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Games with Graph Restricted Communication and Levels Structure of Cooperation

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Abstract: We analyze surplus allocation problems where cooperation between agents is restricted both by a communication graph and by a sequence of embedded partitions of the agent set. For this type of problem, we define and characterize two new values extending the Shapley value and the Banzhaf value respectively. Our results enable the axiomatic comparison between the two values and provide some basic insights for the analysis of fair resource allocation in nowadays fully integrated societies..

JEL Codes: C7.

Keywords: Coalitional games, Restricted cooperation, Graph restricted communication, Levels structure, Shapley value, Banzhaf value.

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1 Introduction

As a general rule, the layout of political administrations consist of a series of embedded layers. For instance, the EU is integrated by countries, which are organized in regions, which in turn are divided into smaller administrative units, say cities, and so on. As far as his/her relation with political institutions (and hence with political power) is concerned, a EU citizen can typically deal with one and only one institution at each level. The overall political and administrative layout typically affects the possibilities of citizens to cooperate, which would be free of administrative hurdles if no political institution existed. For example, it is generally easier to engage in a business venture within a common labor market with a single set of rules than within several markets, each regulated by a different set of rules.[1]

The fact that citizens are integrated in a sequence of embedded layers does not prevent them from interacting in various other ways, some of which go beyond the lines set up by political institutions. For instance, there is a great flow of workers between the major financial markets even though they do not belong to a common political entity. This is possible due to the existence of certain networks, very especially transportation networks and communication networks. More generally, trade between individuals and firms does not necessarily obey political structures, specially after the outset of globalization. Rivers, highways, train networks, among others, are facilitators of business and enable the possibility for different groups of individuals or firms to reach their full potential. Production chains of some major corporations in particular are now dispersed in many different countries thanks to the existence of many global networks. With such non-political linkages boosting economic growth, most of the redistribution policies (say, taxes and public expenditures) are however decided only by political institutions operating mostly at the national or regional level.

In a framework where production is not only intertwined with political institutions

[1] The so-called *border effect* is a well-documented phenomenon in international trade (see e.g. Evans, 2003).

but also with a host of other (binary) linkages such as social networks, is there a reasonable way to address the issue of how should the aggregate spoils generated by all citizens be shared among them? Gaining knowledge about solutions to this problem is the general object of this paper. Taking here mainly a normative approach, we focus in particular on certain properties that may be required for surplus allocation rules. In this vein, it is worth noting that the increase in inequality (see e.g. Atkinson, 2015) that has occurred in the latter years in many countries has raised some concerns with regard to the possibility that policy decisions should be adopted, which would guarantee that all citizens benefit from globalization. Our analysis features some elements of fairness that revolve around such concerns.

To elaborate, we analyze the class of surplus allocation problems where cooperation between agents (say, citizens) is restricted both by a sequence of embedded partitions of the agent set and by a communication graph between agents. Formally, we consider a triple made up of a TU-game, a levels structure, and a non-directed graph, which we call a *Game with Graph Restricted Communication and Levels Structure of Cooperation*. First, the TU-game describes the potential gains that any subset of players can attain on their own assuming that cooperation were unrestricted. Second, the sequence of embedded partitions of the player set (the so-called *levels structure*) represents the different "administrative units" in which players are organized. These given arrangements among agents restrict or hinder the formation of coalitions where some of its members belong to different units at some levels. Third and last, the *communication graph* accounts for the bilateral relations that may exist between players, e.g. due to commercial relations. We assume that a coalition of agents can cooperate only if all of its members are path connected within the graph. Such links extend naturally to higher units: A city is connected with another city if there is a link between citizens of both cities. Likewise the levels structure, a communication graph affects cooperation between all players, whether directly or indirectly.

Games with graph restricted communication and levels structure of cooperation provide an appropriate model to address the normative problem of how to allocate the surplus that all the agents can potentially create in light of the restrictions placed by all layers of political institutions and all communication networks. To make progress in this direction, we introduce two values (or point-valued solutions) for games with graph restricted communication and levels structure of cooperation. The first value focuses on the orderings in which coalitions are formed, while the second value considers coalitions directly, without any reference as to how they are formed. As is standard in the literature, the first one extends the Shapley value and the second one extends the Banzahf value, the two classic solutions for TU-games. We then provide a characterization of each value by means of several properties (or axioms). Such properties are of two types. A first type describes particular ways how the surplus sharing should be affected by alterations of the communication graph. A second set of properties deals with changes in the levels structure. Importantly, the latter properties used in either characterization result are (logically) comparable. This fact may be useful when facing the problem of whether to use a value or another for a given particular situation.[2]

Our contribution belongs to the extensive literature of games with restricted cooperation, which dates back at least to Aumann and Drèze (1974) (see also Owen, 1977; Myerson, 1977). While the former considers TU-games where cooperation is restricted by a partition of the player set (the so-called *games with coalition structure*), the latter consider TU-games where cooperation is restricted by a communication graph (the so-called *games with graph restricted communication*). Several papers have built on or extended these models (see e.g. Owen, 1986; Amer et al., 2002; Alonso-Meijide and Fiestras-Janeiro, 2002, 2006). Singularly, Winter (1989) generalized the model of games with coalition structure to account for the fact that restrictions to cooperation may exist at various levels. He refers to his extended framework as *games with levels structure (of cooperation)*. Both type of restrictions to cooperation, i.e. levels structures and undirected graphs, can however exist simultaneously. To account for this possibility, Vázquez-Brage et al. (1996) and Alonso-Meijide et al. (2009) have already proposed and characterized generalizations of the Shapley value and the Banzhaf value for games

[2]We elaborate more on this issue in Section 4.

with both a coalition structure and a communication graph. The model we analyze here is a natural generalization of the latter, insofar as it considers a levels structure instead of coalition structure (i.e, a levels structure with a single level).

The paper is organized as follows: In Section 2 we set the notation and introduce the main concepts from the literature. In Section 3 we define two new values for games with graph restricted communication and levels structure of cooperation, which we characterize by means of a number of properties and then compare. Section 4 concludes. The proofs are all contained in the Appendix.

2 Notations and Preliminaries

2.1 TU-games

Let Ω denote the (possibly infinite) set of potential players. A *cooperative game with transferable utility* (TU-game) is a pair (N, v), where $\emptyset \neq N \subseteq \Omega$ is a finite set of players and $v\colon 2^N = \{S : S \subseteq N\} \to \mathbb{R}$ is the *characteristic function*, with $v(\emptyset) = 0$. For every coalition $S \subseteq N$, $v(S)$ represents the worth of coalition S, i.e., the total payoff that members of the coalition can obtain by agreeing to cooperate. We denote the collection of all TU-games by $\mathcal{G}$. For the sake of readability, we henceforth abuse notation slightly and write $T \cup i$ and $T \setminus i$ instead of $T \cup \{i\}$ and $T \setminus \{i\}$ for $T \subseteq N$ and $i \in N$, respectively. We use the $|\cdot|$ operator to denote the cardinality of a finite set.

