Black holes and axions

Author: Gastón Creci*
Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain.

Advisor: Helvi Witek†
Departament de Física Quàntica i Astrofísica & Institut de Ciències del Cosmos (ICCUB), Universitat de Barcelona, Martí i Franquès 1, E-08028 Barcelona, Spain

Abstract: We solve the Klein-Gordon equation for an axion field in a Kerr background analytically for $M\mu \ll 1$ and $M\omega \ll 1$ for small spins via matching the near and far region solutions. This yields a hydrogen-like frequency spectrum and, if the superradiance condition is met, an instability. Moreover, we discuss some physically relevant elements of our system such as the potential associated to a black hole, the superradiance rate or the flux at the event horizon, comparing them with the numerical solution where our approximations are no longer valid.

I. INTRODUCTION

Black hole (BH) superradiance is a dissipative effect triggered by the scattering of monochromatic waves with frequency ω off rotating BHs satisfying:

$$\omega < m\Omega_H \tag{1}$$

where Ω_H is the angular velocity of the event horizon and m an azimuthal quantum number. As a consequence the BH loses energy and angular momentum while the wave is amplified [1]. Since the existence of an event horizon in a stationary and axisymmetric spacetime automatically implies the existence of an ergoregion [1], the Kerr metric provides a good background to study superradiance.

For the superradiance instability to be significant enough to manifest in an astrophysical system, the Compton wavelength of a field has to be comparable to the size of the BH [2]. This is realized for ultra-light bosons with small but non zero rest mass. In particular, for masses in the range $10^{-9} - 10^{-21}$eV, astrophysical BHs can become sensitive detectors, serving as tests for fundamental physics theories [3]. One of these particles is the QCD axion with a rest mass (in the QCD frame) of $\mu \leq 6 \cdot 10^{-10}$eV or axion-like particles that are also interesting dark matter candidates. [4].

The superradiance through the axion yields the *Axionic BH Atom* [3], where the BH acts as an hydrogen atom and gravitons would be emitted by transitions between axion energy levels. The gravitational waves emitted in this phenomena can be detected with Advanced LIGO, and therefore we can constrain the axion mass. We model the axion as a (light) massive scalar field [3] and study its interaction with the Kerr BH.

*Electronic address: gcrecike@gmail.com
†Electronic address: hwitek@icc.ub.edu

[1] Superradiance is similar to the Penrose process, where an incoming particle decays within the ergoregion producing an ingoing negative-energy particle (as perceived by an observer at infinity) and an outgoing particle with more energy than the incoming one.

II. DIFFERENTIAL EQUATION

The Klein-Gordon equation reads:

$$(\Box - \mu^2)\phi = 0, \tag{2}$$

where $\mu = \mathcal{M}G/\hbar c$ for a particle of mass $\mathcal{M}$. The metric in Boyer-Lindquist coordinates (t, r, θ, ϕ) is

$$\mathrm{d}s^2 = -\Big(1 - \frac{2Mr}{\Sigma}\Big)\mathrm{d}t^2 - \frac{4Mar\sin^2\theta}{\Sigma}\mathrm{d}t\mathrm{d}\varphi + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2 + \Big(r^2 + a^2 + \frac{2Ma^2r\sin^2\theta}{\Sigma}\Big)\sin^2\theta\mathrm{d}\varphi^2, \tag{3}$$

where

$$\Delta = r^2 + a^2 - 2Mr \quad , \quad \Sigma = r^2 + a^2\cos^2\theta. \tag{4}$$

