WORKING PAPERS

Col.lecció d'Economia E17/369

The incentive core in co-investment problems

Josep Ma. Izquierdo

Carlos Rafels

UNIVERSITAT DE BARCELONA

UB Economics Working Papers 2017/369

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Abstract: We study resource-monotonicity properties of core allocations in co-investment problems: those where a set of agents pool their endowments of a certain resource or input in order to obtain a joint surplus or output that must be allocated among the agents. We analyze whether agents have incentives to raise their initial contribution (resource-monotonicity). We focus not only on looking for potential incentives to agents who raise their contributions, but also in not harming the payoffs to the rest of agents (strong monotonicity property). A necessary and sufficient condition to fulfill this property is stated and proved. We also provide a subclass of co-investment problems for which any core allocation satisfies the aforementioned strong resource-monotonicity property. Moreover, we introduce the subset of core allocations satisfying this condition, namely the incentive core.

JEL Codes: C71, D63, D70.

Keywords: Core, co-investment problems, proportional allocation, resource-monotonicity.

Josep Ma. Izquierdo
Universitat de Barcelona

Carlos Rafels
Universitat de Barcelona

ISSN 1136-8365

1. Introduction

A co-investment problem is described as the one where agents are endowed with some amount of input (resources, labor, capital,...) so that they can pool them and obtain some amount of output through a technology with increasing average returns. The problem is then how to share this output. This is a simple but interesting model of a one input-one output system, already quoted in some papers like Lemaire (1984, 1991), who studies insurance problems, Shubik (1962) or Mas-Colell et al. (1995), where a production problem is analyzed, Izquierdo and Rafels (2001), who propose a financial problem, and Moulin (1990) or Roemer and Silvestre (1993).

The increasing average returns assumed in this model provide incentives to cooperate and the core of this situation turns out to be always a nonempty set. That is, we can find a feasible and efficient allocation of the total output such that no subcoalition of agents can block it upon. The proportional allocation with respect to the amount of input contributed arises as a natural distribution within the core.

In this paper we mainly address the study of the behavior of the core of this model when an agent or some agents vary their contributions of the amount of input, namely *resource-monotonicity*[1].

By its importance, the monotonicity of the core has been already studied in a game theory framework by Young (1985) and Housman and Clark (1998), among others. This author shows the incompatibility between core selection and the fact that the solution increases the payoff to the members of some coalition whenever the worth of this coalition increases, remaining the worth of other coalitions the same (*coalitional*

[1]Resource-monotonicity has already been analyzed in the context of allocation problems: Chun and Thomson (1988) study monotonicity properties of bargaining problems; Thomson (1994) analyzes resource-monotonicity in the context of fair division when preferences are single-peaked.

monotonicity). Meggido (1974) and Calleja et al. (2009) have studied monotonicity properties with respect to the worth of the grand coalition. Other authors has examined monotonicity in restricted domains like convex games (Hokari, 2000) or veto balanced games (Arin and Feltkamp, 2005) reaching compatibility results between this two properties.

After some preliminaries (Section 2), we first concentrate in Section 3 on the monotonicity of the core and we show (see Proposition 1) that if only one agent increases his or her contribution, the whole core behaves monotonically, that is, any core allocation of the initial problem can be represented in the core of the new problem so that the payoff to this agent increases. Surprisingly, this property does not hold when two or more agents increase their contributions at the same time (see Example 1). In Section 4 we study a necessary and sufficient condition that must fulfill an initial allocation to guarantee that not only the payoff of the agents that have eventually increased their contribution increases, but also to avoid the payoff of the rest of agents to be lowered (see Theorem 1). In Section 5, we analyze the set of allocations that satisfies the condition of Theorem 1 for any eventual increasing of resources; we call this set as the *incentive core* and we prove that the proportional distribution is one of these allocations (see Proposition 2). In Example 4 we show that this set of allocations may contain more than the proportional one by cutting the core properly.

In Theorem 2 we state necessary and sufficient conditions for the coincidence of the incentive core and the core of a game. On the opposite extreme, Proposition 3 gives a sufficient condition that makes the incentive core shrink into the proportional allocation.

2. Preliminaries

A *co-investment problem* is represented by a triplet (N, ω, f) where $N = \{1, 2, \ldots, n\}$ is the set of agents, $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \mathbb{R}^N_{++}$ is the vector[2] of individual resource endowments, where ω_i is the amount of resource owned by agent $i \in N$, and the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ represents the technology that transforms x units of input into $f(x)$ units of output, with the following assumptions:

$$\begin{aligned} &\text{(a) } f(0) = 0 \text{ and } f(\textstyle\sum_{i=1}^n w_i) > 0 \\ &\text{(b) for any } 0 < z_1 \leq z_2 \text{ then } \frac{f(z_1)}{z_1} \leq \frac{f(z_2)}{z_2}. \end{aligned} \tag{1}$$

Condition (a) states that no output can be produced with no input contribution and some output is produced if all agents contribute. Condition (b) formalizes the classical idea of increasing average returns.

The problem at issue is how to distribute the output $f(\sum_{i \in N} \omega_i)$ among the members of N. An allocation of $f(\sum_{i \in N} \omega_i)$ is denoted by a vector $x = (x_1, x_2, \ldots, x_n)$, where x_i is the allocation to agent $i \in N$. We write $x(S) = \sum_{i \in S} x_i$, for all $\varnothing \neq S \subseteq N$, $x(\varnothing) = 0$ and x_S denotes the restriction of $x \in \mathbb{R}^N$ to the members of $S \subseteq N$, $S \neq \varnothing$.

The *core* of a co-investment problem (N, ω, f) is the most outstanding set-solution defined as

$$C(N, \omega, f) := \{x \in \mathbb{R}^N \mid x(S) \geq f(\omega(S)), \forall\, S \subseteq N, \text{ and } x(N) = f(\omega(N))\}.$$

It is easy to check from condition (1) that the proportional allocation with respect to ω, $P(N, \omega, f)$, is always a core element where

[2] Given a set $N = \{1, 2, \ldots, n\}$, $\mathbb{R}^N_+$ ($\mathbb{R}^N_{++}$) stands for the n-dimensional space of non-negative (positive) vectors whose components are indexed by N.

$$P_i(N,\omega,f) := \omega_i \cdot \frac{f(\sum_{i\in N}\omega_i)}{\sum_{i\in N}\omega_i}, \text{ for all } i \in N.$$

