EMERGENCE OF LEADERSHIP IN COMPLEX NETWORKS

Joan F. Homs Caballero
Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain.

In this work, we introduce a modified version of the voter model that takes into account heterogeneous distributions of activity and influence present in real social networks, where highly connected agents tend to be both more active and influential. The aim is to study the conditions that lead to the emergence of spontaneous collective leadership in scale-free networks.

I. INTRODUCTION

Networks have proven useful in the description of many real-world systems in a wide variety of fields such as biology, economy, sociology or physics [1–3]. In this work, we explore one of the topics that is best described under this framework, the emergence of collective leadership in real-world social environments in which connectivity, activity and influence play a relevant role [4–6].

With this purpose in mind, we will make use of one of the most widely studied models of opinion dynamics, the standard voter model, which consists of a set of interacting agents endowed with a binary state of opinion, being activated at some predefined frequencies and acting by pure imitation, copying the opinion of a randomly selected neighbor when they get activated. By precisely defining the frequencies at which our agents will interact and how neighbor selection will be performed, we will be able to bring the mechanisms of activity and influence to light.

In addition, we will embed our agent into a scale-free network that will provide them with heterogeneous connectivity, dividing the population into two groups of high and low degree agents that will exhibit remarkable differences in their separate paths to consensus, the final fate of the system in which all agents share the same opinion.

Finally, under the assumption that highly connected agents tend to be both more active and persuasive, we will show how these two mechanisms compete to crown one of the two groups as the effective leaders of opinion.

II. OPINION DYNAMICS

A. Herding Voter Model

To address connectivity, we embed our agents in a a scale-free network so that every agent i is assigned a degree k_i sampled from a distribution of the form $P(k) \sim k^{-\gamma}$, with γ typically in the range $2 \leq \gamma \leq 3$. This provides them with a set of acquaintances and divides the population into a vast majority of similarly connected agents with low degree and a tiny minority of highly connected agents, usually called hubs. The complete description of this structure is encoded in the adjacency matrix A with coefficients a_{ij} being 1 if i and j are connected and 0 otherwise.

To address activity, we introduce a new parameter α and assign every agent an activity rate in consonance with its degree so that agent i revises its opinion at rate $\lambda_i \sim k_i^{\alpha}$. The parameter α modulates the effect that connectivity has over the activity of our agents and allows us to capture the fact that the best connected agents tend to be the also the most active ones.

Finally, influence is addressed introducing a new parameter β and the probability $\mathrm{Prob}(j|i) \sim a_{ij}k_j^{\beta}$ that agent i copies the opinion of agent j. The particular form of $\mathrm{Prob}(j|i)$ makes the influence of j over i a function of k_j and its relevance is controlled by means of β that allows us to capture the fact that best connected agents are also the most influential.

