x_4 & -x_1^2 & -x_2^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -x_4^2 & -x_5^2 & 0 & 0 & 0 \\
-x_4-x_5 & 0 & 0 & x_3^2 & 0 & -x_5^2 & 0 & 0 \\
0 & -x_4-x_5 & 0 & -x_2^2 & 0 & 0 & -x_5^2 & 0 \\
0 & 0 & -x_4-x_5 & x_1^2 & 0 & 0 & 0 & -x_5^2 \\
x_3+x_4 & 0 & 0 & 0 & x_3^2 & x_4^2 & 0 & 0 \\
0 & x_3+x_4 & 0 & 0 & -x_2^2 & 0 & x_4^2 & 0 \\
0 & 0 & x_3+x_4 & 0 & x_1^2 & 0 & 0 & x_4^2 \\
-x_1-x_2 & -x_2-x_3 & 0 & 0 & 0 & -x_2^2 & -x_3^2 & 0 \\
x_0+x_1 & 0 & -x_2-x_3 & 0 & 0 & x_1^2 & 0 & -x_3^2 \\
0 & x_0+x_1 & x_1+x_2 & 0 & 0 & 0 & x_1^2 & x_2^2
\end{array}\right)
$$

$d_4 : R(-6) \oplus R(-7) \oplus R(-8) \oplus R(-9) \to R^5(-5) \oplus R^5(-6) \oplus R^5(-7)$

$$\begin{pmatrix} -x_4 - x_5 & x_5^2 & 0 & 0 \\ x_3 + x_4 & -x_4^2 & 0 & 0 \\ -x_2 + x_4 & x_3^2 - x_4^2 & 0 & 0 \\ x_1 + x_4 & x_3^2 - 2x_4^2 & -x_2x_3^2 - x_3^2x_4 + x_2x_4^2 + x_4^3 & 0 \\ -x_0 - x_1 & x_1^2 & 0 & 0 \\ 0 & x_4 + x_5 & -x_5^2 & 0 \\ 0 & -x_3 - x_4 & x_4^2 & 0 \\ 0 & x_2 - x_4 & -x_3^2 + x_4^2 & 0 \\ 0 & -x_1 - x_2 & x_2^2 & 0 \\ 0 & x_0 + x_1 & -x_1^2 & 0 \\ 0 & 0 & -x_4 - x_5 & x_5^2 \\ 0 & 0 & x_3 + x_4 & -x_4^2 \\ 0 & 0 & -x_2 - x_3 & x_3^2 \\ 0 & 0 & x_1 + x_2 & -x_2^2 \\ 0 & 0 & -x_0 - x_1 & x_1^2 \end{pmatrix}.$$

Finally we want to describe a minimal free resolution of the determinantal ideal I_3 generated by submaximal minors of the generic matrix:

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 & x_7 \\ x_8 & x_9 & x_{10} & x_{11} \\ x_{12} & x_{13} & x_{14} & x_{15} \end{pmatrix}$$

Since I_3 is prime, I_3 is the homogeneous ideal of the determinantal variety M_2. The Gulliksen-Negard complex gives a minimal free resolution of M_2:

$$0 \to R(-8) \xrightarrow{d_4} R^{16}(-5) \xrightarrow{d_3} R^{30}(-4) \xrightarrow{d_2} R^{16}(-3) \xrightarrow{d_1} R \to R/I_3 \to 0$$

As before, we describe the boundary maps d_k, $k = 1, \ldots, 4$.

The matrix associated to $d_1 : R^{16}(-3) \to R$ is the transpose of the matrix:

$$\begin{pmatrix}
x_2x_5x_8 - x_1x_6x_8 - x_2x_4x_9 + x_0x_6x_9 + x_1x_4x_{10} - x_0x_5x_{10} \\
x_3x_5x_8 - x_1x_7x_8 - x_3x_4x_9 + x_0x_7x_9 + x_1x_4x_{11} - x_0x_5x_{11} \\
x_3x_6x_8 - x_2x_7x_8 - x_3x_4x_{10} + x_0x_7x_{10} + x_2x_4x_{11} - x_0x_6x_{11} \\
x_3x_6x_9 - x_2x_7x_9 - x_3x_5x_{10} + x_1x_7x_{10} + x_2x_5x_{11} - x_1x_6x_{11} \\
x_2x_5x_{12} - x_1x_6x_{12} - x_2x_4x_{13} + x_0x_6x_{13} + x_1x_4x_{14} - x_0x_5x_{14} \\
x_3x_5x_{12} - x_1x_7x_{12} - x_3x_4x_{13} + x_0x_7x_{13} + x_1x_4x_{15} - x_0x_5x_{15} \\
x_3x_6x_{12} - x_2x_7x_{12} - x_3x_4x_{14} + x_0x_7x_{14} + x_2x_4x_{15} - x_0x_6x_{15} \\
x_2x_9x_{12} - x_1x_{10}x_{12} - x_2x_8x_{13} + x_0x_{10}x_{13} + x_1x_8x_{14} - x_0x_9x_{14} \\
x_3x_9x_{12} - x_1x_{11}x_{12} - x_3x_8x_{13} + x_0x_{11}x_{13} + x_1x_8x_{15} - x_0x_9x_{15} \\
x_6x_9x_{12} - x_5x_{10}x_{12} - x_6x_8x_{13} + x_4x_{10}x_{13} + x_5x_8x_{14} - x_4x_9x_{14} \\
x_7x_9x_{12} - x_5x_{11}x_{12} - x_7x_8x_{13} + x_4x_{11}x_{13} + x_5x_8x_{15} - x_4x_9x_{15} \\
x_3x_{10}x_{12} - x_2x_{11}x_{12} - x_3x_8x_{14} + x_0x_{11}x_{14} + x_2x_8x_{15} - x_0x_{10}x_{15} \\
x_7x_{10}x_{12} - x_6x_{11}x_{12} - x_7x_8x_{14} + x_4x_{11}x_{14} + x_6x_8x_{15} - x_4x_{10}x_{15} \\
x_3x_6x_{13} - x_2x_7x_{13} - x_3x_5x_{14} + x_1x_7x_{14} + x_2x_5x_{15} - x_1x_6x_{15} \\
x_3x_{10}x_{13} - x_2x_{11}x_{13} - x_3x_9x_{14} + x_1x_{11}x_{14} + x_2x_9x_{15} - x_1x_{10}x_{15} \\
x_7x_{10}x_{13} - x_6x_{11}x_{13} - x_7x_9x_{14} + x_5x_{11}x_{14} + x_6x_9x_{15} - x_5x_{10}x_{15}
\end{pmatrix}$$

$d_2 : R^{30}(-4) \to R^{16}(-3)$

$$\left(\begin{array}{ccccccccccccccc}
-x_3 & x_7 & -x_{11} & 0 & 0 & -x_{12} & 0 & 0 & 0 & x_{13} & 0 & -x_{14} & 0 & 0 & 0 \\
x_2 & -x_6 & x_{10} & 0 & 0 & 0 & -x_{12} & 0 & 0 & 0 & x_{13} & 0 & 0 & 0 & -x_{14} \\
-x_1 & x_5 & -x_9 & 0 & 0 & 0 & 0 & -x_{12} & 0 & 0 & 0 & 0 & 0 & x_{13} & 0 \\
x_0 & -x_4 & x_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_{12} & 0 & 0 \\
0 & 0 & 0 & -x_3 & x_7 & x_8 & 0 & 0 & 0 & -x_9 & 0 & x_{10} & x_{11} & 0 & 0 \\
0 & 0 & 0 & x_2 & -x_6 & 0 & x_8 & 0 & 0 & 0 & -x_9 & 0 & -x_{10} & 0 & x_{10} \\
0 & 0 & 0 & -x_1 & x_5 & 0 & 0 & x_8 & 0 & 0 & 0 & 0 & x_9 & -x_9 & 0 \\
0 & 0 & 0 & 0 & 0 & -x_4 & 0 & 0 & -x_3 & x_5 & 0 & -x_6 & 0 & -x_7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -x_4 & 0 & x_2 & 0 & x_5 & 0 & 0 & x_6 & -x_6 \\
0 & 0 & 0 & 0 & 0 & x_0 & 0 & 0 & 0 & -x_1 & 0 & x_2 & 0 & 0 & x_3 \\
0 & 0 & 0 & 0 & 0 & 0 & x_0 & 0 & 0 & 0 & -x_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_4 & -x_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x_0 & 0 & 0 & 0 & 0 & 0 & -x_1 & x_1 \\
0 & 0 & 0 & x_0 & -x_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_0 & 0 & 0 & 0 & -x_4 & x_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_0 & 0 & -x_0
\end{array}\right|$$