M is the mass and $J = aM$ is the angular momentum of the BH. One of the properties of the Kerr BH is the existence of an ergoregion. The ergoregion is the location at which an observer is forced to co-rotate with the BH and is defined as the region between the ergosphere $r_{ergo} = M + \sqrt{M^2 - a^2\cos^2\theta}$, an infinite red-shift surface, and the event horizon $r_+ = M + \sqrt{M^2 - a^2}$. Using the Laplace-Beltrami operator and taking the ansatz [5]

$$\phi = e^{-i\omega t}e^{im\varphi}R(r)\Theta(\theta), \tag{5}$$

we separate Eq. (2) into

$$\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta)\Theta(\theta) + \left(\lambda - a^2\kappa^2\cos^2\theta - \frac{m^2}{\sin^2\theta}\right)\Theta(\theta) = 0, \tag{6}$$

$$\Delta\partial_r(\Delta\partial_r)R(r) + [\omega^2(r^2+a^2)^2 + a^2m^2 - 4Mar\omega m - \Delta(\lambda + a^2\omega^2 + \mu^2r^2)]R(r) = 0\,, \tag{7}$$

where $\kappa^2 \equiv \mu^2 - \omega^2$. Using $\Theta(\theta) = y(x)$ and $x = \cos\theta$, Eq. (6) becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}[(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}y(x)] + \left(\lambda - a^2\kappa^2x^2 - \frac{m^2}{1-x^2}\right)y(x) = 0. \tag{8}$$

We see that it is the differential equation for the spheroidal wave equation [6]

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\eta}[(1-\eta^2)\frac{\mathrm{d}}{\mathrm{d}\eta}S_{mn}(-ic,\eta)]+\\&\left(\lambda_{mn}+c^2\eta^2-\frac{m^2}{1-\eta^2}\right)S_{mn}(-ic,\eta)=0,\end{aligned} \tag{9}$$

with $\eta = x$, $c^2 = -a^2\kappa^2$, $n = l$ and $\lambda_{mn} = \lambda$. Thus, the eigenfunction is $y(x) = S_{mn}(-ic, \cos\theta)$. Note that for c=0 the eigenfunctions are the Legendre polynomials. Normalization yields the shperoidal harmonics

$$Z_{lm}(\theta,\varphi)=\Big[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\Big]^{1/2}S_{lm}(-ic,\cos\theta)e^{im\varphi}, \tag{10}$$

that satisfies $\int_\Omega Z^*_{lm}Z_{l'm'}\mathrm{d}(\cos\theta)\mathrm{d}\varphi = \delta_{ll'}\delta_{mm'}$. To write Eq. (7) in Schrödinger-like form, we transform from spherical to tortoise coordinates

$$\frac{\mathrm{d}r_*}{\mathrm{d}r}=\frac{r^2+a^2}{\Delta}, \tag{11}$$

and substitute $X(r) = \sqrt{r^2 + a^2}R(r)$. Then, we obtain

$$\Big[\frac{\mathrm{d}^2}{\mathrm{d}r_*^2}+V\Big]X(r)=0, \tag{12}$$

with the effective potential

$$\begin{aligned}V=&\omega^2+\frac{a^2m^2}{(r^2+a^2)^2}-\frac{4Mar\omega m}{(r^2+a^2)^2}\\&-\Delta\left[\frac{\lambda+a^2\omega^2+\mu^2r^2}{(r^2+a^2)^2}-\frac{\Delta(2r^2-a^2)}{(r^2+a^2)^4}+\frac{2r(r-M)}{(r^2+a^2)^3}\right].\end{aligned} \tag{13}$$

In tortoise coordinates the limits to the event horizon and to infinity are, respectively, $\lim_{r\to\infty} r_* = \infty$ and $\lim_{r\to r_+} r_* = -\infty$.

A. Boundary conditions and limit solutions

For large radii we find $X(r) \sim e^{\pm ikr_*} \sim e^{\pm ikr}$ with $k = \sqrt{V(r\to\infty)} = \sqrt{V(r_*\to\infty)} = \sqrt{\mu^2-\omega^2} \equiv \kappa$ $(\forall r, r_*)$. As we want an outgoing wave the corresponding sign is (+). Thus

$$R(r)=\frac{A}{\sqrt{r^2+a^2}}e^{i\sqrt{\mu^2-\omega^2}r}, \tag{14}$$

is the solution for an outgoing wave in the infinity limit.

For the event horizon limit we must discuss the boundary conditions. In the coordinate frame we may think of a solution corresponding to an ingoing wave. However, this is not correct because the wave must be ingoing for a physical observer rather than the coordinate frame.