At the proportional allocation, the return per unit contributed is constant and so different amounts of contributed input receive the same average return. However, the core of a co-investment problem is in general wider and other core-selections can be addressed in order to consider different average rewards depending on different contributions. For instance, we can consider the *average proportional distribution*, $AP(N,\omega,f)$ that mixes the idea of marginal output contribution of agents, $f(\omega(N)) - f(\omega(N\setminus\{i\}))$, and the idea of proportionality. It is defined as follows:

$$AP(N,\omega,f) := \frac{1}{n}\sum_{j\in N} x^j(N,\omega,f),$$

where $x^j(N,\omega,f) = (x_1^j, x_2^j, \ldots, x_n^j) \in \mathbb{R}^N$ is defined as

$$\begin{aligned} x_i^j &:= f(\omega(N)) - f(\omega(N\setminus\{j\})), && \text{if } i = j \\ x_i^j &:= \omega_i \cdot \frac{f(\omega(N\setminus\{j\}))}{\omega(N\setminus\{j\})}, && \text{if } i \in N,\ i \neq j. \end{aligned} \tag{2}$$

It can be shown that $AP(N,\omega,f)$ is a core selection just by checking that each vector $x^j(N,\omega,f)$, $j\in N$, is also a core selection. To this aim, it is easy to see that, for any $j \in N$, $x^j(N) = f(\omega(N))$. Moreover, by (1), we have that for any $S \subseteq N$, $j \notin S$, $x^j(S) = \omega(S)\cdot\frac{f(\omega(N\setminus\{j\}))}{\omega(N\setminus\{j\})} \geq \omega(S)\cdot\frac{f(\omega(S))}{\omega(S)} = f(\omega(S))$. On the other hand, if $j \in S$ we have

$$
\begin{aligned}
x^j(S) &= x^j_j + x^j(S \setminus \{j\}) \\
&= f(\omega(N)) - f(\omega(N \setminus \{j\})) + \omega(S \setminus \{j\}) \cdot \frac{f(\omega(N \setminus \{j\}))}{\omega(N \setminus \{j\})} \\
&= f(\omega(N)) - \omega(N \setminus \{j\}) \cdot \frac{f(\omega(N \setminus \{j\}))}{\omega(N \setminus \{j\})} \\
&\qquad\qquad + \omega(S \setminus \{j\}) \cdot \frac{f(\omega(N \setminus \{j\}))}{\omega(N \setminus \{j\})} \\
&= f(\omega(N)) - \omega(N \setminus S) \cdot \frac{f(\omega(N \setminus \{j\}))}{\omega(N \setminus \{j\})} \\
&= \omega(N) \cdot \frac{f(\omega(N))}{\omega(N)} - \omega(N \setminus S) \cdot \frac{f(\omega(N \setminus \{j\}))}{\omega(N \setminus \{j\})} \\
&\geq \omega(N) \cdot \frac{f(\omega(N))}{\omega(N)} - \omega(N \setminus S) \cdot \frac{f(\omega(N))}{\omega(N)} \\
&= \omega(S) \cdot \frac{f(\omega(N))}{\omega(N)} \geq f(\omega(S)).
\end{aligned}
$$

Notice that both the proportional and the average proportional collapse if $\frac{f(\omega(N))}{\omega(N)} = \frac{f(\omega(N \setminus \{i\}))}{\omega(N \setminus \{i\})}$, for all $i \in N$, but in general they are different core selections. Moreover, the fact that each vector $x^j(N, \omega, f)$ is indeed a core element also proves that the marginal contribution of any player $i \in N$, $f(\omega(N)) - f(\omega(N \setminus \{i\}))$, is attainable within the core of any co-investment problem (N, ω, f).

3. Resource-monotonicity and the core

In the previous section we have defined two rules that both propose core allocations. Furthermore, if some agent increases their initial contribution, it is easy to check that each of these rules also selects an allocation that increases the payoff to this agent. In this section we aim to analyze whether for any core allocation, and when an agent increases his contribution, there exists an allocation in the core that gives a larger reward to this agent. This is an important fact since a positive result will tell that an agent can increase his contribution without expecting that any subcoalition of agents might block this initiative. The increasing average returns nature inherent to the model suggests that this should be always possible, as the next proposition states.

Proposition 1. *Let (N,ω,f) and (N,ω',f) be two co-investment problems with $|N| \geq 2$ such that, for some $i^* \in N$, $\omega'_{i^*} > \omega_{i^*}$ and $\omega'_j = \omega_j$, for all $j \in N \setminus \{i^*\}$. Then, for any $x \in C(N,\omega,f)$ there exists $x' \in C(N,\omega',f)$ such that $x'_{i^*} > x_{i^*}$.*

Proof Since $x \in C(N,\omega,f)$ it holds $x_{i^*} \leq f(\omega(N)) - f(\omega(N \setminus \{i^*\}))$. Taking this fact into account, we take $x' = x^{i^*}(N,\omega',f)$ as it is defined in (2). This is,

$$\begin{aligned} x'_{i^*} &= f(\omega'(N)) - f(\omega'(N \setminus \{i^*\})) \\ x'_i &= \omega'_i \cdot \frac{f(\omega'(N \setminus \{i^*\}))}{\omega'(N \setminus \{i^*\})} \text{ for } i \neq i^*. \end{aligned}$$

It is straightforward that $x'(N) = f(\omega'(N))$ and $f(\omega'(N)) > f(\omega(N))$, since $\omega'(N) > \omega(N) > 0$ and $\dfrac{f(\omega'(N))}{\omega'(N)} \geq \dfrac{f(\omega(N))}{\omega(N)} > 0$. Hence, it follows

$$x'_{i^*} = f(\omega'(N)) - f(\omega'(N \setminus \{i^*\})) > f(\omega(N)) - f(\omega(N \setminus \{i^*\})) \geq x_{i^*},$$

since $\omega(N \setminus \{i^*\}) = \omega'(N \setminus \{i^*\})$.

Since we know that $x' = x^{i^*}(N,\omega',f) \in C(N,\omega',f)$, this ends the proof. □

Unfortunately, this result cannot be generalized to the case where two or more agents increase their initial contribution. Next example shows that, if two agents increase their contribution at the same time, not every initial core allocation can be adapted to the new problem, that is, both players cannot benefit simultaneously from increasing their initial contribution.