$$
\left|\begin{array}{ccccccccccccccc}
-x_{15} & 0 & 0 & -x_{15} & 0 & 0 & 0 & 0 & -x_{15} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -x_{15} & 0 & x_{14} & 0 & 0 & 0 & 0 & x_{14} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & x_{14} & -x_{15} & 0 & 0 & -x_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_{13} & x_{14} & -x_{15} & 0 & 0 & 0 \\
x_{11} & 0 & 0 & x_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_{15} & 0 & 0 \\
0 & x_{11} & 0 & -x_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_{14} & 0 & 0 \\
-x_7 & 0 & 0 & 0 & 0 & 0 & x_{11} & 0 & 0 & 0 & 0 & 0 & 0 & -x_{15} & 0 \\
0 & -x_7 & 0 & 0 & 0 & 0 & -x_{10} & 0 & 0 & 0 & 0 & 0 & 0 & x_{14} & 0 \\
0 & 0 & -x_7 & 0 & 0 & 0 & 0 & x_{11} & 0 & 0 & 0 & 0 & 0 & 0 & -x_{15} \\
x_2 & x_3 & x_6 & 0 & 0 & 0 & 0 & -x_{10} & 0 & 0 & 0 & 0 & 0 & 0 & x_{14} \\
0 & 0 & 0 & x_5 & x_6 & -x_7 & x_9 & 0 & 0 & 0 & 0 & 0 & 0 & -x_{13} & 0 \\
-x_1 & 0 & -x_5 & -x_1 & -x_2 & x_3 & 0 & x_9 & 0 & 0 & 0 & 0 & 0 & 0 & -x_{13} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_8 & x_9 & -x_{10} & x_{11} & x_{12} & 0 & 0 \\
0 & 0 & 0 & -x_4 & 0 & 0 & -x_8 & 0 & -x_4 & -x_5 & x_6 & -x_7 & 0 & x_{12} & 0 \\
x_0 & 0 & x_4 & x_0 & 0 & 0 & 0 & -x_8 & x_0 & x_1 & -x_2 & x_3 & 0 & 0 & x_{12}
\end{array}\right)
$$