With no loss of generality (because all physical observers are related by Lorentz transformations) let us take an observer near the horizon, right in the ergosphere. As a consequence the observer moves whitin the ergosphere with an angular velocity $\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \Omega_H = \frac{a}{2Mr_+}$. The wave solution for the observer takes the form $\phi \propto e^{-i(\omega-m\Omega_H)t}e^{\pm ikr_*}$. The observer must see an ingoing wave and therefore we must choose the sign of k opposite to the sign of $(\omega - m\Omega_H)$. Hence, taking into account $k = \sqrt{V(r\to r_+)} = \sqrt{V(r_*\to -\infty)} = \omega - m\Omega_H$

$$R(r)=\frac{B}{\sqrt{r^2+a^2}}e^{-i(\omega-m\Omega_H)r_*}, \tag{15}$$

The situation when $\omega < m\Omega_H$ corresponds to the superradiance effect and the wave extracts energy from the rotating BH (see Sec. IV).

III. MATCHING SOLUTIONS

In the limit of small spins, $a \ll M$, and of small frequencies, i.e., when the Compton wavelength of the field is much larger than the size of the BH, we can perform a matched asymptotic expansion. We divide the problem in three regions: the far, near and overlapping-region. We take the limits $r - r_+ \gg M$ and $r - r_+ \ll 1/\omega$ for the far and the near-region, respectively. Then we match the solution of both regions in the limit where $M \ll r - r_+ \ll 1/\omega$.

In this limit the separation constant λ of Eqs. (6) and (7) behaves as [7]

$$\lambda=l(l+1)+O(a^2\omega^2). \tag{16}$$

Therefore the spheroidal harmonics (10) reduce to spherical harmonics

$$Y_{lm}(\theta,\varphi)=\Big[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\Big]^{1/2}P_{lm}(\cos\theta)e^{im\varphi}. \tag{17}$$

A. Far region solution

The differential equation (7) within the limits discussed above is

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}[rR(r)]+\Big[\frac{2M\mu^2}{r}-\kappa^2-\frac{l(l+1)}{r^2}\Big]rR(r)=0. \tag{18}$$

By doing the following substitutions [8]

$$\nu=M\mu/\kappa,\qquad \chi=2\kappa r, \tag{19}$$

we arrive to the Whittaker equation [9]

$$\frac{\mathrm{d}^2}{\mathrm{d}\chi^2}[\chi R(r)]+\Big[-\frac{1}{4}+\frac{\nu}{\chi}-\frac{l(l+1)}{\chi^2}\Big]\chi R(r)=0. \tag{20}$$

The solution of this equation is related to the confluent hypergeometric function of first kind and second kind (or Whittaker functions). With analogy with the hydrogen atom equation, we can define the reduced radial function $P(r) \equiv \chi R(r)$. The general solution is

$$P(\chi) = \chi^{l+1} e^{-\chi/2} \times \left[C_1 U(l+1-\nu, 2l+2, \chi) + C_2 L_{\nu-l-1}^{2l+1}(\chi)\right], \tag{21}$$

where C_1 and C_2 are constants, $U(a,b,\chi)$ is the conlfuent hypergeometric function of the second kind and $L_n^k(x) = \sum_{m=0}^{n} (-1)^m \frac{(n+k)!}{(n-m)!(k+m)!m!} x^m$. Assuming $\nu - l - 1 \equiv n + \delta\nu$ to be non negative (since we want the radial function to be non-zero at infinity) and within the limit $M\mu \ll 1$ the solution takes the form

$$R(r) = C_1 (2\kappa r)^l e^{-\kappa/2} U(-n-\delta\nu, 2l+2, 2\kappa r). \tag{22}$$

