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Bilinear forms on weighted Besov spaces

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Abstract. We compute the norm of some bilinear forms on products of weighted Besov spaces in terms of the norm of their symbol in a space of pointwise multipliers related to a space of Carleson measures.

1. Introduction.

The object of this paper is the study of some bilinear forms on products of weighted holomorphic Besov spaces on the unit disk $\mathbb{D}$, and their relationship with Hankel operators and weak products.

In order to introduce our main results, we recall a classical theorem for small Hankel operators on the Hardy space H^2.

Let $d\sigma = d\gamma/2\pi$ be the surface measure on $\mathbb{T}$ and denote by $\mathcal{C}$ the Cauchy projection from L^2 to H^2. For $b \in H^1$, let $\mathfrak{h}_b(f) := \overline{\mathcal{C}(b\overline{f})}$ be the small Hankel operator associated to $\mathcal{C}$, defined on the space of holomorphic functions on $\overline{\mathbb{D}}$, $H(\overline{\mathbb{D}})$. The duality $(H^2)' \equiv H^2$ with respect to the pairing

$$\langle f, h\rangle_0 := \lim_{r\to 1^-} \int_0^{2\pi} f(re^{i\gamma})\overline{h(re^{i\gamma})}\frac{d\gamma}{2\pi}$$

shows that $\mathfrak{h}_b(f)$ is bounded from H^2 to $\overline{H^2}$ if and only if the bilinear form $\Lambda_b(f,g) := \langle fg, b\rangle_0$ is bounded on $H^2 \times H^2$. Since the strong product $H^2 \cdot H^2$ is H^1 and $(H^1)' \equiv BMOA$ (with respect to the pairing $\langle\cdot,\cdot\rangle_0$), we obtain that $\mathfrak{h}_b$ extends to a bounded operator from H^2 to $\overline{H^2}$ if and only if $b \in BMOA$, that is, if and only if the measure $d\mu(z) = |b'(z)|^2(1-|z|^2)d\nu(z)$ is a Carleson measure for H^2. Here $d\nu$ denotes the normalized Lebesgue measure on $\mathbb{D}$. Recall that a positive measure μ is a Carleson measure for H^2 if and only if $H^2 \subset L^2(\mu)$ and that it can be characterized in geometric terms as follows: μ is a Carleson measure for H^2 if and only if there exists a constant $C_\mu > 0$ such that $\mu(S_{\gamma,r}) \le C_\mu r$ for any sector $S_{\gamma,r} := \{z = \rho e^{i\eta} : r < \rho < 1, |\gamma-\eta| < r\}$.

The study of the boundedness of bilinear forms on other classical spaces, such as Hardy spaces H^p or Besov spaces B^p_s, and its connection to the boundedness of Hankel operators have been studied by several authors (see for instance [**12**], [**14**], [**15**], [**8**], [**1**], [**2**], [**7**] and the references therein). Even for the unweighted case, there is not a complete

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characterization for all the possible situations, as we will detail when stating our main results.

Our interest is to extend some of these results to the context of holomorphic weighted Besov spaces with weights in the Békollé class, which will be defined below. It is proved in [**6**, Proposition 3.9] that such weighted Besov spaces can be represented as weighted Besov spaces with weights in the Muckenhoupt classes. These last classes of weights are very useful when studying boundedness of some operators (even for the unweighted case), since the powerful extrapolation theorems reduce the general problem to a weighted problem for the case $p = 2$.

In order to state our main theorems and to detail some of the well-known results in this context, we introduce the following pairings. For $t > 0$, we write $d\nu_t(z) := t(1 - |z|^2)^{t-1} d\nu(z)$. In order to unify the statement of our results we define $d\nu_0 := d\sigma$.

If φ and ψ are measurable functions on $\mathbb{D}$ (on $\mathbb{T}$ if $t = 0$) such that $\varphi\overline{\psi} \in L^1(d\nu_t)$, let

$$\langle\langle \varphi, \psi \rangle\rangle_t := \int_{\mathbb{D}} \varphi\overline{\psi} d\nu_t \qquad \Big(\langle\langle \varphi, \psi \rangle\rangle_0 := \int_{\mathbb{T}} \varphi\overline{\psi} d\sigma\Big). \tag{1.1}$$

We also consider the pairings

$$\langle h, b \rangle_t := \lim_{r \to 1^-} \langle\langle h(rz), b(rz) \rangle\rangle_t, \tag{1.2}$$

whose domain is the subset of $H \times H$ for which the limit exists. In particular, if either $b \in B^1_{-t} := H \cap L^1(d\nu_t)$, $t > 0$, or $b \in H^1$, $t = 0$, then we have that for any $h \in H(\overline{\mathbb{D}})$, $\langle h, b \rangle_t = \langle\langle h, b \rangle\rangle_t$.

If $1 < p < \infty$ and $t > 0$, the Békollé class $\mathcal{B}_{p,t}$ consists of non-negative functions $\theta \in L^1(d\nu_t)$ such that the pair of measures $d\mu_t := \theta d\nu_t$ and $d\mu'_t := \theta^{-p'/p} d\nu_t$ satisfy the following condition

$$\mathcal{B}_{p,t}(\theta) := \sup_{z \in \mathbb{D}} \left(\frac{\mu_t(T_z)}{\nu_t(T_z)} \right)^{1/p} \left(\frac{\mu'_t(T_z)}{\nu_t(T_z)} \right)^{1/p'} < \infty,$$

where p' is the conjugate exponent of p,

$$T_z := \{ w \in \mathbb{D} : |1 - w\overline{z}/|z|| < 2(1 - |z|^2) \}, \ z \neq 0, \quad \text{and} \quad T_0 := \mathbb{D}.$$

If $1 \le p < \infty$, $s \in \mathbb{R}$, $\theta \in \mathcal{B}_{p,t}$ and $d\mu_t = \theta d\nu_t$, then the Besov space $B^p_s(\mu_t)$ consists of holomorphic functions f on $\mathbb{D}$ satisfying

$$\|f\|^p_{B^p_s(\mu_t)} := \int_{\mathbb{D}} \left| (1 + R)^{k_s} f(z) \right|^p (1 - |z|^2)^{(k_s - s)p} d\mu_t(z) < \infty.$$

Here, $k_s := \min\{ k \in \mathbb{N} : k > s \}$ and R denotes the radial derivative.

As it happens for the unweighted case, if we replace k_s by another non-negative integer $k > s$ we obtain equivalent norms (see for instance [**6**, Section 3]). In particular, if $s<0$, then we can take $k=0$, and thus we have that $B^p_s(\mu_t)=H\cap L^p(\mu_{t-sp})$. More properties of these spaces will be stated in Section 2.

The classical unweighted Besov space B^p_s corresponds to $B^p_s(\mu_0)$, where $d\mu_0(z)=d\nu(z)/(1-|z|^2)$. Observe that this space is already included in the scale of weighted Besov spaces that we have considered, simply because for any $t>0$

$$B^p_s(\mu_0)=B^p_{s+t/p}(\nu_t). \tag{1.3}$$

In order to recover some well-known results for the unweighted case and the pairing $\langle\cdot,\cdot\rangle_0$, we define $\mathcal{B}_{p,0}:=\{1\}$.

The pairing $\langle\cdot,\cdot\rangle_t$ can be used to identify the dual of $B^p_s(\mu_t)$ with $B^{p'}_{-s}(\mu'_t)$ (see Proposition 2.11).

Now we introduce a space of holomorphic functions related to the space of Carleson measures for weighted Besov spaces, which plays an analogous role to the space $BMOA$ for the classical problem on H^2.

The space $CB^p_s(\mu_t)$ consists of the functions $g\in B^p_s(\mu_t)$ for which

$$\|g\|_{CB^p_s(\mu_t)}:=\sup_{0\neq f\in B^p_s(\mu_t)}\frac{\|f(1+R)^{k_s}g\|_{B^p_{s-k_s}(\mu_t)}}{\|f\|_{B^p_s(\mu_t)}}$$

is finite.

When $t=0$, that is for the unweighted case, we simply denote the space $CB^p_s(\mu_0)$ by CB^p_s.

The space $CB^p_s(\mu_t)$ can be described either in terms of Carleson measures or in terms of pointwise multipliers. Indeed,

(i) $b\in CB^p_s(\mu_t)$ if and only if the measure

$$d\mu_b(z):=|(1+R)^{k_s}b(z)|^p\,(1-|z|^2)^{(k_s-s)p}\,d\mu_t(z),$$

is a Carleson measure for $B^p_s(d\mu_t)$, that is, if and only if the embedding $B^p_s(\mu_t)\subset L^p(d\mu_b)$ is continuous.

(ii) $b\in CB^p_s(\mu_t)$ if and only if $(1+R)^{k_s}b\in Mult(B^p_s(\mu_t)\to B^p_{s-k_s}(\mu_t))$, where $Mult(B^p_s(\mu_t)\to B^p_{s-k_s}(\mu_t))$ denotes the space of pointwise multipliers from $B^p_s(\mu_t)$ to $B^p_{s-k_s}(\mu_t)$.

The spaces CB^p_s appear naturally when dealing with some problems on operators on B^p_s. For instance, it is well known that $Mult(B^p_s)=H^\infty\cap CB^p_s$. In some special cases it is not difficult to give a full description of the space CB^p_s. For example, if $s>1/p$, the space B^p_s is a multiplicative algebra and consequently $CB^p_s=B^p_s$. For $s=0$ and $p=2$ we have $CB^2_0=BMOA$ and if $s<0$, then it is easy to check that CB^p_s coincides with the Bloch space B^∞_0. For $0\le s\le 1/p$ there exist characterizations of these spaces given in terms of capacities associated to the space. All these results can be found in [**13**], [**14**], [**16**] and the references therein.

One of the main results of this paper is the following theorem.