Example 1. *Let $\omega = (1,2,3)$ be the vector of initial endowments for a three-agent co-investment problem where $f(x) = x$, for $0 \leq x < 5$, and $f(x) = 1.5 \cdot x$, for $5 \leq x$. The allocation $x = (1,5,3)$ is in the core of the problem, $C(N,\omega,f)$. Let us suppose* *that players 1 and 2 increase their initial contribution by 1 unit, that is* $\omega' = (1+1, 2+1, 3) = (2,3,3)$*. Notice that for any core element* $x' \in C(N, \omega', f)$ *it holds* $x'_2 \leq f(\omega'(N)) - f(\omega'(\{1,3\})) = 4.5$*. Therefore, it is not possible to find a core element* $x' \in C(N, \omega', f)$ *such that* $x'_1 > x_1$ *and* $x'_2 > x_2 = 5$*.*

And what is more disappointing, even in the case of only one agent raising its contribution, the increased payoff to this agent (guaranteed in Proposition 1) can be at a cost of diminishing the payoff to some of the rest of players. We illustrate this point by an example.

Example 2. *Consider the same problem than in Example 1 but now* $\omega = (1,3,3)$ *and take the core element* $x = (1.25, 5, 4.25)$*. If agent 1 increases his contribution up to* $\omega'_1 = 3$*, then* $\omega' = (3,3,3)$ *and the core shrinks into a single allocation,* $C(N, \omega', f) = \{x'\} = \{(4.5, 4.5, 4.5)\}$*. Notice* $x'_2 < x_2$*.*

The question of which are the conditions that guarantee strictly better payoffs to those agents who increase the contribution and give at least the same payoff to other agents is addressed in the next section.

4. Strong resource-monotonicity and the core: main results

In a dynamic framework, if a set of agents aims to increase their contribution, the allocation of profits should be revised accordingly. In this case, the previous (old) allocation acts as a starting point (or *status quo*) for revising the sharing of profits. Then, the resource-monotonicity requirement should address not only to improve the payoff to agents with increasing contributions, but not to harm the rest of the agents.

Next theorem states a necessary and sufficient condition for a particular core element to fulfill this kind of strong resource- monotonicity requirement.

Theorem 1. *Let (N, ω, f) and (N, ω', f) be two co-investment problems with $|N| \geq 2$ where for some nonempty $S \subseteq N$ we have $\omega'_i > \omega_i$, for all $i \in S$, while $\omega'_i = \omega_i$, for all $i \in N \setminus S$. For any core allocation $x \in C(N, \omega, f)$, the following statements are equivalent:*

1. *There exists $x' \in C(N, \omega', f)$ such that*
   $x'_i > x_i$*, for all* $i \in S$*,*
   $x'_j \geq x_j$*, for all* $j \in N \setminus S$*.*

2. *For all $\varnothing \neq R \subseteq N$ we have*
   $x(R) \leq f(\omega'(N)) - f(\omega'(N \setminus R))$*, if* $R \subseteq N \setminus S$ *and*
   $x(R) < f(\omega'(N)) - f(\omega'(N \setminus R))$*, if* $R \cap S \neq \varnothing$*.*

Proof 1.$\rightarrow$ 2.) Let $R \cap S \neq \varnothing$. Notice $x'(R) > x(R)$, since $x'_i > x_i$ for all $i \in S$. Moreover, as $x' \in C(N, \omega', f)$, $x'(N) = f(\omega'(N))$ and $x'(N \setminus R) \geq f(\omega'(N \setminus R))$. If we subtract both expressions we obtain $x'(R) = x'(N) - x'(N \setminus R) \leq f(\omega'(N)) - f(\omega'(N \setminus R))$. But then $x(R) = x(R \cap S) + x(R \setminus S) < x'(R \cap S) + x'(R \setminus S) = x'(R) \leq f(\omega'(N)) - f(\omega'(N \setminus R))$. The proof of the case $R \cap S = \varnothing$ follows the same argument but we obtain a non-strict inequality and we are done.

2.$\rightarrow$ 1.) To prove this implication define the cooperative game[3] (N, v') where

$$v'(T) = f(\omega'(T)), \text{for all } T \subseteq N, \tag{3}$$

with $v(\varnothing) = 0$. By (1), this game satisfies average monotonicity with respect to $\omega' \in \mathbb{R}^N$

[3] The function v' is called the *characteristic function* and assigns to every subcoalition of agents $S \subseteq N$, its *worth* $v'(S) \in \mathbb{R}$, with $v'(\varnothing) = 0$. The problem at issue is then to find allocations of the total worth $v'(N)$.

(see Izquierdo and Rafels, 2001), i.e.

$$\varnothing \neq T_1 \subseteq T_2 \subseteq N \Rightarrow \frac{v'(T_1)}{\omega'(T_1)} \leq \frac{v'(T_2)}{\omega'(T_2)}. \tag{4}$$

Given a game (N, v), a vector $x \in \mathbb{R}^N$ and a coalition $\varnothing \neq T \subseteq N$ the reduced game[4] on T at x, $(T, r_x^T(v))$ is defined as

$$r_x^T(v)(S) = \max_{Q \subseteq N \setminus T} \{v(S \cup Q) - x(Q)\},$$

for all $\varnothing \neq S \subseteq T$, with $r_x^T(v)(\varnothing) = 0$. Notice, the reduction of a game is transitive. This is, for all $\varnothing \neq T \subseteq T' \subseteq N$,

$$r_x^T(v) = r_{x_{T'}}^T(r_x^{T'}(v)).$$

It is easy to check, and we leave it to the reader, that if (T, v) is an average monotonic game with respect to some vector $\omega \in \mathbb{R}_{++}^T$, then the reduced game $(T \setminus \{i\}, r_x^{T \setminus \{i\}}(v))$ turns out to be an average monotonic game[5] with respect to $\omega_{T \setminus \{i\}}$, whenever $x_i \geq \omega_i \cdot \frac{v(T)}{\omega(T)}$.

After this preliminaries, take $x \in C(N, \omega, f)$ and suppose item 2 of the theorem holds. Then, let us define vector $z \in \mathbb{R}^N$ as follows:

$$\begin{aligned} z_i &= x_i + \tfrac{\varepsilon}{|S|} \quad \text{for all } i \in S, \\ z_i &= x_i \qquad\quad \text{for all } i \in N \setminus S, \end{aligned}$$

[4] This definition of reduced game only differs from the one given by Davis ans Maschler (1965) in the worth of the grand coalition. In Davis and Maschler's definition the worth of the grand coalition is $v(N) - x(N \setminus T)$ while in our definition is $\max_{Q \subseteq N \setminus T} \{v(T \cup Q) - x(Q)\}$.

[5] A proof of this fact can be checked in Izquierdo and Rafels (2001).

where $0 < \varepsilon < \min_{R \subseteq N, R \cap S \neq \varnothing} \{v'(N) - v'(N \setminus R) - x(R)\}$.

