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An essay on assignment games

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Abstract

This degree project studies the main results on the bilateral assignment game. This is a part of cooperative game theory and models a market with indivisibilities and money. There are two sides of the market, let us say buyers and sellers, or workers and firms, such that when we match two agents from different sides, a profit is made.

We show some good properties of the core of these games, such as its non-emptiness and its lattice structure. There are two outstanding points: the buyers-optimal core allocation and the sellers-optimal core allocation, in which all agents of one sector get their best possible outcome.

We also study a related non-cooperative mechanism, an auction, to implement the buyers-optimal core allocation.

Resumen

Este trabajo de fin de grado estudia los resultados principales acerca de los juegos de asignación bilaterales. Corresponde a una parte de la teoría de juegos cooperativos y proporciona un modelo de mercado con indivisibilidades y dinero. Hay dos lados del mercado, digamos compradores y vendedores, o trabajadores y empresas, de manera que cuando se emparejan dos agentes de distinto lado, se produce un cierto beneficio.

Se muestran además algunas buenas propiedades del núcleo de estos juegos, tales como su condición de ser siempre no vacío y su estructura de retículo. Encontramos dos puntos destacados: la distribución óptima para los compradores en el núcleo y la distribución óptima para los vendedores en el núcleo, en las cuales todos los agentes de cada sector obtienen simultáneamente el mejor resultado posible en el núcleo.

También estudiamos un mecanismo no cooperativo, una subasta, para implementar la distribución óptima para los compradores en el núcleo.

Contents

Chapter 1

Introduction

This degree project covers the study of assignment problems in a game theoretical framework, focusing on assignment games and especially in stability notions, that is the core.

What is game theory about?

Decisions are made every day, by all type of agents, let it be individual persons, firms, governments or any kind of economic agent. The outcomes of the decision do not only depend on the decision of the agent but also on the decisions of others. Therefore Game Theory is a formal approach (mathematical in form) to analyze the process of decision making of several agents in mutually dependent situations.

Von Neumann and Morgenstern (1944) [42] introduces for the first time the term Game Theory in their book "Theory of Games and Economic Behavior". They distinguish in this book two major approaches, non-cooperative game theory and cooperative game theory.

Nash (1951) [23] defines the difference in between the two approaches that in a non-cooperative game "each participant acts independently, without collaboration or communication with any of the others", while in a cooperative game they "may communicate and form coalitions which will be enforced by an umpire", and also "this theory is based on an analysis of the interrelationships of the various coalitions which can be formed by the players of the game". While non-cooperative game theory deals with situations with possibly opposing interests and which actions agents would choose in such situations, cooperative game theory is concerned with what kinds of coalitions would be formed and how much payoff every agent should receive.

A cooperative game with transferable utility, or simply a TU-game, considers the situation in which agents are able to cooperate to form coalitions and the total payoff obtained from their cooperation can be freely distributed among the agents in the coalition.

More precisely, a TU-game is described by a finite set of agents, called players, and a characteristic function. A characteristic function of a TU-game assigns to each coalition the total profit, or worth, which can be obtained by the coalition without cooperating with players outside the coalition. A fundamental question of TU-games is how much payoff each player must receive.

A solution concept for TU-games assigns to each TU-game a set of allocations that satisfy certain properties, or axioms. One of the well-known solution concepts of TU-games is the *core* introduced by Gillies (1959) [13], as the set of allocations that are efficient and exactly distribute the worth of the grand coalition of all players, and are stable in the sense that no group of players has the incentive to leave the grand coalition and obtain the worth of themselves.

Assignment problems and assignment games

One of the earliest works on assignment problems within an economic context is Koopmans and Beckmann (1957) [16]. The authors study a market situation in which industrial plants had to be assigned to the designated locations. The idea is to match two disjoint sets (plants and locations) by mixed-pairs where each possible mixed-pair has a given value. The problem in this context is to find a matching with the highest total valuation of mixed-pairs. Making use of Birkhoff-von Neumann Theorem (Birkhoff (1946) [2]; von Neumann (1953) [43]), they show that an optimal assignment can be obtained by solving a linear program. Furthermore, they introduce a system of rents (prices) on the locations that sustain the optimal assignment by solving the dual linear program. Related to that, Gale (1960) [11] defines competitive equilibrium prices and shows they exist for any assignment problem.

Shapley and Shubik (1971) [36] introduces the assignment problem in a cooperative game framework. The authors study a two-sided (house) market. In their setting, there are two disjoint sets that consist of m buyers and n sellers respectively. Each buyer wants to buy at most one house and each seller has one house on sale. Utility is identified with money, each buyer has a value (which can be different) for every house, and each seller has a reservation value. The valuation matrix represents the joint profit obtained by each mixed-pair. They define the corresponding cooperative game (*assignment game*) for the market. The question is how to share the profit and, to this end, the authors analyze a solution concept: the core (the set of allocations that cannot be improved upon by any coalition). They show that the core of an assignment game is always non-empty. Furthermore, it coincides with the set of dual solutions to the assignment problem, also with the set of competitive equilibrium payoff vectors, and has a lattice structure. Demange (1982) [9] and Leonard (1983) [18] prove that in the buyers-optimal core allocation each buyer attains his/her marginal contribution and in the sellers-optimal core allocation each seller attains his/her marginal contribution.

This monograph is organized as follows. In Chapter 2, we introduce formally the concept of cooperative game, and to this end we introduce a short overview of the notion of utility and a first distinction between non-transferable utility (NTU) cooperative games and transferable utility (TU) cooperative games. Besides that, we introduce the core of a game and several other essential definitions.

In Chapter 3 we discuss the assignment problem as a part of Operations Research. Linear sum assignment problem is the first and most important assignment problem, and immediately this connects with linear programming. Therefore, this chapter also presents the linear programming as a mathematical technique going through the most basic notions until reaching the duality theorem, which is indispensable to enter into assignment markets and games.

Next chapter, Chapter 4, is the central core of this dissertation. It introduces the assignment market and its associated assignment game. This model of cooperative game was introduced by Shapley and Shubik (1971) [36]. We study the model, an outstanding set solution and the core. We show some good properties of the core of these games, such as its non-emptiness and its lattice structure. We also speak of two outstanding points: the buyers-optimal core allocation and the sellers-optimal core allocation. Some single-valued solutions worthy of mention are the τ-value or fair solution (Thompson, 1981) [39], and the nucleolus (Schmeidler, 1969) [35].

An assignment market with only one seller is the setting of an auction, either a single-object auction or a multi-item auction, depending on the number of objects on sale by the seller. In the final chapter of this dissertation, Chapter 5, we study an auction, which is a mechanism non-cooperative in nature, to obtain the buyers-optimal core allocation: the multi-item auction.

Some final conclusions end this dissertation.

Chapter 2

Cooperative games

2.1 Introduction to cooperative games

Game theory can be broadly divided in non-cooperative and cooperative game theory. As opposed to the non-cooperative models, where the main focus is on the strategic aspects of the interaction among the players, the approach in cooperative game theory is completely different. Now, it is assumed that players can commit to behave in a way that is socially optimal, and therefore the benefits can be as big as possible. The reason can be a contract, a law or a custom. The main issue is how to share the benefits arising from cooperation. Important elements in this approach are the different subgroups of players, referred to as *coalitions*, and the set of outcomes that each coalition can get regardless of what the players outside the coalition do[1]. When discussing the different equilibrium concepts for non-cooperative games, we are concerned about whether a given strategy profile is self-enforcing or not, in the sense that no player has incentives to deviate. We now assume that players can make binding agreements and, hence, instead of being worried about issues like self-enforceability, we care about notions like fairness and equity.

Utility

In economics, utility is a measure of preferences over some set of goods. The concept is an important underpinning of rational choice theory in economics and game theory: since one cannot directly measure benefit, satisfaction or happiness from a good or service, economists instead have devised ways of representing and measuring utility in terms of measurable economic choices. Economists have attempted to perfect highly abstract methods of comparing utilities by observing and calculating economic choices; in the simplest sense, economists consider utility to be revealed in people's willingness to pay different amounts for different goods.

In fact it is assumed that any agent has preferences over goods (binary relation, complete

[1] In Peleg and Sudhölter (2003, Chapter 11) [30], the authors discuss in detail some relations between the two approaches and, in particular, they derive the definition of cooperative game without transferable utility (Definition 2.1 below) from a strategic game in which the players are allowed to form coalitions and use them to coordinate their strategies through binding agreements.

and transitive), and if this preference satisfy some assumptions it can be represented by an utility function.

Depending on whether transference of utility between players is restricted or not, we distinguish between nontransferable utility games (NTU-games) and transferable utility games (TU-games), respectively.

Nontransferable Utility Games

In this section we present a brief introduction to the most general class of cooperative games: nontransferable utility cooperative games or NTU-games. The main source of generality comes from the fact that, although binding agreements between the players are implicitly assumed to be possible, utility is not transferable across players. Below, we present the formal definition and then we illustrate it with an example.

Definition 2.1. *A* non-transferable utility game*, NTU-game, is a pair* (N, V) *where* N *is the finite set of players and* V *is a function that assigns, to each coalition* $S \subset N$ *a set* $V(S) \subset \mathbb{R}^S$*. By convention* $V(\emptyset) = \emptyset$*. Moreover, for each* $S \subset N$, $S \neq \emptyset$ *:*

i) $V(S)$ *is a nonempty and closed subset of* $\mathbb{R}^S$,

ii) $V(S)$ *is comprehensive*[2] *. Moreover, for each* $i \in N$, $V(\{i\}) \neq \mathbb{R}$*, i.e., there is* $v_i \in \mathbb{R}$ *such that* $V(\{i\}) = (-\infty, v_i]$,

iii) The set $V(S) \cap \{y \in \mathbb{R}^S$ *: for each* $i \in S$, $y_i \geq v_i\}$ *is bounded.*

In an NTU-game, the following elements are implicitly involved:

i) For each $S \subset N, V(S) \subseteq \mathbb{R}^S$ is the set of outcomes that players in coalition S can obtain by themselves.

ii) For each $S \subset N, \{(\succeq_i^s)_{i \in S}\}$ are the preferences of players in S over outcomes in $\mathbb{R}^S$. They are assumed to be complete, transitive, and can be represented through an utility function.

iii) For each $S \subset N$, $\{U_i^s\}_{i \in S}$ are the utility functions of the players, which represent their preferences to $\mathbb{R}$.

Let (N, V) be an NTU cooperative game. Then, vectors in $\mathbb{R}^N$ are called *allocations*.

Definition 2.2. *Let* (N, V) *be an NTU cooperative game with a finite set of players* N*. An allocation* $x \in \mathbb{R}^N$ *is* feasible *if there is a partition* $\{S_1, \ldots, S_k\}$ *of* N *satisfying that, for each* $l \in \{1, \ldots, k\}$*, there is* $y \in V(S_l)$ *such that, for each* $i \in S_l$, $y_i = x_i$.

Example 2.3. *(The banker game, Owen (1972) [28]) Consider the NTU cooperative game*

[2] Given $S \subset N$ and a set $A \subset \mathbb{R}^S$, we say that A is *comprehensive* if, for each pair $x, y \in \mathbb{R}^S$ such that $x \in A$ and $y \leq x$, we have that $y \in A$.

(N, V) *given by:*

$$
\begin{aligned}
V(\{i\}) &= \{x_i : x_i \leq 0\}, \quad i \in \{1,2,3\}, \\
V(\{1,2\}) &= \{(x_1, x_2) : x_1 + 4x_2 \leq 1000, \quad x_1 \leq 1000\} \\
V(\{1,3\}) &= \{(x_1, x_3) : x_1 \leq 0, \quad x_3 \leq 0\}, \\
V(\{2,3\}) &= \{(x_2, x_3) : x_2 \leq 0, \quad x_3 \leq 0\}, \\
V(\{N\}) &= \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 \leq 1000\}.
\end{aligned}
$$

One can think of this game in the following way. On its own, no player can get anything. Player 1, with the help of player 2, can get 1000 dollars. Player 1 can reward player 2 by sending him money, but the money sent is lost or stolen with probability 0.75. Player 3 is a banker, so player 1 can ensure his transactions are safely delivered to player 2 by using player 3 as intermediary. Hence, the question is how much should player 1 pay to player 2 for his help to get the 1000 dollars and how much to player 3 for helping him to make transactions to player 2 at no cost. The reason for referring to these games as nontransferable utility games is that some transfers among the players may not be allowed. In this example, for instance, $(1000, 0)$ *belongs to* $V(\{1, 2\})$, *but players 1 and 2 cannot agree to the share* $(500, 500)$ *without the help of player 3.*

In the next part, we define games with transferable utility, in which all transfers are assumed to be possible.

Transferable Utility Games

We now move to the most widely studied class of cooperative games: those with transferable utility, in short, TU-cooperative games, or TU-games. Here, the different coalitions that can be formed among the players in N can enforce certain allocations (possibly through binding agreements); the problem is to decide how benefits generated by the cooperation of the players (formation of coalitions) have to be shared among them. However, there is one important departure from the general NTU-games framework.

Definition 2.4. *A* TU-game *is a pair* (N, v), *where* N *is the (finite) set of players and* $v : 2^N \to \mathbb{R}$ *is the characteristic function of the game. By convention,* $v(\emptyset) := 0$.

In general, we interpret $v(S)$, the worth of coalition S, as the benefit that players in S can generate. When no confusion arises, we denote the game (N, v) by v. Also, we denote $v(\{i\})$ and $v(\{i, j\})$ by $v(i)$ and $v(ij)$, respectively. Let G^N be the class of TU-games with player set N.

Example 2.5. *(The glove game, Owen (1975) [29]) Three players are willing to divide the benefits of selling a pair of gloves. Player* 1 *has a left glove and players* 2 *and* 3 *have one right glove each. A left-right pair of gloves can be sold for one euro. This situation can be modeled as the TU-game* (N, v), *where* $N = \{1, 2, 3\}$, $v(1) = v(2) = v(3) = v(23) = 0$, *and* $v(12) = v(13) = v(N) = 1$.

Example 2.6. *(The Parliament of Aragón, González-Díaz et al. (2010) [14]) In this case, we consider the Parliament of Aragón, one of the regions of Spain. After the elections which took place in May 1991, its composition was: PSOE had 30 seats, PP had 17 seats,* *PAR had 17 seats, and IU had 3 seats. In a Parliament, the most relevant decisions are made using the simple majority rule. We can use TU-games to measure the power of the different parties in a Parliament. This can be seen as "dividing" the power among them. A coalition is said to have the power if it collects more than half of the seats of the Parliament, 34 seats in this example. Then, this situation can be modeled as the TU-game* (N, v), *where* $N = \{1, 2, 3, 4\}$ *(we denote 1=PSOE, 2=PP, 3=PAR, 4=IU),* $v(S) = 1$ *if there is* $T \in \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ *with* $T \subset S$ *and* $v(S) = 0$ *otherwise. The objective when dealing with these kind of games is to define power indices that measure how the total power is divided among the players.*

The main solution concept studied for cooperative games is the core. In the next section we introduce this concept and several other notions we need.