THEOREM 1.1. *Let $1 < p < \infty$, $0 < s < 1$, $t \geq 0$ and $\theta \in \mathcal{B}_{p,t}$. For $b \in H(\mathbb{D})$ the following assertions are equivalent*:

- (i) $b \in CB^p_s(\mu_t)$.
- (ii) $\Gamma_1(b) := \sup\limits_{0 \neq f,g \in H(\overline{\mathbb{D}})} \dfrac{|\langle\langle |fg|, |(1+R)b|\rangle\rangle_{t+1}|}{\|f\|_{B^p_s(\mu_t)}\|g\|_{B^{p'}_{-s}(\mu'_t)}} < \infty$.
- (iii) $\Gamma_2(b) := \sup\limits_{0 \neq f,g \in H(\overline{\mathbb{D}})} \dfrac{|\langle fg, b\rangle_t|}{\|f\|_{B^p_s(\mu_t)}\|g\|_{B^{p'}_{-s}(\mu'_t)}} < \infty$.

Moreover, $\|b\|_{CB^p_s(\mu_t)} \approx \Gamma_1(b) \approx \Gamma_2(b)$.

The symbol $\approx$ means here that each term is bounded by constant times the other term, with constants which do not depend on the function b.

If $b \in L^1(d\nu_t)$, then the small Hankel operator $\mathfrak{h}^t_b$, $t \geq 0$, is defined on $H(\overline{\mathbb{D}})$ by

$$\mathfrak{h}^t_b(f)(z) := \int_{\mathbb{D}} f(w)\overline{b(w)}\frac{d\nu_t(w)}{(1-w\overline{z})^{1+t}}, \quad t > 0, \qquad \mathfrak{h}^0_b(f)(z) := \int_{\mathbb{T}} \frac{f(\zeta)\overline{b(\zeta)}}{1-\zeta\overline{z}}d\sigma(\zeta).$$

Notice that, by Fubini's theorem, if $f, g \in H(\overline{\mathbb{D}})$, then $\left\langle g, \overline{\mathfrak{h}^t_b(f)}\right\rangle_t = \langle fg, b\rangle_t$. Thus, we have $\Gamma_2(b) \approx \|\mathfrak{h}^t_b\|_{\mathcal{L}(B^p_s(\mu_t)\to\overline{B^p_s(\mu_t)})}$.

For the unweighted case, the equivalence between (i) and (iii) in Theorem 1.1 has been stated in other reformulations by different authors. See for instance [**12**], [**15**], [**4**] and the references therein.

In [**12**], the authors study the small Hankel operators associated to the inner product $\langle(1+R)f, (1+R)b\rangle_{2-2\alpha}$ in the Besov space B^2_α, $\alpha < 1$. For $p = 2$ the study of the boundedness of such operator is equivalent to the study of the boundedness of the bilinear form $\langle fg, (1+R)b\rangle_{2-2\alpha}$ in $B^2_\alpha \times B^2_{\alpha-1}$. If $\alpha \leq 1/2$ it is easy to check (see Lemma 2.10 below) that this is equivalent to the boundedness of $\langle fg, b\rangle_{1-2\alpha}$ in $B^2_\alpha \times B^2_{\alpha-1}$. Since $B^2_\alpha = B^1_{1/2}(\nu_{1-2\alpha})$ and $B^2_{\alpha-1} = B^1_{-1/2}(\nu_{1-2\alpha})$ (see (1.3)), Theorem 1.1 for $p = 2$, $s = 1/2$ and $t = 1 - 2\alpha$ coincides with the one given in [**12**] and [**14**]. The case $\alpha > 1/2$ follows directly from duality. Indeed, since B^2_α is a multiplicative algebra included in H^∞, it is easy to check that $B^2_\alpha \cdot B^2_\beta = B^2_\beta$ for any $\beta \leq \alpha$. Thus $\langle fg, b\rangle_t$ is bounded on $B^2_\alpha \times B^2_\beta$ if and only if b is in the dual of B^2_β with respect to the pairing $\langle\cdot,\cdot\rangle_t$, that is if and only if $b \in B^2_{-\beta-t}$. This situation does not translate to the weighted case, because it is not clear when $B^2(\mu_t)$ is a multiplicative algebra.

The generalization of these results for small Hankel operators on B^p_α, $\alpha \leq 1/p$, can be found in [**4**]. This corresponds to the case $s = \alpha + (1-2\alpha)/p$ and $t = 1 - 2\alpha$ in Theorem 1.1.

The fact that Theorem 1.1 involves two parameters, s and t, permits to obtain new results, even for the unweighted case. For instance it extends to $p \neq 2$ some results in [**15**], where it is studied the boundedness of the bilinear form $(f, g) \to \langle(1+R)(f\,g), (1+R)b\rangle_{2-\alpha-\beta}$ on $B^2_\alpha \times B^2_\beta$, $\beta < \alpha \leq 1/2$. If $\alpha + \beta < 0$, then $B^2_\alpha \times B^2_\beta = B^2_{(\alpha-\beta)/2}(\nu_{-\alpha-\beta}) \times B^2_{(\beta-\alpha)/2}(\nu_{-\alpha-\beta})$, which corresponds to the case $s = (\alpha-\beta)/2$ and $t = -\alpha-\beta$ in Theorem 1.1, provided $s < 1$.

The techniques used in [**15**] are different to the ones used in [**12**] and [**4**]. Both techniques do not seem to work when studying the above problem for the case $0<\alpha=\beta\le 1/2$. The only result that we know for this situation corresponds to the Dirichlet case, that is $\alpha=\beta=1/2$ (see [**1**] and [**7**]).

In Theorem 1.1, we compute the norm of the bilinear forms on the product $B^p_s(\mu_t)\times B^{p'}_{-s}(\mu'_t)$. However, analogously to the unweighted case, using that the operator $(1+R)^{s'}$ is a bijection from $B^p_s(\mu_t)$ to $B^p_{s-s'}(\mu_t)$, that $\mathcal{B}_{p,t}\subset\mathcal{B}_{p,t+t_0}$, $t_0\ge 0$ and

$$B^p_s(\mu_t)=B^p_{s+t_0/p}(\mu_{t+t_0}), \tag{1.4}$$

we can use Theorem 1.1 to compute norms of bilinear forms on products $B^p_{s_0}(\mu_{t_0})\times B^{p'}_{s_1}(\mu'_{t_1})$ for some particular choices of the indexes s_0, s_1, t_0 and t_1. For instance, we have:

Corollary 1.2. *Let $1<p<\infty$, $t_0,t_1\ge 0$, $\theta\in\mathcal{B}_{p,t_0}$ and $s_0\in\mathbb{R}$. For $s_1\in\mathbb{R}$ satisfying $s_0+s_1<0$ and $0<(s_0/p')-(s_1/p)<1$, let $t=t_0-s_0-s_1$.*

Then we have

$$\|R^{t-t_1}_{1+t}b\|_{CB^p_{s_0/p'-s_1/p}(\mu_t)}\approx\sup_{0\neq f,g\in H(\overline{\mathbb{D}})}\frac{|\langle fg,b\rangle_{t_1}|}{\|f\|_{B^p_{s_0}(\mu_{t_0})}\|g\|_{B^{p'}_{s_1}(\mu'_{t_0})}},$$

where $R^{t-t_1}_{1+t}$ is a fractional differential operator of order $t-t_1$ (see (2.6) below).

For $s_0,s_1<0$, we have the following result:

Theorem 1.3. *If $1<p<\infty$, $t\ge 0$, $\theta\in\mathcal{B}_{p,t}$ and $s_0,s_1<0$, then*

$$\|b\|_{B^\infty_{-s_0-s_1}}\approx\sup_{0\neq f,g\in H(\overline{\mathbb{D}})}\frac{|\langle fg,b\rangle_t|}{\|f\|_{B^p_{s_0}(\mu_t)}\|g\|_{B^{p'}_{s_1}(\mu_t)}}.$$

Here, $B^\infty_s:=(1+R)^{-s}B^\infty_0$. In particular, if $s>0$, B^∞_s is the holomorphic Lipschitz-Zygmund space $H\cap\Lambda_s$.

As it happens in the unweighted case (see for instance [**14**], [**8**] and [**2**] for $p=2$), Theorems 1.1 and 1.3 give the following duality result for weak products.

Recall that the weak product $F\odot G$ of two Banach spaces of functions F and G consists of the completion of finite sums $h=\sum f_jg_j$ using the norm

$$\|h\|_{F\odot G}:=\inf\Big\{\sum\|f_j\|_F\|g_j\|_G:\sum f_jg_j=h\Big\}.$$

Theorem 1.4. *Let $1<p<\infty$, $t\ge 0$ and $\theta\in\mathcal{B}_{p,t}$. If we consider the pairing $\langle\cdot,\cdot\rangle_t$, we then have*:

- (i) *If $0<s<1$, then $(B^p_s(\mu_t)\odot B^{p'}_{-s}(\mu'_t))'\equiv CB^p_s(\mu_t)$.*
- (ii) *If $s_0,s_1<0$, then $\big(B^p_{s_0}(\mu_t)\odot B^{p'}_{s_1}(\mu'_t)\big)'\equiv B^\infty_{-s_0-s_1}$, and consequently, we have $B^p_{s_0}(\mu_t)\odot B^{p'}_{s_1}(\mu'_t)=B^1_{s_0+s_1-t}$.*

The same arguments used to prove Corollary 1.2 from Theorem 1.1, combining the above theorem with (1.4), give a description of the dual of $B^p_{s_0}(\mu_{t_0}) \odot B^{p'}_{s_1}(\mu'_{t_0})$ for s_0, s_1 and t_0 satisfying the conditions in Corollary 1.2. This description covers some results stated in [**14**] and in Section 5 in [**8**] for the unweighted case.

The paper is organized as follows. In Section 2 we give some definitions and we state some properties of the class of weights in $\mathcal{B}_{p,t}$ and its corresponding weighted Besov spaces. In Section 3 we obtain estimates of $\|b\|_{CB^p_s(\mu_t)}$ which in particular give the proof of Theorem 1.3. Section 4 is devoted to the proof of Theorem 1.1 and Corollary 1.2. In Section 5, we use our previous results to prove Theorem 1.4.

