Applications of the Gross-Pitaevskii equation to gravity.

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In this paper the main characteristics of the interior solution of a static black hole as a Bose-Einstein condensate of gravitons are explored. We find that, even up to the boundary of the black hole, quantum corrections cannot be ignored which in turn leads to the conclusion that, in this framework, not even in the classical limit the interior of the black hole is described adequately by General Relativity. Furthermore, a non-singular interior solution is found.

I. INTRODUCTION

In recent years, Dvali and Gomez [1–3] have proposed a new modelling of quantum black hole physics in terms of modern condensed matter physics, namely a quantum N-portrait. The motivation to do so lies in the fact that many phenomena of the standard view of black holes (BHs) arise naturally when considering such objects as Bose-Einstein condensate (BEC) of gravitons, e.g. Hawking evaporation. Furthermore, in this N-portrait framework, BHs would appear to be entirely non-singular, always stuck in the weak-gravity limit. Attempts to reconcile General Relativity with this BEC portrait of BHs have been proposed [4], however we will follow a different approach.

Our starting point to make a connection between gravity and condensed matter physics is the variational equation:

$$\delta E - \mu \delta N = 0 \tag{1}$$

this minimisation of the energy keeping the particle number fixed through the Lagrange multiplier μ leads in many-body quantum mechanics to the Gross-Pitaevskii (GP) equation, which is a generalisation of the Schrödinger equation to bosonic systems with interactions. Therefore, under the conjecture that a BH is a BEC of gravitons, an analogous GP equation should arise for it.

Following [1], one understands μ as the chemical potential conjugated to the number of gravitons in the condensate and that it is inversely proportional to the wavelength of the gravitons, which in turn are of the order of the Schwarzschild radius ($\lambda \sim r_s$). To an outside observer a BH would still look classical in this picture since the emitted gravitons have an infinitely large wavelength, however with a caveat, we find that inside of the black hole quantum corrections are always important, even up to the Schwarzschild radius, a surprising result since r_s is much larger than the Planck length L_p, the range at which quantum correction around the singularity would be expected to dominate. Consequently, the first result we will present is that, in this N-portrait framework, GR is never recovered for the interior of a BH, not even in what one would naively expect to be the classical limit. We believe that this disconnection might be related to the appearance of a wall-like boundary.

The second result is that a non-singular interior solution to the metric is found. Even more so, the classical notion of horizon, i.e. a null hypersurface, is not valid any more since there is no change of sign in the signature between the inner and outer metric.

A. Notation and units

Throughout this paper we will be adhering to the following notation: Greek indices run from 0 to 3, Latin indices run from 1 to 3, the spacetime metric is given by the tensor $g_{\mu\nu}$, while the spatial metric is given by h_{ij}; quantities defined on the spatial submanifold will be noted accordingly when confusion may arise, for instance: ${}^{(3)}R$ is the Ricci scalar on the spatial metric and R is the usual Ricci scalar of the full spacetime. Unit-wise: $\hbar = 1$.

II. THE ADM DECOMPOSITION OF THE METRIC

The first step towards the goal of this paper is to obtain the Hamiltonian of gravity in vacuum, in the sense of the Arnowitt, Deser & Misner formulation (ADM) [5]. To do so we would like to foliate our space in space-like hypersurfaces, essentially we are asking that to obtain a Hamiltonian formulation of GR the considered spacetime has to be globally hyperbolic, i.e. it admits a foliation in Cauchy surfaces (Σ_t), however given that we will only be considering spacetimes that contain a time-like Killing vector field (t^μ), a very natural set of coordinates that comply with the foliation condition arise, i.e. the natural coordinates of the basis. Therefore, under these conditions, we are allowed to split our 4-dimensional metric into a (1+3)-dimensional form, where the time-like vector field normal to Σ_t will be t^μ itself.