2.2 The core and related concepts

In this section we study the most important concept dealing with stability: the core. To this end, we introduce some definitions and properties of the allocations associated with a TU-game.

Definition 2.7. *Let* (N, v) *be a TU-game and* $x \in \mathbb{R}^N$ *an allocation. Then,* x *is* efficient *if* $\sum_{i \in N} x_i = v(N)$.

Definition 2.8. *Let* (N, v) *be a TU-game and* $x \in \mathbb{R}^N$ *an allocation. The allocation* x *is* individually rational *if, for each* $i \in N, x_i \geq v(i)$*, that is, no player get less than what he can get by himself.*

The set of imputations of a TU-game, $I(v)$, consists of all the efficient and individually rational allocations.

Definition 2.9. *Let* (N, v) *be a TU-game. The* set of imputations *of* $v, I(v)$*, is defined by*

$$I(v) := \{x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N) \;\; \Big| \;\; \forall i \in N, x_i \geq v(i)\}.$$

Now, we do have the main concepts to define the core. The core of (N, v) is the set of payoff vectors $x \in \mathbb{R}^N$, where x_i stands for the payoff to agent $i \in N$, that satisfy efficiency and coalitional rationality:

Definition 2.10. *Let* (N, v) *be a TU-game. The* core *of* $v, C(v)$*, is defined by*

$$C(v) := \{x \in I(v) : \;\; \forall S \subset N, \sum_{i \in S} x_i \geq v(S)\}.$$

The elements of $C(v)$ are usually called core allocations. The core is always a subset of the set of imputations. By definition, in a core allocation no coalition receives less than what it can get on its own (coalitional rationality). Hence, core allocations are stable in the sense that no coalition has incentives to secede. Notice that the core may be empty.

Now we will see two examples of TU-games, and we describe their cores.

Example 2.11. *(The glove game from Example 2.5 is a cooperative game with 3 agents) Let $N = \{1, 2, 3\}$ be the set of players and let w be the characteristic function:*

$$\begin{array}{lll} w(\{1\}) = 0 & w(\{1,2\}) = 1 & w(\{1,2,3\}) = 1 \\ w(\{2\}) = 0 & w(\{1,3\}) = 1 & \\ w(\{3\}) = 0 & w(\{2,3\}) = 0 & \end{array}$$

Table 2.1: Characteristic function of a cooperative game with 3 agents

A payoff distribution $x = (x_1, x_2, x_3) \in C(w)$ has to be coalitionally rational and hence it has to satisfy the following inequalities:

$$\begin{array}{lll} x_1 \geq 0 = w(\{1\}) & x_1 + x_2 \geq 1 = w(\{1,2\}) & x_1 + x_2 + x_3 \geq 1 = w(\{1,2,3\}) \\ x_2 \geq 0 = w(\{2\}) & x_1 + x_3 \geq 1 = w(\{1,3\}) & \\ x_3 \geq 0 = w(\{1\}) & x_2 + x_3 \geq 0 = w(\{2,3\}) & \end{array}$$

Table 2.2: Inequalities for a payoff x to be coalitionally rational

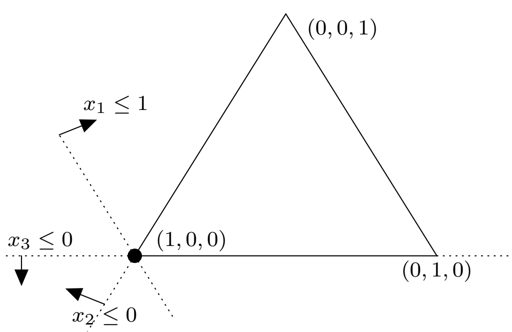
The payoff distribution $x \in \mathbb{R}^3$ also needs to be efficient and distribute the worth of the grand coalition $w(N) = w(\{1,2,3\})$ among the three agents:

$$x_1 + x_2 + x_3 = 1.$$

To obtain a better idea of geometry of the core, we use a diagram. Even though the core $C(w)$ is a set in $\mathbb{R}^3$ the constraint $x_1 + x_2 + x_3 = 1$ makes it possible to draw the core in a two-dimensional subset of $\mathbb{R}^3$ that contains $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. We use the following inequalities to determine the core.

$$\begin{array}{rcl} x_1 + x_2 & \geq & 1 \rightarrow x_3 \leq 0 \\ x_1 + x_3 & \geq & 1 \rightarrow x_2 \leq 0 \\ x_2 + x_3 & \geq & 0 \rightarrow x_1 \leq 1 \end{array}$$

Figure 2.1: The Core of a Cooperative Game with 3 Agents (Example 2.11)



It is easy to see that the only point that meets the constraints is $(1, 0, 0)$ and hence we have $C(w) = \{(1, 0, 0)\}$.

Example 2.12. *(Example with 4 agents) Let us consider another cooperative game with four agents $N = \{1, 2, 3, 4\}$ and the following characteristic function:*

$\omega(\{1\}) = 0$	$\omega(\{1,2\}) = 0$	$\omega(\{1,2,3\}) = 1$	$\omega(\{1,2,3,4\}) = 2$
$\omega(\{2\}) = 0$	$\omega(\{1,3\}) = 1$	$\omega(\{1,2,4\}) = 1$	
$\omega(\{3\}) = 0$	$\omega(\{1,4\}) = 1$	$\omega(\{1,3,4\}) = 1$	
$\omega(\{4\}) = 0$	$\omega(\{2,3\}) = 1$	$\omega(\{2,3,4\}) = 1$	
	$\omega(\{2,4\}) = 1$		
	$\omega(\{3,4\}) = 0$		

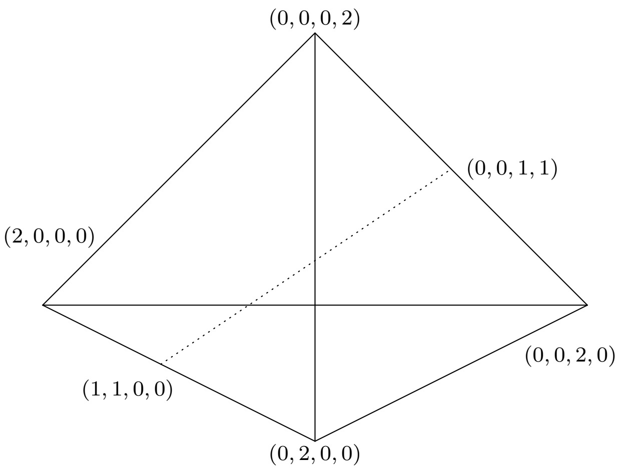
Table 2.3: Characteristic Function of a Cooperative Game with 4 Agents

First we will show that the set $\{(\alpha, \alpha, 1-\alpha, 1-\alpha)|\alpha \in [0,1]\}$ is part of the core, i.e. $\{(\alpha, \alpha, 1-\alpha, 1-\alpha) \mid \alpha \in [0,1]\} \subseteq C(\omega)$. To show it, we just have to prove that $(0,0,1,1)$ and $(1,1,0,0)$ are part of the core. These two payoff distributions are obviously efficient and it is easy to check that they are also coalitionally rational. The core is a convex and compact polyhedron and hence every linear combination of $(0,0,1,1)$ and $(1,1,0,0)$ is also part of the core, i.e.$\{(\alpha, \alpha, 1-\alpha, 1-\alpha) \mid \alpha \in [0,1]\} \subseteq C(\omega)$.

Now we will prove that $C(\omega) \subseteq \{(\alpha, \alpha, 1-\alpha, 1-\alpha) \mid \alpha \in [0,1]\}$. A payoff distribution x in the core has to be efficient, thus $\omega(N) = 2 = x_1 + x_2 + x_3 + x_4$. It also has to be coalitionally rational hence $x_1 + x_3 \geq 1$, $x_2 + x_4 \geq 1$ and $x_1 + x_4 \geq 1$, $x_2 + x_3 \geq 1$. If $x_1 + x_3 > 1$ or $x_2 + x_4 > 1$, then $x_1 + x_2 + x_3 + x_4 > 2 = \omega(N)$, therefore $x_1 + x_3 = 1$ and $x_2 + x_4 = 1$. And if $x_1 + x_4 > 1$ or $x_2 + x_3 > 1$ then $x_1 + x_2 + x_3 + x_4 > 2 = \omega(N)$, hence $x_1 + x_4 = 1$, $x_2 + x_3 = 1$. Since $x_1 + x_3 = 1$, $x_2 + x_3 = 1$, we can conclude $x_1 = x_2$. Since $x_1 + x_3 = 1$ and $x_1 + x_4 = 1$, we can conclude $x_3 = x_4$. Let $x_1 = x_2 = \alpha$ then $\alpha \geq 0$ since $x_1 \geq 0 = \omega(\{1\})$. From $x_1 + x_3 = \alpha + x_3 = 1 \to x_3 = 1 - \alpha$ and $x_3 \geq 0 = \omega(\{3\})$, we see that $\alpha \leq 1$.

Now have proved that $C(\omega) \subseteq \{(\alpha, \alpha, 1-\alpha, 1-\alpha) \mid \alpha \in [0,1]\} \subseteq C(\omega)$ hence the core of our game is $C(\omega) = \{(\alpha, \alpha, 1-\alpha, 1-\alpha) \mid \alpha \in [0,1]\}$.

Figure 2.2: The Core of a Cooperative Game with 4 Agents (Example 2.12)



Chapter 3

Assignment problems and linear programming

3.1 Assignment problems

Assignment problems deal with the question how to assign n items (e.g. jobs) to n machines (or workers) in the best possible way. They consist of two components: the assignment as underlying combinatorial structure and an objective function modeling the "best way".

Mathematically an assignment is nothing else than a bijective mapping of a finite set into itself, i.e., a permutation. Assignments can be modeled and visualized in different ways: every permutation Φ of the set $N = \{1, \ldots, n\}$ corresponds in a unique way to a permutation matrix $A_\Phi = (x_{ij})$ with $x_{ij} = 1$ for $j = \Phi(i)$ and $x_{ij} = 0$ for $j \neq \Phi(i)$.

We can view this matrix as adjacency matrix of a bipartite graph $G_\Phi = (V, W; E)$, where the vertex sets V and W have n vertices, i.e., $|V| = |W| = n$, and there is an edge $(i, j) \in E$ if and only if $j = \Phi(i)$.

Pentico (2007) [31] explains the development of what is called "assignment problems" motivated by the 50th aniversary of the seminal paper by Kuhn. This field is a part of Operations Research, the branch of decision sciences using analytical tools and methods to help making better decisions. Usually is devoted to applied problems related to businesses, engineering and organizations. Kuhn's result allowed a solution of real-world instances, without computers, and the research area is known today as combinatorial optimization.

It is generally recognized that the beginning of the development of practical solution methods for the classic assignment problem was the publication in 1955 of Kuhn's article on the Hungarian method for its solution (Kuhn, 1955) [17]. Naval Research Logistics reprinted it in honor of its 50th anniversary.

There are many different variations corresponding to the assignment problem, and Burkard et al. (2009) [5] is an excellent survey on theoretical methods, algorithms and practical developments.

Assignment problems involve optimally matching the elements of two or more sets, where the dimension of the problem refers to the number of sets of elements to be matched. When there are only two sets, they are referred to as "tasks" and "agents". Thus, for example, "tasks" may be jobs to be done and "agents" the people or machines that can do them, or students to be assigned to schools.

The original version of the assignment problem is discussed in almost every textbook for an introductory course in either management science/operations research or production and operations management. As usually described, the problem is to find a one-to-one matching between n tasks and n agents, the objective being to minimize the total cost of the assignments. Classic examples involve such situations as assigning jobs to machines, jobs to workers, or workers to machines.

The *linear sum assignment problem* (LSAP) is one of the most famous problems in linear programming and in combinatorial optimization. Informally speaking, we are given an $n \times n$ cost matrix $C = (c_{ij})$ and we want to match each row to a different column in such a way that the sum of the corresponding entries is minimized. In other words, we want to select n elements of C so that there is exactly one element in each row and one in each column and the sum of the corresponding costs is a minimum.

Alternatively, one can define it through a graph theory model. Define a bipartite graph $G = (U, V; E)$ having a vertex of U for each row, a vertex of V for each column, and cost c_{ij} associated with edge $[i, j]$ for $i, j = 1, 2, \ldots, n$: The problem is then to determine a minimum cost perfect matching in G (weighted bipartite matching problem: find a subset of edges such that each vertex belongs to exactly one edge and the sum of the costs of these edges is a minimum).

Without loss of generality, we assume that the costs c_{ij} are non-negative. Cases with negative costs can be handled by adding to each element of C a fixed value, the minimum of all entries, ξ. Since we need to select one element per row, any solution of value z for the original cost matrix corresponds to a solution of value $z + n \times \xi$ for the transformed cost matrix. In this way we can manage the maximization version of the problem by solving LSAP on a transformed instance having costs $\tilde{c}_{ij} = -c_{ij}$.

We also assume in general that the values in C are finite, with some c_{ij} possibly having a very large value ($< \infty$) when assigning i to j is forbidden.

The mathematical expression of the linear sum assignment problem is the following one[1]:

$$
\begin{aligned}
\text{Minimize} \qquad & z = \sum_{i \in N} \sum_{j \in N} c_{ij} x_{ij} && (3.1)\\
\text{subject to} \qquad & \sum_{i \in N} x_{ij} = 1, \text{ for all } j \in N, \\
& \sum_{j \in N} x_{ij} = 1, \text{ for all } i \in N, \\
& x_{ij} \in \{0, 1\} \text{ for all } (i, j) \in N \times N.
\end{aligned}
$$

In this dissertation we will use this kind of problems to build a cooperative model used in economics. The optimal (linear sum) assignment problem is that of finding an optimal

[1] N:={1,2,...,n}.

matching, given a matrix that collects the potential profit of each pair of agents. Some examples are the placement of workers to jobs, of students to colleges, of physicians to hospitals or the pairing of men and women in marriage. Once an optimal matching has been found, one question arises: how to share the output among the partners.

Cooperative games arising from Operations Research have been studied by different authors and Curiel (1997) [7] or Borm et al. (2001) [4] are good surveys.

3.2 Linear programming

Linear programming is a mathematical technique for solving constrained maximization and minimization problems when there are many constraints and the objective function to be optimized, as well as the constraints faced, are linear (i.e., can be represented by straight lines).

The subject of linear programming is older than the Second World War. Fourier[2] was among the first to investigate this subject and point outs its importance to mechanics and probability theory. The problem that attracted his attention was that of finding a least maximum deviation fit to a system of linear equations. He reduced the problem to that of finding the lowest point of a polyhedron. His suggested solution to this problem can be viewed as a precursor to the modern day simplex algorithm devised by Dantzig[3]. Dantzig at the time was engaged in a project of an American research program that resulted from the intensive scientific activity during the Second World War, aimed at rationalizing the logistics of the war effort. In the Soviet Union, Kantorovitch[4] had already proposed a similar method for the analysis of economic plans, but his contribution remained unknown to the general scientific community until much later.