Notice, ε is well-defined since we are assuming item 2. It is easy to check that

$$z(R) < v'(N) - v'(N \setminus R), \text{ for all } R \subseteq N,\ R \cap S \neq \varnothing. \tag{5}$$

Moreover, it also holds

$$z(N) = x(N) + \varepsilon < x(N) + v'(N) - x(N) = v'(N). \tag{6}$$

Next, let us define[6]

$$\mathcal{D} = \left\{ \begin{array}{l} \varnothing \neq R \subseteq N : \exists\, \theta = (i_1, \ldots, i_r) \in \Theta^R \text{ such that} \\ \quad z_{i_1} \geq \omega'_{i_1} \cdot \dfrac{v'(N)}{\omega'(N)}, \\ \quad z_{i_2} \geq \omega'_{i_2} \cdot \dfrac{r_z^{N \setminus \{i_1\}}(v')(N \setminus \{i_1\})}{\omega'(N \setminus \{i_1\})}, \\ \qquad \vdots \\ \quad z_{i_r} \geq \omega'_{i_r} \cdot \dfrac{r_z^{N \setminus \{i_1, \ldots, i_{r-1}\}}(v')(N \setminus \{i_1, \ldots, i_{r-1}\})}{\omega'(N \setminus \{i_1, \ldots, i_{r-1}\})} \end{array} \right\}.$$

If $\mathcal{D} = \varnothing$, then just define $x'_i = \omega'_i \cdot \frac{v'(N)}{\omega'(N)}$, for all $i \in N$. Notice $z_i < \omega'_i \cdot \dfrac{v'(N)}{\omega'(N)}$, for all $i \in N$, as in other case there would exist $j \in N$ such that $z_j \geq \omega'_j \cdot \dfrac{v'(N)}{\omega'(N)}$ and so, at least, $R = \{j\} \in \mathcal{D}$, reaching a contradiction. Hence, the proof is done since $x'_i > z_i \geq x_i$ for all $i \in N$.

If $\mathcal{D} \neq \varnothing$, let R^* be a maximal coalition in $\mathcal{D}$ with respect to the inclusion. Since $R^* \in \mathcal{D}$ there exists an ordering of agents in R^*, $\theta^* = (i_1, i_2, \ldots, i_{r^*}) \in \Theta^{R^*}$, that

[6] Given $R \subseteq N$, we denote by Θ^R the set of all permutations of the elements of R.

satisfies the conditions in the definition of $\mathcal{D}$. Notice $R^* \neq N$, since otherwise $R^* = N = \{i_1, i_2, \ldots, i_n\}$ and

$$z_{i_n} \geq \omega'_{i_n} \cdot \frac{r_z^{\{i_n\}}(v')(\{i_n\})}{\omega'(\{i_n\})} = r_z^{\{i_n\}}(v')(\{i_n\}) \geq v'(N) - z(N \setminus \{i_n\}),$$

and so $z(N) \geq v'(N)$ which contradicts (6). Next, we define $x' \in \mathbb{R}^N$ as follows:

$$\begin{aligned}
x'_i &= z_i = x_i + \tfrac{\varepsilon}{|S|} && \text{for all } i \in R^* \cap S, \\
x'_i &= z_i = x_i && \text{for all } i \in R^* \setminus S, \\
x'_i &= \omega'_i \cdot \frac{r_z^{N\setminus R^*}(v')(N \setminus R^*)}{\omega'(N \setminus R^*)} && \text{for all } i \in N \setminus R^*.
\end{aligned}$$

We claim x' is an efficient payoff vector. Notice

$$\begin{aligned}
x'(N) &= z(R^*) + \omega'(N \setminus R^*) \cdot \frac{r_z^{N\setminus R^*}(v')(N \setminus R^*)}{\omega'(N \setminus R^*)} \\
&= z(R^*) + r_z^{N\setminus R^*}(v')(N \setminus R^*) \\
&= z(R^*) + \max_{Q \subseteq R^*}\{v'((N \setminus R^*) \cup Q) - z(Q)\} \\
&= z(R^*) + v'(N) - z(R^*) = v'(N),
\end{aligned}$$

where the penultimate equality follows by the fact that $\max_{Q\subseteq R^*}\{v'((N \setminus R) \cup Q) - z(Q)\} = v'(N) - z(R^*)$, i.e. the maximum is attained at $Q = R^*$. To prove it, let us check that for all $Q \subseteq R^*$, $Q \neq R^*$, it holds that $v'((N \setminus R) \cup Q) - z(Q) \leq v'(N) - z(R^*)$. To this aim consider two cases: (1) if $(R^* \setminus Q) \cap S \neq \varnothing$, then the desired inequality holds just by taking $R = R^* \setminus Q$ in (5); (2) if $(R^* \setminus Q) \cap S = \varnothing$, just take $R = R^* \setminus Q$ in item 2 of the theorem and recall that in this case $z_i = x_i$ and thus

$$\begin{aligned}
z(R^* \setminus Q) &< v'(N) - v'(N \setminus (R^* \setminus Q)) \\
&= v'(N) - v'((N \setminus R^*) \cup Q) + z(R^*) - z(R^*),
\end{aligned}$$

from where we deduce that $v'((N \setminus R^*) \cup Q) - z(Q) < v'(N) - z(R^*)$.

By definition, $x'_i = z_i = x_i + \frac{\varepsilon}{|S|} > x_i$, for all $i \in R^* \cap S$, and if $i \in S \setminus R^*$, we have $x'_i = \omega'_i \cdot \dfrac{r_z^{N\setminus R^*}(v')(N \setminus R^*)}{\omega'(N \setminus R^*)} > z_i > x_i$, for all $i \in S \setminus R^*$ where the first strict inequality follows from the maximality of R^* in $(\mathcal{D}, \subseteq)$. Hence, we conclude $x'_i > x_i$, for all $i \in S$, On the other hand, for all $i \in R^* \setminus S$ we have $x'_i = x_i$. Moreover, for all $i \in N \setminus (S \cup R^*)$, again by the maximality of R^*, we get

$$x'_i = \omega'_i \cdot \frac{r_z^{N\setminus R^*}(v')(N \setminus R^*)}{\omega'(N \setminus R^*)} > z_i \geq x_i,$$

for all $i \in N \setminus (S \cup R^*)$.