The most general ADM-decomposition of a metric takes the form of:

$$ds^2 = L^2 dt^2 + h_{ij}(dx^i + L^i dt)(dx^j + L^j dt)$$

where: L is the lapse function and L^i is the shift vector. Under the considerations of a static and spherically symmetric spacetime, the previous equation reduces to:

$$\begin{aligned} ds^2 &= L(r)^2 dt^2 + h_{ij} dx^i dx^j \\ h_{ij} &= \begin{pmatrix} \phi(r) & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2(\theta) \end{pmatrix} \end{aligned} \tag{2}$$

and now, following [6] the Hamiltonian of gravity is written, up to boundary terms, as:

$$\begin{aligned} H_G &\equiv \int \mathcal{H}_G \sqrt{h} dx^3 \\ \mathcal{H}_G &= \pi^{ij} \dot{h}_{ij} - \mathcal{L}_G \\ &= \sqrt{h} L \left[-^{(3)}R + \frac{1}{h}\left(\pi^{ij}\pi_{ij} - \frac{1}{2}(\pi^i_i)^2 \right) \right] \end{aligned} \tag{3}$$

however, in static conditions π^{ij}, the momentum canonically conjugated to h_{ij}, is null and the variation with respect to L yields the ADM Hamiltonian constraint:

$$\frac{\delta \mathcal{H}_G}{\delta L} = -^{(3)}R = 0 \tag{4}$$

Together with the evolution equation for π^{rr}:

$$\begin{aligned} \dot{\pi}^{rr} &= -\frac{\delta \mathcal{H}_G}{h_{rr}} \equiv -\frac{\delta \mathcal{H}}{\delta \phi} \\ &= -L\sqrt{h}\left(^{(3)}R^{rr} - \frac{1}{2}\, ^{(3)}R h^{rr} \right) \\ &+ \sqrt{h}\left(\nabla^r \nabla^r L - h^{rr} \nabla^k \nabla_k L \right) \end{aligned} \tag{5}$$

the behaviour of our system is completely determined, in a similar manner as the (tt) and (rr) components of the Riemann tensor solve Einstein field equations in vacuum. Now we define:

$$\delta E \equiv \delta H_G$$

so that (1) in conjunction with (4) and (5) read:

$$\begin{cases} -^{(3)}R\, \delta L = \mu \dfrac{\delta N}{\delta L} \delta L \\ \\ \dot{\pi}^{rr} \delta \phi = \mu \dfrac{\delta N}{\delta \phi} \delta \phi \end{cases} \tag{6}$$

where $\dot{\pi}^{rr}$ is to be understood as shorthand for the RHS of (5). The motivation of this definition is that whenever μ is null, our system reduces back again to GR as one would expect.

A. Quantum quantities of the theory

In the main section we have found a pair of equations that determine completely the behaviour of our system, however we still lack a definition of particle number and chemical potential. In a BEC, the only macroscopically occupied state is the ground energy level, that is all particles of the condensate are in the same quantum state and hence they have the same wavelength, then we define the particle number of gravitons as:

$$N \equiv \int d^3x \sqrt{h} \frac{(L^2+1)M_p^2}{\lambda_g} \tag{7}$$

for two reasons: first, due to the intuition that the particle number should scale with the Newtonian potential ($\Phi = L^2 + 1$), which is understood as the strength of gravity, and secondly since all particles are in the same quantum state –at zero-th order– all will have the same energy per particle contribution to the total energy, thus allowing us to calculate the particle number as the quotient on (7), which is saying that:

$$N \sim \frac{\text{total energy}}{\lambda_g}$$

Furthermore, we will suppose as in [1] that the coupling constant of gravity takes the form: $\alpha = \dfrac{1}{N} = \dfrac{1}{r_s^2 M_p^2}$.

On a first approach to solving (6) we will be checking whether or not the standard GR solution is recoverable in the interior of the BH for this particular model, under these assumptions one can integrate (7) up to the boundary r_s, where the lapse function L and the spatial volume element $\sqrt{h}$ will take on the form of the usual Schwarzschild interior solution, with that we find:

$$N = \frac{3M_p^2}{2\lambda_g}\pi^2 r_s^3 = r_s^2 M_p^2 \quad \longrightarrow \quad \lambda_g \propto r_s \equiv \tau\, r_s$$

which is in agreement with Dvali's and Gomez's proposal that $\lambda_g \sim r_s$ and we have redefined the λ_g in terms of τ, a finite constant that will depend upon the exact structure of our condensate, simply to generalise the result as we expect that even if the interior metric is not GR the proportionality $\lambda_g \sim r_s$ still will hold. Finally, the chemical potential is taken as in [1]:

$$\mu = \frac{1}{\lambda_g} = \frac{1}{\tau}\frac{1}{r_s} \tag{8}$$

Notice now that sending: $\mu \to 0$ implies $r_s \to \infty$, one might at first believe this to be the limit of classicality, however this is actually sending the particle number to infinity faster than the chemical potential goes to zero, so the product of μN is never null in the presence of a BEC condensate of gravitons. This will have very important repercussions in the theory, as we will see in the following section.