A *value on* $\mathcal{G}$ is a map, f, that assigns a unique vector $\mathsf{f}(N, v) \in \mathbb{R}^N$ to every $(N, v) \in \mathcal{G}$. A *permutation* of N is a bijective map $\pi : N \to N$. Let $\Pi(N)$ denote the set of permutations of N. Given $\pi \in \Pi(N)$ and $i \in N$, let $\pi^{-1}[i]$ indicate the set of players ordered before i in permutation π, i.e., $\pi^{-1}[i] = \{j \in N : \pi(j) < \pi(i)\}$. Next, we present the formal definitions of two well-known values on $\mathcal{G}$, namely the Shapley and Banzhaf values.

Definition 2.1. *The* Shapley value *(Shapley, 1953),* Sh, *is the value on* $\mathcal{G}$ *defined for* *every* $(N,v) \in \mathcal{G}$ *and* $i \in N$ *by*

$$\mathsf{Sh}_i(N,v) = \frac{1}{|\Pi(N)|} \sum_{\pi \in \Pi(N)} \left[v\left(\pi^{-1}[i] \cup i\right) - v\left(\pi^{-1}[i]\right) \right].$$

The Banzhaf value *(Banzhaf, 1965),* Ba*, is the value on* $\mathcal{G}$ *defined for every* $(N,v) \in \mathcal{G}$ *and* $i \in N$ *by*

$$\mathsf{Ba}_i(N,v) = \frac{1}{2^{|N|-1}} \sum_{S \subseteq N \backslash i} \left[v(S \cup i) - v(S) \right].$$

The differences between the two above values are well know from an axiomatic viewpoint (see e.g. Young, 1985; Feltkamp, 1995; Nowak, 1997).

2.2 Games with graph restricted communication

A *communication graph* is an undirected graph without loops defined on a finite set of nodes. That is, (N,C) is a communication graph if N is a finite set of nodes and C is a set of links among the nodes. A *link* between i and j is denoted by $\{i:j\}$ (note that $\{i:j\} = \{j:i\}$). Given $i,j \in S \subseteq N$, we say that i and j are *connected* in S by C if there is a *path* in S connecting them, i.e., for some $k \geq 0$, there is a subset $\{i_0,\ldots,i_k\} \subseteq S$ such that $i_0 = i$, $i_k = j$, and for every $l \in \{1,\ldots k\}$, $\{i_{l-1}:i_l\} \in C$. By S/C we denote the partition of S into maximal connected components. The complete graph on a finite set N is denoted by (N,C^N), i.e., $C^N = \{\{i:j\} : i,j \in N, i \neq j\}$. The set of all communication graphs is denoted by $\mathcal{C}$. Given a communication graph $(N,C) \in \mathcal{C}$, a node $i \in N$ is said to be *isolated* in the graph if there is no link from her, i.e., for every $j \in N \backslash i$, $\{i:j\} \notin C$. The communication graph (N,C^{-i}) is obtained from (N,C) by dissolving the links where i is involved, i.e., $C^{-i} = \{\{k:l\} \in C : k,l \in N \backslash i\}$. Similarly, given a link $\{i:j\}$, the communication graph (N,C^{-ij}) is obtained from (N,C) by eliminating the link $\{i:j\}$, i.e., $C^{-ij} = \{\{k:l\} \in C : \{k:l\} \neq \{i:j\}\}$.

A *game with graph restricted communication* is a triple (N,v,C) where $(N,v) \in \mathcal{G}$ and $(N,C) \in \mathcal{C}$. The set of all games with graph restricted communication is denoted by $\mathcal{GC}$. A *value on* $\mathcal{GC}$ is a map, f, that assigns a unique vector $\mathsf{f}(N,v) \in \mathbb{R}^N$ to every $(N,v,C) \in \mathcal{GC}$.

For every $(N, v, C) \in \mathcal{GC}$, the *graph restricted game* $(N, v^C) \in \mathcal{G}$ assumes that players can only communicate through (N, C). In other words, a coalition can cooperate only if it is connected through the communication graph. Formally, for every $S \subseteq N$,

$$v^C(S) = \sum_{T \in S/C} v(T).$$

Two well-known values on $\mathcal{GC}$ that generalize the Shapley and Banzhaf values are presented below.

Definition 2.2. *The* Myerson value *(Myerson, 1977),* SG*, is the value on* $\mathcal{GC}$ *defined for every* $(N, v, C) \in \mathcal{GC}$ *by*

$$\mathsf{SG}(N, v, C) = \mathsf{Sh}\left(N, v^C\right).$$

The Banzhaf graph value *(Owen, 1986),* BG*, is the value on* $\mathcal{GC}$ *defined for every* $(N, v, C) \in \mathcal{GC}$ *by*

$$\mathsf{BG}(N, v, C) = \mathsf{Ba}\left(N, v^C\right).$$

2.3 Games with levels structure of cooperation

Consider now that the cooperation among agents is restricted by means of a finite sequence of partitions defined on the player set, each of them being coarser than the previous one. Formally, Winter (1989) introduced a *levels structure of cooperation* (or simply a *levels structure*) being a pair $(N, \underline{B})$ where $N \subseteq \Omega$ is a finite set of players and $\underline{B}$ is a sequence of partitions of N, $\underline{B} = \{B_0, \ldots, B_{k+1}\}$, with the following properties: $B_0 = \{\{i\} : i \in N\}$, $B_{k+1} = \{N\}$, and, for each $r \in \{0, \ldots, k\}$, B_{r+1} is coarser than B_r. That is to say, for each $r \in \{1, \ldots, k+1\}$ and each $S \in B_r$, there is $B \subseteq B_{r-1}$ such that $S = \cup_{U \in B} U$. Each $S \in B_r$ is called a *union* and B_r is called the r^{th} *level of* $\underline{B}$. The levels B_0 and B_{k+1} are added for notational convenience. We denote by $\left(N, \underline{B_0}\right)$ the trivial levels structure with $k = 0$, i.e., $\underline{B_0} = \{B_0, B_1\}$ where $B_0 = \{\{i\}\}_{i \in N}$ and $B_1 = \{N\}$. We further denote by $\mathcal{L}$ the set of all levels structures. Let $(N, \underline{B}) \in \mathcal{L}$ with $\underline{B} = \{B_0, \ldots, B_{k+1}\}$ and $i \in N$. Then, $(N, \underline{B^{-i}}) \in \mathcal{L}(N)$ is the levels structure obtained from $(N, \underline{B})$ by isolating player i from the union she belongs to at each level, i.e., $\underline{B^{-i}} = \{B_0, B_1^{-i}, \ldots, B_k^{-i}, B_{k+1}\}$, where, for all $r \in \{1, \ldots, k\}$, $B_r^{-i} = \{U \in B_r : i \notin U\} \cup \{S \setminus i, \{i\}\}$ with $i \in S \in B_r$.