Just like the hydrogen atom, we expect a free oscillation of the scalar field to behave like a bound state of the hydrogen atom. The main difference to solving the hydrogen equation is that, instead of imposing boundary conditions at the origin, we impose them at the event horizon. In general it yields complex frequencies $\omega = \omega_R + i\delta$, with $\delta < 0$ or $\delta > 0$ for stable and unstable modes respectively. Therefore, assuming slowly growing instabilities, we treat the imaginary part of the frequency as a first-order perturbation, i.e., $\kappa = \kappa\left(1 - \frac{\omega}{\kappa^2} i\delta\right) + O(\delta^2)$. The relation between κ, n, l and μ is

$$\mu^2 - \omega_R^2 = \mu^2 \left(\frac{\mu M}{n+l+1}\right)^2. \tag{23}$$

This is, taking $\mu M \ll 1$ we find

$$\omega_R \approx \mu\left[1 - \frac{1}{2}\left(\frac{\mu M}{n+l+1}\right)^2\right] = \mu + O(\mu M)^2. \tag{24}$$

The perturbation yields $\nu = \nu + \frac{\mu M \omega}{\kappa^3} i\delta \equiv \nu + \delta\nu$ and therefore

$$i\delta = \frac{\delta\nu}{M}\left(\frac{\mu M}{n+l+1}\right)^3. \tag{25}$$

Using the Maclaurin series of the confluent hypergeometric function of second kind and taking the behavior at $z \to 0$ we only take into account the following terms

$$U(a,b,z) = \frac{\Gamma(-b+1)}{\Gamma(1+a-b)} + z^{1-b}\frac{\Gamma(b-1)}{\Gamma(a)}. \tag{26}$$

Therefore, the solution in the far region, in the limit of small r, becomes

$$R(r) = C_1 \frac{(2\kappa)^l \Gamma(-2l-1)}{\Gamma(-2l-n-1)} r^l + C_1 (2\kappa)^{-l-1} \frac{\Gamma(2l+1)}{\Gamma(-n-\delta\nu)} r^{-l-1} \tag{27}$$

B. Near region solution

In the near horizon region, we take the approximation $r - r_+ \ll 1/\omega$, the differential radial equation within this limit is

$$\Delta\partial_r(\Delta\partial_r)R(r) + [r_+^4(\omega - m\Omega_H)^2 - l(l+1)\Delta]R(r) = 0. \tag{28}$$

Substituting $z = \frac{r-r_+}{r-r_-}$ we find

$$z(1-z)\partial_z^2 R(z) + (1-z)\partial_z R(z) + \varpi^2\left(\frac{1-z}{z}\right)R(z) - \frac{l(l+1)}{(1-z)}R(z) = 0, \tag{29}$$

with ϖ the superradiant factor defined as

$$\varpi \equiv (\omega - m\Omega_H)\frac{r_+^2}{r_+ - r_-}. \tag{30}$$

Since every second-order ordinary differential equation with at most three regular singular points can be transformed into the hypergeometric differential equation [10], we can express the radial function as $R(z) = z^{i\varpi}(1-z)^{l+1}F(z)$ [11] . Then, Eq. (29) becomes

$$z(1-z)\partial_z^2 F(z) - [(l+1+2i\varpi)(l+1)]F(z) + \Big[(1+2i\varpi) - [2(l+1)+2i\varpi+1]z\Big]\partial_z F(z) = 0. \tag{31}$$

As expected, this differential equation corresponds to the hypergeometric differential equation with the following parameters:

$$a = l+1+2i\varpi, \quad b = l+1, \quad c = 1+2i\varpi, \tag{32}$$

The most general solution near $z = 0$ (i.e. $r = r_+$) is

$$R(z) = Az^{-i\varpi}(1-z)^{l+1}\ {}_2F_1(a+1-c, b+1-c; 2-c; z) + Bz^{i\varpi}(1-z)^{l+1}\ {}_2F_1(a,b;c;z) \tag{33}$$

This solution represents an ingoing and outgoing wave. As discussed in Sec. II we only want an ingoing wave and therefore we set $B = 0$. Since we are looking for the behavior at large r (i.e. $z \to 1$), we must shift the equation from z to $1-z$. This can be done with Euler's hypergeometric transformations. Therefore, taking the limit $r \to \infty$ i.e. $(1-z) \to \frac{r_+ - r_-}{r}$

$$R(r) = A\Gamma(1-2i\varpi)\Big[\frac{(r_+ - r_-)^{-l}\Gamma(2l+1)}{\Gamma(l+1)\Gamma(l+1-2i\varpi)} r^l + \frac{(r_+ - r_-)^{l+1}\Gamma(-2l-1)}{\Gamma(-l-2i\varpi)\Gamma(-l)} r^{-l-1}\Big], \tag{34}$$

where we have used the property ${}_2F_1(a,b;c;0) = 1$.