2. Notations and preliminaries.

Throughout this paper, the expression $F \lesssim G$ means that there exists a positive constant C independent of the essential variables and such that $F \le CG$. If $F \lesssim G$ and $G \lesssim F$ we will write $F \approx G$.

2.1. Differential and integral operators.

We denote the partial derivatives of first order by $\partial := \partial/\partial z$ and $\overline{\partial} := \partial/\partial \overline{z}$ respectively . Let $R := z\partial$ be the radial derivative.

For $s, t \in \mathbb{R}$, $t > 0$ and k a non-negative integer, we consider the differential operator R^k_t of order k defined by

$$R^k_t f := \left(1 + \frac{R}{t+k-1}\right) \cdots \left(1 + \frac{R}{t}\right) f.$$

If we need to specify the variable of differentiation, then we write ∂_z, R_z and $R^k_{t,z}$, respectively.

The operators R^k_t satisfy the following formula:

$$R^k_t \frac{1}{(1 - z\overline{w})^t} = \frac{1}{(1 - z\overline{w})^{t+k}}. \tag{2.5}$$

DEFINITION 2.1. For $N > 0$ and $M \ge 0$, we consider the following integral operators:

$$\mathcal{P}^{N,M}(\varphi)(z) := \int_{\mathbb{D}} \varphi(w) \mathcal{P}^{N,M}(z,w)\, d\nu(w), \quad \text{where} \quad \mathcal{P}^{N,M}(z,w) := N \frac{(1 - |w|^2)^{N-1}}{(1 - z\overline{w})^{1+M}}.$$

$$\mathbb{P}^{N,M}(\varphi)(z) := \int_{\mathbb{D}} \varphi(w) \mathbb{P}^{N,M}(z,w)\, d\nu(w), \quad \text{where} \quad \mathbb{P}^{N,M}(z,w) := |\mathcal{P}^{N,M}(z,w)|.$$

We extend the definition to the case $N = 0$ by writing

$$\mathcal{P}^{0,M}(\varphi)(z) := \int_{\mathbb{T}} \frac{\varphi(\zeta)}{(1 - z\overline{\zeta})^{1+M}} d\sigma(\zeta), \quad \mathbb{P}^{0,M}(\varphi)(z) := \int_{\mathbb{T}} \frac{\varphi(\zeta)}{|1 - z\overline{\zeta}|^{1+M}} d\sigma(\zeta).$$

If $N = M$, then we denote $\mathcal{P}^{N,N}$ and $\mathbb{P}^{N,N}$ by $\mathcal{P}^N$ and $\mathbb{P}^N$, respectively.

For $N \geq 0$, we also define

$$\mathcal{K}^N(\overline{\partial}\varphi)(z) := \int_{\mathbb{D}} \overline{\partial}\varphi(w)\mathcal{K}^N(w,z)d\nu(w), \quad \text{where} \quad \mathcal{K}^N(w,z) := \frac{(1-|w|^2)^N}{(1-z\overline{w})^N}\frac{1}{w-z}.$$

The weighted Cauchy-Pompeiu representation formula is given by:

Theorem 2.2. *Let $N \geq 0$ and $\varphi \in C^1(\overline{\mathbb{D}})$. Then $\varphi(z) = \mathcal{P}^N(\varphi)(z) + \mathcal{K}^N(\overline{\partial}\varphi)(z)$.*

Since $R^k_{1+N}f = R^k_{1+N}\mathcal{P}^N(f) = \mathcal{P}^{N,N+k}(f)$, it is natural to extend the definition of R^k_t for a non-integer order by considering

$$R^s_{1+N}f := \mathcal{P}^{N,N+s}(f), \qquad s,\, N > 0. \tag{2.6}$$

Note that by Theorem 2.2 we have

$$\int_{\mathbb{D}} \mathcal{P}^{N+s,N}(w,z)\mathcal{P}^{N,N+s}(u,w)d\nu(w) = \mathcal{P}^N(u,z).$$

Therefore, for $s > 0$ we can define the inverse of R^s_{1+N} by $R^{-s}_{1+N}f := \mathcal{P}^{N+s,N}(f)$.

Let us recall the following estimate.

Lemma 2.3. *If $q < 2$, $N > 0$, $M \neq N - q$ and $z \in \mathbb{D}$, then*

$$\int_{\mathbb{D}} \frac{\mathbb{P}^{N,M}(w,z)}{|w-z|^q}d\nu(w) \lesssim (1 + (1-|z|^2)^{N-M-q}).$$

Proof. The case $q = 0$ is well known (see for instance [**17**, Lemma 4.2.2]). The case $q \neq 0$ can be reduced to $q = 0$ using the change of variables $w = \varphi_z(u) := (z - u)/(1 - u\overline{z})$. Indeed, we have

$$\int_{\mathbb{D}} \frac{\mathbb{P}^{N,M}(w,z)}{|w-z|^q}d\nu(w) = (1-|z|^2)^{N-M-q}\int_{\mathbb{D}} \frac{(1-|u|^2)^{N-1}}{|1-u\overline{z}|^{1+2N-M-q}}\frac{d\nu(u)}{|u|^q},$$

which ends the proof. □

2.2. Békollé weights.

In this section we recall some properties of the Békollé weights $\mathcal{B}_{p,t}$. We refer to [**3**] for more details. Recall that if $t > 0$ and $\theta \in \mathcal{B}_{p,t}$, then $d\mu_t = \theta d\nu_t$ and $d\mu'_t = \theta^{-p'/p}d\nu_t$.

Since, for any $w \in T_z$, $1 - |w|^2 \leq 4(1-|z|^2)$, we have:

Lemma 2.4. *If $1 < p < \infty$, $0 < t_0 < t_1$ and $\theta \in \mathcal{B}_{p,t_0}$, then $\mathcal{B}_{p,t_1}(\theta) \lesssim \mathcal{B}_{p,t_0}(\theta)$. Thus, $\mathcal{B}_{p,t_0} \subset \mathcal{B}_{p,t_1}$.*

The next result was proved in [**3**, Theorem 1 and Propositions 3, 5]

Theorem 2.5. *Let $1 < p < \infty$, $t > 0$ and let θ be a positive locally integrable function θ on $\mathbb{D}$. Then, the following assertions are equivalent*:

(i) $\theta \in \mathcal{B}_{p,t}$.
(ii) *The integral operator* $\mathbb{P}^t$ *is bounded on* $L^p(d\mu_t)$.
(iii) *The integral operator* $\mathcal{P}^t$ *is bounded on* $L^p(d\mu_t)$.

It is well known that any weight in the Muckenhoupt class A_p satisfies a doubling condition. Similarly to what happens for these classes of weights, any weight in $\mathcal{B}_{p,t}$ satisfies a doubling type condition with respect to tents. We also have a characterization of weights in $\mathcal{B}_{p,t}$ in terms of the kernels $\mathbb{P}^{t,M}$, which is analogous to the one satisfied for the weights in A_p (see [**11**], [**5**]). This is the content of the following proposition.

Proposition 2.6. *Let* $1 < p < \infty$, $t > 0$ *and* $\theta \in \mathcal{B}_{p,t}$. *We then have*:

(i) *The measure* μ_t *satisfies the following doubling type condition*: *if* $0 < r_1 < r_2 < 1$ *and* $\zeta \in \mathbb{T}$, *then*

$$\frac{\mu_t(T_{r_1\zeta})}{\mu_t(T_{r_2\zeta})} \le \mathcal{B}_{p,t}(\theta)^p \left(\frac{\nu_t(T_{r_1\zeta})}{\nu_t(T_{r_2\zeta})}\right)^p \approx \mathcal{B}_{p,t}(\theta)^p \left(\frac{1-r_1}{1-r_2}\right)^{(1+t)p}.$$

(ii) *If* $M > (1+t)(\max\{p,p'\}-1)$, *the following equivalence holds*:

$$\mathcal{B}_{p,t}(\theta) \lesssim \sup_{z\in\mathbb{D}} (1-|z|^2)^M \left(\mathbb{P}^{t,t+M}(\theta)(z)\right)^{1/p} \left(\mathbb{P}^{t,t+M}(\theta^{-p'/p})(z)\right)^{1/p'} \lesssim \mathcal{B}_{p,t}(\theta)^2.$$

Proof. Part (i) follows easily from Hölder's inequality and the fact that $\theta \in \mathcal{B}_{p,t}$. Indeed, the embedding $T_{r_2\zeta} \subset T_{r_1\zeta}$ gives

$$\nu_t(T_{r_2\zeta}) \le \left(\int_{T_{r_2\zeta}} d\mu_t\right)^{1/p} \left(\int_{T_{r_1\zeta}} d\mu_t'\right)^{1/p'} \le \mu_t(T_{r_2\zeta})^{1/p} \frac{\mathcal{B}_{p,t}(\theta)\ \nu_t(T_{r_1\zeta})}{(\mu_t(T_{r_1\zeta}))^{1/p}}.$$

Since $\nu_t(T_{r\zeta}) \approx (1-r)^{1+t}$, we conclude the proof.

In order to prove (ii) it is enough to prove the following estimates, valid for $z \in \mathbb{D}$:

$$\frac{\mu_t(T_z)}{\nu_t(T_z)} \lesssim (1-|z|^2)^M \mathbb{P}^{t,t+M}(\theta)(z) \lesssim \mathcal{B}_{p,t}(\theta)^p \frac{\mu_t(T_z)}{\nu_t(T_z)}, \tag{2.7}$$

$$\frac{\mu_t'(T_z)}{\nu_t(T_z)} \lesssim (1-|z|^2)^M \mathbb{P}^{t,t+M}(\theta^{-p'/p})(z) \lesssim \mathcal{B}_{p,t}(\theta^{-p'/p})^{p'} \frac{\mu_t'(T_z)}{\nu_t(T_z)}. \tag{2.8}$$

Observe that (2.8) follows from (2.7) since $\theta \in \mathcal{B}_{p,t}$ if and only if $\theta^{-p'/p} \in \mathcal{B}_{p',t}$.