Finally, we have to check that x' is a core element, $x' \in C(N, \omega', f)$. Let us first recall $R^* = \{i_1, \ldots, i_{r^*}\}$ where $\theta^* = (i_1, \ldots, i_{r^*})$ is an ordering of the members of R^* that fulfills the condition expressed in $\mathcal{D}$. Then, take a nonempty $T \subseteq N$ and consider first the case $T \subseteq R^*$. If $T = \{i_1\}$ it trivially holds that

$$x'_{i_1} = z_{i_1} \geq \omega'_{i_1} \cdot \frac{v'(N)}{\omega'(N)} \geq v'(\{i_1\}),$$

since the game satisfies (4). Take now $T \subseteq R^*$, $T \neq \{i_1\}$ and select the unique player $i_k \in T$ such that

$$T \setminus \{i_k\} \subseteq \{i_1, i_2 \ \ldots, i_{k-1}\}. \tag{7}$$

Notice i_k is the last player in T according to the ordering θ^*. Then, by definition of the set $\mathcal{D}$ we have

$$\begin{aligned}
x'_{i_k} \quad &= z_{i_k} \geq \omega'_{i_k} \cdot \frac{r_z^{N\setminus\{i_1,\ldots,i_{k-1}\}}(v')(N \setminus \{i_1, \ldots, i_{k-1}\})}{\omega'(N \setminus \{i_1, \ldots, i_{k-1}\})} \\
&\geq \omega'_{i_k} \cdot \frac{r_z^{N\setminus\{i_1,\ldots,i_{k-1}\}}(v')(\{i_k\})}{\omega'(\{i_k\})} \\
&\geq r_z^{N\setminus\{i_1,\ldots,i_{k-1}\}}(v')(\{i_k\}) \\
&= \max_{Q \subseteq \{i_1,\ldots,i_{k-1}\}} \{v'(\{i_k\} \cup Q) - z(Q)\} \\
&\geq v'(T) - z(T \setminus \{i_k\}) = v'(T) - x'(T \setminus \{i_k\}),
\end{aligned}$$

where the second inequality follows since $r_z^{N\setminus\{i_1,\ldots,i_{k-1}\}}(v')$ also satisfies (4) by the definition of the set $\mathcal{D}$, and the last inequality by taking $Q = T \setminus \{i_k\}$. Hence, it follows $x'(T) \geq v'(T)$.

Let now $T \subseteq N$ be such that $T \cap (N \setminus R^*) \neq \varnothing$. Then, we have

$$
\begin{aligned}
x'(T \cap (N \setminus R^*)) &= \cdot\omega'(T \cap (N \setminus R^*)) \cdot \frac{r_z^{N\setminus R^*}(v')(N \setminus R^*)}{\omega'(N \setminus R^*)} \\
&\geq \omega'(T \cap (N \setminus R^*)) \cdot \frac{r_z^{N\setminus R^*}(v')(T \cap (N \setminus R^*))}{\omega'(T \cap (N \setminus R^*))} \\
&= r_z^{N\setminus R^*}(v')(T \cap (N \setminus R^*)) \\
&= \max_{Q \subseteq R^*}\{v'((T \cap (N \setminus R^*)) \cup Q) - z(Q)\} \\
&\geq v'(T) - z(T \cap R^*) = v'(T) - x'(T \cap R^*),
\end{aligned}
$$

where the first inequality follows again since $r_z^{N\setminus R^*}(v')$ satisfies (4), and the last one taking $Q = T \cap R^*$. We conclude, $x'(T) \geq v'(T)$, for all $T \subseteq N$ and so $x' \in C(N, \omega', f)$. $\square$

Next example shows that the result of Theorem 1 cannot be improved. It proposes a co-investment problem where the unique core allocation compatible with the strong resource-monotonicity property gives an strictly higher payoff to the unique agent that increases her initial contribution, while for the rest of players it is compulsory to receive exactly the same payoff. That is, we cannot in general guarantee strictly better payoffs to all agents when only a proper subcoalition of them raise their contributions.

Example 3. *Let* $\omega = (1, 2, 3)$ *and* $f(x) = x$*, for* $0 \leq x < 5$*, and* $f(x) = 1.5 \cdot x$*, for* $5 \leq x$*. The allocation* $x = (1.5, 3, 4.5)$ *is in the core* $C(N, \omega, f)$*. Let us suppose that player 1 increases his initial contribution up to 4 units, that is* $\omega' = (1 + 4, 2, 3) = (5, 2, 3)$*. Notice the core of the new problem shrinks into the point* $(7.5, 3, 4.5)$*. This vector assigns an strictly better payoff to agent 1, but the same payoff to agents 2 and*

3, and it is the unique possibility

For a particular core allocation and for some modification in the initial contributions of the agents, the above result provides the tool to check whether it is possible to make compatible the core requirements with the resource-monotonicity property. However, it is also interesting to check which core allocations of a co-investment problem satisfy the strong resource-monotonicity requirement. This point will be addressed in the next section.

5. Incentive core allocations

If some agreement is reached about how to share profits, but some agents aims to contribute more and want to revise this agreement, this must be with the consent of the rest of agents. To pick up a core element is a sensible requirement for the stability of the new agreement; requiring strong resource-monotonicity favors the support of all agents. The incentive core are formed by those core allocations meeting these two properties.

Definition 1. Given a co-investment problem (N, ω, f), the *Incentive Core* is the set[7]

$$IC(N,\omega,f) = \left\{ x \in C(N,\omega,f) \;\middle|\; \begin{array}{l} \forall\, \omega' \in \mathbb{R}^N_{++},\, \omega' \geq \omega, \exists\, x' \in C(N,\omega',f) \\ \text{ such that, for all } i \in N, \\ (1)\; x'_i > x_i \;\text{ if } \omega'_i > \omega_i, \;\text{ and} \\ (2)\; x'_i \geq x_i \;\text{ if } \omega'_i = \omega_i \end{array} \right\}.$$

The first issue to be addressed is the non-emptiness of the incentive core. Next proposition solves this question by showing that the proportional allocation is a vector

[7] We denote by $\geq$ the usual order on $\mathbb{R}^n$.

within the incentive core, and thus proving the compatibility between core selection and strong resource-monotonicity. As a consequence, this result also highlights new normative properties of the proportional solution.

Proposition 2. *For any co-investment problem, the proportional allocation is an element of the incentive core. That is,*

$$P(N, \omega, f) \in IC(N, \omega, f).$$

Proof First recall that $P(N, \omega, f) \in C(N, \omega, f)$. Then take $\omega' \geq \omega$, $\omega' \neq \omega$. As $\omega'(N) > \omega(N) > 0$ then $\dfrac{f(\omega'(N))}{\omega'(N)} \geq \dfrac{f(\omega(N))}{\omega(N)} > 0$ and thus the proportional allocations $P(N, \omega, f)$ and $P(N, \omega', f)$ satisfy, for all $i \in N$,

$$(1) \text{ if } \omega'_i > \omega_i \text{ then } P_i(N, \omega', f) > P_i(N, \omega, f)$$

$$(2) \text{ if } \omega'_i = \omega_i \text{ then } P_i(N, \omega', f) \geq P_i(N, \omega, f),$$

which implies $P(N, \omega, f) \in IC(N, \omega, f)$. $\square$

By definition, the $IC(N, \omega, f)$ is a subset of the core. The following example shows that not all core allocations are in the incentive core.