Finally, recollecting the results of (6 - 8) we obtained the main equations of our theory, in leading order:

$$\begin{cases} \frac{1}{(r\phi)^2}\left[(\phi-1)+r\partial_r\right]\phi = -\mu\frac{L}{\lambda_g} \equiv -2\mathcal{C}L & \text{(9a)} \\ \frac{1}{\phi r}\left[\frac{\phi-1}{r}-2\partial_r\right]L = \mu\frac{L^2+1}{2\lambda_g} \equiv \mathcal{C}(L^2+1) & \text{(9b)} \end{cases}$$

with: $\mathcal{C} \equiv \frac{\mu}{2\lambda_g} = \frac{1}{2\tau^2 r_s^2}$.

III. ANALYSIS OF THE EQUATIONS.

At this point we would like to solve our main set of equations, however an analytical solution for the whole interior of the BH is not possible, thus we resorted to the study of the behaviour of such equations close to the horizon and near the would-be singularity.

At first, we will be working with the supposition that GR solution is recoverable in the interior of the BH at leading order for some limit in the parameters of our theory that will take our quantum BH to classicality, and for the boundary that there is a matching condition between the exterior (GR) and the interior at leading order. We propose the ansatz:

$$\phi(r) = \frac{1}{1-\frac{R(r)}{r}} \tag{10}$$

for the simple reason that it simplifies the form of the differential equations, while being completely general. Solving (9a) in terms of the lapse function yields:

$$\partial_r R(r) = -\mathcal{C}r^2 L(r) \tag{11}$$

If the interior and exterior solution match on the horizon, then, at r_s, the interior lapse function should take the form: $L = \sqrt{\frac{r_s}{r}-1} + \mathcal{O}\left[\frac{r_s}{r}\right]$, from which one can solve (11):

$$\begin{aligned} R(r) \equiv &-\frac{\mathcal{C}}{24}\sqrt{\frac{r_s}{r}-1}\left(-3r_s^2 r - 2r_s r^2 + 8r^3\right) \\ &+\frac{\mathcal{C}}{16}r_s^3 \arctan\left(\sqrt{\frac{r_s}{r}-1}\,\frac{(2r-r_s)}{2(r-r_s)}\right) \\ &+R_p - \mathcal{C}^2\eta(r) \end{aligned} \tag{12}$$

where, R_p is a constant of integration and we have added $\eta(r)$, a small perturbation of the solution to generalise around the boundary, of course the matching condition forces from (10) that:

$$R(r_s) = r_s \quad \rightarrow \quad R_p = r_s - \frac{\mathcal{C}\pi}{32}r_s^3 + \mathcal{C}^2\eta(r_s)$$

so that we still comply with the initial hypothesis. One can now see that $\eta(r)$ acts as a shift. The problem of the matching reduces to checking the metric components and its first derivatives for continuity conditions at: $r \to r_s$. Thus, substituting the results of (12) in (10) and (11) we proceed to study the solution in two regimes. To ease the reader into the subsequent arguments, we show the explicit final form of the interior lapse function below:

$$L(r)\big|_{\text{int}} = \sqrt{\frac{r_s}{r}-1} + \frac{\mathcal{C}\eta'(r)}{r^2}$$

A. Horizon solution.

First, we will work in the limit: $r \to r_s$. In this particular situation the continuity conditions read, up to leading order:

i. For $g_{tt} \equiv L^2$, one obtains by forcing the match, with the interior on the l.h.s. and the exterior on the r.h.s. of the equation below:

$$L^2\big|_{\text{int}} = -1 + \frac{r_s}{r}\Big|_{\text{ext}} \;\rightarrow\; \frac{\mathcal{C}^2\eta'(r_s)^2}{r^4} + \mathcal{O}\left[\sqrt{\frac{r_s}{r}-1}\right] = 0$$

so, in a neighbourhood close enough to the boundary: $\eta'(r_*) \ll 1$, with $\eta'(r_s)$ null in leading order of $\left(\frac{r_s}{r}\right)$. From which we would expect that:

$$\eta(r_s) \sim 0; \quad \eta(r) \sim \mathcal{O}\left[\sqrt{\frac{r_s}{r}-1}\right]$$

at most.