A *game with levels structure of cooperation* is a triple $(N, v, \underline{B})$, where $(N, v) \in \mathcal{G}$ and $(N, \underline{B}) \in \mathcal{L}$. We denote by $\mathcal{GL}$ the set of all games with levels structure of cooperation. Let $(N, v, \underline{B}) \in \mathcal{GL}$ with $\underline{B} = \{B_0, \ldots, B_{k+1}\}$, for each $r \in \{0, \ldots, k\}$ we define the r^{th} *level union game* $(B_r, v^r, \underline{B^r}) \in \mathcal{GL}$ induced from $(N, v, \underline{B})$ as the game with levels structure of cooperation with the elements of B_r as players, characteristic function v^r given by $v^r(S) = v(\bigcup_{U \in S} U)$ for any coalition $S \subseteq B_r$, and with levels structure $\underline{B^r} = \{B_0^r, \ldots, B_{k-r+1}^r\}$ given by $B_0^r = \{\{U\} : U \in B_r\}$, $B_s^r = \{\{U \in B_r : U \subseteq U'\} : U' \in B_{r+s}\}$ for $s \in \{1, \ldots, k-r\}$, and $B_{k-r+1}^r = \{\{U : U \in B_k\}\}$. Note that $\underline{B^r} = \underline{B}$ if $r = 0$, whereas $\underline{B^k}$ is the trivial levels structure $\underline{B_0}$ on the player set $\{U : U \in B_k\}$.

A *value on* $\mathcal{GL}$ is a map, f, that assigns to every game with levels structure of cooperation $(N, v, \underline{B}) \in \mathcal{GL}$ a vector $\mathsf{f}(N, v, \underline{B}) \in \mathbb{R}^N$. To present two values on $\mathcal{GL}$, we need to introduce some further notation. On the one hand, for every $(N, \underline{B}) \in \mathcal{L}$ with $\underline{B} = \{B_0, \ldots, B_{k+1}\}$, the set of permutations of N that respect the levels structure $(N, \underline{B})$ is denoted by $\Omega(\underline{B})$ and is defined by

$$\begin{aligned}\Omega(\underline{B}) = \{&\pi \in \Pi(N) : \forall B_r \in \underline{B}, \forall T \in B_r, \forall i, j \in T, \\ &\text{and } k \in N, \text{ if } \pi(i) < \pi(k) < \pi(j) \text{ then } k \in T\}.\end{aligned}$$

On the other hand, for every $(N, \underline{B}) \in \mathcal{L}$ and $i \in N$, let $i \in U_0 = \{i\} \subseteq U_1 \subseteq \cdots \subseteq U_k$ such that $U_r \in B_r$ for all $r \in \{0, \ldots, k\}$. Then, the *partition induced by* $\underline{B}$ *on* i is defined as

$$P(i, \underline{B}) = \bigcup_{r=0}^{k} (B_r)_{|U_{r+1} \setminus U_r},$$

where $U_{k+1} = N$ by convenience. Then, $P(i, \underline{B}) \in \mathcal{P}(N \setminus i)$. We denote $|P(i, \underline{B})|$ by m_i, and the unions of the partition induced by $\underline{B}$ on i, by $P(i, \underline{B}) = \{T_1, \ldots, T_{m_i}\}$. Finally, the set of indices of the partition induced by $\underline{B}$ on i is denoted by $M_i = \{1, \ldots, m_i\}$ and for every $R \subseteq M_i$, $T_R = \cup_{r \in R} T_r$.

Definition 2.3. *The* Shapley levels value *(Winter, 1989),* Sh^{L}*, is the value on* $\mathcal{GL}$ *defined for every* $(N, v, \underline{B}) \in \mathcal{GC}$ *and* $i \in N$ *by*

$$\mathsf{Sh}^{\mathsf{L}}_i(N, v, \underline{B}) = \frac{1}{|\Omega(\underline{B})|} \sum_{\sigma \in \Omega(\underline{B})} \left[v\left(\sigma^{-1}[i] \cup i\right) - v\left(\sigma^{-1}[i]\right)\right].$$

The Banzhaf Levels value *(Álvarez-Mozos and Tejada, 2011),* Ba^{L}*, is the value on* $\mathcal{GL}$ *defined for every* $(N, v, C) \in \mathcal{GC}$ *and* $i \in N$ *by*

$$\mathsf{Ba}^{\mathsf{L}}_i(N, v, \underline{B}) = \sum_{R \subseteq M_i} \frac{1}{2^{m_i}} \left[v(T_R \cup i) - v(T_R)\right].$$

The two values above yield the Shapley value and the Banzhaf value, respectively if the levels structure is trivial. They also generalize the Owen value (Owen, 1977) and the Banzhaf-Owen value (Owen, 1982).[3]

3 Two New Solutions

Finally, we are in a position to introduce the main mathematical objects we are dealing with in this paper. A *game with graph restricted communication and levels structure of cooperation* is a four-tuple $(N, v, C, \underline{B}) \in \mathcal{GCL}$ where $N \subseteq \Omega$, $(N, v) \in \mathcal{G}$, $(N, C) \in \mathcal{C}$ and $(N, \underline{B}) \in \mathcal{L}$. We let $\mathcal{GCL}$ denote the set of all such games. A *value on* $\mathcal{GCL}$ is then a map, f, that assigns to every game with graph restricted communication and levels structure of cooperation $(N, v, C, \underline{B}) \in \mathcal{GCL}$ a vector $\mathsf{f}(N, v, C, \underline{B}) \in \mathbb{R}^N$.

In this section we do three things. First, we introduce and characterize a new value for games with graph restricted communication and levels structure of cooperation, which generalizes the Shapley value. Second, we introduce and characterize another new value for games with graph restricted communication and levels structure of cooperation, which generalizes the Banzhaf value. Third, we build on the two characterizations to compare the two proposed values.

[3]These values are defined in the framework of games with a coalition structure.

3.1 The Myerson levels value

We start by proposing the natural generalization to our setting of the Shapley value, the Myerson value, and the Shapley levels value.

Definition 3.1. *The* Myerson levels value, SG^{L}, *is the value on $\mathcal{GCL}$, defined for every $(N, v, C, \underline{B}) \in \mathcal{GCL}$ by*

$$\mathsf{SG}^{\mathsf{L}}(N, v, C, \underline{B}) = \mathsf{Sh}^{\mathsf{L}}\left(N, v^{C}, \underline{B}\right).$$

That is, the Myerson levels value of a game with graph restricted communication and levels structure of cooperation, say $(N, v, C, \underline{B})$, is the result of a two-stage procedure. First, the (unrestricted) possibilities of cooperation captured by (N, v) are reduced in accordance with the communication graph (N, C). This means that if there exist returns to scale—i.e., the TU-game is strictly superadditive—, only coalitions whose members are all connected within the graph can obtain their full potential.[4] The outcome of this restriction procedure is (N, v^C). Second, the Shapley levels value is applied to the resulting game with levels structure of cooperation $\left(N, v^C, \underline{B}\right)$. This implies that cooperation among agents is further restricted, so that only permutations in which all agents are ordered in a way that respects the levels structure $(N, \underline{B})$ are payoff-relevant.

In the following, we introduce a number of properties that a value for games with graph restricted communication and levels structure of cooperation may satisfy.