The estimate on the left hand side of (2.7) is valid for any $M > 0$ and $t > 0$, and follows from

$$\begin{aligned}\frac{\mu_t(T_z)}{\nu_t(T_z)} = \frac{1}{\nu_t(T_z)} \int_{T_z} \theta d\nu_t \lesssim (1-|z|^2)^M \int_{T_z} \frac{\theta(w)}{|1-w\bar{z}|^{1+t+M}} d\nu_t(w) \\ = (1-|z|^2)^M \mathbb{P}^{t,t+M}(\theta)(z).\end{aligned}$$

Let us prove the estimate on the right hand side of (2.7). If $z = 0$ then $T_0 = \mathbb{D}$ and thus the result is clear. If $z \neq 0$ then let $\zeta = z/|z|$ and J_z the integer part of $-\log_2(1-|z|)$. Consider the sequence $\{z_k\} \subset \mathbb{D}$ defined by

$$z_k = (1 - 2^k(1-|z|))\zeta \text{ if } k = 0, 1, \dots, J_z, \quad \text{and} \quad z_k = 0 \ \text{ if } k > J_z.$$

Observe that $z_0 = z$ and that $1 - |z_k|^2 \approx |1 - w\overline{z}|$ for $w \in T_{z_k} \setminus T_{z_{k-1}}$. Therefore,

$$\begin{aligned}(1-|z|^2)^M \mathbb{P}^{t,t+M}(\theta)(z) &= (1-|z|^2)^M \sum_{k=0}^{J_z+1} \int_{T_{z_k} \setminus T_{z_{k-1}}} \frac{\theta(w)\, d\nu_t(w)}{|1-w\overline{z}|^{1+t+M}} \\ &\lesssim \sum_{k=0}^{J_z+1} \frac{(1-|z|^2)^M}{(2^k(1-|z|^2))^{1+t+M}} \mu_t(T_{z_k}).\end{aligned}$$

By the doubling property (i), we have

$$\mu_t(T_{z_k}) \lesssim \mathcal{B}_{p,t}(\theta)^p \frac{(1-|z_k|)^{(1+t)p}}{(1-|z|)^{(1+t)p}} \mu_t(T_z) \approx \mathcal{B}_{p,t}(\theta)^p 2^{k(1+t)p} \mu_t(T_z).$$

Since $M > (1+t)(p-1)$ and $\nu_t(T_z) \approx (1-|z|^2)^{1+t}$ we obtain

$$(1-|z|^2)^M \mathbb{P}^{t,t+M}(\theta)(z) \lesssim \mathcal{B}_{p,t}(\theta)^p \frac{\mu_t(T_z)}{\nu_t(T_z)},$$

which concludes the proof of the right hand side estimate in (2.7). □

As a consequence of the above proposition and the estimate $1 - |w|^2 \leq 2|1 - z\overline{w}|$, we obtain:

Corollary 2.7. *If $1 < p < \infty$, $t \geq 0$, $N > 0$, $M > (1+t+N)(\max\{p, p'\} - 1)$ and $\theta \in B_{p,t}$, then*

$$\sup_{z \in \mathbb{D}} (1-|z|^2)^M \left(\mathbb{P}^{t+N,t+N+M}(\theta)(z)\right)^{1/p} \left(\mathbb{P}^{t+N,t+N+M}(\theta^{-p'/p})(z)\right)^{1/p'} \lesssim \mathcal{B}_{p,t}(\theta)^2.$$

2.3. Weighted Besov spaces.

In this section we recall some properties of the weighted Besov spaces $B^p_s(\mu_t)$ introduced in Section 1.

The next result is well known for the unweighted case (see for instance [**18**, Chapters 2, 6]). The proof for the weighted Besov spaces can be done following the same arguments used to prove Theorem 3.1 in [**6**].

Proposition 2.8. *Let $1 < p < \infty$, $s \in \mathbb{R}$, $t \geq 0$ and $\theta \in \mathcal{B}_{p,t}$. If $k > s$ is a non-negative integer, then*

$$\int_{\mathbb{D}} |D^k f(z)|^p (1-|z|^2)^{(k-s)p} d\mu_t(z) \quad \textit{and} \quad \sum_{m=0}^{k} \int_{\mathbb{D}} |\partial^k f(z)|^p (1-|z|^2)^{(k-s)p} d\mu_t(z)$$

provide equivalent norms on $B^p_s(\mu_t)$*, where* D^k *is either* $(1+R)^k$ *or* R^k_L*.*

The next embedding relates weighted and unweighted Besov spaces.

Lemma 2.9. *If* $1<p<\infty$*,* $s\in\mathbb{R}$*,* $t\geq 0$ *and* $\theta\in\mathcal{B}_{p,t}$*, then* $B^p_s(\mu_t)\subset B^1_{s-t}$*.*

Proof. Since for any positive integer k we have $B^p_s(\mu_t)=(1+R)^{-k}B^p_{s-k}(\mu_t)$ and $B^1_{s-t}=(1+R)^{-k}B^1_{s-t-k}$, it is sufficient to prove the above embedding for $s<0$.

In this case, Hölder's inequality gives

$$\|f\|_{B^1_{s-t}} \leq \Big(\int_{\mathbb{D}} |f|^p d\mu_{t-sp}\Big)^{1/p} \Big(\int_{\mathbb{D}} d\mu'_t\Big)^{1/p'},$$

which proves the result. □

In order to state a duality relation between weighted Besov spaces, we need the next lemma.

Lemma 2.10. *The pairing* $\langle\cdot,\cdot\rangle_\delta$ *defined in* (1.2) *satisfies that for* $f,g\in H(\overline{\mathbb{D}})$*:*

- (i) $\langle f,g\rangle_\delta=\langle f,R^k_{\delta+1}g\rangle_{\delta+k}=\langle R^k_{\delta+1}f,g\rangle_{\delta+k}$.
- (ii) *If* $\tau\in\mathbb{R}$ *then we have* $\langle f,g\rangle_\delta=\langle(1+R)^\tau f,(1+R)^{-\tau}g\rangle_\delta$.

Proof. Let us prove (i) for $k=1$, that is

$$\langle f,g\rangle_\delta=\Big\langle f,\Big(1+\frac{R}{\delta+1}\Big)g\Big\rangle_{\delta+1}=\Big\langle\Big(1+\frac{R}{\delta+1}\Big)f,g\Big\rangle_{\delta+1}.$$

Observe that the second equality can be deduced from the first one by conjugation.

If $\delta=0$, then Stokes' theorem gives

$$\begin{aligned}\langle f,g\rangle_0&=\frac{1}{2\pi i}\lim_{r\to1^-}\int_{\mathbb{T}} f(r\zeta)\overline{g(r\zeta)}\,\overline{\zeta}d\zeta=\lim_{r\to1^-}\int_{\mathbb{D}}\overline{\partial}\big(\overline{z}f(rz)\overline{g(rz)}\big)\,d\nu(z)\\&=\lim_{r\to1^-}\int_{\mathbb{D}} f(rz)\overline{((1+R)g)(rz)}\,d\nu(z)=\langle f,(1+R)g\rangle_1.\end{aligned}$$

The case $\delta>0$ follows from the identity

$$\delta(1-|z|^2)^{\delta-1}=(\delta+1)(1-|z|^2)^\delta-\overline{\partial}(\overline{z}(1-|z|^2)^\delta),$$

and integration by parts.

A simple iteration of these identities gives (i).

Assertion (ii) follows from the facts that $(1+R)^\tau z^m=(1+m)^\tau z^m$ and that

$\langle z^k, z^m\rangle_\delta = 0$, $k \neq m$. □

The next result extends the well known duality $(B^p_s)' \equiv B^{p'}_{-s}$ for the case $t = 0$ (see [**9**]).

PROPOSITION 2.11. *Let $1 < p < \infty$, $t \geq 0$ and $\theta \in \mathcal{B}_{p,t}$. If $s \in \mathbb{R}$, then, the dual of $B^p_s(\mu_t)$ with respect to the pairing $\langle\cdot,\cdot\rangle_t$ is the Besov space $B^{p'}_{-s}(\mu'_t)$.*

PROOF. As in the unweighted case, from the duality $(L^p(\mu_t))' \equiv L^{p'}(\mu'_t)$, with respect to the pairing $\langle\langle\cdot,\cdot\rangle\rangle_{t+1}$, Theorem 2.5 and the Hahn-Banach theorem, we obtain

$$\left(B^p_{-1/p}(\mu_t)\right)' = (H \cap L^p(\mu_t))' \equiv H \cap L^{p'}(\mu'_t) = B^{p'}_{-1/p'}(\mu'_t),$$

with respect to the pairing $\langle\langle\cdot,\cdot\rangle\rangle_{t+1}$, and consequently with respect to the pairing $\langle\cdot,\cdot\rangle_{t+1}$.

Next, we use the above result and Lemma 2.10 to prove the general case.

If $g \in B^{p'}_{-s}(\mu'_t)$ and $f \in B^p_s(\mu_t)$, then

$$|\langle f, g\rangle_t| = |\langle R^1_t f, g\rangle_{t+1}| \leq \|g\|_{L^{p'}(\mu'_{sp'+t})}\|R^1_t f\|_{L^p(\mu_{(1-s)p+t})} \approx \|g\|_{B^{p'}_{-s}(\mu'_t)}\|f\|_{B^p_s(\mu_t)}.$$

Thus, the map $g \to \langle\cdot, g\rangle_t$ is an injective map from $B^{p'}_{-s}(\mu'_t)$ to $(B^p_s(\mu_t))'$.

Let us prove that this map is surjective. If Λ is a linear form on $B^p_s(\mu_t)$, then $\Lambda \circ (1+R)^{-s-1/p}$ is also a linear form on $B^p_{-1/p}(\mu_t)$. Thus, there exists $g \in B^{p'}_{-1/p'}(\mu'_t)$ such that for any $h \in B^p_{-1/p}(\mu_t)$,

$$\begin{aligned}\Lambda \circ (1+R)^{-s-1/p}(h) &= \langle h, g\rangle_{t+1} = \langle (1+R)^{-s-1/p}h, (1+R)^{s+1/p}g\rangle_{t+1}\\ &= \langle (1+R)^{-s-1/p}h, R^{-1}_{1+t}(1+R)^{s+1/p}g\rangle_t,\end{aligned}$$

where in the second identity we have used (ii) in Lemma 2.10 and in the last one (i) in the same lemma.