ii. For $g_{rr} \equiv \phi$, we find:

$$\frac{r}{r_s + \frac{\mathcal{C}}{32}\pi r_s^3 - r + \mathcal{C}^2[\eta(r_s)-\eta(r)]}\Bigg|_{\text{int}} + \mathcal{O}\left[\sqrt{\frac{r_s}{r}-1}\right] = \phi(r)\big|_{\text{ext}}$$

then, in leading order on the boundary one obtains:

$$\frac{32}{\mathcal{C}\pi r_s^2}\Bigg|_{\text{int}} = \frac{64\tau^2}{\pi}\Bigg|_{\text{int}} = \phi(r_s)\big|_{\text{ext}} \tag{13}$$

iii. Similar conditions as (i) are found for the first derivatives.

Therefore, from (13) we find that there cannot ever be a matching at the boundary with GR at leading order of $\left(\frac{r_s}{r}\right)$, as we would have to ask for a divergence in the interior solution which will not happen, since taking $\mathcal{C} \sim \frac{\mu}{\lambda_g} \to 0$ is taking $r_s \to \infty$, as discussed previously, and the quotient in (13) becomes finite.

B. Interior solution.

Now turning back to checking the interior in a similar manner, we will be working in the range: $r_s \gg r \to 0$. In the previous subsection we checked that a matching with GR on the boundary is not possible. We now would like to see whether or not is possible to recover GR close to the singularity.

We will be solving (9b) under the ansatz:

$$\eta(r) = -\frac{2\sqrt{r_s r^5}}{5\mathcal{C}} + \mathcal{D}$$

which corresponds to the scenario where the lapse function is non-singular at $r \to 0$ in the leading order of r, with $\mathcal{D}$ a constant of integration left to determine. We chose this ansatz because we would like to see precisely the case where the metric is non-singular as $r \to 0$. Otherwise, if we cannot solve the equations of the theory or we are only able to find a singular solution, this model is somewhat useless regarding BH theory as there would be no benefit on working with it over regular GR.

As mentioned before, solving (9b) in leading order leads to the equation:

$$\sqrt{r_s}\left[\mathcal{C}^2\mathcal{D} - \mathcal{C}^2\eta(r_s) - r_s\right]\sqrt{r^3} + \mathcal{O}\left[\sqrt{r^5}\right] = 0$$

Solving for $\eta(r_s)$, again in leading order, yields:

$$\eta(r_s) = \frac{\mathcal{C}^2\mathcal{D} - r_s}{\mathcal{C}^2} = \mathcal{D} - \frac{r_s}{\mathcal{C}^2} \sim \mathcal{D} - 4\tau^4 r_s^5 \tag{14}$$

which will only be divergent when $\mathcal{C}$ is null, however this is again not possible and is related to the meaning of the product μN, as mentioned previously: for any value of the chemical potential the quantity μN always remains non-null as the particle number grows faster than the chemical potential decreases. Furthermore, we have left the constant $\mathcal{C}$ as is to show that not even by fine tuning any parameter in the theory, i.e. giving different values for the chemical potential μ or the wavelength λ_g, this conclusion differs.

Comparison of the results of both analysis leads to the following inconsistency: by asking for a match with GR, in the boundary analysis we have seen that at leading order the solution must have $\eta(r) = 0$ $(\forall r \leq rs)$, which is non-divergent in the boundary, and when studying the interior solution we find again a finite value for $\eta(r_s)$. Therefore we conclude that the initial hypothesis that GR is matched with our theory anywhere on the inside or boundary of the BH is wrong, which implies that, in this model, quantum corrections must be important up to r_s.

IV. NON-SINGULARITY OF THE MODEL

So far, we have seen that the interior solution is never GR, not even sending the chemical potential to zero is enough as the particle number diverges faster than the chemical potential goes to zero, meaning that quantum corrections are always important, albeit one should re-evaluate the exact value of the chemical potential now that we have shown that GR is not valid inside. Nevertheless, the qualitative result derived from (13 - 14) is independent from the value one chooses to give to the chemical potential and the conclusion from the previous section still holds.