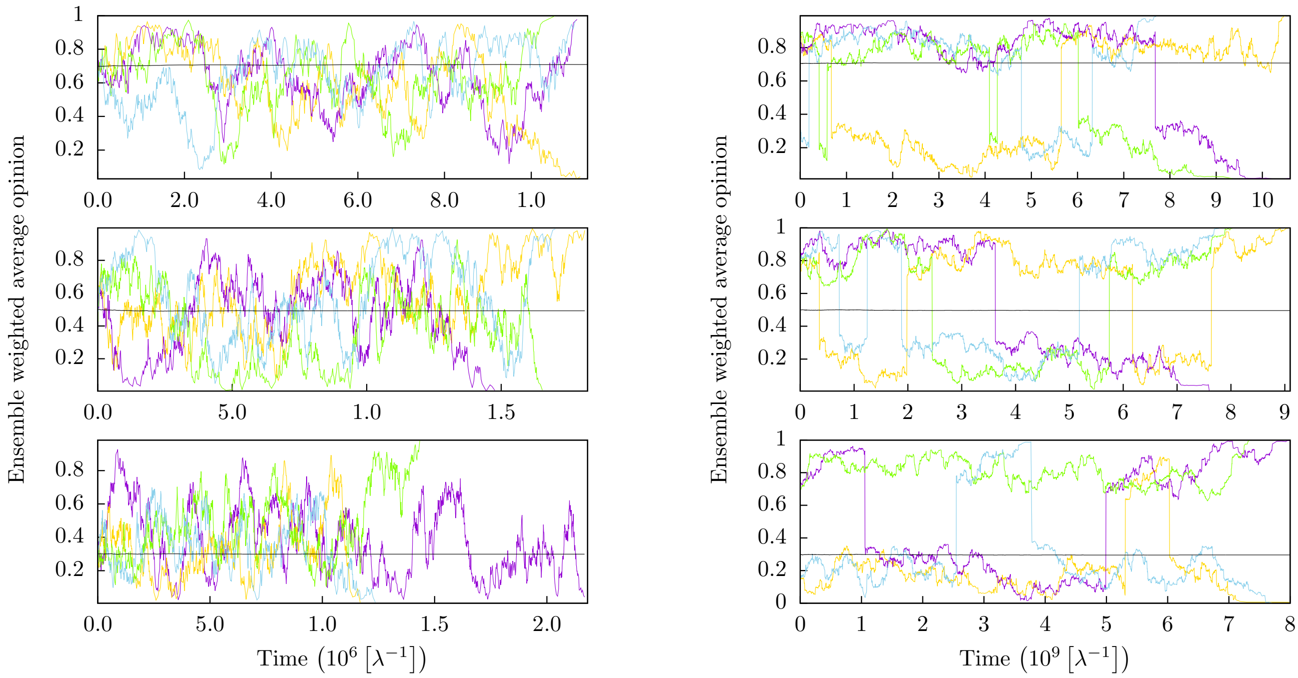
This modified version of the voter model can be considered as a major generalization that retains some valuable features present in real systems and that can be reduced to the standard version by simply setting $\alpha = \beta = 0$. The dynamics of the state of the system can be completely described using a set of N stochastic processes, $\{n_i(t)\}$ that takes value 0 or 1 according to the opinions of each agent at time t. Assuming that all temporal processes follow Poisson statistics, the evolution of $n_i(t)$ after an increment of time dt satisfies the stochastic equation

$$n_i(t+dt) \;=\; n_i(t)\left[1-\xi_i(t)\right] + \eta_i(t)\xi_i(t) \qquad (1)$$

where $\xi_i(t)$ is a random variable that controls whether node i is activated during the interval $(t, t+dt)$ and takes value

$$\xi_i(t) = \begin{cases} 1 & \text{with probability } \lambda_i\, dt \\ 0 & \text{with probability } 1-\lambda_i\, dt. \end{cases} \qquad (2)$$

When $\xi_i = 0$, node i is not activated and thus, preserves its opinion state.



(a) Conservation of Φ for $\alpha = 0$ and consequently, all rates fixed at $\lambda_i = 1$.

(b) Conservation of Φ for λ_i uniformly distributed in the range $1 \leq \lambda_i \leq 10^3$

FIG. 1: Conservation of $\Phi(t)$ computed over 10^4 realizations on a network with $\gamma = 2.5$, $N = 10^3$ and $\beta = 0$ assigning initial opinion "1" with probability 0.7, 0.5, and 0.3 from top to bottom, respectively. Colored traces correspond to evolutions of Φ in particular realizations while black traces correspond to the average $\bar{\Phi}$.

When $\xi_i = 1$, instead, i gets activated and copies the opinion of one of its neighbors j according to Prob$(j|i)$ so that

$$\eta_i(t) = \begin{cases} 1 & \text{with probability } \sum_{j=1}^{N} \text{Prob}(j|i)\eta_j(t) \\ 0 & \text{with probability } 1 - \sum_{j=1}^{N} \text{Prob}(j|i)\eta_j(t). \end{cases} \tag{3}$$

This set of equations represent the exact stochastic evolution of the system and allows us to derive the instantaneous average opinion of agent i, $\rho_i ::= \langle n_i(t) \rangle$ by averaging Eq.(1) over $\xi_i(t)$, $\eta_i(t)$ and then, over the ensemble leading to

$$\frac{d\rho_i}{dt} = \lambda_i \left[\sum_{j=1}^{N} \text{Prob}(j|i)\rho_j - \rho_i \right]. \tag{4}$$

Given its particular form, one can define a weighted average opinion $\Phi ::= \sum_i \phi_i \rho_i$ where the weights ϕ_i are characteristic of every agent and can be computed to keep Φ preserved.