Since for any $f \in B^p_s(\mu_t)$, we have that $h = (1+R)^{s+1/p}(f) \in B^p_{-1/p}(\mu_t)$, we deduce that $\Lambda(f) = \langle f, G\rangle_t$ with $G := R^{-1}_{1+t}(1+R)^{s+1/p}g \in B^{p'}_{-s}(\mu'_t)$. □

COROLLARY 2.12. *Let $1 < p < \infty$, $t' > t \geq 0$ and $\theta \in \mathcal{B}_{p,t}$. If $s \in \mathbb{R}$, then $(B^p_s(\mu_t))' = B^{p'}_{-s+t-t'}(\mu'_t)$ with respect to the pairing $\langle\cdot,\cdot\rangle_{t'}$.*

In particular, if $t = 0$, then $(B^p_s)' \equiv B^{p'}_{-s-t'}$, with respect to the pairing $\langle\cdot,\cdot\rangle_0$.

PROOF. By the above proposition, we have

$$(B^p_s(\mu_t))' \equiv \left(B^p_{s+(t'-t)/p}(\mu_{t'})\right)' \equiv \left(B^{p'}_{-s-(t'-t)/p}(\mu'_{t'})\right) = \left(B^{p'}_{-s+t-t'}(\mu'_t)\right)$$

which ends the proof. □

3. Estimates of $\|b\|_{CB^p_s(\mu_t)}$ and proof of Theorem 1.3.

We introduce a variation in the definition of the constants $\Gamma_1(b)$ and $\Gamma_2(b)$ in Theorem 1.1, which allow us to cover some general situations.

DEFINITION 3.1. If $1<p<\infty$, $s_0, s_1\in\mathbb{R}$, $t\geq 0$, $\theta\in\mathcal{B}_{p,t}$ and $b\in H$, then

$$\Gamma_3(b)=\Gamma(b,p,s_0,s_1,t):=\sup_{0\neq f,g\in H(\overline{B})}\frac{|\langle fg,b\rangle_t|}{\|f\|_{B^p_{s_0}(\mu_t)}\,\|g\|_{B^{p'}_{s_1}(\mu'_t)}}.$$

We will start proving the following theorem.

THEOREM 3.2. *Let* $1<p<\infty$*,* $s_0,s_1\in\mathbb{R}$*,* $t\geq 0$ *and* $\theta\in\mathcal{B}_{p,t}$*. Then* $\|b\|_{B^\infty_{-s_0-s_1}}\lesssim\Gamma_3(b)$*.*

If $s_0,s_1<0$*, then the converse inequality holds.*

The proof of this result will be a consequence of Lemmas 3.4 and 3.6.

LEMMA 3.3. *Let* $1<p<\infty$*,* $s_0,s_1\in\mathbb{R}$*,* $t\geq 0$ *and* $\theta\in\mathcal{B}_{p,t}$*. Let*

$$\tau>\lambda:=(1+t)(\max\{p,p'\}-1)+\max\{0,-s_0p,-s_1p'\}. \tag{3.9}$$

For $z\in\mathbb{D}$*, we consider the functions*

$$f_z(w)=\frac{1}{(1-w\overline{z})^{(1+t+\tau)/p}}\quad\textit{and}\quad g_z(w)=\frac{1}{(1-w\overline{z})^{(1+t+\tau)/p'}}.$$

Then

$$\|f_z\|_{B^p_{s_0}(\mu_t)}\,\|g_z\|_{B^{p'}_{s_1}(\mu'_t)}\lesssim\mathcal{B}_{p,t}(\theta)^2(1-|z|^2)^{-\tau-s_0-s_1}.$$

PROOF. If $m>s_0$ is a non-negative integer, then

$$\|f_z\|^p_{B^p_{s_0}(\mu_t)}\approx\int_{\mathbb{D}}\frac{(1-|w|^2)^{t+(m-s_0)p-1}}{|1-z\overline{w}|^{1+t+\tau+mp}}\theta(w)d\nu(w).$$

Analogously, if $m>s_1$, then

$$\|g_z\|^{p'}_{B^{p'}_{s_1}(\mu'_t)}\approx\int_{\mathbb{D}}\frac{(1-|w|^2)^{t+(m-s_1)p'-1}}{|1-z\overline{w}|^{1+t+\tau+mp'}}\theta^{-p'/p}(w)d\nu(w).$$

Therefore, if N,M satisfy $0<N<\min\{(m-s_0)p,(m-s_1)p'\}$ and $(1+t+N)(\max\{p,p'\}-1)<M<\min\{k\tau+s_0p,\tau+s_1p'\}$, then the estimate $1-|z|^2\leq 2|1-w\overline{z}|$ and Corollary 2.7 give

$$
\begin{aligned}
&\|f_z\|_{B^p_{s_0}(\mu_t)}\|g_z\|_{B^{p'}_{s_1}(\mu'_t)}\\
&\quad\lesssim (1-|z|^2)^{M-\tau-s_0-s_1}(\mathbb{P}^{t+N,t+N+M}(\theta)(z))^{1/p}\big(\mathbb{P}^{t+N,t+N+M}(\theta^{-p'/p})(z)\big)^{1/p'}\\
&\quad\lesssim \mathcal{B}_{p,t}(\theta)^2(1-|z|^2)^{-\tau-s_0-s_1},
\end{aligned}
$$

which ends the proof. □

Lemma 3.4. *Let* $1<p<\infty$, $s_0,s_1\in\mathbb{R}$, $t\geq 0$, $\theta\in\mathcal{B}_{p,t}$ *and* $b\in H$. *Then* $\|b\|_{B^{\infty}_{-s_0-s_1}}\lesssim\Gamma_3(b)$.

Proof. We want to prove that for some positive integer k, we have

$$\|b\|_{B^{\infty}_{-s_0-s_1}}\approx\sup_{z\in\mathbb{D}}(1-|z|^2)^{k+s_0+s_1}|R^k_{1+t}b(z)|\lesssim\Gamma_3(b).$$

By Cauchy formula, we have

$$
\begin{aligned}
R^k_{1+t}b(z)&=t\lim_{r\to1^-}R^k_{1+t}\int_{\mathbb{D}}b(rw)\frac{(1-|w|^2)^{t-1}}{(1-rz\overline{w})^{1+t}}d\nu(w)\\
&=t\lim_{r\to1^-}\int_{\mathbb{D}}b(rw)\frac{(1-|w|^2)^{t-1}}{(1-rz\overline{w})^{1+t+k}}d\nu(w).
\end{aligned}
$$

Assume that k is a positive integer satisfying (3.9), and let

$$f_z(w)=\frac{1}{(1-w\overline{z})^{(1+t+k)/p}}\quad\text{and}\quad g_z(w)=\frac{1}{(1-w\overline{z})^{(1+t+k)/p'}}.$$

Since $|R^k_{1+t}g(z)|=|\langle f_zg_z,b\rangle_t|$, Lemma 3.3 gives

$$|R^k_{1+t}b(z)|\leq\Gamma_3(b)\|f_z\|_{B^p_{s_0}(\mu_t)}\,\|g_z\|_{B^{p'}_{s_1}(\mu'_t)}\lesssim\Gamma_3(b)(1-|z|^2)^{-k-s_0-s_1},$$

which concludes the proof. □

Corollary 3.5. *Let* $1<p<\infty$ *and* $0<s<1$. *If* b *satisfies condition* (iii) *in Theorem* 1.1, *that is* $\Gamma_3(b,p,s,-s,t)<\infty$, *then* $b\in B^p_s(\mu_t)\cap B^{\infty}_0$.

Proof. The above lemma gives $b\in B^{\infty}_0$. The fact that $b\in B^p_s(\mu_t)$ follows from the estimate $|\langle g,b\rangle_t|\leq C_b\|1\|_{B^p_s(\mu_t)}\|g\|_{B^{p'}_{-s}(\mu'_t)}$ and the duality result in Proposition 2.11. □

Lemma 3.6. *If* $1<p<\infty$ *and* $s_0,s_1<0$, *then* $\Gamma_3(b)\lesssim\|b\|_{B^{\infty}_{-s_0-s_1}}$.