The problem of optimizing a linear function subject to linear inequality and equality constraints is called linear programming (LP). Every linear programming problem can be written in the following standard form:

$$\begin{aligned} \max \quad & c \cdot x \\ \text{s.t.} \quad & Ax = b, \\ & x \geq 0. \end{aligned} \tag{3.2}$$

Here 's.t.' is an abbreviation for 'subject to'. In this standard form, we are given two vectors $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ with a matrix $A \in \mathbb{R}_{m\times n}$. In this LP problem, $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ are

[2]Joseph Fourier (1768-1830) was a French mathematician and physicist best known for initiating the investigation of Fourier series and their applications to problems of heat transfer and vibrations. The Fourier transform and Fourier's law are also named in his honor. Fourier is also generally credited with the discovery of the greenhouse effect.

[3]George Bernard Dantzig (November 8, 1914 – May 13, 2005) was an American mathematical scientist who made important contributions to operations research, computer science, economics, and statistics. Dantzig is known for his development of the simplex algorithm, an algorithm for solving linear programming problems.

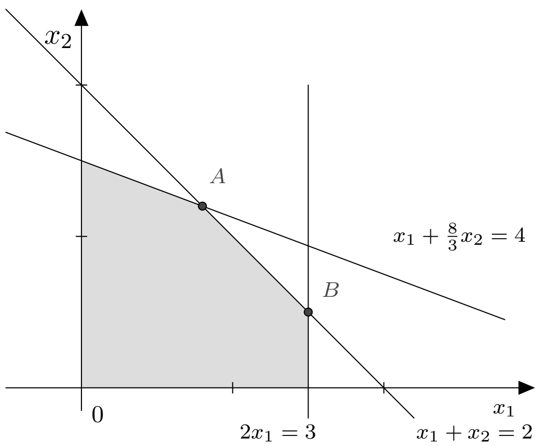
[4]Leonid Kantorovitch (1912-1986) was a Soviet mathematician and economist, known for his theory and development of techniques for the optimal allocation of resources. He is regarded as the founder of linear programming. He was the winner of the Stalin Prize in 1949 and the Nobel Memorial Prize in Economics in 1975.

the variables that satisfy the constraints which form a polyhedron. This polyhedron is called the feasible region of the LP.

Example 3.1. *Here is an example of a Linear Program (LP).*

$$\begin{aligned} \max \quad & x_1 + 2x_2 \\ s.t. \quad & x_1 + \tfrac{8}{3}x_2 \le 4, \\ & x_1 + x_2 \le 2, \\ & 2x_1 \le 3, \\ & x_1, x_2 \ge 0. \end{aligned}$$

Figure 3.1: The feasible region of Example 3.1



This polyhedron, the shaded part of Figure 3.1 is called the feasible region of the LP. In this case, the feasible region is a polytope. A geometrical rendition of our optimization problem is to find a point in the feasible region that maximizes $f(x_1, x_2) = x_1 + 2x_2$.

Observe that the optimal solution cannot be in the interior of the feasible region.

Suppose it were. Call it (a, b). Let $\varepsilon \ge 0$ be sufficiently small such that $(a + \varepsilon, b + \varepsilon)$ is feasible. Such an ε exists because (a, b) is in the interior of the feasible region. Notice that $f(a+\varepsilon, b+\varepsilon) = f(a, b)+3\varepsilon > f(a, b)$, contradicting the optimality of (a, b). Therefore that the optimal solution must lie on the boundary of the feasible region.

Last remark suggests that one of the extreme points of the feasible region must be an optimal solution.

Suppose there is an optimal solution on the boundary between the points A and B marked on the figure but not the extreme points A, B. Call it (a, b). Since this point is on the boundary our previous argument does not apply because $(a + \varepsilon, b + \varepsilon)$ need not be feasible. The idea is to perturb (a, b) to a new feasible point that is still on the same boundary segment. Consider the point $(a + \mu_1, b + \mu_2)$. We want this to be on the same boundary *segment that (a, b) is on. That boundary is defined by the equation $x_1 + x_2 = 2$. So we need $a + \mu_1 + b + \mu_2 = 2$. Since $a + b = 2$ it follows that $\mu_1 + \mu_2 = 0$. We must ensure that the μ_1 and μ_2 are chosen so that $(a + \mu_1, b + \mu_2)$ is feasible. Given the location of (a, b) we know that all the other inequalities are satisfied strictly. That is $a + \frac{8}{3}b < 4$, $2a < 3$ and $a, b > 0$. So, for $|\mu_1|$, $|\mu_2|$ sufficiently small $(a + \mu_1, b + \mu_2)$ will be feasible. Notice that $f(a + \mu_1, b + \mu_2) = a + 2b + \mu_1 + 2\mu_2 = a + b + \mu_2$ because $\mu_1 = -\mu_2$. If we choose $\mu_2 \geq 0$ then $f(a + \mu_1, b + \mu_2) \geq f(a, b)$ which contradicts the optimality of (a, b).*

In this example, the optimal solution is at the point A. It is formed by the intersections of the lines $x_1 + x_2 = 2$ and $x_1 + \frac{8}{3}x_2 = 4$.

If an LP has equality constraints, the constraints that are satisfied at equality by a feasible solution are said to bind at the solution. In our example, the constraints $x_1 + x_2 \leq 2$ and $x_1 + \frac{8}{3} \leq 4$ bind at an optimal solution. They will be called *binding constraints.* The function $c \cdot x$ being optimized is called *objective function* and the matrix A defining the feasible region is called the *constraint matrix.* The vector b is called the vector of right-hand sides.

To convert any LP into the standard form, the following modifications listed below are performed:

- If variable x_j is unrestricted, then substitute $x_j = x_j^+ - x_j^-, \quad x_j^+, x_j^- \geq 0$.
- If a constraint is in the form $\sum_{j=1}^n a_{ij}x_j \leq b_i$ then add a slack variable $s_i \geq 0$ such that $\sum_{j=1}^n a_{ij}x_j + s_i = b_i$.
- If a constraint is in the form $\sum_{j=1}^n a_{ij}x_j \geq b_i$ then subtract a surplus variable $s_i \leq 0$ such that $\sum_{j=1}^n a_{ij}x_j - s_i = b_i$.
- If the objective is $\min cx$ then replace it with: $\max -cx$.
- To change $\sum_{j=1}^n a_{ij}x_j = b_i$ to an inequality constraint, replace equality with these two sets of inequality constraints: $\sum_{j=1}^n a_{ij}x_j \leq b_i$ and $-\sum_{j=1}^n a_{ij}x_j \leq -b_i$.

Example 3.2. *The standard form of the LP above is*

$$\begin{aligned}
\max \quad & x_1 + 2x_2 \\
s.t. \quad & x_1 + \tfrac{8}{3}x_2 + s_1 = 4, \\
& x_1 + x_2 + s_2 = 2, \\
& 2x_1 + s_3 = 3, \\
& x_1, x_2, s_1, s_2, s_3 \geq 0.
\end{aligned}$$

Now we define what is called a basic solution. To this end, first consider the rank of matrix $A \in \mathbb{R}_{m\times n}$ in the LP. If its rank (number of linear independent rows and/or number of linearly independent columns) is less than the number of rows, this means that some equations are redundant and can be eliminated. Therefore we can suppose that the number of rows and the rank of matrix A coincide, and they are less or equal than $n + 1$.

Definition 3.3. *Consider the LP given in (3.2), with $b \in \mathbb{R}^m, c \in \mathbb{R}^n, A \in \mathbb{R}_{m\times n}$ and $x \in \mathbb{R}^n$. Let B be a basis formed from m linearly independent columns of matrix A, that* *is the corresponding submatrix. Choose* $x \in \mathbb{R}^n$ *so as for* x_j *such that* $j \in B$ *is to solve* $Bx^B = b$, *and* $x_j = 0$ *if* $j \notin B$. *The resulting solution is called a* basic solution.

Notice the choice will be unique because B is a non-singular square matrix.

If a basic solution x associated with the basis $B, x = [x^B|0] = [B^{-1}b|0]$, is non-negative then x is a *basic feasible solution* to the LP.

Example 3.4. *Consider the LP*

$$\begin{aligned} x_1 + x_2 + x_3 &= 1, \\ 2x_1 + 3x_2 &= 1, \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

The constraint matrix is

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \end{pmatrix},$$

and here is one basis:

$$\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}.$$

To find the basic solution associated with this basis, we set $x_2 = 0$ *and solve*

$$\begin{aligned} x_1 + x_3 &= 1, \\ 2x_1 + 0x_3 &= 1. \end{aligned}$$

So, the basic solution is $x_1 = \frac{1}{2}$, $x_2 = 0$ *and* $x_3 = \frac{1}{2}$, *which also happens to be a basic feasible solution.*

Another basis is

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix},$$

The basic solution associated with this basis is found by setting $x_3 = 0$ *and solving*

$$\begin{aligned} x_1 + x_2 &= 1, \\ 2x_1 + 3x_2 &= 1. \end{aligned}$$

The basic solution is $x_1 = 2$, $x_2 = -1$ *and* $x_3 = 0$ *which is not a basic feasible solution.*

Now we prove that the solution of the LP is found in an extreme point if the program is feasible.

Lemma 3.5. *Consider the LP given in (3.2), with* $b \in \mathbb{R}^m, c \in \mathbb{R}^n, A \in \mathbb{R}_{m\times n}$ *and* $x \in \mathbb{R}^n$. *If the set* $\{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$ *is feasible, then it has a basic feasible solution.*

Proof. Let $x' \in \mathbb{R}^n$ be a feasible solution. Then $x'_j \geq 0$, $j \in \{1, 2, \ldots, n\}$ and $\sum_{j \in S} a_{ij} x'_j = b_i$, for $i \in \{1, 2, \ldots, m\}$ where $A = (a_{ij})$. We can ignore terms such that $x'_j = 0$ and take $S = \{j \in \{1, \ldots, n\} : x'_j \neq 0\}$. Let $\{a^j\}$ be the columns of matrix A, for $j \in \{1, \ldots, n\}$. If the set $\{a^j : j \in S\}$ are linearly independent we are done: if the cardinality of this set is less than m, throw in some additional columns of the A matrix to produce a set of m linearly independent vectors. The variables associated with these extra columns take the value zero. Then x' is a basic feasible solution.

Assume $\{a^j : j \in S\}$ are not linearly independent. Then there exists $\{\lambda_j\}$ not all zero s.t. $\sum_{j\in S} \lambda_j a^j = 0$. Let $x'' = x' - \theta\lambda \geq 0$ by picking θ as small as necessary. The columns of A associated with the positive components of x'' involve one fewer independent column. Next, we verify that x'' is feasible.

$$Ax'' = A(x' - \theta\lambda) = Ax' - \theta A\lambda = Ax' - \theta \sum_{j\in S} \lambda_j * a^j = Ax' - \theta * 0 = Ax' = b$$

If the columns associated with the non-zero components of x'' are linearly dependent, repeat the argument above. As there are finite number of columns and the method eliminates one column at each iteration, it will terminate after a finite number of steps. □

Lemma 3.6. *Consider the LP given in (3.2), with $b \in \mathbb{R}^m, c \in \mathbb{R}^n, A \in \mathbb{R}_{m\times n}$ and $x \in \mathbb{R}^n$. If x^* is a basic feasible solution of the set $\{x : Ax = b, x \geq 0\}$, then x^* is an* extreme point *of the set.*

Proof. If x* is not an extreme point there exist feasible y and z, distinct from x*, such that $x* = \lambda y + (1-\lambda)z$. Let B the basis associated with x* and set $x* = [x^B|x^N]$, $A = [B|N]$, $y = [y^B|y^N]$, $z = [z^B|z^N]$, where N is the rest of the columns. From the definitions we have $\lambda y^N + (1-\lambda)z^N = x^N = 0 \Rightarrow y^N = z^N = 0 = x^N$.

Feasibility implies

$$Ay = b \Rightarrow By^B = b$$

and

$$Az = b \Rightarrow Bz^B = b,$$

but x^B is the unique solution to $Bx = b$. Then $x^B = z^B = y^B$, so $x*= z = y$. As a result there do not exist z, y different than x^*. Therefore x^* is an extreme point. □

Theorem 3.7. *Consider the LP given in (3.2), with $b \in \mathbb{R}^m, c \in \mathbb{R}^n, A \in \mathbb{R}_{m\times n}$ and $x \in \mathbb{R}^n$ and let $P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$. If A is of full row rank and* $\max_{x\in P} cx$ *has a finite optimal solution, there is an optimal solution at one of the extreme points of P.*

Proof. In order to prove this theorem, Lemma 3.5 can be used. The reader is referred to Vohra (2005) [40] for its complete proof. □

Associated with each LP is another LP called its dual. The original LP is called the primal.

Upper bounds on the optimal objective function value can be found by taking appropriate linear combinations of constraints (yA) that dominate the objective function c, i.e., $c \leq yA \Rightarrow cx \leq yAx$ since $x \geq 0$. Using the fact that $Ax = b$ allows one to conclude that

$$cx \leq yAx = yb \Rightarrow cx \leq yb$$

Thus yb is an upper bound on the objective function value.

Definition 3.8. *The* dual *is the problem of finding the smallest function value such upper bound from the primal LP.*

$$\left.\begin{array}{l} \text{Primal(P)} \\ \\ Z_p = \max cx \\ \quad s.t. \;\; Ax = b \\ \qquad\quad x \geq 0 \end{array}\right\} \Longrightarrow \qquad \begin{array}{l} \text{Dual(D)} \\ \\ Z_p = \min yb \\ \quad s.t. \;\; yA \geq c \\ \qquad\quad y \;\; \text{unrestricted.} \end{array}$$

Example 3.9 (Example 3.2 continued)**.** *We derive the dual to the Example 3.2 above.*

$$\begin{array}{ll} \max & x_1 + 2x_2 \\ s.t. & x_1 + \frac{8}{3}x_2 + s_1 = 4, \\ & x_1 + x_2 + s_2 = 2, \\ & 2x_1 + s_3 = 3, \\ & x_1, x_2, s_1, s_2, s_3 \geq 0. \end{array}$$

The dual of the example problem will be

$$\begin{array}{ll} \min & 4y_1 + 2y_2 + 3y_3 \\ s.t. & y_1 + y_2 + 2y_3 \geq 1, \\ & \frac{8}{3}y_1 + y_2 \geq 2, \\ & y_1, y_2, y_3 \geq 0. \end{array}$$

Now we introduce Farkas' Lemma. It is used for our LP problem and it can also be used in the proof of the Karush-Kuhn-Tucker Theorem. It simply says that a vector is either in a convex cone or there is an hyperplane separating the vector from the cone (separating hyperplane).