Example 4. *Let $\omega = (1, 2, 3)$ be the vector of initial endowments of the agents $N = \{1, 2, 3\}$ and*

$$f(x) = \begin{cases} x & 0 \leq x \leq 3, \\ 2x & 3 < x \leq 4, \\ 3x & 4 < x \leq 6, \\ 4x & 6 < x \end{cases}$$

be the co-investment function. The output of coalitions are

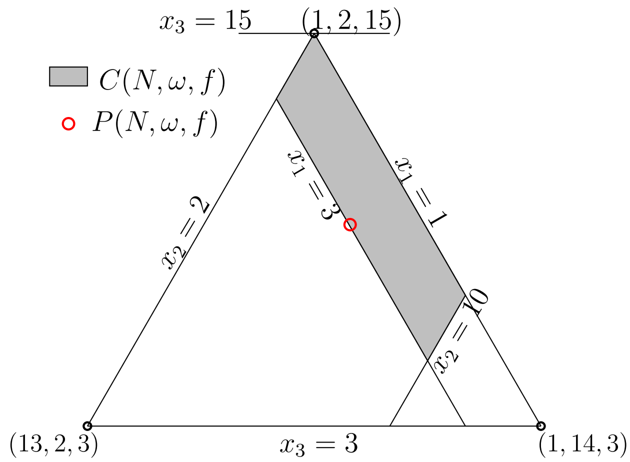

Figure 1: The core and the proportional allocation.

$$\begin{array}{lll} f(\omega_1) = 1, & f(\omega_1 + \omega_2) = 3, & \\ f(\omega_2) = 2, & f(\omega_1 + \omega_3) = 8, & f(\omega_1 + \omega_2 + \omega_3) = 18. \\ f(\omega_3) = 3, & f(\omega_2 + \omega_3) = 15, & \end{array}$$

The core of this game can be described as

$$C(N, \omega, f) = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x_1 + x_2 + x_3 = 18,\ 1 \leq x_1 \leq 3, \\ 2 \leq x_2 \leq 10 \textit{ and } 3 \leq x_3 \leq 15 \end{array} \right\},$$

and the proportional allocation is $P(N, \omega, f) = (3, 6, 9)$*. In Figure 1, we depict the core ( ) and the proportional allocation ( ) . It is interesting to point out that the proportional allocation assigns the first agent the maximum payoff within the core of the game; graphically, the proportional allocation is located on the left border of the core where agent 1 receives the largest possible reward within the core. This example shows one of the potential drawbacks of the proportional solution.*

Now we **claim** *that the incentive core is given by*

$$IC(N,\omega,f) = \{x \in C(N,\omega,f) \mid x_1 \le 4, x_2 \le 8, x_3 \le 12\}.$$

To see this, let $(x_1,x_2,x_3) \in C(N,\omega,f)$ *be a core allocation such that* $x_1 \le 4$, $x_2 \le 8$ *and* $x_3 \le 12$ *and take* $\omega' \in \mathbb{R}^3_{++}$, $\omega' \ge \omega = (1,2,3)$ *and* $\omega' \ne \omega$*. We can describe* $\omega' = (1+\varepsilon_1, 2+\varepsilon_2, 3+\varepsilon_3)$ *where* $\varepsilon_1,\varepsilon_2,\varepsilon_3 \ge 0$ *and* $\varepsilon_1+\varepsilon_2+\varepsilon_3 > 0$*.*

Since $\omega'_1 + \omega'_2 + \omega'_3 = 6 + \varepsilon$*, where* $\varepsilon = \varepsilon_1+\varepsilon_2+\varepsilon_3 > 0$*, we have* $f(\omega'(N)) = 4 \cdot \omega'(N) = 24 + 4\varepsilon$*.*

Now take the proportional allocation w.r.t. ω'*, that is,* $P(N,\omega',f) = (4+4\varepsilon_1, 8+4\varepsilon_2, 12+4\varepsilon_3)$*. As the allocation* $x = (x_1,x_2,x_3)$ *satisfies* $x_1 \le 4$*,* $x_2 \le 8$ *and* $x_3 \le 12$ *it follows*

$$\begin{aligned} x_1 &\le P_1(N,\omega',f),\\ x_2 &\le P_2(N,\omega',f),\\ x_3 &\le P_3(N,\omega',f). \end{aligned}$$

Moreover, for any $i = 1,2,3$*, if* $\varepsilon_i > 0$ *we also have* $x_i < P_i(N,\omega',f)$ *and* $P(N,\omega',f) \in C(N,\omega',f)$ *which implies* $(x_1,x_2,x_3) \in IC(N,\omega,f)$*.*

To justify the other inclusion, let $x = (x_1,x_2,x_3) \in IC(N,\omega,f)$*. We just take* $\omega' = (1,6,6)$ *that satisfies* $\omega' \ge \omega$ *and* $\omega' \ne \omega$*. Therefore as* $C(N,\omega',f) = \{P(N,\omega',f)\} = \{(4,24,24)\}$*, we obtain* $x_1 \le 4$*. Repeating the same argument but taking* $\omega'' = (5,2,5)$ *and* $\omega''' = (4,4,3)$ *we deduce* $x_2 \le 8$ *and* $x_3 \le 12$*, respectively.*

In Figure 2, you can check that the incentive core it is larger than the proportional allocation, but strictly smaller than the core. Indeed, it is the convex hull of its four extreme points, $A = (3,3,12)$*,* $B = (1,5,12)$*,* $C = (1,8,9)$ *and* $D = (3,8,7)$*. In this numerical example, and in order to solve the drawbacks of the proportional allocation, it seems appropriate to take the average of these four extreme points* $I = \frac{1}{4}(A+B+C+D) = (2,6,10)$ *obtaining a central element within the incentive core.*