Proof. Let k be a positive integer such that $k>-s_0-s_1$. Then

$$\begin{aligned}|\langle fg,b\rangle_t| &= |\langle fg,R^k_{1+t}b\rangle_{t+k}| \lesssim \|b\|_{B^\infty_{-s_0-s_1}}\|fg\|_{L^1(d\nu_{t-s_0-s_1})}\\ &\le \|b\|_{B^\infty_{-s_0-s_1}}\|f\|_{L^p(\theta d\nu_{t-s_0p})}\|g\|_{L^{p'}(\theta^{-p'/p}d\nu_{t-s_1p'})}\\ &\approx \|b\|_{B^\infty_{-s_0-s_1}}\|f\|_{B^p_{s_0}(\mu_t)}\|g\|_{B^{p'}_{s_1}(\mu'_t)},\end{aligned}$$

which ends the proof. □

PROOF OF THEOREM 1.3. The proof is an immediate consequence of Lemmas 3.4 and 3.6. □

THEOREM 3.7. *Let $1<p<\infty$, $s<1$, $t\ge 0$ and $\theta\in\mathcal{B}_{p,t}$. Then, $CB^p_s(\mu_t)\subset B^p_s(\mu_t)\cap B^\infty_0$. If $s<0$, then $CB^p_s(\mu_t)=B^\infty_0$.*

PROOF. The first inclusion follows from the same arguments used to prove Lemma 3.4. For a non-negative integer $k>s$ which we precise later, we have

$$\begin{aligned}&|R^k_{1+t+(1-s)p}(I+R)b(z)| = |R^k_{1+t+(1-s)p}\mathcal{P}^{t+(1-s)p}((I+R)b)(z)|\\ &\quad= |\mathcal{P}^{t+(1-s)p,t+(1-s)p+k}((I+R)b)(z)|\\ &\quad\le \left(\int_{\mathbb{D}}\frac{(1-|w|^2)^{(1-s)p+t-1}|(I+R)b(w)|^p}{|1-w\overline{z}|^{1+t+(1-s)p+k}}\theta(w)d\nu(w)\right)^{1/p}\\ &\qquad\cdot\left(\int_{\mathbb{D}}\frac{(1-|w|^2)^{(1-s)p+t-1}}{|1-w\overline{z}|^{1+t+(1-s)p+k}}\theta^{-p'/p}(w)d\nu(w)\right)^{1/p'}\\ &\quad\lesssim \|b\|_{CB^p_s(\mu_t)}\left(\int_{\mathbb{D}}\frac{(1-|w|^2)^{(1-s)p+t-1}}{|1-w\overline{z}|^{1+t+(1-s)p+k+p}}\theta(w)d\nu(w)\right)^{1/p}\\ &\qquad\cdot\left(\int_{\mathbb{D}}\frac{(1-|w|^2)^{(1-s)p+t-1}}{|1-w\overline{z}|^{1+t+(1-s)p+k}}\theta^{-p'/p}(w)d\nu(w)\right)^{1/p'}\\ &\quad\le \|b\|_{CB^p_s(\mu_t)}(1-|z|^2)^{-1}\left(\mathbb{P}^{t+(1-s)p,t+(1-s)p+k}(\theta)(z)\right)^{1/p}\\ &\qquad\cdot\left(\mathbb{P}^{t+(1-s)p,t+(1-s)p+k}(\theta^{-p'/p})(z)\right)^{1/p'}.\end{aligned}$$

If $k>(1+t+(1-s)p)(\max\{p,p'\}-1)$, then Corollary 2.7 with $N=(1-s)p$ and $M=k$, gives $|R^k_{1+t+(1-s)p}(I+R)b(z)|\lesssim\|b\|_{CB^p_s(\mu_t)}(1-|z|^2)^{-1-k}$ which proves that $b\in B^\infty_0$.

Next, if $s<0$, then we have $k_s=1$ and the inequality $\|b\|_{CB^p_s(\mu_t)}\lesssim\|b\|_{B^\infty_0}$ follows from

$$\int_{\mathbb{D}}|f(z)|^p\,|(1+R)b(z)|^p\,(1-|z|^2)^{(1-s)p}\,d\mu_t(z)\lesssim\|b\|^p_{B^\infty_0}\|f\|^p_{B^p_s(\mu_t)},$$

which concludes the proof. □

Remark 3.8. Observe that if $0 < s < 1$, $0 < \varepsilon < 1 - s$ and $\|g\|_{B^\infty_{s+\varepsilon-1}} < \infty$, then

$$\|gf\|_{B^p_{s-1}(\mu_t)} \lesssim \|g\|_{B^\infty_{s+\varepsilon-1}} \|f\|_{B^p_{-\varepsilon}(\mu_t)} \lesssim \|g\|_{B^\infty_{s+\varepsilon-1}} \|f\|_{B^p_s(\mu_t)}.$$

Therefore, $g \in Mult(B^p_s(\mu_t) \to B^p_{s-1}(\mu_t))$. In particular,

$$B^\infty_0 \subset B^\infty_{s+\varepsilon-1} \subset Mult(B^p_s(\mu_t) \to B^p_{s-1}(\mu_t)).$$

This gives that $g \in CB^p_s(\mu_t)$ if and only if for some (any) $l > 0$, $(l + R)g \in Mult(B^p_s(\mu_t) \to B^p_{s-1}(\mu_t))$.

4. Proof of Theorem 1.1 and Corollary 1.2.

4.1. Proof of (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) in Theorem 1.1.

The fact that (i) $\Longrightarrow$ (ii) is a consequence of Hölder's inequality. Indeed, since $0 < s < 1$, we have

$$\begin{aligned} \langle\langle |fg|, |(1+R)b| \rangle\rangle_{t+1} &\le \|g\|_{L^{p'}(\theta^{-p'/p} d\nu_{sp'+t})} \|f(1+R)b\|_{L^p(\theta d\nu_{(1-s)p+t})} \\ &\le \|g\|_{B^{p'}_{-s}(\mu'_t)} \|f\|_{B^p_s(\mu_t)} \|b\|_{CB^p_s(\mu_t)}. \end{aligned}$$

Clearly (ii) $\Longrightarrow$ (iii) is a consequence of Lemma 2.10 (i). Indeed, if $|\langle\langle |fg|, |(1+R)b| \rangle\rangle_{t+1}| < \infty$ for any $f, g \in H(\overline{\mathbb{D}})$, then by Corollary 3.5 (see also Remark 3.8) we have $|\langle\langle |fg|, |R^1_{t+1} b| \rangle\rangle_{t+1}| < \infty$. Thus

$$|\langle fg, b \rangle_t| = |\langle fg, R^1_{t+1} b \rangle_{t+1}| \le |\langle\langle |fg|, |R^1_{t+1} b| \rangle\rangle_{t+1}|.$$

which concludes the proof.

Observe that if $b \in CB^p_s(\mu_t)$, the above estimates give

$$|\langle fg, b \rangle_t| \le \|b\|_{CB^p_s(\mu_t)} \|f\|_{B^p_s(\mu_t)} \|g\|_{B^{p'}_{-s}(\mu'_t)}. \tag{4.10}$$

Thus we have $\Gamma_2(b) \le \Gamma_1(b) \le \|b\|_{CB^p_s(\mu_t)}$.

4.2. Proof of (iii) $\Longrightarrow$ (i) in Theorem 1.1 for the unweighted case $t = 0$.

In the next proposition we use Corollary 3.5 and the weighted Cauchy-Pompeiu's formula, to give a simple proof of (iii) $\Longrightarrow$ (i) in Theorem 1.1 for the unweighted case $t = 0$. This last case has been proved using different methods in [**12**] for $p = 2$ and in [**4**] for any $p > 1$. Our approach follows the techniques in [**15**].

Proposition 4.1. *Let* $1 < p < \infty$ *and* $0 < s < 1$. *Assume that* $b \in H$ *satisfies* $|\langle fg, b \rangle_0| \le C_b \|f\|_{B^p_s} \|g\|_{B^{p'}_{-s}}$ *for any* $f, g \in H(\overline{\mathbb{D}})$. *Then* $b \in CB^p_s$.

Proof. By Lemma 2.9 we have $b \in B^p_s \subset B^1_0$. Therefore, for $f \in H(\overline{\mathbb{D}})$, the weighted Cauchy-Pompeiu's representation formula in Theorem 2.2 gives

$$(1+R)b(z)\overline{f} = \mathcal{P}^1((1+R)b\overline{f}) + \mathcal{K}^1((1+R)b\overline{\partial f}). \tag{4.11}$$

In order to prove this proposition it is enough to show that the $L^p(d\nu_{(1-s)p})$-norms of the two terms in the right hand side in (4.11) are bounded by a constant times $\|f\|_{B^p_s}$.

The first term $h = \mathcal{P}^1((1+R)b\overline{f})$ is a holomorphic function on $\mathbb{D}$. Thus, by Corollary 2.12, it suffices to prove that $|\langle h, g\rangle_1| \le C\|f\|_{B^p_s}\|g\|_{B^{p'}_{-s}}$ for any $g \in H(\overline{\mathbb{D}})$.

By Lemma 2.10, this follows from $\langle h, g\rangle_1 = \langle (1+R)b, fg\rangle_1 = \langle b, fg\rangle_0$ and the hypotheses.

In order to estimate the $L^p(d\nu_{(1-s)p})$-norm of $\mathcal{K}^1((1+R)b\overline{\partial f})$, note that by Corollary 3.5 we have $b \in B^\infty_0$. This fact, Hölder's inequality and the estimates of Lemma 2.3, with $\varepsilon > 0$ small enough to be chosen later on, we have

$$\begin{aligned}
|\mathcal{K}^1((1+R)b\overline{\partial f})(z)|^p &\le \|b\|^p_{B^\infty_0}\left(\int_{\mathbb{D}} \frac{|\partial f(w)|}{|1-z\overline{w}||w-z|}d\nu(w)\right)^p \\
&\le \|b\|^p_{B^\infty_0}\int_{\mathbb{D}} \frac{|\partial f(w)|^p(1-|w|^2)^{(1-\varepsilon)p-1}}{|1-z\overline{w}|^{(1-2\varepsilon)p}|w-z|}d\nu(w)\left(\int_{\mathbb{D}} \frac{(1-|w|^2)^{\varepsilon p'-1}}{|1-z\overline{w}|^{2\varepsilon p'}|w-z|}d\nu(w)\right)^{p/p'} \\
&\lesssim \|b\|_{B^\infty_0}\int_{\mathbb{D}} \frac{|\partial f(w)|^p(1-|w|^2)^{(1-\varepsilon)p-1}}{|1-z\overline{w}|^{(1-2\varepsilon)p}|w-z|}d\nu(w)(1-|z|^2)^{-\varepsilon p}.
\end{aligned}$$

Therefore, if $0 < \varepsilon < \min\{s, 1-s\}$, then the above estimate, Fubini's theorem and Lemma 2.3 give

$$\|\mathcal{K}^1((1+R)b\overline{\partial f})\|_{L^p(d\nu_{(1-s)p})} \lesssim \|b\|_{B^\infty_0}\|\partial f\|_{L^p(d\nu_{(1-s)p})} \lesssim \|b\|_{B^\infty_0}\|f\|_{B^p_s},$$

which ends the proof. □

4.3. Proof of (iii) $\Longrightarrow$ (i) in Theorem 1.1 for the general case.

Observe that if we use the same arguments of the above section to prove the unweighted case, then in the estimate of $\mathcal{K}^{t+1}((1+R)b\,\overline{\partial f})$ we will end up with integrals of the type

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^{N-1}}{|1-z\overline{w}|^{1+M}|w-z|}\,\theta(w)d\nu(w),$$

which are difficult to estimate because we do not have precise information on θ near the diagonal $z = w$. One method to avoid this difficulty is based in the use of the following modification of the Cauchy-Pompeiu's formula, which on one hand avoid the singularity on the diagonal and in other hand increases the power of $(1-|w|^2)$.