Lemma 3.10. *(Farkas'* [5] *Lemma) Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$, and $F = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$. Then either $F \neq \emptyset$ or there exists $y \in \mathbb{R}^m$ such that $yA \geq 0$ and $yb < 0$ but not both.*

Proof. The proof of Farkas' Lemma can be found in several books under different forms. The reader is referred to Vohra (2005) [40]. □

Lemma 3.11. *If problem (P) is infeasible then (D) is either infeasible or unbounded. If (D) is unbounded then (P) is infeasible.*

[5] Farkas Gyula, or Julius Farkas (1847–1930) was a Hungarian mathematician and physicist. The Hungarian Academy of Science elected him corresponding member May 6, 1898. He has made contribution to linear algebra with Farkas' lemma, which is named after him for his derivation of it.

Proof. Suppose for a contradiction that (D) has a finite optimal solution, y^*, say. Infeasibility of (P) implies by Lemma 3.10 (Farkas' Lemma) that there exists a vector $\hat{y}$ such that $\hat{y}A \geq 0$ and $\hat{y} \cdot b < 0$. Let $t > 0$. The vector $y^* + t\hat{y}$ is a feasible solution for (D) since $(y^* + t\hat{y})A \geq y^*A \geq c$. Its objective function value is $(y^* + t\hat{y}) \cdot b < y^*b$, contradicting the optimality of y^*. Since (D) cannot have a finite optimal, it must be infeasible or unbounded.

Now suppose (D) is unbounded. Because of the feasible set is a polyhedron, we can write any solution of (D) as $y + r$ where y is a feasible solution to the dual and r is a ray, i.e., $yA \geq c$ and $rA \geq 0$. Furthermore $r \cdot b < 0$ since (D) is unbounded. By Farkas' Lemma, the existence of r implies the primal is infeasible. □

Theorem 3.12. *(Duality theorem) Let Z_P, Z_D be the sets of optimal solutions for (P) and (D) respectively. If a finite optimal solution for either the primal or dual exists, then $Z_P = Z_D$.*

Proof. The reader can find two proofs of this theorem in Vohra (2005) [40]. □

Chapter 4

Assignment games

The aim of this chapter is to present formally the assignment market, focusing on the associated cooperative game, introduced by Shapley and Shubik (1971). The assignment problem has been analyzed in operations research long before the assignment game was investigated.

The assignment game is a model for a two-sided market in which a product that comes in indivisible units (e.g., houses, cars, etc.) is exchanged for money, and in which each participant either supplies or demands exactly one unit. The units need not be alike, and the same unit may have different values to different participants.

4.1 The assignment model

An assignment game is a model for a two-sided market introduced by Shapley[1] and Shubik[2] (1971). There are two disjoint sets of agents, let us call them buyers and sellers and denote them by M and M' respectively. In this market, there are m buyers and m' sellers. Therefore, the assignment market is integrated by a finite set of agents M of cardinality $|M| = m$ which has to be assigned to a set of tasks M' of cardinality $|M'| = m'$. Each buyer $i \in M$ is willing to buy at most one good and each seller $j \in M'$ has exactly one good on sale. Assume $h_{ij} \geq 0$ is how much buyer $i \in M$ values the good of seller $j \in M'$ and $c_j \geq 0$ is the reservation value of this seller, meaning j will not sell his good for a lower price. Then, whenever $h_{ij} \geq c_j$, there is room too agree on some price $h_{ij} \geq p \geq c_j$ and the joint profit of this trade is $(h_{ij} - p) + (p - c_j)$. As a consequence, we consider a valuation matrix $A = (a_{ij})_{(i,j)\in M\times M'}$ that represents the joint profit obtained by a mixed-pair of a buyer and a seller that is $a_{ij} = \max\{h_{ij} - c_j, 0\}\ \forall i \in M, \forall j \in M'$.

Formally, we denote this market by $\gamma = (M, M'; A)$.

[1] Lloyd Stowell Shapley (June 2, 1923 - March 12, 2016) was a distinguished American mathematician and Nobel Prize winning economist (2012). He was a Professor Emeritus at University of California, Los Angeles (UCLA), affiliated with departments of Mathematics and Economics. He contributed to the fields of mathematical economics and especially game theory.

[2] Martin Shubik (born March 24, 1926) is an American economist, who is Professor Emeritus of Mathematical Institutional Economics at Yale University. Shubik specializes in strategic analysis, the study of financial institutions, the economics of corporate competition, and game theory.

4.2 The assignment game

Shapley and Shubik (1971) [36] associates to each assignment market (M, M', A) a cooperative game which is called the assignment game.

Definition 4.1. *Let $\gamma = (M, M'; A)$ be an assignment market. The associated* assignment game *$(M \cup M', \omega_A)$ is defined by a set of agents (the union of buyers and sellers: $M \cup M'$) and the characteristic function ω_A which associates to each coalition of agents the maximum benefit they can get by assigning buyers and sellers inside this coalition.*

Definition 4.2. *Let $\gamma = (M, M'; A)$ be an assignment market. A* matching *μ between M and M' is a subset of the cartesian product, $M \times M'$, such that each agent belongs to at most one pair.*

We denote by $\mathcal{M}(M, M')$ the set of all possible matchings.

Definition 4.3. *A matching $\mu \in \mathcal{M}(M, M')$ is* optimal *for the market $(M, M'; A)$ if $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$ for all $\mu' \in \mathcal{M}(M, M')$.*

The set of all optimal matchings for the market $(M, M'; A)$ is denoted by $\mathcal{M}_A(M, M')$. An optimal matching μ can be found by solving the so-called linear assignment problem.

Definition 4.4. *Let $\gamma = (M, M'; A)$ be an assignment market and $(M \cup M', \omega_A)$ its associated assignment game. The value for the total coalition $\omega_A(M \cup M')$ is the optimum value of the linear program:*

$$
\begin{aligned}
\max \quad & z = \sum_{i\in M} \sum_{j\in M'} a_{ij}\mu_{ij} && (4.1)\\
s.t. \quad & \sum_{i\in M} \mu_{ij} \leq 1, \ \textit{for all } j \in M', \\
& \sum_{j\in M'} \mu_{ij} \leq 1, \ \textit{for all } i \in M, \\
& \mu_{ij} \in \{0, 1\} \ \textit{for all } (i, j) \in M \times M'.
\end{aligned}
$$

Notice that this is an integer linear program, and by the definition of the linear program, matrix $(\mu_{ij})_{(i,j)\in M\times M'}$ has at most only one non-zero entry for each row and column. If $\mu \in \{0, 1\}^{M\times M'}$ is a solution of (4.1), then $\mu = \{(i, j) \mid \mu_{ij} = 1\}$ is an optimal matching.

We now consider the continuous relaxation, or continuous case of this integer linear program. This is our next linear program (4.2) and we will solve it using several well known algorithms. Notice that matrices $(\mu_{ij})_{(i,j)\in M\times M'}$ which are solutions of our first program are also solutions of the continuous relaxation program:

$$\begin{aligned}
\max \quad & z = \sum_{i \in M} \sum_{j \in M'} a_{ij} \mu_{ij} \\
\text{s.t.} \quad & \sum_{i \in M} \mu_{ij} \leq 1, \text{ for all } j \in M', \\
& \sum_{j \in M'} \mu_{ij} \leq 1, \text{ for all } i \in M, \\
& \mu_{ij} \geq 0 \text{ for all } (i, j) \in M \times M'.
\end{aligned} \tag{4.2}$$

One of most well-known solutions of the assignment problem, the Hungarian method, was provided by Harold Kuhn [3] [17] in 1955, even though Carl Gustav Jacobi already discovered the same solution in the 19th century[4].

In fact, the assignment problem is a special case of the transportation problem. Other solutions e.g. the simplex method provided by Dantzig (1963) [8] can also be used to find an optimal matrix that maximizes z. The solution of the assignment problem (see Dantzig (1963), p. 318) shows that the optimal value for 4.2 is attained with all $\mu_{ij} \in \{0, 1\}$, for all $(i, j) \in M \times M'$. This result was independently proved in Birkhoff[5] (1946) [2] and von Neumann[6] (1953) [43]. Hence this implies a solution to the assignment problem 4.1.

Since the solution of the assignment problem deals with a linear program, it allows us to consider the linear program that is dual to the first program (4.2):

$$\begin{aligned}
\min \quad & z = \sum_{i \in M} u_i + \sum_{j \in M'} v_j \\
\text{s.t.} \quad & u_i + v_j \geq a_{ij} \text{ for all } (i, j) \in M \times M', \\
& u_i \geq 0, \text{ for all } i \in M, \\
& u_j \geq 0, \text{ for all } j \in M'.
\end{aligned} \tag{4.3}$$

Therefore, because of the Duality Theorem (Theorem 3.12) for linear programming, we can state the following corollary.

Corollary 4.5. *The solution of the dual program (4.3) coincides with the solution of the linear program (4.1).*

[3] Harold William Kuhn (July 29, 1925 – July 2, 2014) was an American mathematician known for the Karush–Kuhn–Tucker conditions, for Kuhn's theorem, for developing Kuhn poker as well as the description of the Hungarian method for the assignment problem.

[4] Jacobi's solution was rediscovered in 2006. Further information can be found in Cariñena, J. et al. (2006) [6].

[5] George David Birkhoff (March 21, 1884 – November 12, 1944) was an American mathematician, best known for what is now called the ergodic theorem. He introduced the chromatic polynomiale and proved Poincaré's "Last Geometric Theorem," a special case of the three-body problem.

[6] John von Neumann (December 28, 1903 – February 8, 1957) was a Hungarian-American pure and applied mathematician, physicist, inventor, computer scientist, and polymath. He was a pioneer of the application of operator theory to quantum mechanics, in the development of functional analysis, and a key figure in the development of game theory.

To finish the description of the game, now we can define the characteristic function. Recall that $N = M \cup M'$, The characteristic function $\omega_A(S)$ defines the benefit that can be obtained by each coalition.

Definition 4.6. *Let $\gamma = (M, M'; A)$ be an assignment market and $(M \cup M', \omega_A)$ its associated assignment game. The characteristic function $\omega_A(S)$ defines the benefit that can be obtained by each coalition and it is expressed in the following form*

$$\omega_A(S) = \max_{\mu \in \mathcal{M}(M \cap S, M' \cap S)} \sum_{(i,j) \in \mu} a_{ij} \quad \textit{for all } S \subseteq N.$$

Notice that $\omega_A(S)$ is the optimal value of the linear program (4.1) restricted to $i \in S \cap M$ and $j \in S \cap M'$.

The concept of solution more studied for cooperative games in general, and for the assignment games in particular, is the core.

4.3 The core of the assignment game

Definition 4.7. *Given an assignment game* $(M \cup M', \omega_A)$*, an imputation is a vector of payments* $(u, v) \in R_+^m \times R_+^{m'}$ *where* $u_i \geq 0$ *is the payment to buyer* $i \in M$ *and* $v_j \geq 0$ *is the payment to seller* $j \in M'$*, such that*

$$\sum_{i \in M} u_i + \sum_{j \in M'} v_j = \omega_A(M \cup M').$$

We denote $I(\omega_A)$ the set of imputations of the assignment game.

Now we define the core of the assignment game.

Definition 4.8. *The core of an assignment game* $(M \cup M', \omega_A)$ *is the set of those imputations such that every coalition receives, at least, its value according to the characteristic function:*

$$C(\omega_A) = \left\{ (u, v) \in I(\omega_A) \,\middle|\, \sum_{i \in S \cap M} u_i + \sum_{j \in S \cap M'} v_j \geq \omega_A(S) \;\; \forall S \subseteq M \cup M' \right\}.$$

Theorem 4.9. *(Shapley and Shubik, 1971 [36]) Let* $\gamma = (M, M'; A)$ *be an assignment market. Then, its corresponding assignment game* (N, ω_A) *has a non-empty core. Moreover, the core coincides with the set of dual solutions to the linear assignment problem.*

Proof. Consider the assignment market $\gamma = (M, M'; A)$ and its corresponding game (N, ω_A). An optimal matching μ can be found by solving the so-called linear assignment problem:

$$\begin{aligned} \max & \sum_{i \in M} \sum_{j \in M'} a_{ij} x_{ij} \qquad (4.4) \\ \text{s.t.} & \sum_{i \in M} x_{ij} \leq 1, \text{ for all } j \in M', \\ & \sum_{j \in M'} x_{ij} \leq 1, \text{ for all } i \in M, \\ & x_{ij} \in \{0, 1\} \text{ for all } (i, j) \in M \times M'. \end{aligned}$$

By the Birkhoff-von Neumann Theorem the solution of the above integer linear program coincides with its LP relaxation, which is the related continuous linear program with $x_{ij} \geq 0$ for all $(i, j) \in M \times M'$. The fundamental duality theorem states that every linear program can be transposed into a dual form and, if the primal program has a solution, then the optimal values of both programs coincide. Then, the dual of the LP relaxation of the primal program (4.4) is:

$$\begin{aligned} \min \quad & \sum_{i \in M} u_i + \sum_{j \in M'} v_j \qquad (4.5) \\ \text{s.t.} \quad & u_i + v_j \geq a_{ij} \text{ for all } (i, j) \in M \times M', \\ & u_i \geq 0 \text{ for all } i \in M, \\ & v_j \geq 0 \text{ for all } j \in M'. \end{aligned}$$

In our case, the fundamental duality theorem tells that (4.5) has a solution and, over the respective sets of constraints, $\min \sum_{i\in M} u_i + \sum_{j\in M'} v_j = \max \sum_{i\in M}\sum_{j\in M'} a_{ij}x_{ij} = \omega_A(M\cup M')$. Hence, a payoff vector (u,v) is a solution of the dual program (4.5) if and only if it is an element of the core of (N,ω_A). As a consequence, the core is non-empty. □

Shapley and Shubik (1971) [36] shows that it is sufficient to take into account mixed-pair coalitions to describe the core. Then, for each optimal matching $\mu \in \mathcal{M}_A(M,M')$, the core of the corresponding assignment game (N,ω_A) is described by

$$C(\omega_A) = \left\{ (u,v) \in \mathbb{R}^M_+ \times \mathbb{R}^{M'}_+ \;\middle|\; \begin{array}{l} u_i + v_j = a_{ij} \text{ for all } (i,j)\in\mu \text{ and} \\ u_i + v_j \geq a_{ij} \text{ for all } (i,j)\in M\times M' \end{array} \right\}.$$

By the nature of the assignment game, the core only considers $u_i, u_j \geq 0$ for all $i \in M, j \in M'$ where $u_i = 0$ if i is unmatched by μ and $v_j = 0$ if j unmatched by μ.

Shapley and Shubik prove that the core of an assignment game is always non-empty, that is, assignment games are balanced[7].

The set of dual solutions of the assignment problem had already been analyzed by Gale (1960) [11] and related to his notion of competitive equilibrium. As in Roth and Sotomayor (1990) [33], let us assume that M' contains as many copies as necessary of a null object $o \in \mathcal{O}$ such that $a_{iO} = 0$ for all $i \in M$. Then, for any matching μ, all buyers can be assumed to be matched either to a real object or to a null object O.