The computation of those weights imply solving

$$\sum_{i=1}^{N} \lambda_i \phi_i \text{Prob}(j|i) = \lambda_j \phi_j \tag{5}$$

which follows from computing the weighted sum of Eq.(4) and imposing conservation, the solutions are

$$\phi_i = \frac{k_i^\beta}{\lambda_i} \sum_{j=1}^{N} a_{ij} k_j^\beta. \tag{6}$$

Φ is therefore conserved, its value depends entirely on the initial conditions and reads

$$\Phi = \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{a_{ij} k_i^\beta k_j^\beta}{\lambda_i} \rho_i(t) = \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{a_{ij} k_i^\beta k_j^\beta}{\lambda_i} \rho_i(t_0). \tag{7}$$

Although the particular evolution of Φ can be completely different in every single realization, its average over a large number of realizations is preserved over time. Under this framework, the final fate of a finite system is always a frozen consensus state in which all agents share the same opinion. This implies that $\rho_i(t \to \infty)$ should be the same for every agent and equal to the probability of ending up absorbed in the "1" consensus state. The conservation of Φ becomes useful here allowing us to express it as $\Phi/\sum_i \phi_i$ which equals $\rho_i(t_0)$ when we let the last be the same for every agent. A graphical representation of this fact is given in figure 1 for two different distributions of λ_i.

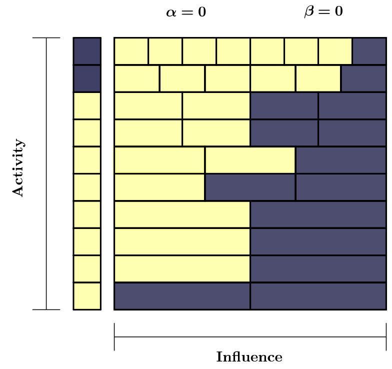
B. Emergence of Leadership

In order to gain insight about how α and β shape the evolution towards consensus, we can build a representation of the system encoding all the relevant information that consists of a set of N stacked rows fulfilling a square of area 1. Each row is assigned an agent i and divided into k_i cells representing every one of its neighbors. The height of a row will measure the activity of the corresponding agent, will be given by $h_i = k_i^\alpha / \sum_j k_j^\alpha$ and its area will represent the probability of activation of agent i in the time interval $(t, t+dt)$. The width of each cell will measures the relative influence that a particular neighbor j has over the opinion of a given agent i, its width will be given by $w_{ij} = k_j^\beta / \sum_j a_{ij} k_j^\beta$ and its area will represents the probability of agent i copying the opinion of its neighbor j.

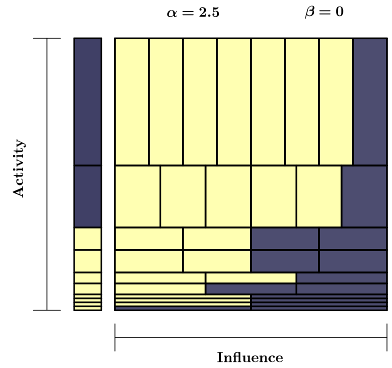
As an example, we show a picture of a small system with $N = 10$, and $\alpha = \beta = 0$ in figure 2. For readability, rows are ordered from bottom to top according to connectivity so that hubs, represented as blue, are placed top and low degree agents, represented as yellow, bottom. The same applies for cells, with highly connected neighbors placed right and low degree neighbors left.

Under this representation, the evolution of the system can be visualized as sampling a uniform random variable over the square such that when the sample falls within the area of the cell c_{ij}, agent i, copies the opinion of its neighbor j and the state of the system changes accordingly. In figure 2.a we provide a representation of a system shaped according to standard voter model dynamics, with $\alpha = \beta = 0$. In this representation, every agent interacts with his neighborhood with the same frequency, reflected as a uniform distribution of heights in the rows, and every neighbor of a particular agent has the same influence over his opinion, reflected as a uniform distribution of areas over the cells of each row.