LEMMA 4.2. *Let $t > 0$, $b \in B^\infty_0$ and $f \in H(\overline{\mathbb{D}})$. For any integer $m \ge 2$, we have*

$$\begin{aligned}
\mathcal{K}^{t+1}((1+R)b\overline{\partial f}) &= \mathcal{K}^{t+m}_0((1+R)b\,\overline{\partial^2 f}) + \mathcal{K}^{t+m-1}_1((1+R)b\,\overline{\partial f}) \\
&\quad + \sum_{j=1}^{m-1} Q^{t+j}((1+R)b\overline{Rf}),
\end{aligned}$$

where

$$\mathcal{K}_0^{t+m}((1+R)b\overline{\partial^2 f})(z) := -\int_{\mathbb{D}} \frac{((1+R)b\overline{\partial^2 f})(w)}{(1-z\overline{w})^{t+m}}\frac{\overline{w-z}}{w-z}d\nu_{t+m+1}(w),$$

$$\mathcal{K}_1^{t+m-1}((1+R)b\overline{\partial f})(z) := (t+m)\int_{\mathbb{D}} \frac{((1+R)b\overline{\partial f})(w)\,(\overline{w-z})}{(1-z\overline{w})^{t+m+1}}d\nu_{t+m}(w),$$

$$Q^{t+j}((1+R)b\overline{Rf})(z) := \int_{\mathbb{D}}((1+R)b\overline{Rf})(w)\frac{d\nu_{t+j+1}(w)}{(1-z\overline{w})^{t+j+1}}.$$

Proof. Recall that

$$\mathcal{K}^t(w,z) = \frac{(1-|w|^2)^t}{(1-z\overline{w})^t}\frac{1}{w-z}.$$

Since $1 = (1-|w|^2)/(1-z\overline{w}) + (\overline{w}(w-z))/(1-z\overline{w})$, we have

$$\mathcal{K}^{t+1}((1+R)b\overline{\partial f})(z) = \mathcal{K}^{t+2}((1+R)b\overline{\partial f})(z) + Q^{t+1}((1+R)b\overline{Rf})(z).$$

Iterating this formula, we obtain

$$\mathcal{K}^{t+1}((1+R)b\overline{\partial f})(z) = \mathcal{K}^{t+m}((1+R)b\overline{\partial f})(z) + \sum_{j=1}^{m-1} Q^{t+j}((1+R)b\overline{Rf})(z).$$

An easy computation shows that

$$\begin{aligned}\mathcal{K}^{t+m}(w,z) &= \frac{(1-|w|^2)^{t+m}}{(1-z\overline{w})^{t+m}}\frac{1}{w-z}\\ &= \overline{\partial_w}\left(\frac{(1-|w|^2)^{t+m}}{(1-z\overline{w})^{t+m}}\frac{\overline{w-z}}{w-z}\right) + (t+m)\frac{(1-|w|^2)^{t+m-1}(\overline{w-z})}{(1-z\overline{w})^{t+m+1}}.\end{aligned}$$

Fixed $z\in\mathbb{D}$ and $0<\varepsilon<1-|z|$, let $\Omega_{z,\varepsilon} := \mathbb{D}\setminus\{w\in\mathbb{D} : |w-z|<\varepsilon\}$. If we apply Stokes' theorem to the region $\Omega_{z,\varepsilon}$ and let $\varepsilon\to 0$, we obtain

$$\begin{aligned}\mathcal{K}^{t+m}((1+R)b\overline{\partial f})(z) = &-\int_{\mathbb{D}}((1+R)b\overline{\partial^2 f})(w)\frac{(1-|w|^2)^{t+m}}{(1-z\overline{w})^{t+m}}\frac{\overline{w-z}}{w-z}d\nu(w)\\ &+ (t+m)\int_{\mathbb{D}}((1+R)b\overline{\partial f})(w)\frac{(1-|w|^2)^{t+m-1}(\overline{w-z})}{(1-z\overline{w})^{t+m+1}}d\nu(w),\end{aligned}$$

which concludes the proof. □

Proposition 4.3. *Let $1<p<\infty$, $0<s<1$, $t>0$, $b\in B_0^\infty$, $f\in H(\overline{\mathbb{D}})$ and*

$$\varphi_f(w) := |\partial^2 f(w)|(1-|w|^2)^{2-s} + |\partial f(w)|(1-|w|^2)^{1-s}.$$

Then we have

$$|\mathcal{K}^{t+1}((1+R)b\overline{\partial f})(z)| \lesssim \|b\|_{B_0^\infty}\,\mathbb{P}^{t+s,t+1}(\varphi_f)(z). \tag{4.12}$$

Therefore, if $\theta \in \mathcal{B}_{p,t}$, *then*

$$\|(1-|z|^2)^{1-s}\mathcal{K}^{t+1}((1+R)b\overline{\partial f})(z)\|_{L^p(\mu_t)} \lesssim \|b\|_{B_0^\infty}\,\|f\|_{B_s^p(\mu_t)}. \tag{4.13}$$

PROOF. The pointwise estimate (4.12) follows from Lemma 4.2. Since $1-|w|^2 \leq 2|1-z\bar{w}|$ and $|z-w| \leq |1-z\bar{w}|$, then for $m \geq 3$, we have

$$\begin{aligned}
&|\mathcal{K}^{t+1}((1+R)b\overline{\partial f})(z)| \\
&\lesssim \|b\|_{B_0^\infty}\left(\mathbb{P}^{t+m-2+s,t+m-1}(\varphi_f)(z) + \sum_{j=1}^{m-1}\mathbb{P}^{t+j+s-1,t+j}(\varphi_f)(z)\right) \\
&\lesssim \|b\|_{B_0^\infty}\,\mathbb{P}^{t+s,t+1}(\varphi_f)(z).
\end{aligned}$$

In order to prove the $L^p(\mu_t)$-norm estimate (4.13), from (4.12) we have

$$(1-|z|^2)^{1-s}|\mathcal{K}^{t+1}((1+R)b\overline{\partial f})(z)| \lesssim \|b\|_{B_0^\infty}\,\mathbb{P}^{t+s}(\varphi_f)(z)$$

and thus $\|\mathbb{P}^{t+s}(\varphi_f)\|_{L^p(\mu_t)} \lesssim \|\varphi_f\|_{L^p(\mu_t)} \lesssim \|f\|_{B_s^p}(\mu_t)$, which is a consequence of Theorem 2.5 and Proposition 2.8. □

Now we can prove (iii) $\Longrightarrow$ (i) in Theorem 1.1.

PROPOSITION 4.4. *If* b *satisfies condition* (iii) *in Theorem* 1.1, *then* $b \in CB_s^p(\mu_t)$.

PROOF. We want to prove that

$$\int_{\mathbb{D}} |f(z)|^p |R_{t+1}^1 b(z)|^p (1-|z|^2)^{(1-s)p} d\mu_t(z) \lesssim C_b \|f\|_{B_s^p(\mu_t)}^p.$$

To do so, by the Cauchy-Pompeiu's formula in Theorem 2.2,

$$R_{t+1}^1 b(z)\overline{f} = \mathcal{P}^{t+1}(R_{t+1}^1 b\overline{f}) + \mathcal{K}^{t+1}(R_{t+1}^1 b\overline{\partial f}), \tag{4.14}$$

we will show that the two terms in the right hand side in (4.14) are both in $L^p(\theta d\nu_{(1-s)p+t})$ and that these norms are bounded up to a constant by $\|f\|_{B_s^p(\mu_t)}$.

Since $h = \mathcal{P}^{t+1}(\overline{f}\,R_{t+1}^1 b)$ is a holomorphic function on $\mathbb{D}$, the norm estimate of h is similar to the one for the unweighted case. Indeed, for $g \in H(\overline{\mathbb{D}})$ Lemma 2.10 gives

$$|\langle h, g\rangle_{t+1}| = |\langle R_{t+1}^1 b, fg\rangle_{t+1}| = |\langle b, fg\rangle_t| \leq \Gamma_2(b)\|f\|_{B_s^p(\mu_t)}\|g\|_{B_{-s}^{p'}(\mu_t')}$$

which, by Corollary 2.12, proves that $\|h\|_{L^p(\theta d\nu_{(1-s)p+t})} \leq \Gamma_2(b)\|f\|_{B_s^p(\mu_t)}$.

Using the $L^p(\theta d\nu_{(1-s)p+t})$-norm estimate of $\mathcal{K}^{t+1}(R_t^1 b\overline{\partial f})$ given in Proposition 4.3, we conclude the proof. $\square$

4.4. Proof of Corollary 1.2.

Using $B_s^p(\mu_\delta) = B^p_{s+\tau/p}(\mu_{\delta+\tau})$, for $\tau > 0$, we will deduce the result from Theorem 1.1.