C. Matching

Once we have computed the far and near-region solutions we match them to get the solution in the overlapping region. The matching yields

$$\frac{(r_+ - r_-)^{-l}}{\Gamma(l+1-2i\varpi)}\frac{\Gamma(2l+1)}{\Gamma(l+1)}\frac{\Gamma(-2l-n-1)}{\Gamma(-2l-1)} = (2\kappa)^l, \tag{35a}$$

$$\frac{(r_+ - r_-)^{l+1}}{\Gamma(-l-2i\varpi)} \frac{\Gamma(-2l-1)}{\Gamma(-l)} \frac{\Gamma(-n-\delta\nu)}{\Gamma(2l+1)} = (2\kappa)^{-l-1}. \tag{35b}$$

Dividing (35b) by (35a) and using the gamma function property [10] $\Gamma(1+x) = x\Gamma(x)$ one can derive (for $n = 1, 2, 3\ldots$) $\Gamma\left(\frac{1}{2}+n\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$ and $\Gamma\left(\frac{1}{2}-n\right) = \frac{(-1)^n 2^n}{(2n-1)!!}\sqrt{\pi}$. Then, the imaginary part of the frequency is

$$\delta_{nlm} = -2r_+(\mu - m\Omega_H)\mu(\mu M)^{4l+4}\gamma_{nlm}, \tag{36}$$

where

$$\gamma_{nlm} \equiv \frac{2^{4l+2}(2l+n+1)!}{(l+n+1)^{2l+4}(n!)}\left[\frac{l!}{(2l)!(2l+1)!}\right]^2 \times \\ \times \prod_{k=1}^{l}\left[k^2\left(1-\frac{a^2}{M^2}\right) + 4r_+^2(\mu - m\Omega_H)^2\right]. \tag{37}$$

As expected, for $\omega_R \sim \mu > m\Omega_H$ the imaginary part decays. However, if the superradiance condition $\omega_R < m\Omega_H$ is satisfied, the imaginary part becomes positive and indicates an instability. As the field has an exponential dependence $\phi \propto e^{-i\omega t} = e^{-iRe(\omega)}e^{\delta t}$ the growth is exponential with a growth timescale $\tau = \frac{1}{\delta}$. Then, the frequency is

$$\omega_{nlm} \simeq \mu - \frac{1}{2}\left(\frac{\mu M}{n+l+1}\right)^2 - \\ - 2ir_+(\mu - m\Omega_H)\mu(\mu M)^{4l+4}\gamma_{nlm}. \tag{38}$$

As we can see, the real part of the frequency follows, as expected, a hydrogen-like spectra with $\tilde{n} = n + l + 1$.

IV. ENERGY FLUX ACROSS THE HORIZON

In this section we investigate the effect of superradiance on the energy balance. In particular, the energy flux across the horizon must be negative within the superradiant regime (see Sec. II). It is given by [12]

$$\frac{d^2E_{hole}}{dtd\Omega} = \frac{\omega}{k_H}2Mr_+T^{\mu\nu}n_\mu n_\nu, \tag{39}$$

where $T_{\mu\nu}$ is the energy-momentum tensor, $k_H = \lim_{r\to r_+} V(r)$ and n_μ the normal vector pointing in the inward direction. For a massive complex field the energy-momentum tensor is given by

$$T_{\mu\nu} = \nabla_{(\mu}\phi^*\nabla_{\nu)}\phi - g_{\mu\nu}(\nabla_\rho\phi^*\nabla^\rho\phi - \mu^2|\phi|^2). \tag{40}$$

Taking the normal vector to the horizon to be $n^\mu = -\chi^\mu$, where $\chi^\mu = (\partial_t, 0, 0, \Omega_H\partial_\varphi)$ is the Killing field associated to the Kerr spacetime, we find

$$\frac{d^2E_{hole}}{dtd\Omega} = 2Mr_+k_H|S(\theta)|^2|A_{hole}|^2, \tag{41}$$

where A_{hole} is the amplitude of the ingoing wave at the event horizon. As we can see, within the superradiance regime $k_H = (\omega - m\Omega_H) < 0$ the energy flux across the horizon is negative and there is energy extracted from the BH, evincing the dissipative nature of this phenomena.