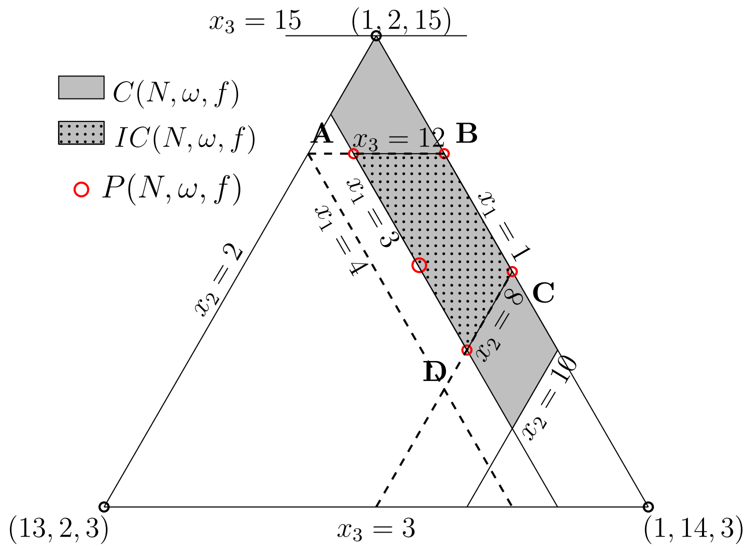

Figure 2: The incentive core

Now we characterize a larger subclass of co-investment problems where the incentive core coincides with the core. This means that any core allocation can be adapted monotonically as a reaction to any increasing of the resources invested.

Theorem 2. *Let (N, ω, f) be a co-investment problem with $|N| \geq 2$. Then, the following statements are equivalent:*

1. *f is ultramodular*[8] *on $\mathbb{R}_+$, i.e. for all $0 < x < y$ and $z > 0$,*

$$f(y) - f(x) \leq f(y + z) - f(x + z). \tag{8}$$

2. *The incentive core coincides with the core, i.e.*

$$IC(N, \omega, f) = C(N, \omega, f), \text{ for any } \omega \in \mathbb{R}^N_{++}.$$

[8]See Marinacci and Montrucchio, 2005. Convex functions from $\mathbb{R}_+$ to $\mathbb{R}$ with $f(0) = 0$ and $f(x) > 0$, for all $x > 0$, satisfy (1) and (8).

Proof 1. $\rightarrow$ 2.) Let (N, ω, f) be a co-investment problem and take a coalition $S \subseteq N$, $S \neq \varnothing$. Let us consider $\omega' \in \mathbb{R}^N$ such that $\omega'_i > \omega_i$ for all $i \in S$ and $\omega'_i = \omega_i$, for all $i \in N \setminus S$, and $x \in C(N, \omega, f)$.

We must first prove that item 2 of Theorem 1 holds, for any $x \in C(N, \omega, f)$. To this aim we consider three cases depending on an arbitrary coalition $\varnothing \neq R \subseteq N$:

(a) If $R \subseteq N \setminus S$, let us suppose to the contrary that $x(R) > f(\omega'(N)) - f(\omega'(N \setminus R))$. Then

$$\begin{aligned} x(R) \; &> f(\omega'(N)) - f(\omega'(N \setminus R)) \\ &\geq f(\omega(N)) - f(\omega(N \setminus R)), \end{aligned}$$

where the second inequality holds since $R \subseteq N \setminus S$ and taking in (8) $y = \omega(N) > x = \omega(N \setminus R) > 0$ and $z = \omega'(S) - \omega(S) > 0$. Now, taking the above inequality into account and $x \in C(N, \omega, f)$ we obtain

$$\begin{aligned} x(R) > \; & f(\omega(N)) - f(\omega(N \setminus R)) \\ & \geq x(N) - x(N \setminus R) = x(R), \end{aligned}$$

getting a contradiction. We conclude $x(R) \leq f(\omega'(N)) - f(\omega'(N \setminus R))$.

(b) If $R \cap S \neq \varnothing$ and $S \setminus R \neq \varnothing$, let suppose $x(R) \geq f(\omega'(N)) - f(\omega'(N \setminus R))$. Then,

$$\begin{aligned} x(R) \; &\geq f(\omega'(N)) - f(\omega'(N \setminus R)) \\ &\geq f(\omega'(N) - (\omega'(S \setminus R) - \omega(S \setminus R))) \\ &\quad - f(\omega'(N \setminus R) - (\omega'(S \setminus R) - \omega(S \setminus R))) \\ &= f(\omega'(N) - (\omega'(S \setminus R) - \omega(S \setminus R))) \\ &\quad - f(\omega(N \setminus R)) \\ &> f(\omega(N)) - f(\omega(N \setminus R)) \\ &\geq x(N) - x(N \setminus R) = x(R), \end{aligned}$$

where the second inequality comes from (8) taking $y = \omega'(N) - [\omega'(S \setminus R) - \omega(S \setminus R)] > x = \omega'(N \setminus R) - [\omega'(S \setminus R) - \omega(S \setminus R)] > 0$ and $z = \omega'(S \setminus R) - \omega(S \setminus R)] > 0$. Moreover, the strict inequality holds since $\omega'(N) - (\omega'(S \setminus R) - \omega(S \setminus R)) > \omega(N)$ and $\frac{f(\omega(N))}{\omega(N)} > 0$, and the last equality, since $x \in C(N, \omega, f)$. Hence, we reach a contradiction. We conclude $x(R) < f(\omega'(N)) - f(\omega'(N \setminus R))$.

(c) If $R \cap S \neq \varnothing$ and $S \setminus R = \varnothing$, or equivalently $S \subseteq R$, then

$$\begin{aligned} x(R) &\leq f(\omega(N)) - f(\omega(N \setminus R)) \\ &< f(\omega'(N)) - f(\omega'(N \setminus R)), \end{aligned}$$

where the first inequality comes from $x \in C(N, \omega, f)$ and the last one since $\omega'(N) > \omega(N)$ and $\omega'(N \setminus R) = \omega(N \setminus R)$.

2. $\rightarrow$ 1.) Let $0 < x < y$ and $0 < z$. Then, select an arbitrary agent $k \in N$ and take a vector $\omega \in \mathbb{R}^N_{++}$ such that $\omega(N) = y$, and $\omega(N \setminus \{k\}) = x$. On the other hand, as $|N| \geq 2$, select an arbitrary agent $k_1 \in N$, $k_1 \neq k$ and take another vector $\omega' \in \mathbb{R}^N_{++}$ such that $\omega'_j = \omega_j$, for all $j \neq k_1$ and $\omega'_{k_1} = \omega_{k_1} + z$. Let $u \in C(N, \omega, f)$ such that $u_k = f(\omega(N)) - f(\omega(N \setminus \{k\}))$ and $u_i = \omega_i \cdot \frac{f(\omega(N \setminus \{k\}))}{\omega(N \setminus \{k\})}$, for all $i \neq k$ (see (2)).