CE A value on $\mathcal{GCL}$, f, satisfies *component efficiency* if for every $(N, v, C, \underline{B}) \in \mathcal{GCL}$ and every $S \in N/C$,

$$\sum_{i \in S} \mathsf{f}_i\left(N, v, C, \underline{B}\right) = v(S).$$

FG A value on $\mathcal{GCL}$, f, satisfies *fairness in the graph* if for every $(N, v, C, \underline{B}) \in \mathcal{GCL}$ and every $i, j \in U \in B_1 \in \underline{B}$ such that $\{i : j\} \in C$,

$$\mathsf{f}_i\left(N, v, C, \underline{B}\right) - \mathsf{f}_i\left(N, v, C^{-ij}, \underline{B}\right) = \mathsf{f}_j\left(N, v, C, \underline{B}\right) - \mathsf{f}_j\left(N, v, C^{-ij}, \underline{B}\right).$$

[4] A TU-game (N, v) is *strictly superadditive* if $v(S \cup T) > v(S) + v(T)$ for $\emptyset \neq S, T$ such that $S \cap T = \emptyset$.

BC A value on $\mathcal{GCL}$, f, satisfies *balanced contributions* if for every $(N, v, C, \underline{B}) \in \mathcal{GCL}$ and every $i, j \in U \in B_1 \in \underline{B}$,

$$\mathsf{f}_i(N, v, C, \underline{B}) - \mathsf{f}_i\left(N, v, C, \underline{B^{-j}}\right) = \mathsf{f}_j(N, v, C, \underline{B}) - \mathsf{f}_j\left(N, v, C, \underline{B^{-i}}\right).$$

CLG A value on $\mathcal{GCL}$, f, satisfies the *communication level game property* if for every $(N, v, C, \underline{B}) \in \mathcal{GCL}$ and every $U \in B_r \in \underline{B}$,

$$\sum_{i \in U} \mathsf{f}_i(N, v, C, \underline{B}) = \mathsf{f}_U\left(B_r, \left(v^C\right)^r, C^{B_r}, \underline{B_r}\right).$$

The four properties above are inspired by similar properties from the literature. CE and FG were introduced by Myerson (1977) in the framework of games with graph restricted communication. BC was proposed by Vázquez-Brage et al. (1996) and CLG by Alonso-Meijide et al. (2009), in both cases for games with graph restricted communication and a coalition structure.

The first property, CE, requires the spoils generated by unrestricted cooperation be always attainable for coalitions that are maximally connected in the communication graph, regardless of the restrictions placed by the levels structure. The latter may affect the payoff to specific players but not the total payoff of a maximal connected coalition. Accordingly, there is a sense in which CE builds on the idea that "connected markets" are always efficient. Hence, CE is more a descriptive property of our setting rather than a normative requirement.

By contrast, FG is a property with a strong flavor of equity: it requires that the introduction or removal of a communication link between two agents has to affect both players' payoff equally. Redistribution tools such as taxes and public spending could be used to ensure that this condition is fulfilled. Unlike CE, FG does not apply independently of the restrictions placed by the levels structure, for the two players considered have to belong to the same administrative unit (i.e., union) at all levels. FG is silent with regard to changes in the communication structure that affect citizens who live in different cities, regions or countries.

Likewise FG, BC applies to any pair of players who belong to the same union at all levels of the levels structure. For such a pair of players, BC requires that the when one of the players becomes completely isolated from the administrative structure—as modelled by the levels structure—, the effect on the other player is always the same (for that union). Hence, BC equalizes for each citizen—or, more generally, for the minimal unit in our model, be it citizens or groups of citizens—the value of the current political structure with respect to the threat of any of her fellow citizens (who are closest to her in the levels structure) that they will leave the structure. In a sense, BC can be seen as a stability property for the lowest non-trivial level of the entire administrative structure: for any two players that belong to the same union, the threat of one against the other has to be the same.[5] Note that BC assumes that changes in the levels structure induce in turn no changes in the communication graph.

Finally, CLG also deals with the stability of the levels structure, keeping the communication graph fixed. While BC is concerned with changes in the levels structure from a horizontal perspective (it compares the payoff change of two citizens who belong to the same union at all levels), CLG is concerned with changes in the levels structure from a vertical perspective. Specifically, the latter property demands that aggregating units cannot have an effect on the total payoff of the administrative units (i.e., unions) being aggregated.

It turns out that the four properties just introduced single out SG^{L}.

Theorem 3.1. *The Myerson levels value is the only value on $\mathcal{GCL}$ that satisfies* CE*,* FG*,* BC*, and* CLG*. Moreover, the four properties are independent.*[6]

We point out that the result remains valid if CE and FG are only demanded when the levels structure is trivial. What is more, we could replace this two properties by any set

[5]In combination with CLG, such a notion of stability translates into all other levels of the administrative structure.

[6]For the independence of the properties, we weaken CE and require its applicability only to games where the levels structure is trivial.

that characterizes the Myerson value (see e.g. Myerson, 1980, and the characterization therein).

3.2 The Banzhaf levels graph value

Next, we propose *one* natural generalization to our setting of the Banzhaf value, the Banzhaf graph value, and the Banzhaf levels value.

Definition 3.2. *The* Banzhaf Levels graph value*,* BG^{L}*, is the value on* $\mathcal{GCL}$*, defined for every* $(N, v, C, \underline{B}) \in \mathcal{GCL}$ *by*

$$\mathsf{BG}^{\mathsf{L}}(N, v, C, \underline{B}) = \mathsf{Ba}^{\mathsf{L}}\left(N, v^{C}, \underline{B}\right).$$

The Banzhaf Levels graph value is obtained by a two-stage procedure similar to the one used to build the Myerson levels value. The main difference is that after the transformation of the players' cooperation possibilities set by the communication graph, which yields a game with levels structure of cooperation, all admissible coalitions are now assumed to be equally likely. This property is characteristic of the Banzhaf value—upon which BG^{L} is based— and is in sharp contrast with the assumption that lies at the foundation of the Shapley value—upon which SG^{L} is based—, namely that all admissible permutations are equally likely. The differences between the two approaches are further discussed in Section 4.

In the following, we introduce more properties that a value for games with graph restricted communication and levels structure of cooperation may satisfy.

GI A value on $\mathcal{GCL}$, f, satisfies *graph isolation* if for every $(N, v, C, \underline{B}) \in \mathcal{GCL}$ and every $\{i\} \in N/C$,

$$\mathsf{f}_i\left(N, v, C, \underline{B}\right) = v(i).$$

2-E A value on $\mathcal{GCL}$, f, satisfies *2-efficiency* if for every $(N, v, C, \underline{B}) \in \mathcal{GCL}$ and every $i, j \in U \in B_1 \in \underline{B}$ such that $\{i : j\} \in C$,

$$\mathsf{f}_i\left(N, v, C, \underline{B}\right) + \mathsf{f}_j\left(N, v, C, \underline{B}\right) = \mathsf{f}_i\left(N, v_{ij}, C_{ij}, \underline{B}\right) + \mathsf{f}_j\left(N, v_{ij}, C_{ij}, \underline{B}\right),$$

where $C_{ij} = C^{-j} \cup \{\{i : k\} : k \in N \setminus i \text{ with } \{k : j\} \in C\}$ and for every $S \subseteq N$,
$$v_{ij}(S) = \begin{cases} v(S \cup j) & \text{if } i \in S \\ v(S \setminus j) & \text{if } i \notin S \end{cases}.$$