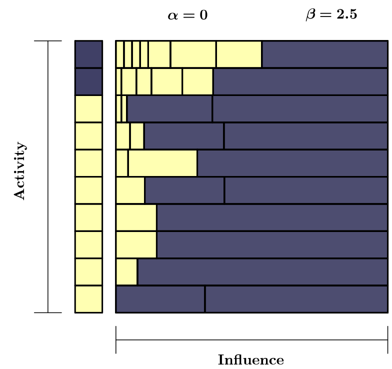
The parameter α determines how connectivity maps into activity. Its increase introduces a distribution of heights in the rows of the system so that when α is high enough, rows representing highly connected nodes tend to expand over the square intensifying their activity. The rest of the rows tend to collapse to the bottom making the low-degree agents, to whom they represent, difficult to target by the random variable and, consequently, less active.



(a) When $\alpha = \beta = 0$, row heights and cell widths are evenly distributed.



(b) When α increases, hub rows tend to cover greater area while low degree agents tend to collapse to the bottom.



(c) When β increases, hub cells tend to cover greater areas while low degree cells tend to collapse to the left.

FIG. 2: Schematic representation of a system with $N = 10$ and different values of α and β. In this representation, hubs are colored blue and low-degree agents, yellow.

This situation corresponds to a system in which hubs are completely open-minded, willing to copy the opinion of their neighbors with higher frequency, and quasi-frozen low degree agents, reticent to change their opinion and, therefore, making it prevail in the system for longer periods of time. Its schematic representation is shown in figure 2.b.

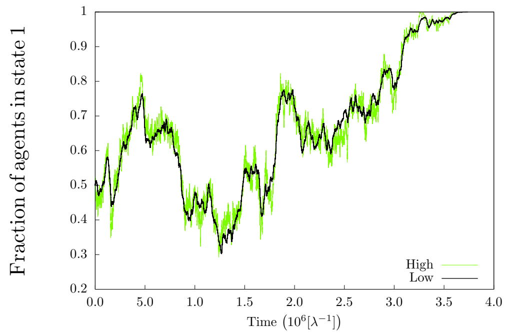
The parameter β, instead, controls how connectivity maps into influence. Its increase introduces a distribution of widths in the cells of each row so that when β is high enough, cells representing highly connected agents tend to expand over the row enhancing their influence. The rest of the cells tend to collapse to the left, making the agents, whom they represent, difficult to target by the random variable and, consequently, less influential. This situation translates into a system in which hubs govern the opinion of the system while low-degree agents become widely ignored. Its schematic representation is shown in figure 2.c.

Under this framework, the relevance of the opinion of a given agent in the final fate of the system is completely determined by three mechanisms that are also reflected in the weights of equation (6): Connectivity, that empowers higher connected agents to spread their opinion over a wider part of the population; influence, that enables the opinion of a given agent to be copied with higher frequency; and activity, that allows the opinion of the less active nodes to prevail for longer periods of time.

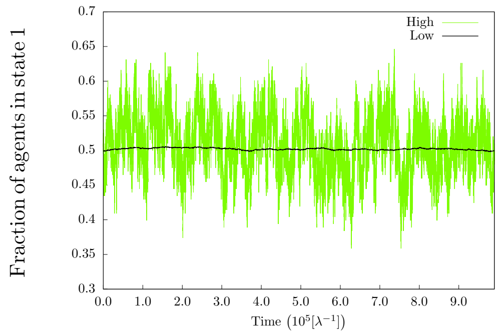
Following this reasoning, if both mechanisms, activity and influence, are turned off (α and β close to zero), both groups diffuse together to one of the consensus states as in the standard voter model. In this situation only connectivity plays a role and, since highly connected nodes are individually more relevant for the final fate but also significantly less numerous, there is no clear dominance of one group over the other in terms of opinion (figure 3.a).

When our agents interact at widely separate frequencies but influence is distributed evenly over the neighbors (high α but small β), the final fate of the system is entirely governed by low-degree agents, who remain quasi-frozen in their states pulling the highly adaptable group of active hubs towards their opinion, constituting a relevant part of their neighborhoods and exerting remarkable influence. In this situation, consensus becomes virtually impossible since it requires that a numerous group of low-frequency interacting agents agree (figure 3.b).