V. MODEL DISCUSSION AND RESULTS

As we have derived the solutions within a certain range (see Sec. III), it is of interest to plot some physically relevant quantities such as the superradiance rate and the potential associated to our solution.

A. Potential

In Fig. 1 we plot the potential, Eq. (13), exemplarily for the $l = m = 1$ mode of a field with mass coupling $M\mu = 0.5$ and for different values of the spin. As the spin increases, we see that the potential seems to develop a pronounced well in which modes with $\omega_R \lesssim \mu$ are trapped. These are the modes that are reflected back onto the BH by the potential wall, and that are superradiantly amplified as they return to the BH. Note that, strictly speaking, our approach is only valid for small spins and small masses. Therefore, we compare our results to numerical and analytical calculations for arbitrary mass coupling $M\mu$ and near extremal spins. As computed in Ref. [13], the form of the potential for high spins is very similar to the one that we have obtained. Also, we have seen that the centrifugal barrier gets high as l increases. Therefore, we have the maximum instability, associated to the maximum probability of tunneling, for minimum $l \neq 0$, i.e. $l = 1$. On the other hand, the maximum m is associated to a thicker barrier. Therefore, the $l = m = 1$ mode has the largest instability growth rate, in accordance with numerical calculations [2].

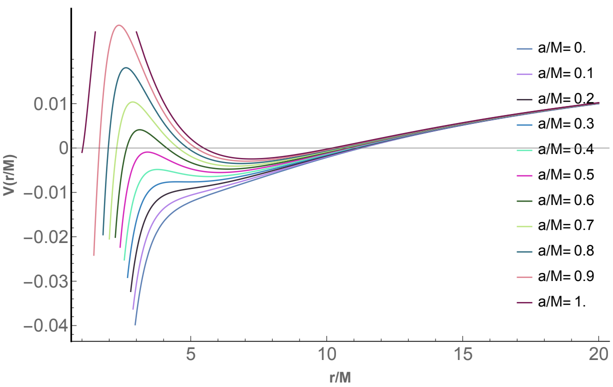

FIG. 1: Plot of the potential (13) for different spin values, $l = m = 1$, $n = 0$ and $M\mu = 0.5$.

B. Superradiance rate

In Fig. 2 we show the imaginary part of the frequency, which indicates the growth or decay rate of a mode, as a function of the mass coupling $M\mu$ and for different values of the spin. We focus on the $l = m = 1$ mode. We see that the growth rate increases with increasing $M\mu$ until the superradiant condition $\omega_R \sim \mu = m\Omega_H$ is saturated.

The maximum value of the imaginary part is $M\delta = 8.42 \cdot 10^{-9}$ at $M\mu = 0.279$ for $a/M = 0.9$. Comparing with numerical calculations for high spins [2], the maximum value for the same spin and mass is $M\delta \sim 1.55 \cdot 10^{-8}$ at $M\mu \sim 0.231$, i.e., a relative error of 21% for $M\mu$ and 46% for $M\delta$.

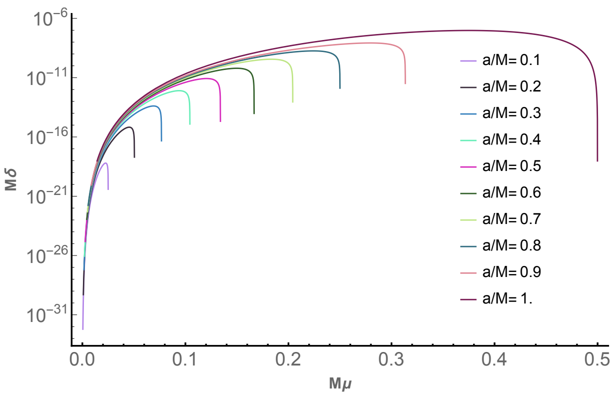

FIG. 2: Log-plot of the imaginary part of the frequency (36), rescaled by M to make it dimensionless, as a function of the field mass μ for the $l = m = 1$ mode.