Definition 4.10 (Gale, 1960 [11]). *Given a vector of non-negative prices $p \in \mathbb{R}^{M'}$, with $p_{\mathcal{O}} = 0$, the* demand set *of buyer $i \in M$ at prices p is*

$$D_p(i) = \{ j \in M' \quad | \quad a_{ij} - p_j = \max_{k\in M'}\{a_{ik} - p_k\}\}.$$

This means that buyer i asks for those objects that give him the maximum profit, given by the difference of valuation and price.

Then, a pair (p,μ) formed by a vector of prices and a matching is a *competitive equilibrium* if $\mu(i) \in D_i(p)$ for all $i \in M$ and $p_j = 0$ whenever $j \in M'$ is unassigned by μ. In this case, p is said to be a *competitive equilibrium price vector.* Given a competitive equilibrium (p,μ), the payoff vector (u,v) where $u_i = a_{i\mu(i)} - p_{\mu(i)}$ for all $i \in M$ and $v_j = p_j$ for all $j \in M'$ is a *competitive equilibrium payoff vector.*

Theorem 4.11 (Gale, 1960 [11]). *For any assignment game, the set of solutions of the dual program of (4.1) coincides with the set of competitive equilibrium payoff vectors.*

Proof. Given a solution (u,v) of the dual program, define $p = v \in \mathbb{R}^{M'}_+$. Take μ an optimal matching. From $\sum_{(i,j)\in\mu} a_{ij} = \sum_{i\in M} u_i + \sum_{j\in M'} v_j$ and $u_i + v_j \geq a_{ij}$ for all $(i,j)\in\mu$ it follows that $p_j = v_j = 0$ for all unassigned object $j \in M'$ and $u_i + v_j = a_{ij}$ if $(i,j)\in\mu$. Moreover, for all $i \in M$,

$$a_{i\mu(i)} - p_{\mu(i)} = u_i \geq a_{ij} - p_j \quad \text{for all } j \in M',$$

[7] A game (N,v) is said to be balanced if it has a non-empty core.

where the inequality follows from the dual program constraints. Hence, p is a competitive price vector.

Conversely, if p is a competitive price vector, then there exists $\mu \in \mathcal{M}(M, M')$ such that $p_j = 0$ if j is unassigned by μ and for all $i \in M$,

$$\mu(i) \in D_i(p).$$

Define now $(u, v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$ by $v_j = p_j$ for all $j \in M'$ and $u_i = a_{i\mu(i)} - p_{\mu(i)}$ for all $i \in M$. Notice that if $i \in M$ is assigned to a null object, then $u_i = 0$. Also, $v_j = 0$ if $j \notin \mu(M)$. Let us check that (u, v) is a solution of the dual problem.

We see first that if (p, μ) is a competitive equilibrium, then μ is an optimal matching. Indeed, take another matching $\mu' \in \mathcal{M}(M, M')$. Now, since $a_{i\mu(i)} - p_{\mu(i)} \geq a_{i\mu'(i)} - p_{\mu'(i)}$ for all $i \in M$,

$$\begin{aligned}
\sum_{(i,j)\in\mu} a_{ij} = \sum_{i\in M} a_{i\mu(i)} &\geq \sum_{i\in M} (a_{i\mu'(i)} - p_{\mu'(i)}) + \sum_{i\in M} p_{\mu(i)} \\
&= \sum_{i\in M} a_{i\mu'(i)} - \sum_{j\in\mu'(M)} p_j + \sum_{j\in\mu(M)} p_j \\
&= \sum_{i\in M} a_{i\mu'(i)} - \sum_{j\in\mu'(M)\setminus\mu(M)} p_j + \sum_{j\in\mu(M)\setminus\mu'(M)} p_j \\
&\geq \sum_{i\in M} a_{i\mu'(i)}
\end{aligned}$$

where the last inequality follows from the fact that (p, μ) is a competitive equilibrium and hence $p_j = 0$ for all $j \notin \mu(M)$.

Since μ is an optimal matching and agents assigned to the null object receive zero,

$$\omega_A(M \cup M') = \sum_{i\in M} a_{i\mu(i)} = \sum_{i\in M} u_i + v_{\mu(i)} = \sum_{i\in M} u_i + \sum_{j\in M'} v_j,$$

which means (u, v) is efficient.

Finally, for all $i \in M$ and for all $j \in M'$,

$$\begin{aligned}
u_i + v_j = u_i + p_j &= a_{i\mu(i)} - p_{\mu(i)} + p_j \\
&\geq a_{ij} - p_j + p_j = a_{ij},
\end{aligned}$$

which concludes the proof that (u, v) is a solution of the dual program. □

Theorem 4.12. *Let $\gamma = (M, M'; A)$ be an assignment market and $(M \cup M', \omega_A)$ its associated assignment game. For the assignment game, the four sets below coincide:*

- *The core, $C(w_A)$.*
- *The set of dual solutions to the assignment problem (4.1).*
- *The set of competitive equilibrium payoff vectors of the market.*

- *The set of pairwise-stable payoff vectors.*

Now we put several examples of an assignment game and its core.

Example 4.13. *Consider an assignment game with two sellers* $M = \{1, 2\}$ *and two buyers* $M' = \{1', 2'\}$. *The worth of a mixed-pair coalition is put in the next table, that is matrix* A, *where the optimal matching is shown.*

	$1'$	$2'$
1	⑤	4
2	2	③

Once we have found an optimal matching, we can determine the core directly without calculating the characteristic function. We just make use of the constraints of the dual problem of the corresponding assignment problem.

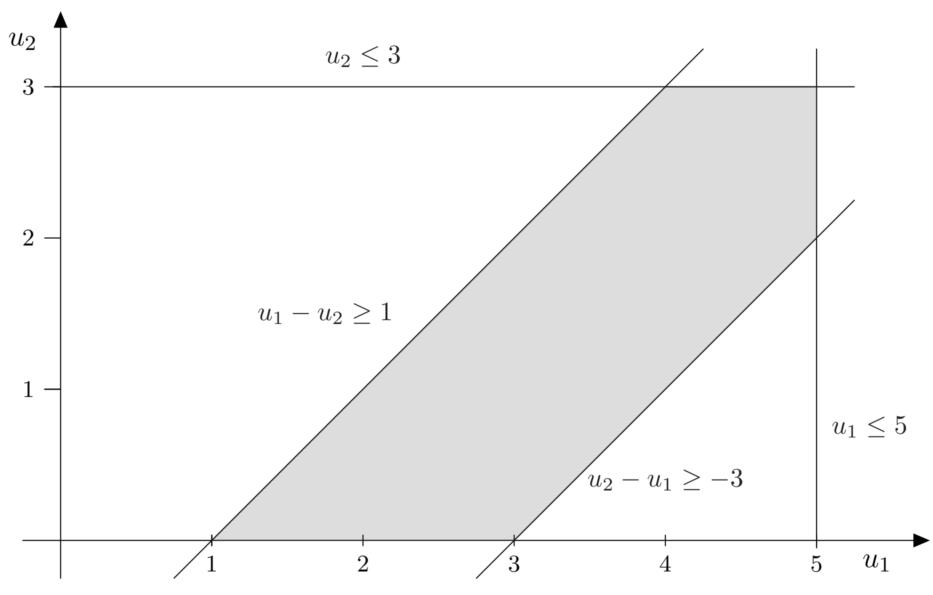
$$C(\omega_A) = \left\{ (u_1, u_2; v_1, v_2) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+ \;\middle|\; \begin{array}{l} u_1 + v_1 = 5,\ u_2 + v_2 = 3 \\ u_1 + v_2 \geq 4,\ u_2 + v_1 \geq 2 \end{array} \right\}$$

Let us now represent the core in a two-dimensional space, that is, its projection on the first two coordinates:

$$C_u(\omega_A) = \{u \mid (u, v) \in C(\omega_A)\}.$$

From $u_1 + v_1 = 5$ *and* $u_1, v_1 \geq 0$, *we obtain* $0 \leq u_1 \leq 5$. *Similarly from* $u_2 + v_2 = 3$ *and* $u_2, v_2 \geq 0$, *we have* $0 \leq u_2 \leq 3$. *In* $u_1 + v_2 \geq 4$ *we substitute* $v_2 = 3 - u_2$ *and hence have* $u_1 - u_2 \geq 1$. *In* $u_2 + v_1 \geq 2$ *we substitute* $v_1 = 5 - u_1$ *and obtain* $u_2 - u_1 \geq -3$. *Now we can draw the core in a two-dimensional space.*

Figure 4.1: The Core of Example 4.13 in a two-dimensional space



Example 4.14 (Shapley and Shubik, 1971 [36]). *Consider an assignment game with three sellers $M = \{1, 2, 3\}$ and three buyers $M' = \{1', 2', 3'\}$. The worth of a mixed-pair coalitions is put in the next table, that is matrix A, where the optimal matching is shown.*

	$1'$	$2'$	$3'$
1	5	⑧	2
2	7	9	⑥
3	②	3	0

$\mu = \{(1, 2'), (2, 3'), (3, 1')\}$ *is an optimal matching for this game. Once we have found* μ*, we can determine the core directly without calculating the characteristic function as in Example 4.13 above. We just make use of the constraints of the dual problem of the corresponding assignment problem.*

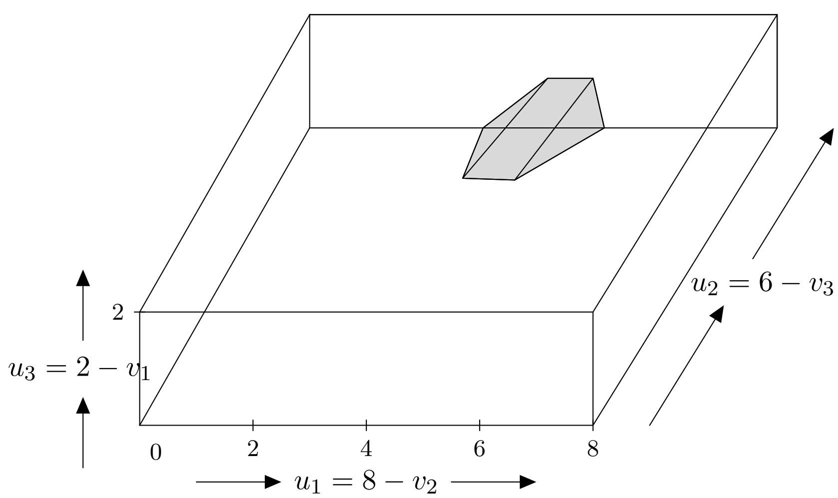
$$C(\omega_A) = \left\{ (u_1, u_2, u_3; v_1, v_2, v_3) \in \mathbb{R}^3_+ \times \mathbb{R}^3_+ \;\middle|\; \begin{array}{l} u_1 + v_1 \geq 5,\ u_1 + v_2 = 8,\ u_1 + v_3 \geq 3 \\ u_2 + v_1 \geq 7,\ u_2 + v_2 \geq 9,\ u_2 + v_3 = 6 \\ u_3 + v_1 = 2,\ u_3 + v_2 \geq 3,\ u_3 + v_3 \geq 0 \end{array} \right\}$$

Let us now represent the core in a three-dimensional space, that is, its projection on the first three coordinates:

$$C_u(\omega_A) = \{u \mid (u, v) \in C(\omega_A)\}.$$

The same procedure applied in Example 4.13 provides us the lines and constraints to draw the core in a three-dimensional space.

Figure 4.2: The Core of Example 4.14 in a three-dimensional space



The reader might observe that buyers and sellers are upside down according to the way we have been defining the games in the examples above. This is due to the fact we have wanted to keep the original form from Shapley and Shubik (1971) [36].

Notice that some examples seen as cooperative games from Chapter 2 are actually assignment games. Let us show them down below.

Example 4.15. *(Example 2.5: The glove game) Consider an assignment game with 1 buyer (one person with a left glove) $M = \{1\}$ and 2 sellers (two people with a right glove each) $N = \{2, 3\}$. The worth of a mixed-pair coalition is put in the next matrix.*

$$\begin{array}{cc} & \begin{array}{cc} 2 & 3 \end{array} \\ 1 & \begin{pmatrix} 1 & 1 \end{pmatrix} \end{array}$$

This is because the glove market only considers a positive utility when a left glove is allocated to a right glove (and the other way around).

Example 4.16. *(Example 2.12) Let us consider another assignment game with 2 buyers $M = \{1, 2\}$ and 2 sellers $N = \{3, 4\}$. The worth of a mixed-pair coalition is represented in the following matrix.*

$$\begin{array}{cc} & \begin{array}{cc} 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{array}$$

4.4 Lattice structure of the core of the assignment game

The second main contribution in the paper of Shapley and Shubik (1971) [36] is the study of the structure of the core of the assignment game. If we consider on the core elements the partial order defined by one side of the market, for instance $(u, v) \leq_M (u', v')$ if and only if $u_i \leq u'_i$ for all $i \in M$, it results that the core has the structure of a complete lattice with respect to this order. Indeed, Shapley and Shubik prove the following theorem.

Theorem 4.17. *(Shapley and Shubik, 1971 [36]) Let $\gamma = (M, M'; A)$ be an assignment market. Given two core elements $(u, v) \in C(\omega_A)$ and $(u', v') \in C(\omega_A)$, the join*

$$(u, v) \vee (u', v') = ((\max\{u_i, u'_i\})_{i \in M}, (\min\{v_j, v'_j\})_{j \in M'})$$

and the meet

$$(u, v) \wedge (u', v') = ((\min\{u_i, u'_i\})_{i \in M}, (\max\{v_j, v'_j\})_{j \in M'})$$

also belong to the core.

A consequence of this lattice structure of the core is the existence of two special extreme core points, one for each side of the market, namely, *buyers-optimal* core allocation and *sellers-optimal* core allocation. In one of them, $(\overline{u}^A, \underline{v}^A)$ each buyer maximizes her payoff in the core, while each seller minimizes his. This core element is related to the minimum competitive equilibrium price vector. In the other one, $(\underline{u}^A, \overline{v}^A)$, each seller maximizes his core payoff while buyers get their minimum one, and this is related to the maximum competitive equilibrium price vector. Moreover, these are the two more distant points inside the core. What is remarkable is that all agents on the same side of the market, despite being competing for the best deal, obtain their maximum core payoff in the same core element.