$d_3 : R^{16}(-5) \to R^{30}(-4)$

$$
\begin{pmatrix}
x_{12} & 0 & -x_{13} & x_{14} & 0 & x_{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x_{12} & 0 & 0 & -x_{13} & 0 & x_{15} & 0 & -x_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x_{12} & 0 & -x_{14} & x_{15} & x_{13} & 0 & 0 & 0 & 0 \\
-x_8 & 0 & x_9 & -x_{10} & 0 & -x_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -x_8 & 0 & 0 & x_9 & 0 & -x_{11} & 0 & x_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-x_3 & x_7 & 0 & 0 & 0 & 0 & 0 & -x_{11} & 0 & 0 & 0 & 0 & x_{15} & 0 & 0 & 0 \\
x_2 & -x_6 & 0 & 0 & 0 & 0 & 0 & x_{10} & 0 & 0 & 0 & 0 & -x_{14} & 0 & 0 & 0 \\
-x_1 & x_5 & 0 & 0 & 0 & 0 & 0 & -x_9 & 0 & 0 & 0 & 0 & x_{13} & 0 & 0 & 0 \\
x_4 & 0 & -x_5 & x_6 & 0 & x_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -x_3 & 0 & x_7 & 0 & 0 & 0 & 0 & 0 & 0 & x_{11} & 0 & -x_{15} & 0 & 0 \\
0 & 0 & x_2 & 0 & -x_6 & 0 & 0 & 0 & 0 & 0 & 0 & -x_{10} & 0 & x_{14} & 0 & 0 \\
0 & 0 & 0 & -x_3 & 0 & 0 & 0 & 0 & -x_7 & x_{11} & 0 & 0 & 0 & 0 & -x_{15} & 0 \\
x_0 & -x_4 & 0 & 0 & 0 & 0 & 0 & x_8 & 0 & 0 & 0 & 0 & 0 & x_{13} & -x_{14} & x_{15} \\
x_0 & -x_4 & -x_1 & 0 & x_5 & 0 & 0 & 0 & 0 & x_{10} & -x_{11} & 0 & 0 & 0 & -x_{14} & x_{15} \\
x_0 & 0 & -x_1 & x_2 & 0 & 0 & x_7 & 0 & 0 & 0 & -x_{11} & 0 & 0 & 0 & 0 & x_{15} \\
-x_0 & 0 & x_1 & 0 & 0 & -x_3 & 0 & 0 & x_6 & -x_{10} & 0 & 0 & 0 & 0 & x_{14} & 0 \\
0 & 0 & 0 & 0 & 0 & x_2 & -x_6 & 0 & 0 & 0 & x_{10} & 0 & 0 & 0 & 0 & -x_{14} \\
0 & x_0 & 0 & 0 & -x_1 & 0 & x_3 & 0 & -x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x_4 & -x_1 & 0 & 0 & 0 & x_7 & 0 & -x_6 & 0 & 0 & x_9 & 0 & -x_{13} & 0 & 0 \\
0 & 0 & 0 & x_1 & 0 & 0 & 0 & 0 & x_5 & -x_9 & 0 & 0 & 0 & 0 & x_{13} & 0 \\
0 & 0 & 0 & 0 & 0 & -x_1 & x_5 & 0 & 0 & 0 & -x_9 & 0 & 0 & 0 & 0 & x_{13} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_4 & 0 & x_6 & -x_7 & -x_5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x_0 & 0 & -x_2 & x_3 & x_1 & 0 & 0 & 0 & 0 \\
x_0 & -x_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_{10} & -x_{11} & -x_9 & -x_{12} & 0 & 0 & 0 \\
0 & 0 & -x_0 & 0 & x_4 & 0 & 0 & 0 & 0 & 0 & 0 & x_8 & 0 & -x_{12} & 0 & 0 \\
0 & 0 & 0 & -x_0 & 0 & 0 & 0 & 0 & -x_4 & x_8 & 0 & 0 & 0 & 0 & -x_{12} & 0 \\
0 & 0 & 0 & 0 & 0 & x_0 & -x_4 & 0 & 0 & 0 & x_8 & 0 & 0 & 0 & 0 & -x_{12} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_8 & x_9 & -x_{10} & x_{11} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_4 & -x_5 & x_6 & -x_7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_0 & x_1 & -x_2 & x_3
\end{pmatrix}
$$

$d_4 : R(-8) \to R^{16}(-5)$

$$
\begin{pmatrix}
-x_7x_{10}x_{13} + x_6x_{11}x_{13} + x_7x_9x_{14} - x_5x_{11}x_{14} - x_6x_9x_{15} + x_5x_{10}x_{15} \\
-x_3x_{10}x_{13} + x_2x_{11}x_{13} + x_3x_9x_{14} - x_1x_{11}x_{14} - x_2x_9x_{15} + x_1x_{10}x_{15} \\
-x_7x_{10}x_{12} + x_6x_{11}x_{12} + x_7x_8x_{14} - x_4x_{11}x_{14} - x_6x_8x_{15} + x_4x_{10}x_{15} \\
-x_7x_9x_{12} + x_5x_{11}x_{12} + x_7x_8x_{13} - x_4x_{11}x_{13} - x_5x_8x_{15} + x_4x_9x_{15} \\
-x_3x_{10}x_{12} + x_2x_{11}x_{12} + x_3x_8x_{14} - x_0x_{11}x_{14} - x_2x_8x_{15} + x_0x_{10}x_{15} \\
x_6x_9x_{12} - x_5x_{10}x_{12} - x_6x_8x_{13} + x_4x_{10}x_{13} + x_5x_8x_{14} - x_4x_9x_{14} \\
x_2x_9x_{12} - x_1x_{10}x_{12} - x_2x_8x_{13} + x_0x_{10}x_{13} + x_1x_8x_{14} - x_0x_9x_{14} \\
-x_3x_6x_{13} + x_2x_7x_{13} + x_3x_5x_{14} - x_1x_7x_{14} - x_2x_5x_{15} + x_1x_6x_{15} \\
x_3x_9x_{12} - x_1x_{11}x_{12} - x_3x_8x_{13} + x_0x_{11}x_{13} + x_1x_8x_{15} - x_0x_9x_{15} \\
x_3x_5x_{12} - x_1x_7x_{12} - x_3x_4x_{13} + x_0x_7x_{13} + x_1x_4x_{15} - x_0x_5x_{15} \\
x_2x_5x_{12} - x_1x_6x_{12} - x_2x_4x_{13} + x_0x_6x_{13} + x_1x_4x_{14} - x_0x_5x_{14} \\
x_3x_6x_{12} - x_2x_7x_{12} - x_3x_4x_{14} + x_0x_7x_{14} + x_2x_4x_{15} - x_0x_6x_{15} \\
-x_3x_6x_9 + x_2x_7x_9 + x_3x_5x_{10} - x_1x_7x_{10} - x_2x_5x_{11} + x_1x_6x_{11} \\
x_3x_6x_8 - x_2x_7x_8 - x_3x_4x_{10} + x_0x_7x_{10} + x_2x_4x_{11} - x_0x_6x_{11} \\
x_3x_5x_8 - x_1x_7x_8 - x_3x_4x_9 + x_0x_7x_9 + x_1x_4x_{11} - x_0x_5x_{11} \\
x_2x_5x_8 - x_1x_6x_8 - x_2x_4x_9 + x_0x_6x_9 + x_1x_4x_{10} - x_0x_5x_{10}
\end{pmatrix}
$$