By hypothesis we know $IC(N, \omega, f) = C(N, \omega, f)$. Then, by Theorem 1 we know the existence of $u' \in C(N, \omega', f)$ satisfying (i) $u'_{k_1} > u_{k_1}$ and (ii) $u'_j \geq u_j$, for all $j \in N \setminus \{k_1\}$. Using these inequalities we get

$$\begin{aligned} f(y) - f(x) &= f(\omega(N)) - f(\omega(N \setminus \{k\})) = u_k \leq u'_k \\ &\leq f(\omega'(N)) - f(\omega'(N \setminus \{k\}) = f(y + z) - f(x + z), \end{aligned}$$

where the second inequality comes from the fact that $u' \in C(N, \omega', f)$. hence, we conclude f is ultramodular.

□

There is an interesting case where the incentive core shrinks to the proportional allocation. The condition that supports this case requires that the average worth of the investment function $\frac{f(x)}{x}$ remains constant beyond the level of total investment $x = \omega(N)$.

Proposition 3. *Let (N, ω, f) be a co-investment problem with $|N| \geq 2$ where $\frac{f(x)}{x} = \frac{f(\omega(N))}{\omega(N)}$, for all $x \geq \omega(N)$. Then*

$$IC(N, \omega, f) = \{P(N, \omega, f)\}.$$

Proof Since $P(N, \omega, f) \in IC(N, \omega, f)$, it only remains to prove that $IC(N, \omega, f) \subseteq \{P(N, \omega, f)\}$. Let $x \in IC(N, \omega, f)$. Then, select an arbitrary agent $i \in N$ and define $\omega' \in \mathbb{R}^N_{++}$ as $\omega'_i = \omega(N)$ and $\omega'_j = \omega_j$, for all $j \in N \setminus \{i\}$. By Theorem 1 part 2. we know that, for any $k \in N$, $k \neq i$,

$$x_k \leq f(\omega'(N)) - f(\omega'(N \setminus \{k\})) = \omega_k \cdot \frac{f(\omega(N))}{\omega(N)},$$

where the last equality comes from $\frac{f(\omega(N))}{\omega(N)} = \frac{f(x)}{x}$, for all $x \geq \omega(N)$. Since agent has been selected arbitrarily we get, by effciency, $x = P(N, \omega, f)$.

$\square$

Remark 1. Given a co-investment problem (N, ω, f), and just by looking the proof of Proposition 3, we can realize that if, by increasing the contribution of some agents, it turns out that the average return of the grand coalition of agents and the average return of the coalitions of all but one agent become the same, then the incentive core reduces to the proportional distribution, i.e. if there exists $\omega' \geq \omega$, $\omega' \neq \omega$, such that $\frac{f(\omega(N))}{\omega(N)} = \frac{f(\omega'(N))}{\omega'(N)} = \frac{f(\omega'(N \setminus \{i\}))}{\omega'(N \setminus \{i\})}$, for all $i \in N$, then $IC(N, \omega, f) = \{P(N, \omega, f)\}$.

6. Conclusions

The model we discuss in the paper reflects a very sensitive requirement when some agents aim to cooperate. If we increase our contribution, if we put more effort, it is natural to expect a larger reward. Furthermore, if we need the consent of other agents, we must take care not to harm them. It is evident that the new and interesting positive results we get in the paper mainly relies on the increasing average returns assumption of the model. However, in our opinion, the revealed negative results are also surprising.

In spite of its simplicity, some interesting lines of research remains still open. The first one is to relax the requirement on other agents: what about if we just require other agents (the ones that do not increase the contribution) to receive a joint payoff not smaller than the initial one. Assuming this, a second natural question is to review whether well known solutions as the nucleolus, the per capita nucleolus and other core selections satisfy resource-monotonicity requirements. The third one might be to study the more complex problem of adding several interrelated co-investment problems and the strategic analysis that arises when agents must decide where to invest their resources. And finally, it is interesting, but not easy to deal with, the problem of considering multi-dimensional input contributions.

References

[1] Arin, J., Feltkamp, V., 2005. Monotonicity properties of the nucleolus on the domain of veto balanced games. TOP 13(2), 331–342.

[2] Calleja, P., Rafels, C., Tijs, S., 2009. The aggregate-monotonic core. Games and Econ. Behav. 66(2), 742–748.

[3] Chun, Y., Thomson, W., 1988. Monotonicity properties of bargaining solutions when applied to economics. Math. Soc. Sci 15(1), 11–27.

[4] Davis, M., Maschler, M., 1965. The kernel of a cooperative game. Naval Research Logistics Quarterly 2, 223–259.

[5] Hokari, T., 2000. The nucleolus is not aggregate-monotonic on the domain of convex games. Int. J. Game Theory 29, 133–137.

[6] Housman, D., Clark, L., 1998. Core and Monotonic Allocation Methods. Int. J. Game Theory 27, 611–616.

[7] Izquierdo, J.M., Rafels, C., 2001. Average Monotonic Cooperative Games. Games and Econ. Behav. 36, 174–192.

[8] Lemaire, J., 1984. An Application of Game Theory: Cost Allocation. Astin Bull. 14, 61–81.

[9] Lemaire, J., 1991. Cooperative Game Theory and its Insurance Applications. Astin Bull. 21, 17–40.

[10] Marinacci, M., Montrucchio, L., 2005. Ultramodular functions. Math. Op. Research 30, 311–332.

[11] Mas-Colell, A., Whinston, M.D., Green, J.R., 1995. Microeconomic Theory. Oxford University Press, Oxford.

[12] Meggido, N., 1974. On the monotonicity of the bargaining set, the kernel and the nucleolus of a game. SIAM J. Applied Math. 27, 355–358.

[13] Moulin, H., 1990. Joint Ownership of a Convex Technology: Comparison of Three Solutions. Rev. Econ. Studies 57, 439–452.

[14] Roemer, E., Silvestre, J., 1993. The Proportional Solution in Economies with Both Private and Public Ownership. J. Econ. Theory 59(2), 426–444.

[15] Shubik, M., 1962. Incentives, Decentralized Control, the Assignment of Joint Costs and Internal Pricing. Management Sci. 8, 325–343.

[16] Thomson, W., 1994. Resource-monotonic solutions to the problem of fair division when preferences are single-peaked. Soc. Choice and Welfare 11(3), 205–223.

[17] Young, H.P., 1985. Monotonic Solutions of Cooperative Games. Int. J. Game Theory 14, 65–78.