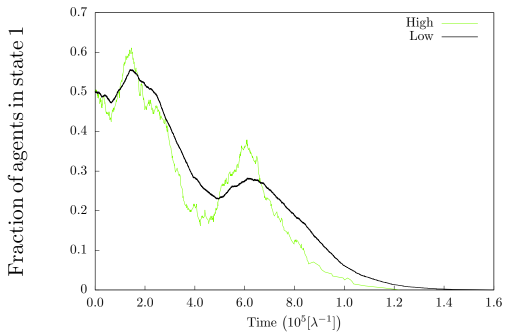
In contrast, when highly connected agents are much more influential than low-degree ones (high β but small α), despite being considerably less numerous, their individual opinions are so heavily weighted in the final fate of the



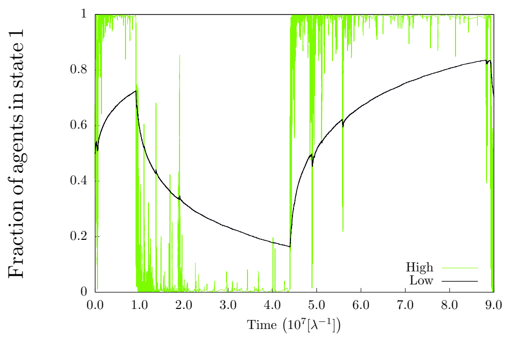
(a) Simulation with $\alpha = 0$ and $\beta = 0$



(b) Simulation with $\alpha = 2.5$, $\beta = 0$



(c) Simulation with $\alpha = 0$, $\beta = 2.5$



(d) Simulation with $\alpha = 2.5$, $\beta = 2.4$

FIG. 3: Different behaviors exhibited by a system with $\gamma = 2.1$, $N = 10^4$ and dissassortative structure according to α and β. Green traces correspond to high-degree agents and black, to low-degree agents.

system that they become the effective leaders of opinion. Since they exert remarkable influence over the group of low-degree agents and their opinion is almost completely determined by the rest of the highly connected agents, they are able to reach consensus on their own ignoring the rest of the population who will deterministically adopt their opinion (figure 3.c).

When activity and influence shape together the dynamics of the system, a competition between both groups is established. β provides increasing amounts of persuasive power to the highly connected group but α provides prevalent opinions to low-degree agents, making them ignore what their peers have to say about the particular topic and, therefore, nullifying their persuasive power. There exists a regime in which the group of highly connected agents behaves effectively as a two-state system remaining in one of the consensus states during relatively long periods of time and being eventually pulled out abruptly towards the opposite one.

In this situation, the highly connected group is able to reach consensus almost freely thanks to the influence provided by β but there is still a chance that, once in consensus, a highly connected agent adopts the opposite opinion from a low-degree agent, introducing some noise in the system that can eventually pull the whole group towards the opposite consensus state. In order to give rise to this event, the value of α should allow the group of highly connected agents to perceive low-degree agents as frozen in their states during long periods of time. Now, if the value of α is extremely high, the group of hubs will keep jumping abruptly between consensus states while the group of low-degree agents remains completely frozen in their initial state but, in contrast, if the value α is not extremely high, the final fate of the system will be governed by the group of highly connected agents that will be able to pull the whole group of low-degree agents towards their opinion (figure 3.d).

III. CONCLUSIONS

The introduction of a modified version of the voter model has allowed the incorporation of heterogeneous distributions of activity and influence that has led to the emergence of collective leadership and herding. The division of the system into two separate groups of highly connected and low-degree agents and the study of their separate evolution towards consensus, has revealed how these two social mechanisms of interaction compete to rise one of the two groups as effective leaders of opinion. The results presented in this work are, nonetheless, derived from computational simulations and further analysis is required in order to clarify with precision the exact conditions that lead to this phenomenon.

IV. ACKNOWLEDGMENT

Special thanks to my advisor, Marián Boguñá, for his valuable guidance and support.

[1] S. Wasserman and K. Faust, *Social Network Analysis: Methods and Applications*, Cambridge University Press, Cambridge (1994).

[2] R. Albert and A.-L. Barabási, *Review of Modern Physics* **74**, 47-97 (2002).

[3] S. N. Dorogovtsev and J. F. F. Mendes, *Advances in Physics* **51**, 1079-1187 (2002).

[4] L. Rozanova and M. Boguñá. arXiv:1704.06473 (2017).

[5] M. Catanzaro, M. Boguñá and R.Pastor-Satorras. Phys. Rev. E **71**: 027103 (2005).

[6] G. Mosquera and M. Boguñá. Phys. Rev. **91**: 052804 (2015).