Bibliography

[1] K. Akin, D. A. Buchsbaum and J. Weyman, *Schur functors and Schur complexes*, Adv.in Math. **44** (1982), 207–278.

[2] K. Akin, D.A.Buchsbaum and J.Weyman, *Resolutions of determinantal ideals: the submaximal minors.* Adv. in Math. **39** 1 (1981), 1–30.

[3] H. Boerner, *Representations of Groups* Second Edition, North Holland Pub. Co., (1970).

[4] W. Bruns, U. Vetter, *Determinantal Rings.*, Springer-Verlag, Berlin, (1988).

[5] L. Busé, *Resultants of determinantal varieties.*, J. Pure Appl. Algebr. **193** (2004), 71–97.

[6] E.J.Choi, Y.H.Kim, H.J.Ko and S.J.Won, *GL_n-The decomposition of the Schur complex $S_r(\Lambda^2\varphi)$.*, Bull. Korean Math. Soc. **40**, 1 (2003), 29–51.

[7] J. A. Eagon and D. G. Northcott *Ideals defined by matrices and certain complex associated with them.* Proc. R. Soc. A **269** (1962), 188–204.

[8] D. Eisenbud, *The Geometry of syzygies. A second course in commutative algebra and algebraic geometry.*, Springer-Verlag, Berlin, (2005).

[9] W. Fulton, J. Harris, *Representation Theory: a first course*, Springer-Verlag, New York, Berlin, Paris, (1991).

[10] D. R. Grayson and M.E.Stillman, *Macaulay2, a software system for research in algebraic geometry*, `http://www.math.uiuc.edu/Macaulay2/`.

[11] H. T. Gulliksen and G. O. Negard, *Un complexe résolvant pour certains ideaux determinantiels.* C.R. Acad. Sci. Paris Ser A **274** (1972) 16–18.

[12] J. Harris, *Algebraic geometry: a first course*, Springer-Verlag, (1992).

[13] J. O. Kleppe and R. M. Miró-Roig, *Draft*, Preprint.

[14] H. J. Ko, *The decompositions of Schur Complexes*, Trans. Amer. Math. Soc. **324** (1899), 388–413.

[15] S. Lang *Algebra. Revised third edition.*, Springer-Verlag, New York, (2005).

[16] A. Lascoux, *Syzygies des variétés déterminantales*, Adv.in Math. **30** (1978), 202–237.

[17] R. M. Miró-Roig, *Determinantal Ideals.*, Birkhäuser, Basel, Boston, Berlin, (2008).

[18] P. C. Roberts, *A minimal free complex associated to the minors of a matrix.*, Preprint.

[19] P. C. Roberts, *On the construction of generic resolutions of determinantal ideals.* Young Tableaux and Schur Functors in Algebra and Geometry (Toruń 1980), Asterisque **87-88** 353–378 Soc. Math. France, Paris (1981).

[20] J. Weyman, *Cohomology of vector bundles and syzygies*, Cambridge University Press, Cambridge, New York, (2003).