NID A value on $\mathcal{GCL}$, f, satisfies *neutrality under individual desertion* if for every $(N, v, C, \underline{B}) \in \mathcal{GCL}$ and every $i, j \in U \in B_1 \in \underline{B}$,

$$\mathsf{f}_i\left(N, v, C, \underline{B}\right) = \mathsf{f}_i\left(N, v, C, \underline{B^{-j}}\right).$$

1-CLG A value on $\mathcal{GCL}$, f, satisfies the *1-communication level game property* if for every $(N, v, C, \underline{B}) \in \mathcal{GCL}$ and every $\{i\} \in B_r \in \underline{B}$,

$$\mathsf{f}_i\left(N, v, C, \underline{B}\right) = \mathsf{f}_{\{i\}}\left(B_r, \left(v^C\right)^r, C^{B_r}, \underline{B_r}\right).$$

These four properties are again inspired by similar properties from the literature. GI was introduced by Alonso-Meijide and Fiestras-Janeiro (2006) in the framework of games with graph restricted communication. 2-E was originally proposed by Lehrer (1988) for TU-games (see also Haller, 1994). The last two properties, NID and 1-CLG, were introduced by Alonso-Meijide et al. (2007) for games with a coalition structure.

Because GI is weaker than CE, it admits a similar interpretation. Specifically, it requires that regardless of any consideration, a player who is isolated in the graph cannot establish any cooperation with any other player, and thus her payoff must be equal to her stand-alone worth. Hence, this property builds on the positive assumption that connectedness (in the graph) is a necessary condition for cooperation (in the game). GI also admits a normative interpretation: no player should be allowed to rip off someone else's production if she is isolated in the graph. We stress that GI holds irrespective of the levels structure.

2-E is concerned with the payoff of two players who not only belong together in all levels of the level structure of cooperation but are also directly connected in the graph. Specifically, this property requires that if one of these players delegates her role to another player, their aggregate payoff has to remain the same. Such an internal reorganization may, however, have a re-distributive impact in how much each player is eventually allocated. Note that 2-E assumes both a change in the game and a change in the graph. A rationale for this simultaneous premise is the following: the fact that one player delegates her role to other player means that the former has to be able to transfer to the latter a property right on both the production (i.e., on the potential contributions of the game where the delegating player is involved, provided that the delegated player is also involved) and the network (i.e. on all links in which the delegating player is involved). In particular, if 2-E is satisfied and either transferring the property rights on production (e.g., by signing a contract) or transferring the links (e.g., when they are physical, geographically-based networks) is costly, such internal reorganizations should in principle not take place, and thus we should always observe that all players take an active role in the negotiations.

NID is stronger than BC. Indeed, while the latter property is a symmetry condition that equalizes the effect that the departure of one another player has on any other player who belongs to the same union at all levels, NID requires these changes be zero. NID thus builds on the normative assumption that the threat of any citizen claiming to unilaterally deviate from the current administrative structure should be void for the fellow citizens who belong to the same unions at all levels.

Finally, 1-CLG is weaker than CLG. Indeed, the former property results from applying the latter property to singleton unions of any level. In particular, the interpretation of 1-CLG is essentially the same as that of CLG.

It turns out that in combination with FG, the four properties just introduced single out BG^{L}.

Theorem 3.2. *The Banzhaf Levels graph value is the only value on $\mathcal{GCL}$ that satisfies* GI*,* FG*,* 2-E*,* NID*, and* 1-CLG*. Moreover, the five properties are independent.*[7]

We point out that the result remains valid if GI, 2-E, and FG are only demanded

[7]For the independence of the properties, we weaken GI and require its applicability only to games where the levels structure is trivial.

when the levels structure is trivial. What is more, we could replace this two properties by any set that characterizes the Banzhaf graph value.

4 Conclusion

In this paper we have introduced and characterized two new values for games with graph restricted communication and levels structure of cooperation. To the best of our knowledge, such an analysis was absent in the literature despite the fact that it follows naturally from all previous contributions regarding games with restricted cooperation.

For our results, we have used properties of two kinds. The first kind of property describe how a value should behave with respect to changes in the communication graph, and they can be replaced by any set of properties characterizing the corresponding value on $\mathcal{GC}$ (the Myerson value in the case of SG^{L} and the Banzhaf graph value in the case of BG^{L}). The second type of property, namely BC and CLG for the Myerson levels value (SG^{L}) and NID and 1-CLG for the Banzhaf levels graph value (BG^{L}), are parallel in the following sense: First, BC is a weakening of NID and hence SG^{L} is less restrictive than BG^{L} regarding the effect of one agent leaving the levels structure on another. Second, CLG is a stronger requirement than 1-CLG. Accordingly, SG^{L} could be more reasonable than BG^{L} in those situations where the stability of the levels structure is high.

Several aspects of our model deserve scrutiny in the future, of which we mention two. On the one hand, a closer mapping of our model to real data may help us discern whether it is the Shapley-like or the Banzhaf-like approach, if any, that is more prevalent. On the other hand, a purely non-cooperative model of public finance that implements our two values could guide policy-making based on the normative considerations raised by our paper.

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Appendix

Proof of Theorem 3.1.

The proof proceeds in three steps. First, we prove that the Myerson levels value satisfies CE, FG, BC, and CLG. Second, we prove that there is at most one value on $\mathcal{GCL}$ satisfying CE, FG, BC, and CLG. Third, we prove that the four properties are logically independent.

Existence

First, we check that SG^{L} satisfies CE. Let $(N, v, C, \underline{B}) \in \mathcal{GCL}$ and $S \in N/C$. By definition of v^C, for every $i \in S$ and $T \subseteq N \setminus i$,

$$v^C(T \cup i) - v^C(T) = v^C((T \cap S) \cup i) - v^C(T \cap S). \tag{1}$$

Then,

$$\sum_{i\in S}\mathsf{SG}_i^{\mathsf{L}}(N,v,C,\underline{B}) = \frac{1}{|\Omega(\underline{B})|}\sum_{\sigma\in\Omega(\underline{B})}\sum_{i\in S}\left[v^C\left(\sigma^{-1}[i]\cup i\right) - v^C\left(\sigma^{-1}[i]\right)\right]$$
$$= \frac{1}{|\Omega(\underline{B})|}\sum_{\sigma\in\Omega(\underline{B})}\sum_{i\in S}\left[v^C\left((\sigma^{-1}[i]\cap S)\cup i\right) - v^C\left(\sigma^{-1}[i]\cap S\right)\right]$$
$$= \frac{1}{|\Omega(\underline{B})|}\sum_{\sigma\in\Omega(\underline{B})} v^C(S) = v(S),$$

where the first equality is obtained by switching the order of the summations, the second equality results from applying Eq. (1), and the last equality follows by definition of v^C and the observation that for every ordering of the agents in S, adding the marginal contributions of every player to her set of predecessors yields $v^C(S) - v^C(\emptyset) = v^C(S)$. Indeed, let $S = \{i_1, \ldots, i_s\}$, with $s = |S|$, be such that $\sigma^{-1}[i_j] < \sigma^{-1}[i_l]$ if and only if $j < l$. Then

$$\sum_{i\in S}\left[v^C\left((\sigma^{-1}[i]\cap S)\cup i\right) - v^C\left(\sigma^{-1}[i]\cap S\right)\right]$$
$$= \sum_{l=1}^{s}\left[v^C\left(i_1\cup\ldots\cup i_l\right) - v^C\left(i_1\cup\ldots\cup i_{l-1}\right)\right] = v^C\left(i_1\cup\ldots\cup i_s\right).$$