Theorem 4.18. *(Shapley and Shubik, 1971 [36]) Let $\gamma = (M, M'; A)$ be an assignment market. The core of the assignment game $C(\omega_A)$ always contains a buyer and a seller optimum.*

Proof. The basic idea behind the proof is related to the special form of the restrictions of the core, where each variable has 1 as a coefficient. Figure of Example 4.13 might help visualizing this idea. We will show that for any allocations (u', v') and (u'', v'') that are part of the core the allocations $(\underset{\sim}{u}, \tilde{v})$ and $(\tilde{u}, \underset{\sim}{v})$ with

$$\begin{aligned} \underset{\sim}{u}_i &= \min\{u'_i, u''_i\} \qquad i \in M, \\ \underset{\sim}{v}_j &= \min\{v'_j, v''_j\} \qquad j \in M', \\ \tilde{u}_i &= \max\{u'_i, u''_i\} \qquad i \in M, \\ \tilde{v}_j &= \max\{v'_j, v''_j\} \qquad j \in M'. \end{aligned}$$

are themselves in the core. This is equivalent to say that the core points form a lattice (Birkhoff, 1973) [3]. We first prove that $(\underset{\sim}{u}, \tilde{v})$ is coalitionally rational. For all $i \in M$, $j \in M'$, we have

$$\begin{aligned} \underset{\sim}{u}_i + \tilde{v}_j &= \min\{u'_i + \tilde{v}_j, u''_i + \tilde{v}_j\} \\ &\geq \min\{u'_i + v'_j, u''_i + v''_j\} \\ &\geq a_{ij}. \end{aligned}$$

Now we have to show that $(\underset{\sim}{u}, \tilde{v})$ is an imputation. Let μ be an optimal assignment, then we have

$$\begin{aligned} \underset{\sim}{u}_i &= \min\{u'_i, u''_i\} \\ &= \min\{a_{i\mu(i)} - v'_{\mu(i)}, a_{i\mu(i)} - v''_{\mu(i)}\} \\ &= a_{i\mu(i)} - \max\{v'_{\mu(i)}, v''_{\mu(i)}\} \\ &= a_{i\mu(i)} - \tilde{v}_{\mu(i)}. \end{aligned}$$

For any player that is not assigned we have $\underset{\sim}{u}_i = 0$ and $\tilde{v}_j = 0$, and hence

$$\sum_{i \in M} \underset{\sim}{u}_i + \sum_{j \in M'} \tilde{v}_j = \sum_{i \in M} a_{i\mu(i)} = w(M \cup M').$$

Similarly $(\underset{\sim}{u}, \tilde{v}) \in C(\omega_A)$.

Let us consider $\underline{u}_i = \min_{(u,v) \in C(\omega_A)}\{u_i\}$ for all $i \in M$ and $\overline{u}_i = \max_{(u,v) \in C(\omega_A)}\{u_i\}$ for all $i \in M$. In the same way we define $\underline{v}_j$ and $\overline{v}_j$. Since the core is compact (for any general cooperative game), there exists a vector that contains the minimum $\underline{u}_k$ and another vector which contains the minimum $\underline{u}_l$. With these two vectors we can construct a new vector, using the lattice structure, which contains $\underline{u}_k$ and $\underline{u}_l$ and is also part of the core. If we continue this process we obtain a core allocation $(\underline{u}, v)$ where all sellers receive there minimum payoff. We know that in the core each optimal pair $(i, \mu^*(i))$ receives $a_{i\mu^*(i)}$, i.e. $u_i + v_{\mu^*(i)} = a_{i\mu^*(i)}$. So if u_i is minimum $v_{\mu^*(i)}$ is maximum, i.e. $\overline{v}_{\mu^*(i)} = a_{i\mu^*(i)} - \underline{u}_i$. Therefore we have $(\underline{u}, v) = (\underline{u}, \overline{v})$. In the same way we can show that $(\overline{u}, \underline{v})$ is part of the core, too.

It is quite obvious that there are no other core allocations which are further away from each other than $(\underline{u}, \overline{v})$ and $(\overline{u}, \underline{v})$ since for any core elements (u', v') and (u'', v'') the following inequalities hold true:

$$\begin{aligned} |\overline{u}_i - \underline{u}_i| &\geq |u'_i - u''_i| \quad \forall i \in M, \\ |\overline{v}_j - \underline{v}_j| &\geq |v'_j - v''_j| \quad \forall j \in M'. \end{aligned}$$

We have proved that the core is a lattice with a maximum and a minimum. Since the total payoff of any core vector is always the same, a maximum in this context refers to a maximum payoff for all buyers or all sellers. If the payoff for all buyers is maximum, the payoff for sellers is also determined and minimum. □

Roth and Sotomayor (1990) [33] provides a neat proof of a result presented, independently, in Demange [9] (1982) and Leonard [18] (1983). It says that the maximum core payoff of an agent, be it a buyer or a seller, is her/his marginal contribution to the grand coalition. That is the following lemma.

Lemma 4.19. *Given an assignment game $(M \cup M', w_A)$, the maximum core payoff of an agent is his/her marginal contribution to the grand coalition, that is,*

$$\begin{aligned} \overline{u}_i^A &= w_A(M \cup M') - w_A((M \setminus \{i\}) \cup M'), \quad \forall i \in M, \\ \overline{v}_j^A &= w_A(M \cup M') - w_A(M \cup (M' \setminus \{j\})), \quad \forall j \in M'. \end{aligned}$$

Then we can state the following theorem.

Theorem 4.20. *There exists an element in the core of the assignment game where each buyer receives her marginal contribution to the grand coalition.*

Demange proves that in any mechanism that, from the valuation matrix, implements the buyers-optimal core allocation, truth telling is dominant strategy for each buyer.

Another question that was first studied by Mo [22] (1988) in the assignment game, and later also consider by Roth and Sotomayor [33] (1990) for the marriage market, is concerned with the effect on the core of changing the market by introducing a new agent.

Remark 4.21. *Let $(M \cup M', \omega_A)$ be an assignment game and assume a new buyer i^* enters the market. The new game will be $((M \cup \{i^*\}) \cup M', \omega_{A'})$ where $a'_{ij} = a_{ij}$ for all $i \in M$ and $j \in M'$. Then,*

$$\begin{aligned} &\overline{u}_i^A \geq \overline{u}_i^{A'} \text{ and } \underline{u}_i^A \geq \underline{u}_i^{A'} \quad \text{for all } i \in M, \\ &\overline{v}_j^A \leq \overline{v}_j^{A'} \text{ and } \underline{v}_j^A \leq \underline{v}_j^{A'} \quad \text{for all } j \in M'. \end{aligned}$$

This means that each of the previously existing buyers is worse off in the market with the new entrant and each of the sellers is better off in this new market.

We find more precise conclusions in Mo (1988) [22] about the changes in the core when the market faces a new entrant.

Remark 4.22. *If the new buyer i^* gets matched by some optimal assignment for the new market, then there exists a non-empty set of agents such that for all buyer i in this set $\overline{u}_i^{A'} = \underline{u}_i^A$; and for all seller j in this set $\underline{u}_j^{A'} = \overline{u}_j^A$. That is, all buyers in this set are so punished by the presence of the new entrant that their best payoff in the core of the new market equals their worst payoff in the core of the original market.*

Earlier, we have seen the proof which proves the core of these kind of games is always a lattice. Observing the core of a 2×2 assignment game, when projected to the space of payoffs to one side of the market, has a quite particular shape: it is a 45-degree lattice.[8] This fact is extensively analyzed in Núñez and Rafels (2015) [27].

Definition 4.23. *Let $\gamma = (M, M'; A)$ be a square assignment market, and μ an optimal matching. Denote by $i' = \mu(i)$ the i-th seller and then $\mu = \{(i, i') \mid i \in M\}$. Then, the projection of $C(\omega_A)$ to the space of the buyers' payoff is*

$$C_u(\omega_A) = \left\{ u \in \mathbb{R}^M \;\middle|\; \begin{array}{l} a_{ij} - a_{jj} \leq u_i - u_j \leq a_{ij} - a_{ji} \text{ for all } i, j \in \{1, 2, \ldots, m\} \\ 0 \leq u_i \leq a_{ii} \text{ for all } i \in \{1, 2, \ldots, m\} \end{array} \right\}.$$

Notice that $C_u(\omega_A)$ is a 45-degree lattice[9].

Theorem 4.24. *(Quint, 1991 [32]; Characterization of the core) Given any 45-degree lattice L, there exists an assignment game (M, M', A) such that $C_u(\omega_A) = L$.*

For $m = 2$, the lattice L determines either a unique valuation matrix A with two rows and two columns such that $C(\omega_A) = L$, or a unique valuation matrix A with two rows and three columns such that $C(\omega_A) = L$.

Example 4.25. *With 3 or more agents, the same 45-degree lattice may represent the core of several valuation matrices of the same dimension. The two following matrices define markets with the same core.*

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$C_u(\omega_A) = C_u(\omega_B) = \{(t, t, t; 1-t, 1-t, 1-t) \mid 0 \leq t \leq 1\}.$$

[8] Quint (1991) proves that this also holds for markets with more agents on each side and in fact this property gives a geometric characterization of the core of the assignment game

[9] L is a non-empty 45-degree lattice in $\mathbb{R}^M$ if can be expressed as:

$$L = \left\{ u \in \mathbb{R}^M \;\middle|\; \begin{array}{l} u_i - u_k \geq d_{ik} \text{ for all } i, k \in \{1, \ldots, m\}, i \neq k \\ b_i \leq u_i \leq e_i \text{ for all } i \in \{1, \ldots, m\} \end{array} \right\}.$$

where $d_{ik}, b_i, e_i \in \mathbb{R}, b_i, e_i \geq 0$.

Quint (1991) [32] poses the question of which is the maximum matrix $\overline{A}$ among those matrices with $C(\omega_A) = L$, given a 45-degree lattice L. Clearly, this maximality requires that no matrix entry can be raised without modifying the core, that is, for each $(i, j) \in M \times M'$ there must be a core element $(u, v) \in C(\omega_A)$ with $u_i + v_j = \overline{a}_{ij}$. This is a weaker form of exactness[10].

Theorem 4.26. *A square assignment game* $(M \cup M', \omega_A)$*, with an optimal matching placed on the main diagonal, is exact if and only if matrix* A *has:*

- *a dominant diagonal:* $a_{ii} \geq a_{ij}$ *and* $a_{ii} \geq a_{ji}$*, for all* $i, j \in M$ *and*
- *a doubly dominant diagonal:* $a_{ij} + a_{kk} \geq a_{ij} + a_{jk}$*, for all* $i, j, k \in M$*.*

Proof. The reader is referred to Solymosi and Raghavan (2001) [37]. □

If an assignment game is exact, its valuation matrix is maximum among those leading to the same core.

Theorem 4.27. *Given an assignment game* $(M \cup M', \omega_A)$*, there exists a unique matrix* $\overline{A}$ *that is buyer-seller exact and give rise to the same core,* $C(\omega_A) = C(\omega_{\overline{A}})$*.*

Proof. The theorem above is proved in Núñez and Rafels (2002a) [24]. □

In Núñez and Rafels (2002a) [24], under the assumption that A is square and μ an optimal matching, for all $(i, j) \in M \times M'$, the entry in $\overline{A}$ is given by

$$\overline{a}_{ij} = a_{i\mu(i)} + a_{\mu^{-1}(j)j} + \omega_A \left(M \cup M' \setminus \{\mu^{-1}(j), \mu(i)\} \right) - \omega_A(M \cup M').$$

Corollary 4.28. *An assignment game* $(M \cup M', \omega_A)$ *is buyer-seller exact if and only if* A *has a doubly dominant diagonal.*

Example 4.29. *The buyer-seller exact representative of matrices* A *and* B *in Example 4.25 is the matrix with entries* $\overline{a}_{ij} = 1$ *for all* $i, j = 1, 2, 3$*. This is a glove market, of the same type that the one mentioned in Example 2.5.*

[10] A coalitional game (N, v) is exact if for all $S \subset N$ there exists $x \in C(v)$ such that $\sum_{i \in S} x_i = v(S)$.

4.5 Buyer and seller optima

In 1982/83 Leonard [18] and Demange [9] discovered independently from each other a simple procedure to find the buyer or seller optimum of an assignment market. We will explain this procedure briefly, and first we will explain what this concept means.

Let $(M, M'; A)$ be an assignment market and $(M \cup M', \omega_A)$ be its associated assignment game. We start with calculating the maximum payoff each buyer j in the core can receive. This is equal to $\overline{v}_j$, and Leonard and Demange prove that it is the value that buyer j adds to the grand coalition $N = M \cup M'$ in case he joins as the last member:

$$\overline{v}_j = \omega_A(N) - \omega_A(N \backslash \{j\}) \quad \forall j \in M'.$$

Roth and Sotomayor [33] have a rather simpler proof to show this equality. Once each buyer received his maximum payoff, it easy to determine the minimum payoff for each seller i hence for any optimal assignment μ the equation $u_i + v_{\mu(i)} = a_{i\mu(i)}$ holds true. Hence each seller i receives

$$\underline{u}_i = a_{i\mu(i)} - \overline{v}_{\mu(i)} = a_{i\mu(i)} - \omega_A(N) + \omega_A(N \backslash \{\mu(i)\}) \quad \forall i \in M.$$

To calculate the seller optimum we use the same procedure but starting with calculating the sellers' maximum payoffs

$$\overline{u}_i = \omega_A(N) - \omega_A(N \backslash i) \quad \forall i \in M,$$

and afterward the buyers' minimum payoffs

$$\underline{v}_j = a_{\mu^{-1}(j)j} - \overline{u}_{\mu^{-1}(j)} = a_{\mu^{-1}(j)j} - \omega_A(N) + \omega_A(N \backslash \mu^{-1}(j)) \quad \forall j \in M'.$$

Example 4.30. *We apply this last formula to Example 4.14 and we obtain the next buyer and seller optima. The calculations can be found in Example 4.34.*

$$(\overline{u}, \underline{v}) = (5, 6, 1; 1, 3, 0), \qquad (\underline{u}, \overline{v}) = (3, 5, 0; 2, 5, 1).$$

The buyers-optimal core allocation corresponds to the minimal competitive price vector. The sellers-optimal core allocation corresponds to the maximal competitive price vector.

4.6 The extreme core allocations of the assignment game

The core of a cooperative game can be described with a set of equalities and inequalities and it is always a compact and convex set. Therefore it can also be described by its extreme points. There does not exist a simple procedure to find these extreme points in general. Due to the special form of the core of the assignment game, it is a lattice, there exists a quite intuitive way, similarly to the finding of the sellers-optimal and buyers-optimal allocation, to find the extreme allocations of the core of the assignment game.

Núñez and Rafels (2003) [26] shows that the set of extreme core allocations of the assignment game coincides with the set of reduced marginal worth vectors. It is a complex but complete procedure to calculate the set of reduced marginal worth vectors[11]. However, in 2007, Izquierdo, Núñez and Rafels [15] discovered another and simpler method to calculate the extreme core allocations of an assignment game.

To this end, let us introduce a set of vectors called max-payoff vectors. with one max-payoff vector $x^\theta(A)$ for each possible ordering of the agents. An ordering $\theta = (k_1, k_2, \ldots, k_n)$ is

a bijection from $N = M \cup M'$ to $N = M \cup M'$. Each agent $k_i \in N$ is assigned to a position $i \in \{1, 2, \ldots, n\}$. The function $\theta(i)$ returns agent k_i for all $i \in N$ and the inverse function $\theta^{-1}(k_i) = i$ gives back the position i for each agent $k_i \in N$. The set of orderings on N is called Θ. The set of antecessors of an agent $k \in N$ in the ordering θ is $P_k^\theta = \{j \in M \cup M' | \theta^{-1}(j) < \theta^{-1}(k)\}$.