This realisation leaded to the question of whether or not the *singularity* in: $r \to 0$; is an artefact of this *GR limit* in which we are, wrongly, enforcing a null μN inside. To check the solution close to zero we propose the following ansatz:

$$\begin{aligned} \phi &= 1 + \alpha_\phi r^2 \\ L &= 1 + \alpha_1 r + \alpha_2 r^2 \end{aligned}$$

solving for (9a, 9b), yields:

$$\begin{aligned} \phi &= 1 - \frac{2}{3}\mathcal{C}r^2 \\ L &= 1 - \frac{7}{6}\mathcal{C}r^2 \end{aligned} \tag{15}$$

Notice that with this solutions there is no change of signature in the metric between the interior and exterior of BH, unless a double change in the signature is produced at some point between the solution close to the would-be singularity and the exterior, which seems unlikely. With respect to the Kretschmann scalar, we now find that:

$$\mathcal{K} \equiv R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = 212\frac{\mathcal{C}^2}{3} + \mathcal{O}\left[r^2\right] \tag{16}$$

which, first and foremost, shows a non-singular curvature-squared scalar, unlike the standard Schwarzschild result (i.e. $\mathcal{K}_{std} \propto \frac{G^2M^2}{r^6}$), secondly, if one rewrites the Kretschmann scalar in terms of the structure constant τ and r_s:

$$\mathcal{C} \equiv \frac{\mu}{2\lambda_g} = \frac{1}{2\tau^2 r_s^2} \quad \to \quad \mathcal{K} = \frac{53}{3\tau^4}\frac{1}{r_s^4} \propto \frac{1}{r_s^4}$$

the curvature is much smaller than the Planck scale. Even more so, comparing the standard result with our own:

$$\mathcal{K}_{std} \propto \frac{G^2M^2}{r^6} \sim \frac{G^2M^2}{L_P^6} \gg M_P^4 \to \text{Strong gravity}$$

$$\mathcal{K} \propto \frac{1}{r_s^4} \ll M_P^4 \to \text{Weak gravity}$$

in this approach we are always in a weak gravity regime, which is in agreement with what was expected by [1]. The interpretation of this result is that a BH behaves as a large number of softly interacting long-wavelength gravitons.

V. CONCLUSIONS AND OUTLOOK

Throughout this paper, one has shown that in the quantum N-portrait model proposed by Dvali and Gomez the classical Schwarzschild interior metric, as a solution to Einstein equations, is never regained in the interior of the BH, quantum correction are always important, even in what would appear to be the classicality limit of: $r_s \to \infty$. Furthermore, the would-be singularity at: $r \to 0$; appears to be an artefact of ignoring such quantum corrections in GR.

Therefore, one concludes that in the present model the interior of a BH is entirely quantum and non-singular, up to Planck distances where the model stops being appropriate. In the future, the characteristics of the natural boundary that appears by considering a BH as a fluffy ball of gravitons will be studied, this might bring interesting results since we have seen that the classical sense of horizon disappears for our theory, i.e. there is no change of sign in the signature between the inner and outer metric. In the future we will be checking whether a double change of sign in the signature is possible, albeit it seems quite unlikely.

Furthermore, as of the moment of writing this paper, we still have to understand the exact nature of such boundary given that from the interior the infinite expected redshift at the boundary is not obtained. This would seem to point again to some discontinuity in the metric between the interior and the exterior, a possible explanation for this point is that maybe some kind of *wall* appears at the boundary. Finally, one would like to obtain an exact expression for the chemical potential, under the new finding that GR is not valid in the interior, while hopefully recovering the $\lambda_g \sim r_s$ proportionality.

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[1] G. Dvali and C. Gomez, Fortsch. Phys. **61**, 742 (2013), 1112.3359.
[2] G. Dvali and C. Gomez, Eur. Phys. J. **C74**, 2752 (2014), 1207.4059.
[3] G. Dvali and C. Gomez, Phys. Lett. **B716**, 240 (2012), 1203.3372.
[4] J. Alfaro, D. Espriu, and L. Gabbanelli (2016), 1609.01639.
[5] R. L. Arnowitt, S. Deser, and C. W. Misner, Gen. Rel. Grav. **40**, 1997 (2008), gr-qc/0405109.
[6] R. M. Wald, *General relativity* (Univ. of Chicago Press, 1984).