Second, we check that SG^{L} satisfies FG. Let $(N,v,C,\underline{B}) \in \mathcal{GCL}$ and $i,j \in U \in B_1 \in \underline{B}$ be such that $\{i:j\} \in C$. Define $w = v^C - v^{C^{-ij}}$. Since SG^{L} is additive (see pp 229-230 in Winter, 1989),

$$\mathsf{SG}_i^{\mathsf{L}}(N,v,C,\underline{B}) - \mathsf{SG}_i^{\mathsf{L}}\left(N,v,C^{-ij},\underline{B}\right) = \mathsf{Sh}_i^{\mathsf{L}}(N,v^C,\underline{B}) - \mathsf{Sh}_i^{\mathsf{L}}(N,v^{C^{-ij}},\underline{B})$$
$$= \mathsf{Sh}_i^{\mathsf{L}}(N,w,\underline{B}). \tag{2}$$

We show that i and j are symmetric players in (N,w). Indeed, let $S \subseteq N\setminus\{i,j\}$. Then, because $(S\cup i)/C = (S\cup i)/C^{-ij}$ and $(S\cup j)/C = (S\cup j)/C^{-ij}$, it must be that

$$w(S\cup i) = w(S\cup j) = 0.$$

Next, since $i,j \in U \in B_1 \in \underline{B}$ and Sh^{L} satisfies symmetry (see pp 23-24 in Álvarez-Mozos and Tejada, 2011), $\mathsf{Sh}_i^{\mathsf{L}}(N,w,\underline{B}) = \mathsf{Sh}_j^{\mathsf{L}}(N,w,\underline{B})$. Finally, observe that in Eq.

(2) the roles of i and j can be replaced. Combining the last equality with these two versions of Eq. (2), we obtain that SG^{L} satisfies FG.

Third, SG^{L} satisfies BC by its definition and the fact that Sh^{L} satisfies the level balanced contributions property (see p 24 in Álvarez-Mozos and Tejada, 2011).

Fourth and last, SG^{L} satisfies CLG because for every $U \in B_r \in \underline{B}$,

$$\begin{aligned}\sum_{i\in U} \mathsf{SG}^{\mathsf{L}}_i(N, v, C, \underline{B}) &= \sum_{i\in U} \mathsf{Sh}^{\mathsf{L}}_i(N, v^C, \underline{B}) = \mathsf{Sh}^{\mathsf{L}}_U\left(B_r, (v^C)^r, \underline{B_r}\right) \\ &= \mathsf{Sh}^{\mathsf{L}}_U\left(B_r, \left((v^C)^r\right)^{C^{B_r}}, \underline{B_r}\right) = \mathsf{SG}^{\mathsf{L}}_U\left(B_r, (v^C)^r, C^{B_r}, \underline{B_r}\right),\end{aligned}$$

where the first and last equalities are by definition of SG^{L}, the second one is due to the fact that Sh^{L} satisfies the level game property (see p 24 in Álvarez-Mozos and Tejada, 2011), and the third holds from the observation that when the communication graph is complete, the graph restricted game is the game itself.

Uniqueness

Let f be a value on $\mathcal{GCL}$ satisfying the four properties and let $(N, v, C, \underline{B}) \in \mathcal{GCL}$ be a game with k levels. We show uniqueness of $\mathsf{f}(N, v, C, \underline{B})$ by induction on k.

First, let $k = 0$, i.e., $\underline{B} = \underline{B_0}$. Define fc, a value on $\mathcal{GC}$, by $\mathsf{fc}(N, v, C) = \mathsf{f}(N, v, C, \underline{B_0})$ for every $(N, v, C) \in \mathcal{GC}$. Since f satisfies CE and FG, fc satisfies the two properties of the characterization of Myerson (1977). Then $\mathsf{fc} = \mathsf{SG}$, and $\mathsf{f}(N, v, C, \underline{B_0})$ is uniquely determined. Note that we only use CE for games (with graph restricted communication and levels structure of cooperation) $(N, v, C, \underline{B})$ where $\underline{B} = \underline{B_0}$, i.e., where the levels structure is trivial.

Second, suppose that the payoffs according to f are unique for any game with less than k levels. Then, let $(N, v, C, \underline{B}) \in \mathcal{GCL}$ be a game with k levels, so that we have $\underline{B} = \{B_0, \ldots, B_{k+1}\}$. Let $i \in U \in B_1$. We show uniqueness of $\mathsf{f}_i(N, v, C, \underline{B})$ by a second induction on the cardinality of U.

On the one hand, suppose that $U = \{i\}$. Then, by CLG,

$$\mathsf{f}_i(N, v, C, \underline{B}) = \mathsf{f}_U\left(B_1, (v^C)^1, C^{B_1}, \underline{B_r}\right).$$

Note that $\left(B_1, (v^C)^1, C^{B_1}, \underline{B_r}\right)$ is a game with $k-1$ levels. Then, the payoff above is unique by the first induction hypothesis. On the other hand, suppose that the payoff according to f is unique for any player that belongs to a union of the first level with a cardinality smaller than $u > 1$, and let $i \in U \in B_1 \in \underline{B}$ be such that $|U| = u$. Take $j \in U \setminus i$. By BC,

$$\mathsf{f}_i(N, v, C, \underline{B}) - \mathsf{f}_j(N, v, C, \underline{B}) = \mathsf{f}_i(N, v, C, \underline{B^{-j}}) - \mathsf{f}_j(N, v, C, \underline{B^{-i}}). \tag{3}$$

Note that the payoffs in the right-hand side of the above equation are unique by the second induction hypothesis. Adding up Eq. (3) for every $j \in U \setminus i$, we obtain that

$$u \cdot \mathsf{f}_i(N, v, C, \underline{B}) - \sum_{j \in U} \mathsf{f}_j(N, v, C, \underline{B}) \tag{4}$$

is uniquely determined. Finally, by CLG,

$$\sum_{j \in U} \mathsf{f}_j(N, v, C, \underline{B}) = \mathsf{f}_{[U]}\left(B_1, (v^C)^1, C^{B_1}, \underline{B_r}\right),$$

which is also unique by the first induction hypothesis. Together with Eq. (4), this fact concludes this step of the proof.

Logical independence

It only remains to check that the four properties are independent.

(i) Let f be the value on $\mathcal{GCL}$ defined for every $(N, v, C, \underline{B}) \in \mathcal{GCL}$ and $i \in N$ by $\mathsf{f}_i(N, v, C, \underline{B}) = 0$.

   Then, f satisfies FG, BC, and CLG but not CE.