Definition 4.31. *Given an assignment game $(M \cup M', \omega_A)$ we recursively define a payoff vector $x^\theta(A) \in \mathbb{R}^M \times \mathbb{R}^{M'}$, named a max-payoff vector, for each possible order θ on the player set in the following way: $x_{k_1}^\theta(A) = 0$ and*

$$x_{k_r}^\theta(A) = \begin{cases} \max_{i \in P_{k_r}^\theta \cap M}\{0, a_{ik_r} - x_i^\theta(A)\} & \text{if} \quad k_r \in M', \\ \max_{j \in P_{k_r}^\theta \cap M'}\{0, a_{k_r j} - x_j^\theta(A)\} & \text{if} \quad k_r \in M. \end{cases}$$

Example 4.32. *As an exercise, take the order $\theta = (1, 2', 1', 2)$ on the player set of the assignment game of Example 4.13. The associated max-payoff vector is $x_1^\theta(A) = 0$, $x_{2'}^\theta(A) = \max\{0, 4-0\} = 4$, $x_{1'}^\theta(A) = \max\{0, 5-0\} = 5$ and $x_2^\theta(A) = \max\{0, 2-5, 3-4\} = 0$. Then $x^\theta(A) = (0, 0, 5, 4)$. It is not efficient, which is not surprising since A has not a dominant diagonal. Some other orders, take for instance $\theta' = (2, 2', 1, 1')$, lead to an extreme core point.*

The total number of orderings $|\Theta| = (m + m')!$ is large even for small games. It is possible to reduce to the number of required orderings to a subset Θ^μ. Here each buyer or seller

[11] Given a coalitional game (N, v) and any order k on the player set N, the marginal worth vector $m^{k,v} \in \mathbb{R}^N$ pays each agent his/her contribution to the set of predecessors according to the order k. That is, $m_{k(1)}^{k,v} = v(\{k(1)\})$ and, for all $i \in \{2, \ldots, n\}$,

$$m_{k(i)}^{k,v} = v(\{k(1), \ldots, k(i)\}) - v(\{k(1), \ldots, k(i-1)\})$$

who occupies an odd number is followed by his optimal match, i.e. for $\mu \in \mathcal{M}^*_{\mathrm{A}}(M, M')$:

$$\Theta^\mu = \left\{ \theta = (k_1, \dots, k_n) \in \Theta \,\middle|\, \begin{array}{l} \text{for all } r \in \left\{0, 1, 2, \dots, \frac{n}{2} - 1\right\} \\ \text{if } k_{2r+1} \in M \text{ then } k_{2r+2} = \mu(k_{2r+1}) \\ \text{if } k_{2r+1} \in M' \text{ then } k_{2r+2} = \mu^{-1}(k_{2r+1}) \end{array} \right\}.$$

The number of required orderings $|\Theta^\mu|$ for an assignment game with an equal number of buyers and sellers ($m = m'$) still increases rapidly since there are m optimal buyer-seller pairs, hence $m!$ different pair orderings. Each optimal pair can be switched around $(k, \mu(k)) \to (\mu(k), k)$ thus another 2^m different combinations are possible. If we combine these two numbers we obtain the number of orderings:

$$|\Theta^\mu| \quad = \quad m! \times 2^m.$$

This number is already much smaller than the total number of orderings $|\Theta| = (2m)!$ but still quite large. To show the increasing number of orderings, we calculate the number of orderings for games with different size. It becomes quite obvious that for large games e.g. a used car market with 30 different cars the max-payoff vectors are so numerous that it becomes impossible to calculate them.

Table 4.1: Number of Orderings $|\Theta^\mu|$ and $|\Theta|$

m	$\|\Theta^\mu\|$	$\|\Theta\|$
1	2	2
2	8	24
3	48	720
4	384	40,320
5	3,840	3,628,800
6	46,080	479,001,600
7	645,120	87,178,291,200
8	10,321,920	20,922,789,888,000
9	185,794,560	6.402,373,705,728,000
10	3,715,891,200	2,432,902,008,176,640,000
11	81,749,606,400	1,124,000,727,777,607,680,000
12	1,961,990,553,600	620,448,401,733,239,439,360,000

Izquierdo, Núñez, and Rafels [15] prove that max-payoff vectors are not, in general, efficient and that only the efficient max-payoff vectors are extreme core allocations.

Theorem 4.33. *Let $(M \cup M', \omega_A)$ be an assignment game with an equal number of buyers and sellers and let μ^* be an optimal matching, $\mu^* \in \mathcal{M}^*_{\mathrm{A}}(M, M')$. Then,*

$$\mathit{Ext}(C(\omega_A)) = \{x^\theta \mid \theta \in \Theta^\mu, x^\theta_i + x^\theta_j = a_{ij} \quad \textit{for all} \quad (i,j) \in \mu^*\}.$$

This is a full characterization of the extreme core allocations and therefore closes the search for the extreme points.

Example 4.34. *(A numerical example) Let us consider an example that Shapley and Shubik (1971) [36] use in the first analysis of the assignment game, to illustrate the geometry of the core. This example consists of a house market with three sellers* $(1, 2, 3)$ *and three buyers* $(1', 2', 3')$.

The reader will notice this is an assignment game whose core has been studied before (see Example 4.14).

The features of the market, in other words the valuation of each house by its corresponding seller and the valuation of each house by each buyer, are shown in the next Table 4.2.

Table 4.2: A House Market

Houses	Sellers' basis in \$	Buyers' valuation in \$		
i	c_i	h_{i1}	h_{i2}	h_{i3}
1	18,000	23,000	26,000	20,000
2	15,000	22,000	24,000	21,000
3	19,000	21,000	22,000	17,000

Based on this information we can calculate the benefit of each buyer-seller pair $a_{ij} = \max\{0, h_{ij} - c_i\}$*, which is shown in matrix A (Table 4.3). Furthermore the optimal assignment and the worth of the grand coalition, which is* 16*, can be determined.*

Table 4.3: The Benefit of each Buyer-Seller Pair

(units of \$ 1000)

(sellers M) \ (buyers M')	$1'$	$2'$	$3'$
1	5	(8)	2
2	7	9	(6)
3	(2)	3	0

We will first find the buyers- and sellers- optima and later use the max-payoff vectors to calculate all extreme core allocations. We start with calculating $\overline{v}_1$*. To do so, we have to know the value of the grand coalition* $\omega(N)$ *with* $N = \{1, 2, 3, 1', 2', 3'\}$ *which we have already determined and the value of the grand coalition without seller* 1*:* $\omega(N \backslash \{1\})$*. The latter can be found by erasing row* 1 *of matrix A and looking for the new optimal assignment.*

Table 4.4: A House Market without Seller 1

(units of \$ 1000)

(sellers M) \ (buyers M')	$1'$	$2'$	$3'$
2	7	(9)	6
3	(2)	3	0

(units of \$ 1000)

From $\omega(N \backslash \{1\}) = \$9,000 + \$2,000 = \$11,000$ *we obtain:*

$$\overline{u}_1 = \omega(N) - \omega(N \backslash \{1\}) = \$16,000 - \$11,000 = \$5,000$$

In the optimal matching seller 1 *trades with buyer* $2'$. *Therefore their payoffs must equal* a_{12}, *i.e.* $u_1 + v_2 = a_{12} = \$8,000$. *We see that buyer* $2'$ *receives* $\$8,000 - \$5,000 = \$3,000$.

From

$$\begin{aligned}\overline{u}_2 &= \$16,000 - \$10,000 = \$6,000\\ \overline{u}_3 &= \$16,000 - \$15,000 = \$1,000\\ \overline{v}_1 &= \$16,000 - \$14,000 = \$2,000\\ \overline{v}_2 &= \$16,000 - \$11,000 = \$5,000\\ \overline{v}_3 &= \$16,000 - \$15,000 = \$1,000\end{aligned}$$

we obtain the following data: Now the sellers-optimum $(\overline{u}, \underline{v}) = (5, 6, 1; 1, 3, 0)$ *is found.*

Table 4.5: The Sellers-optimum

i	$\mu(i)$	$a_{i\mu(i)}$	$\overline{u}_i$	$\underline{v}_{\mu(i)}$
1	$2'$	8	5	3
2	$3'$	6	6	0
3	$1'$	2	1	1

(units of \$ 1000)

Table 4.6: The Buyers-optimum

j	$\mu^{-1}(j)$	$a_{\mu^{-1}(j)j}$	$\overline{v}_j$	$\underline{u}_{\mu^{-1}(j)}$
1’	3	2	2	0
2’	1	8	5	3
3’	2	6	1	5

(units of \$ 1000)

The buyers-optimum can be stated: $(\underline{u}; \overline{v}) = (3, 5, 0; 2, 5, 1)$.

In this example we already have to deal with 48 *max-payoff vectors which makes it quite cumbersome to calculate the extreme core allocations. To show the procedure, we will calculate one max-payoff vector for the ordering* $(1\ , 2', 2\ , 3', 3\ , 1')$ *and provide the other max-payoff vectors without calculation.*

Let us put $x^\theta = (x_1, x_2, x_3, x_{1'}, x_{2'}, x_{3'}) = (u_1, u_2, u_3; v_1, v_2, v_3)$. *By definition the first agent in the ordering, seller* 1, *receives a payoff equal to* 0. *Hence the first number of the max-payoff vector is already determined:* $x^\theta = (0, u_2, u_3; v_1, v_2, v_3)$. *In second place comes buyer* $2'$ *with a payoff equal to*

$$v_2 = \max\{0, a_{12} - x_1\} = \max\{0, 8 - 0\} = 8,$$

hence $x^\theta = (0, u_2, u_3; v_1, 8, v_3)$. *In third place comes seller* 2 *with a payoff of*

$$u_2 = \max\{0, a_{22} - x_{2'}\} = \max\{0, 9 - 8\} = 1,$$

thus $x^\theta = (0, 1, u_3; v_1, 8, v_3)$. *Next in the ordering is buyer* $3'$ *who can trade with seller* 1 *and* 2. *His payoff is*

$$v_3 = \max\{0, a_{13} - x_1, a_{23} - x_2\} = \max\{0, 2 - 0, 6 - 1\} = 5,$$

and therefore we obtain $x^\theta = (0, 1, u_3; v_1, 8, 5)$*. Then seller 3 receives his payoff*

$$u_3 = \max\{0, a_{32} - x_{2'}, a_{33} - x_{3'}\} = \max\{0, 3 - 8, 0 - 5\} = 0,$$

and we have $x^\theta = (0, 1, 0; v_1, 8, 5)$*. Last buyer* $1'$ *who can trade with seller* $1, 2$ *and* 3 *receives*

$$v_1 = \max\{0, a_{11} - x_1, a_{21} - x_2, a_{31} - x_3\} = \max\{0, 5 - 0, 7 - 1, 2 - 0\} = 6,$$

and we obtain $x^\theta = (0, 1, 0; 6, 8, 5)$*. The max-payoff vector is not efficient since* $\sum x_i^\theta = 20 \neq 16 = \omega(N)$ *and therefore it is not an extreme core allocation. All max-payoff vectors are shown in Table 4.7. The ones that are efficient, and hence are extreme core allocations, are shown in boldface.*

Table 4.7: Extreme Core Allocations

$\theta \in \Theta^\mu$	$x^\theta = (u_1, u_2, u_3; v_1, v_2, v_3)$	$\sum x_i^\theta$
$(1, 2', 2, 3', 3, 1')$	$(0, 1, 0; 6, 8, 5)$	*20*
$(1, 2', 2, 3', 1', 3)$	$(0, 1, 0; 6, 8, 5)$	*20*
$(1, 2', 3', 2, 3, 1')$	$(0, 4, 0; 5, 8, 2)$	*19*
$(1, 2', 3', 2, 1', 3)$	$(0, 4, 0; 5, 8, 2)$	*19*
$(1, 2', 3, 1', 2, 3')$	$(0, 2, 0; 5, 8, 4)$	*19*
$(1, 2', 3, 1', 3', 2)$	$(0, 4, 0; 5, 8, 2)$	*19*
$(1, 2', 1', 3, 2, 3')$	$(0, 2, 0; 5, 8, 4)$	*19*
$(1, 2', 1', 3, 3', 2)$	$(0, 4, 0; 5, 8, 2)$	*19*
$(2', 1, 2, 3', 3, 1')$	$(8, 9, 3; 0, 0, 0)$	*20*
$(2', 1, 2, 3', 1', 3)$	$(8, 9, 3; 0, 0, 0)$	*20*
$(2', 1, 3', 2, 3, 1')$	$(8, 9, 3; 0, 0, 0)$	*20*
$(2', 1, 3', 2, 1', 3)$	$(8, 9, 3; 0, 0, 0)$	*20*
$(2', 1, 3, 1', 2, 3')$	$(8, 9, 3; 0, 0, 0)$	*20*
$(2', 1, 3, 1', 3', 2)$	$(8, 9, 3; 0, 0, 0)$	*20*
$(2', 1, 1', 3, 2, 3')$	$(8, 9, 3; 0, 0, 0)$	*20*
$(2', 1, 1', 3, 3', 2)$	$(8, 9, 3; 0, 0, 0)$	*20*
$(2, 3', 1, 2', 3, 1')$	$(0, 0, 0; 7, 9, 6)$	*22*
$(2, 3', 1, 2', 1', 3)$	$(0, 0, 0; 7, 9, 6)$	*22*
$(2, 3', 2', 1, 3, 1')$	$(0, 0, 0; 7, 9, 6)$	*22*
$(2, 3', 2', 1, 1', 3)$	$(0, 0, 0; 7, 9, 6)$	*22*
$(2, 3', 3, 1', 1, 2')$	$(0, 0, 0; 7, 9, 6)$	*22*
$(2, 3', 3, 1', 2', 1)$	$(0, 0, 0; 7, 9, 6)$	*22*
$(2, 3', 1', 3, 1, 2')$	$(0, 0, 0; 7, 9, 6)$	*22*
$(2, 3', 1', 3, 2', 1)$	$(0, 0, 0; 7, 9, 6)$	*22*
$(3', 2, 1, 2', 3, 1')$	$(2, 6, 0; 3, 6, 0)$	*17*
$(3', 2, 1, 2', 1', 3)$	$(2, 6, 0; 3, 6, 0)$	*17*
$(3', 2, 2', 1, 3, 1')$	$\mathbf{(5, 6, 0; 2, 3, 0)}$	*16*
$(3', 2, 2', 1, 1', 3)$	$\mathbf{(5, 6, 1; 1, 3, 0)}$	*16*
$(3', 2, 3, 1', 1, 2')$	$\mathbf{(3, 6, 0; 2, 5, 0)}$	*16*
$(3', 2, 3, 1', 2', 1)$	$\mathbf{(5, 6, 0; 2, 3, 0)}$	*16*
$(3', 2, 1', 3, 1, 2')$	$\mathbf{(4, 6, 1; 1, 4, 0)}$	*16*
$(3', 2, 1', 3, 2', 1)$	$\mathbf{(5, 6, 1; 1, 3, 0)}$	*16*