VI. CONCLUSIONS

We have solved the Klein-Gordon equation analytically within the range $M\mu \ll 1$ and $M\omega \ll 1$, for small spins via a matched asymptotic expansion. The obtained hydrogen-like frequency spectrum shows that superradiance can occur when the condition (1) is met. Moreover, we can associate a growth time-scale due to the exponential dependence of the field and compute the superradiance rate of our physical system.

While our results agree well with numerical calculations [2] when the spin and mass coupling are small, they deviate by about 21% for $a/M \sim 0.9$ and $M\mu \sim 0.231$, where our approximation breaks down. Also, our potential has the same form as the one obtained by [13] for all the spin values. This allows us to make some deductions about the maximum values of the instability that also matches the numerical computation by [2].

Most studies of the evolution of the superradiant instability have focused on the fastest growing mode; see e.g. Ref. [14] for an adiabatic approximation. This is a specific setup, and in the future it would be interesting to investigate the BH response to a more generic field configuration. For example, fully nonlinear evolutions [15] indicate a more complex behaviour of the system in the presence of both stable and unstable modes. Moreover, several improvements are done in the field of gravitational waves detection [16], so we expect to have enough data within the next decade in order to clarify and constrast our different theories about axions and BHs in general.

Acknowledgments

I am very greatful to Christiano Germani for his amability and interest. I want to thank Helvi Witek for all her support, the uncountable meetings and to believe in me from the very first moment. Also, thank my family for their unstoppable support and love. Finally, to my friends and classmates for giving me the best four years of my life, sharing countless hours at the library.

[1] V. Cardoso and P. Pani, Class. Quant. Grav. **30**, 045011 (2013), [1205.3184].

[2] S. R. Dolan, Phys. Rev. **D76**, 084001 (2007), [0705.2880].

[3] A. Arvanitaki and S. Dubovsky, Phys. Rev. **D83**, 044026 (2011), [1004.3558].

[4] L. Hui, J. P. Ostriker, S. Tremaine and E. Witten, Phys. Rev. **D95**, 043541 (2017), [1610.08297].

[5] D. R. Brill, P. L. Chrzanowski, C. M. Pereira, E. D. Fackerell and J. R. Ipser, Phys. Rev. D **5**, 1913 (1972).

[6] C. Flammer, *Spheroidal Wave Functions*Dover Books on Mathematics (Dover Publications, 2014).

[7] J. M. Bardeen, W. H. Press and S. A. Teukolsky, Astrophys. J. **178**, 347 (1972).

[8] S. Detweiler, Phys. Rev. D **22**, 2323 (1980).

[9] Abramowitz and Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 10th ed. (Cambridge University Press, 1972).

[10] H. J. W. George B. Arfken and F. E. Harris, *Mathematical Methods for Physicists. A Comprehensive Guide*, 7th ed. (Elseiver, 2013).

[11] V. Cardoso, O. J. C. Dias, J. P. S. Lemos and S. Yoshida, Phys. Rev. **D70**, 044039 (2004), [hep-th/0404096], [Erratum: Phys. Rev.D70,049903(2004)].

[12] R. Brito, V. Cardoso and P. Pani, Lect. Notes Phys. **906**, pp.1 (2015), [1501.06570].

[13] A. Arvanitaki, S. Dimopoulos, S. Dubovsky, N. Kaloper and J. March-Russell, Phys. Rev. **D81**, 123530 (2010), [0905.4720].

[14] R. Brito, V. Cardoso and P. Pani, Class. Quant. Grav. **32**, 134001 (2015), [1411.0686].

[15] H. Okawa, H. Witek and V. Cardoso, Phys. Rev. **D89**, 104032 (2014), [1401.1548].

[16] Virgo, LIGO Scientific, B. P. Abbott *et al.*, Phys. Rev. Lett. **116**, 061102 (2016), [1602.03837].