(ii) Let f be the value on $\mathcal{GCL}$ defined for every $(N, v, C, \underline{B}) \in \mathcal{GCL}$ as follows: If $N = \{i, j\}$ and $C = C^N$, then

$$\mathsf{f}_i(N, v, C, \underline{B}) = \frac{3}{4}\left(v(N) - v(j)\right) + \frac{1}{4}v(i)$$
$$\mathsf{f}_j(N, v, C, \underline{B}) = \frac{1}{4}\left(v(N) - v(i)\right) + \frac{3}{4}v(j)$$

Otherwise, $\mathsf{f}(N, v, C, \underline{B}) = \mathsf{SG}^{\mathsf{L}}(N, v, C, \underline{B})$.[8]

Then, f satisfies CE, BC, and CLG but not FG.

(iii) Let f be the value on $\mathcal{GCL}$ defined for every $(N, v, C, \underline{B}) \in \mathcal{GCL}$ by $\mathsf{f}(N, v, C, \underline{B}) = \mathsf{SG}^{\mathsf{L}}(N, v, C, \underline{B_0})$.

Then, f satisfies CE, FG, and BC, but not CLG.

(iv) Let f be the value on $\mathcal{GCL}$ defined for every $(N, v, C, \underline{B}) \in \mathcal{GCL}$ and $i \in N$ by $\mathsf{f}_i(N, v, C, \underline{B}) = \dfrac{\mathsf{SG}^{\mathsf{L}}_U \left(B_1, (v^C)^1, C^{B_1}, \underline{B_1}\right)}{|U|}$, where $i \in U \in B_1$.

Then, f satisfies CE (for $\underline{B_0}$), FG, and CLG, but not BC.

□

Proof of Theorem 3.2.

The proof proceeds in three steps. First, we prove that the Banzhaf levels graph value satisfies GI, FG, 2-E, NID, and 1-CLG. Second, we prove that there is at most one value on $\mathcal{GCL}$ satisfying GI, FG, 2-E, NID, and 1-CLG. Third, we prove that the five properties are logically independent.

Existence

First, we check that BG^{L} satisfies GI. Let $(N, v, C, \underline{B}) \in \mathcal{GCL}$ be such that $\{i\} \in N/C$. Then, by definition of the graph restricted game, for every $S \subseteq N \setminus i$, it holds that $v^C(S \cup i) - v^C(S) = v(i)$. Accordingly, i is a dummy player in (N, v^C) and GI then follows from the fact that Ba^{L} satisfies the dummy player property (see pp 23-25 in Álvarez-Mozos and Tejada, 2011).

Second, we check that BG^{L} satisfies 2-E. Let $(N, v, C, \underline{B}) \in \mathcal{GCL}$ and $i, j \in U \in B_1 \in \underline{B}$ be such that $\{i : j\} \in C$. By definition of $P(i, \underline{B})$, $P(i, \underline{B}) \setminus \{j\} = P(j, \underline{B}) \setminus \{i\}$. Then, define $P(ij, \underline{B}) = P(i, \underline{B}) \setminus \{j\} = \{T_1, \ldots, T_{m_{ij}}\}$ and $M_{ij} = \{1, \ldots, m_{ij}\}$, and

[8] Here we assume that i and j can never be the label of any player that results from the merging of "core" players of the lowest level of the levels structure considered in CLG.

note that $m_{ij} = m_i - 1$. On the one hand,

$$
\begin{aligned}
&\mathsf{BG}^{\mathsf{L}}_i(N, v, C, \underline{B}) + \mathsf{BG}^{\mathsf{L}}_j(N, v, C, \underline{B}) = \mathsf{Ba}^{\mathsf{L}}_i\left(N, v^C, \underline{B}\right) + \mathsf{Ba}^{\mathsf{L}}_j\left(N, v^C, \underline{B}\right) \\
&= \sum_{R \subseteq M_{ij}} \frac{1}{2^{m_i}} \left[v^C(T_R \cup i) - v^C(T_R) + v^C(T_R \cup j \cup i) - v^C(T_R \cup j)\right] \\
&+ \sum_{R \subseteq M_{ij}} \frac{1}{2^{m_i}} \left[v^C(T_R \cup j) - v^C(T_R) + v^C(T_R \cup i \cup j) - v^C(T_R \cup i)\right] \\
&= \sum_{R \subseteq M_{ij}} \frac{1}{2^{m_i - 1}} \left[v^C(T_R \cup j \cup i) - v^C(T_R)\right].
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\mathsf{BG}^{\mathsf{L}}_i(N, v_{ij}, C_{ij}, \underline{B}) = \mathsf{Ba}^{\mathsf{L}}_i\left(N, (v_{ij})^{C_{ij}}, \underline{B}\right) \\
&= \sum_{R \subseteq M_i} \frac{1}{2^{m_i}} \left[(v_{ij})^{C_{ij}}(T_R \cup i) - (v_{ij})^{C_{ij}}(T_R)\right] \\
&= \sum_{R \subseteq M_{ij}} \frac{1}{2^{m_i}} \left[(v_{ij})^{C_{ij}}(T_R \cup i) - (v_{ij})^{C_{ij}}(T_R) + (v_{ij})^{C_{ij}}(T_R \cup j \cup i) - (v_{ij})^{C_{ij}}(T_R \cup j)\right] \\
&= \sum_{R \subseteq M_{ij}} \frac{1}{2^{m_i - 1}} \left[v^C(T_R \cup j \cup i) - v^C(T_R)\right].
\end{aligned}
$$

where the last equality follows from the following four observations:

$$
\begin{aligned}
&(v_{ij})^{C_{ij}}(T_R \cup i) = \sum_{S \in (T_R \cup i)/C_{ij}} v_{ij}(S) = \sum_{S \in (T_R \cup i)/C_{ij}} v_{ij}(S \setminus j) \\
&= \sum_{S \in (T_R \cup i \cup j)/C} v_{ij}(S \setminus j) = \sum_{S \in (T_R \cup i \cup j)/C} v(S) = v^C(T_R \cup i \cup j), \\
&(v_{ij})^{C_{ij}}(T_R) = \sum_{S \in T_R/C} v_{ij}(S) = \sum_{S \in T_R/C} v(S) = v^C(T_R), \\
&(v_{ij})^{C_{ij}}(T_R \cup i \cup j) = \sum_{S \in (T_R \cup i \cup j)/C_{ij}} v_{ij}(S) = v_{ij}(j) + \sum_{S \in (T_R \cup i \cup j)/C} v_{ij}(S \setminus j) \\
&= \sum_{S \in (T_R \cup i \cup j)/C} v(S) = v^C(T_R \cup i \cup j), \\
&(v_{ij})^{C_{ij}}(T_R \cup j) = \sum_{S \in (T_R \cup j)/C_{ij}} v_{ij}(S) = v_{ij}(j) + \sum_{S \in T_R/C} v_{ij}(S) \\
&= \sum_{S \in T_R/C} v(S) = v^C(T_R).
\end{aligned}
$$

Taking into account that j is isolated in C_{ij}, that BG^{L} satisfies GI and that $v_{ij}(\{j\}) = 0$, it then follows that BG^{L} satisfies 2-E.

Third, to verify that BG^{L} satisfies FG, we can replicate the argument in the proof of Theorem 3.1. Indeed, note that Ba^{L} satisfies additivity and symmetry (see pp 24-25 in Álvarez-Mozos and Tejada, 2011).