Table 4.7: Extreme Core Allocations

$\theta \in \Theta^{\mu}$	$x^{\theta} = (u_1, u_2, u_3; v_1, v_2, v_3)$	$\sum x_i^{\theta}$
$(3, 1', 1, 2', 2, 3')$	$(\mathbf{3}, \mathbf{5}, \mathbf{0}; \mathbf{2}, \mathbf{5}, \mathbf{1})$	*16*
$(3, 1', 1, 2', 3', 2)$	$(\mathbf{3}, \mathbf{6}, \mathbf{0}; \mathbf{2}, \mathbf{5}, \mathbf{0})$	*16*
$(3, 1', 2', 1, 2, 3')$	$(\mathbf{5}, \mathbf{6}, \mathbf{0}; \mathbf{2}, \mathbf{3}, \mathbf{0})$	*16*
$(3, 1', 2', 1, 3', 2)$	$(\mathbf{5}, \mathbf{6}, \mathbf{0}; \mathbf{2}, \mathbf{3}, \mathbf{0})$	*16*
$(3, 1', 2, 3', 1, 2')$	$(\mathbf{3}, \mathbf{5}, \mathbf{0}; \mathbf{2}, \mathbf{5}, \mathbf{1})$	*16*
$(3, 1', 2, 3', 2', 1)$	$(\mathbf{4}, \mathbf{5}, \mathbf{0}; \mathbf{2}, \mathbf{4}, \mathbf{1})$	*16*
$(3, 1', 3', 2, 1, 2')$	$(\mathbf{3}, \mathbf{6}, \mathbf{0}; \mathbf{2}, \mathbf{5}, \mathbf{0})$	*16*
$(3, 1', 3', 2, 2', 1)$	$(\mathbf{5}, \mathbf{6}, \mathbf{0}; \mathbf{2}, \mathbf{3}, \mathbf{0})$	*16*
$(1', 3, 1, 2', 2, 3')$	$(5, 7, 2; 0, 3, 0)$	*17*
$(1', 3, 1, 2', 3', 2)$	$(5, 7, 2; 0, 3, 0)$	*17*
$(1', 3, 2', 1, 2, 3')$	$(7, 8, 2; 0, 1, 0)$	*18*
$(1', 3, 2', 1, 3', 2)$	$(7, 8, 2; 0, 1, 0)$	*18*
$(1', 3, 2, 3', 1, 2')$	$(5, 7, 2; 0, 3, 0)$	*17*
$(1', 3, 2, 3', 2', 1)$	$(6, 7, 2; 0, 2, 0)$	*17*
$(1', 3, 3', 2, 1, 2')$	$(5, 7, 2; 0, 3, 0)$	*17*
$(1', 3, 3', 2, 2', 1)$	$(6, 7, 2; 0, 2, 0)$	*17*

We have found 14 *efficient max-payoff vectors. However, many of them coincide and therefore* 6 *max-payoff vectors are sufficient to describe the core:*

$C(\omega_A) = \text{Co}\{(5,6,0;2,3,0)(5,6,1;1,3,0)(3,6,0;2,5,0)(4,6,1;1,4,0)(3,5,0;2,5,1)(4,5,0;2,4,1)\}.$

4.7 Some single-valued solutions

Other cooperative solutions have been studied for the assignment game. Among the single-valued solutions, that are defined for arbitrary cooperative TU-games, the *nucleolus* stands out. We will briefly give its definition applied to the assignment game.

Let $\gamma = (M, M'; A)$ be an assignment market and the corresponding assignment game (N, w_A) where $N = M \cup M'$. Consider all basic coalitions $\mathcal{B}$ (singletons and mixed-pairs) and at each imputation $x \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$ the *excess of x at coalition $S \in \mathcal{B}$*, is defined as $e(S, x) := w_A(S) - \sum_{i \in S} x_i$. Let us denote by $\theta(x)$ the vector formed by the decreasingly ordered excesses of all basic coalitions at imputation $x \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$. Then, the *nucleolus* (Schmeidler, 1969) [35] is the imputation that lexicographically minimizes this vector of excesses: $\theta(\eta) \leq_L \theta(x)$ for all $x \in I(w_A)$. The computation of the nucleolus for arbitrary cooperative TU-games involves solving a series of linear programs, and it is not an easy task. An algorithm to compute the nucleolus of the assignment game is given in Solymosi and Raghavan (1994) [37] and a simple procedure in Martínez-de-Albéniz et al. (2013) [21]. A geometric characterization is in Llerena and Núñez (2011) [19], and an axiomatization in Llerena et al. (2015)[20].

Another single-valued solution for the assignment game was introduced by Thompson (1981) [39] with the name of *fair division point*, since it is the midpoint of the segment between the buyers-optimal and the sellers-optimal core allocation. If $(M \cup M', w_A)$ is an assignment game, the fair division point is given by

$$\tau(w_A) = \frac{1}{2}(\overline{u}^A, \underline{v}^A) + \frac{1}{2}(\underline{u}^A, \overline{v}^A).$$

Nuñez and Rafels (2002b) [25] proves that the fair division point coincides with the τ-value (Tijs, 1981) [38].

Chapter 5

Multi-item auctions

As seen in Chapter 4, the core of the assignment game coincides with the competitive equilibria of the assignment market. There is a one-to-one correspondence between each core allocation and competitive prices. A very natural question arises from this fact, if it is possible to design a mechanism (non-cooperative in nature) such that the equilibrium of the associated non-cooperative game correspond to some of these competitive equilibria[1]. In particular we care about the buyers-optimal core allocation.

5.1 Multi-item auction mechanism

In this section we describe a mechanism to determine the buyers-optimal core allocation given the data of the market. The question above is addressed in Demange et al. (1986) [10]. Studies by Demange (1982) [9], Leonard (1983) [18] and Gale and Demange (1985) [12] consider an allocation mechanism that turns out to be a generalization of the well-known "second-price[2]" auction first described by Vickrey (1961) [41].

In order to describe the multi-item generalization of this mechanism it is convenient to think of the second-price scheme as an ordinary competitive equilibrium, when there is only one item on sale. The most important property of the second highest bid is that it is the smallest equilibrium price since for any smaller price at least two bidders would demand the item. In the multi-item generalization it is assumed that each bidder is interested in acquiring at most one item, as might be the case if, for example, the auction was designed to assign individuals to positions, as considered by Leonard (1983) [18].

The multi-item auction mechanism requires each bidder (buyer) to submit in a sealed bid listing his valuation of all the items. The auctioneer then allocates the items in accordance with the minimum price equilibrium.

A main point of the papers cited is that the important "incentive capability" of the single-item auction carries over to the multi-item case, meaning that submitting true valuation is

[1] Reader can check the definition of competitive equilibrium in Definition 4.10.

[2] In this auction, the participants submit sealed bids for a single item. That item is sold to the highest bidder at a price equal to the second highest bid.

a dominant strategy for the bidders. Therefore, by jointly falsifying valuations, no subset of bidders can improve the outcome for all its members.

In this chapter we are going to study the *exact auction mechanism* from Demange et al. (1986) [10] applied to a second-price auction.

Clearly, a second-price auction can be read into as an assignment game within the next framework.

- A finite set $N$of agents and a finite set $O'$of indivisible objects.
- The finite set of objects $O \subset \mathbb{N}$ that includes O' and as many copies of a null object as agents in N. This null object is denoted by $\emptyset$.
- Each $i \in N$ can acquire at most one object and each $i \in N$ has a valuation $a_{ij} \in \mathbb{R}_+$ for each $j \in O$. Null objects have a valuation of zero.
- Agent i's valuation $a_i = (a_{ij})_{j \in O} \in \mathcal{A}$ and a valuation profile $a = (a_i)_{i \in N} \in \mathcal{A}^N$ where $\mathcal{A}^N = \mathcal{A} \times ... \times \mathcal{A}$.
- An assignment $\mu = (\mu_i)_{i \in N}$ is a vector that assigns $\mu_i \in O$ to i and each object is assigned at most to one agent.
- Let $\mathcal{M}(S, Q)$ be the set of all allocations of objects $Q \subseteq O$ to agents $S \subseteq N$.
- A price vector is $p \in \mathbb{R}^O_+$ such that $p_\emptyset = 0$ for each $\emptyset$.

Thus, as we can see, we have an assignment problem where the question is how to assign objects to agents in the best possible way. In Vickrey (1961) [41], it is shown a mechanism to find the maximum competitive equilibrium and the minimum competitive equilibrium under certain circumstances: *the gross-substitutes (GS) condition*[3] *and quasi-linear preferences*[4].

In other words, the GS condition supposes that given an initial price vector, an agent wants to consume a bundle of objects. Assume that some prices increase and considers a new vector of prices, then the agent still wants to consume those objects belonging to the bundle with the same price in both price vectors. This is exactly what is happening in the framework above. The quasi-linear preferences are satisfied by the assignment game we are considering. We have seen in Chapter 4 that competitive equilibrium price vectors of an assignment game belong to the core of the assignment game. And we have also seen that the core of the assignment game is non-empty and forms a complete lattice.

Once checked these two conditions, it is possible to apply which is known as VCG mechanism[5] to obtain the maximum competitive equilibrium and the minimum competitive

[3]A valuation $a_i = (a_{ij})_{j \in O} \in \mathcal{A}$ satisfies *gross-substitutes condition* if, and only if, for every price $p \in \mathbb{R}^O_+$, every set $S \in D_i(p)$, and every $q > p, q \in \mathbb{R}^O_+$, there is a set $T \subseteq A$ with $(S \setminus A) \cup T \in D_i(p)$ where $A = \{j : q(j) > p(j)\}, \quad j \in O$.

[4]The set of competitive equilibrium price vectors of the market is non-empty and forms a complete lattice.

[5]VCG mechanisms allow for the selection of a socially optimal outcome out of a set of possible outcomes in a Vickrey–Clarke-Groves (VCG) auction. This is a type of sealed-bid auction of multiple items. The auction is named after Vickrey (1961) [41], Clarke (1971) and Groves (1973) for their papers that successively generalized the idea. The VCG auction is a specific use of the more general VCG mechanism.

equilibrium.

This mechanism is not shown in this dissertation[6] but two interesting results come out from this process.

- The payoff obtained via VCG belongs to the core of the assignment game.
- The payoff obtained via VCG is supported by the minimum competitive equilibrium in the replicated market.

The exact auction mechanism derived in order to find the best possible outcome for all the agents works as follows:

- *Step* 1*:* The auctioneer announces the initial price vector. Then, every agent "bids" by announcing which object or objects are in his demand set given the prices.

- *Step t* ($t \geq 2$)*:* If it is possible to allocate to each agent an object belonging to his demand set given the price vector, the algorithm stops. If no such allocation exists, then there is an overdemanded set of objects. The auctioneer chooses a minimal overdemanded set of objects and raises the price of each object in the set by one unit. The price of all other objects remain at the same level.

The example below represents the operating of this mechanism.

Example 5.1. *Consider a second-price auction with a set of 3 agents* $\{1, 2, 3\}$ *and a set of 3 objects* $\{1', 2', 3'\}$*. Suppose the valuations are* $a_1 = (10, 9, 8)$*,* $a_2 = (10, 9, 4)$ *and* $a_3 = (10, 4, 2)$*.*

The auction begins, consider the initial price vector $p^1 = (0, 0, 0)$*. The demand sets of the agents are:*

- $D_1(p^0) = \{1'\}$*,* $D_2(p^0) = \{1'\}$*,* $D_3(p^0) = \{1'\}$*.*

Since there is an overdemanded set, the auctioneer raises the price. Consider now $p^1 = (1, 0, 0)$ *(*$p^1 > p^0$*, satisfies the GS condition).*

- $D_1(p^1) = \{1', 2'\}, D_2(p^1) = \{1', 2'\}, D_3(p^1) = \{1'\}$*.*

Notice that $\{1'\}$ *is not an overdemanded set because*

$$|\{i \in N | D_i(p^1) \subseteq \{1'\}\}| = |\{3\}| = |\{1'\}|,$$

but, note that $\{1', 2'\}$ *is overdemanded:*

$$|\{i \in N | D_i(p^1) \subseteq \{1', 2'\}\}| = |\{1, 2, 3\}| > |\{1', 2'\}|.$$

Now prices in the overdemanded set are raised. Consider $p^2 = (2, 1, 0)$ *(*$p^2 > p^1$*, satisfies the GS condition again).*

- $D_1(p^2) = \{1', 2', 3'\}$*,* $D_2(p^2) = \{1', 2'\}$*,* $D_3(p^2) = \{1'\}$*.*

Now we can assign object $1'$ *to agent* 3*, object* $2'$ *to agent* 2 *and object* $3'$ *to agent* 1*.*

[6]Interested reader is referred to Vickrey (1961) [41].

Chapter 6

Conclusions

Matching markets is an active area of research. This research consists of the study of resource allocation problems in which two sets of agents, or a set of agents and a set of goods, have to be assigned to one another in a way the respects the preferences of the market participants. There are several types of allocation problems. For example, two-sided matching markets where prices cannot be used; markets without money where we desire to assign or re-allocate indivisible goods; and assignment markets where we want to find prices that clear markets for potentially distinct and heterogeneous indivisible goods.

These are marriage markets, college admission problem, roommate problem, depending on the precise characteristics of preferences, and so on. There are mechanisms that have been succesfully applied for the allocation of students to universities, resident doctors to hospitals (see Roth and Sotomayor (1990) [33]) or kidneys to patients in need of a transplant (Roth et al. (2005) [34]).

In this dissertation we have focused on markets with money, that is markets where preferences are quasilinear in money, and agents ask for one unit each. Money is the common utility for all agents.

These markets allow the modeling of auctions of multiple objects (Demange et al. (1986) [10]) as it has been introduced in Chapter 5: Multi-items auctions. In the case of auctions where agents are allowed to bid for bundles of objects, it is also possible to model through the assignment markets. For instance, licenses for mobile telephony in different states which are represented by what is known as package assignment game (Bikhchandani and Ostroy (2002) [1]).

The multi-item auction model seen in this project is "unsymmetrical" which means that each seller specifies only one number, his reservation price, while buyers specify their valuations for each of the items. There are, however, economically natural situations in which sellers "discriminate" specifying different reservation prices to different buyers. The job assignment problem is an example of this situation. Here the sellers are workers who are selling their services to employers. Clearly, the minimum salary a worker would accept might vary depending on the job; for example, the more disagreeable the job, the higher the minimum acceptable salary.

The model of the assignment game, as defined by Shapley and Shubik (1971) [36], relies on several economic assumptions: there are two sides in the market, utility is identified with money, side-payments are allowed and the supply of each seller and the demand of each buyer are unitary. We could consider two different extensions that follow from relaxing the two-sided nature of the market and the unitary demand and supply of the agents. It results that some of the nice properties and structure of solutions do not hold when we consider these generalized assignment markets. The extension of the assignment market with multiple sides and multiple partners is called *multi-sided assignment markets* and is studied in Núñez and Rafels (2015) [27].

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