Let $1 < p < \infty$, $s_0, s_1 \in \mathbb{R}$, $t_0 \geq 0$ and $\theta \in \mathcal{B}_{p,t_0}$. Then, for $t = t_0 - s_0 - s_1 > t_0$ we have

$$\begin{aligned} &B^p_{s_0}(\mu_{t_0}) = B^p_{s_0+(-s_0-s_1)/p}(\mu_t) = B^p_{s_0/p'-s_1/p}(\mu_t), \quad \text{and} \\ &B^{p'}_{s_1}(\mu'_{t_0}) = B^{p'}_{s_1/p-s_0/p'}(\mu'_t). \end{aligned}$$

Moreover, since $\langle fg, b\rangle_{t_1} = \langle \mathcal{P}^t(fg), b\rangle_{t_1} = \langle fg, \mathcal{P}^{t_1,t} b\rangle_t$, we have

$$\frac{|\langle fg, b\rangle_{t_1}|}{\|f\|_{B^p_{s_0}(\mu_{t_0})}\,\|g\|_{B^{p'}_{s_1}(\mu'_{t_0})}} = \frac{|\langle fg, \mathcal{P}^{t_1,t} b\rangle_t|}{\|f\|_{B^p_{s_0/p'-s_1/p}(\mu_t)}\,\|g\|_{B^{p'}_{s_1/p-s_0/p'}(\mu'_t)}}.$$

Thus, Theorem 1.1, with $0 < s := s_0/p' - s_1/p < 1$, gives

$$\|\mathcal{P}^{t_1,t} b\|_{CB^p_{s_0/p-s_1/p'}(\mu_t)} \approx \sup_{0\neq f,g\in H(\overline{\mathbb{D}})} \frac{|\langle fg, b\rangle_{t_1}|}{\|f\|_{B^p_{s_0}(\mu_{t_0})}\|g\|_{B^{p'}_{s_1}(\mu_{t_0})}},$$

which concludes the proof.

5. Proof of Theorem 1.4.

We will determine the predual of $CB_s^p(\mu_t)$ generalizing some results for the unweighted case (see for instance [**12**], [**2**], [**8**] and the references therein).

5.1. Weak products and the predual of $CB_s^p(\mu_t)$.

Definition 5.1. Given two Banach spaces X and Y of holomorphic functions on $\mathbb{D}$, let $X \odot Y$ be the completion of finite sums $h = \sum_{j=1}^M f_j g_j$, $f_j \in X$, $g_j \in Y$, using the norm

$$\|h\|_{X\odot Y} := \inf\left\{\sum_{k=1}^N \|\tilde{f}_k\|_X \|\tilde{g}_k\|_Y : \sum_{k=1}^N \tilde{f}_k\,\tilde{g}_k = h\right\}.$$

The following well-known proposition will be used to prove our duality results.

Proposition 5.2. *The norm of a linear form Λ on $X \odot Y$ coincides with the norm of the bilinear form on $X \times Y$ on defined by $\tilde{\Lambda}(f, g) = \Lambda(fg)$.*

5.2. Proof of Theorem 1.4.

Proof. The embedding $i : B^{p'}_{-s}(\mu'_t) \to B^p_s(\mu_t) \odot B^{p'}_{-s}(\mu'_t)$, shows that any linear form $\Lambda \in (B^p_s(\mu_t) \odot B^{p'}_{-s}(\mu'_t))'$ produces a linear form $\Lambda_i = \Lambda \circ i$ on $B^{p'}_{-s}(\mu'_t)$, which by

Proposition 2.11 can be expressed as $\Lambda_i(f) = \langle f, b\rangle_t$, for some $b \in B^p_s(\mu_t)$.

Consequently, $\Lambda(h) = \langle h, b\rangle_t$ for $h \in H(\overline{\mathbb{D}})$. Since $H(\overline{\mathbb{D}})$ is dense in both spaces $B^p_s(\mu_t)$ and $B^{p'}_{-s}(\mu'_t)$, then it is also dense in $B^p_s(\mu_t) \odot B^{p'}_{-s}(\mu'_t)$, and thus the norm of Λ coincides with the norm of the bilinear form $(f, g) \to \langle fg, b\rangle_t$ on $B^p_s(\mu_t) \times B^{p'}_{-s}(\mu'_t)$. Therefore, the equivalence between (i) and (iii) in Theorem 1.1 concludes the proof.

The same arguments used in the first part show that the norm of a linear form Λ on $B^p_{s_0}(\mu_t) \odot B^{p'}_{s_1}(\mu'_t)$ is equivalent to the norm of the bilinear form $(f, g) \to \langle fg, b\rangle_t$, where $b \in B^p_{-s_1}(\mu_t)$. By Theorem 3.2 this norm is equivalent to $\|b\|_{B^\infty_{-s_0-s_1}}$ which proves the first statement.

The second statement follows from the computation by duality of the norms $\|h\|_{B^1_{s_0+s_1-t}}$ and $\|h\|_{B^p_{s_0}(\mu_t)\odot B^{p'}_{s_1}(\mu'_t)}$. Indeed, if $h \in H(\overline{\mathbb{D}})$, then

$$\|h\|_{B^1_{s_0+s_1-t}} \approx \sup_{0\neq b\in B^\infty_{-s_0-s_1}} \frac{|\langle h, b\rangle_t|}{\|b\|_{B^\infty_{-s_0-s_1}}} \approx \|h\|_{B^p_{s_0}(\mu_t)\odot B^{p'}_{s_1}(\mu'_t)}.$$

Since $h \in H(\overline{\mathbb{D}})$ is dense in both spaces, we obtain the result. □

5.3. Further remarks.

Combining Theorem 1.4 with (1.4) we can obtain characterizations of weak products of type $B^p_{s_0}(\mu_t) \odot B^{p'}_{s_1}(\mu_t)$ which generalize some of the results stated in Section 5 in [**8**].

For instance, if $0 < s < p$, then

$$\big(B^p_0(\mu_t) \odot B^{p'}_{-s}(\mu_t)\big)' = \big(B^p_{s/p}(\mu_{t+s}) \odot B^{p'}_{-s/p}(\mu_{t+s})\big)' \equiv CB^p_{s/p}(\mu_{t+s}) = CB^p_0(\mu_t),$$

with respect to the pairing $\langle\cdot,\cdot\rangle_{t+s}$.

Observe that in the particular case $p = 2$ and $t = 0$, we have $CB^2_0 = BMOA \equiv (H^1_{-s})'$, with respect to the pairing $\langle\cdot,\cdot\rangle_s$. Therefore, the above duality result and the fact that $B^2_s = H^2_s$ give $H^2 \odot H^2_{-s} = H^1_{-s}$.

This unweighted weak factorization result can be generalized to the case $1 < p < 2$. In this case $B^p_0 \subset H^p$, and we have that $CB^p_0 = F^{\infty,p}_0$, where $F^{\infty,p}_0$ denotes the Triebel-Lizorkin space of holomorphic functions on $\mathbb{D}$ such that the measure $d\mu_g(z) = |\partial g(z)|^p(1-|z|^2)^{p-1}$ is a Carleson measure for H^p, that is $\mu_g(T_z) \lesssim (1-|z|^2)$ for any $z \in \mathbb{D}$ (see [**10**], p.178). Since $F^{\infty,p} \equiv (F^{1,p'}_{-s})'$, with respect to the pairing $\langle\cdot,\cdot\rangle_s$, we have $B^p_0 \odot B^{p'}_{-s} = F^{1,p'}_{-s}$. Here, $F^{1,p'}_{-s}$ is the Triebel-Lizorkin space of holomorphic functions g on $\mathbb{D}$ satisfying

$$\int_{\mathbb{T}} \left(\int_{|1-\zeta\overline{w}|<1-|w|^2} |g(w)|^{p'}(1-|w|^2)^{sp'-2} d\nu(w) \right)^{1/p'} d\sigma(\zeta) < \infty.$$

References

[1] N. Arcozzi, R. Rochberg, E. Sawyer and B. D. Wick, Bilinear forms on the Dirichlet space, Anal. PDE, **3** (2010), 21–47.

[2] N. Arcozzi, R. Rochberg, E. Sawyer and B. D. Wick, Function spaces related to the Dirichlet space, J. Lond. Math. Soc. (2), **83** (2011), 1–18.

[3] D. Békollé, Inégalités à poids pour le projecteur de Bergman dans la boule unité de $\mathbb{C}^n$, Stud. Math., **71** (1981), 305–323.

[4] D. Blasi and J. Pau, A characterization of Besov-type spaces and applications to Hankel-type operators, Michigan Math. J., **56** (2008), 401–417.

[5] C. Cascante, J. Fàbrega and J. M. Ortega, Toeplitz operators on weighted Hardy and Besov spaces, Math. Z., **263** (2009), 787–812.

[6] C. Cascante and J. M. Ortega, Carleson Measures for weighted Besov spaces, Ark. Mat., **49** (2011), 31–59.

[7] C. Cascante and J. M. Ortega, On a characterization of bilinear forms on the Dirichlet space, Proc. Amer. Math. Soc., **140** (2012), 2429–2440.

[8] W. S. Cohn and I. E. Verbitsky, Factorization of tent spaces and Hankel operators, J. Funct. Anal., **175** (2000), 308–329.

[9] D. H. Luecking, Representation and duality in weighted spaces of analytic functions, Indiana Univ. Math. J., **34** (1985), 319–336.

[10] V. G. Maz'ya and T. O. Shaposhnikova, Theory of multipliers in spaces of differentiable functions, Monographs and Studies in Mathematics, **23**, Pitman (Advanced Publishing Program), Boston, MA, 1985.

[11] S. Petermichl and B. Wick, A weighted estimate for the square function on the unit ball, Arkiv Math., **45** (2007), 337–350.

[12] R. Rochberg and Z. J. Wu, A new characterization of Dirichlet type spaces and applications, Illinois J. Math., **37** (1993), 101–122.

[13] D. A. Stegenga, Multipliers of the Dirichlet space, Illinois J. Math., **24** (1980), 113–139.

[14] Z. Wu, The predual and second predual of W_α, J. Funct. Anal., **116** (1993), 314–334.

[15] Z. Wu, A class of bilinear forms on Dirichlet type spaces, J. London Math. Soc. (2), **54** (1996), 498–514.

[16] Z. Wu, Carleson Measures and Multipliers for Dirichlet Spaces, J. Funct. Anal., **169** (1999), 148–163.

[17] K. H. Zhu, Operator theory in function spaces, Monographs and Textbooks in Pure and Applied Mathematics, **139**, Marcel Dekker, Inc., New York, 1990.

[18] K. H. Zhu, Spaces of holomorphic functions in the unit ball, Graduate Texts in Mathematics, **226**, Springer-Verlag, New York, 2005.

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