Fourth, BG^{L} satisfies NID by definition and by the fact that Ba^{L} satisfies the level neutrality under individual desertion property (see pp 24-25 in Álvarez-Mozos and Tejada, 2011).

Fifth and last, to verify that BG^{L} satisfies 1-CLG, we can replicate the argument in the proof of Theorem 3.1, because Ba^{L} satisfies the singleton level game property (see pp 24-25 in Álvarez-Mozos and Tejada, 2011).

Uniqueness

Let f be a value on $\mathcal{GCL}$ satisfying the five properties and $(N, v, C, \underline{B}) \in \mathcal{GCL}$ be a game with k levels. We show the uniqueness of $\mathsf{f}(N, v, C, \underline{B})$ by induction on k.

First, let $k = 0$, i.e., $\underline{B} = \underline{B_0}$. We show the uniqueness of $\mathsf{f}(N, v, C, \underline{B_0})$ by induction on the number of links $|C|$. In the case $|C| = 0$, every $i \in N$ is isolated in C, and hence $\mathsf{f}_i(N, v, C, \underline{B_0}) = v(i)$ follows from GI. Next, suppose that payoffs are unique for any game with a trivial levels structure and less than $c > 0$ links, and let $(N, v, C, \underline{B_0})$ be a game with c links. Take $i \in N$. If i is isolated in the graph, uniqueness follows again from GI. Otherwise, i.e. if i is not isolated in the graph, let $j \in N$ be such that $\{i : j\} \in C$. On the one hand, by FG

$$\mathsf{f}_i\left(N, v, C, \underline{B_0}\right) - \mathsf{f}_j\left(N, v, C, \underline{B_0}\right) = \mathsf{f}_i\left(N, v, C^{-ij}, \underline{B_0}\right) - \mathsf{f}_j\left(N, v, C^{-ij}, \underline{B_0}\right). \tag{5}$$

On the other hand, by 2-E

$$\mathsf{f}_i\left(N, v, C, \underline{B_0}\right) + \mathsf{f}_j\left(N, v, C, \underline{B_0}\right) = \mathsf{f}_i\left(N, v, C_{ij}, \underline{B_0}\right) + \mathsf{f}_j\left(N, v, C_{ij}, \underline{B_0}\right). \tag{6}$$

By definition, both modified communication graphs above, namely C^{-ij} and C_{ij}, have less than c links. Then, the payoffs in the right-hand side of Eqs. (5) and (6) are uniquely determined by the second induction hypothesis. Consequently, the payoffs in the left-hand side are also determined, thereby implying the uniqueness of $\mathsf{f}_i\left(N, v, C, \underline{B_0}\right)$ and $\mathsf{f}_j\left(N, v, C, \underline{B_0}\right)$—and hence that of $\mathsf{f}(N, v, C, \underline{B_0})$. Note that uniqueness follows because Eqs. (5) and (6) define a determinate compatible system of equations in the two unknowns $\mathsf{f}_i\left(N, v, C, \underline{B_0}\right)$ and $\mathsf{f}_j\left(N, v, C, \underline{B_0}\right)$.

Second, suppose that the payoffs according to f are unique for any game with less than $k > 0$ levels. Let $(N, v, C, \underline{B}) \in \mathcal{GCL}$ be a game with k levels, i.e., $\underline{B} = \{B_0, \ldots, B_{k+1}\}$. Take $i \in U \in B_1$. We show the uniqueness of $\mathsf{f}_i(N, v, C, \underline{B})$ by a second induction on the cardinality of U. On the one hand, suppose that $U = \{i\}$. Then, by 1-CLG

$$\mathsf{f}_i(N, v, C, \underline{B}) = \mathsf{f}_{[U]}\left([B_1], (v^C)^1, C^{[B_1]}, \underline{B_r}\right).$$

Note that $\left([B_1], (v^C)^1, C^{[B_1]}, \underline{B_r}\right)$ is a game with $k-1$ levels. Then, the payoff above is unique by the first induction hypothesis. On the other hand, suppose that the payoff according to f is unique for any player that belongs to a union of the first level with a cardinality smaller than $u > 1$ and let $i \in U \in B_1$ be such that $|U| = u$. Take $j \in U \setminus i$. Then, by NID

$$\mathsf{f}_i(N, v, C, \underline{B}) = \mathsf{f}_i(N, v, C, \underline{B^{-j}}).$$

The payoff $\mathsf{f}_i(N, v, C, \underline{B^{-j}})$ is unique by the second induction hypothesis, which concludes this step of the proof.

Logical independence

It only remains to check that the five properties are independent.

(i) Let f be the value on $\mathcal{GCL}$ defined for every $(N, v, C, \underline{B}) \in \mathcal{GCL}$ and $i \in N$ by $\mathsf{f}_i(N, v, C, \underline{B}) = 0$.

   Then, f satisfies FG, 2-E, NID, and 1-CLG but not GI.

(ii) Let f be the value on $\mathcal{GCL}$ defined for every $(N, v, C, \underline{B}) \in \mathcal{GCL}$ as follows: If

$N = \{i, j\}$ and $C = C^N$, then

$$\mathsf{f}_i(N, v, C, \underline{B}) = \frac{3}{4}\left(v(N) - v(j)\right) + \frac{1}{4}v(i)$$
$$\mathsf{f}_j(N, v, C, \underline{B}) = \frac{1}{4}\left(v(N) - v(i)\right) + \frac{3}{4}v(j)$$

Otherwise, $\mathsf{f}(N, v, C, \underline{B}) = \mathsf{BG}^{\mathsf{L}}(N, v, C, \underline{B})$.

Then, f satisfies GI, 2-E, NID, and 1-CLG[9] but not FG.

(iii) Let f be the value on $\mathcal{GCL}$ defined for every $(N, v, C, \underline{B}) \in \mathcal{GCL}$ and $i \in N$ by $\mathsf{f}_i(N, v, C, \underline{B}) = \sum_{R \subseteq M_i} \frac{|R|!(m_i - |R| - 1)!}{m_i!}\left[v^C(T_R \cup i) - v^C(T_R)\right]$.

Then, f satisfies GI, FG, NID and 1-CLG, but not 2-E.

(iv) Let f be the value on $\mathcal{GCL}$ defined for every $(N, v, C, \underline{B}) \in \mathcal{GCL}$ by $\mathsf{f}(N, v, C, \underline{B}) = \mathsf{BG}^{\mathsf{L}}(N, v, C, \underline{B_0})$.

Then, f satisfies GI, FG, 2-E, and NID but not 1-CLG.

(v) Let f be the value on $\mathcal{GCL}$ defined for every $(N, v, C, \underline{B}) \in \mathcal{GCL}$ and $i \in N$ by $\mathsf{f}_i(N, v, C, \underline{B}) = \frac{\mathsf{BG}^{\mathsf{L}}_U\left(B_1, (v^C)^1, C^{B_1}, \underline{B_1}\right)}{|U|}$, where $i \in U \in B_1$.

Then, f satisfies GI (for $\underline{B_0}$), FG, 2-E, and 1-CLG but not NID.

$\square$

[9] The same remark as in Footnote 8 applies here.