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Induced aggregation operators in decision making with the Dempster-Shafer belief structure

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Abstract: We study the induced aggregation operators. The analysis begins with a revision of some basic concepts such as the induced ordered weighted averaging (IOWA) operator and the induced ordered weighted geometric (IOWG) operator. We then analyze the problem of decision making with Dempster-Shafer theory of evidence. We suggest the use of induced aggregation operators in decision making with Dempster-Shafer theory. We focus on the aggregation step and examine some of its main properties, including the distinction between descending and ascending orders and different families of induced operators. Finally, we present an illustrative example in which the results obtained using different types of aggregation operators can be seen.

JEL Classification: C44, C49, D81, D89.

Keywords: Decision making; aggregation operators; Dempster-Shafer belief structure; uncertainty; IOWA operator.

Resumen: En este trabajo se estudian los operadores de agregación inducidos como son el induced ordered weighted averaging (IOWA) operator y el induced ordered weighted geometric (IOWG) operator. También se analiza el proceso de toma de decisiones mediante la estructura de credibilidad de Dempster-Shafer. La principal propuesta del trabajo es la utilización de operadores de agregación inducidos en la toma de decisiones mediante la estructura de credibilidad de Dempster-Shafer. Se da especial atención al proceso de agregación estudiando algunas de sus principales propiedades como son la distinción entre órdenes ascendentes y descendentes, y el estudio de diferentes familias de operadores inducidos. Finalmente, se desarrolla un ejemplo ilustrativo en donde se pueden observar los diferentes resultados obtenidos según el tipo de operador utilizado.

Palabras clave: Toma de decisiones; operadores de agregación; estructura de credibilidad de Dempster-Shafer; incertidumbre; operador IOWA.

1. Introduction

The Dempster-Shafer (D-S) theory of evidence was developed by Dempster (Dempster, 1967; 1968) and Shafer (Shafer, 1976) and has subsequently been used in an astonishingly wide range of applications (see, among others, Yager *et al.*, 1994; Srivastava and Mock, 2002). It provides a unifying framework for representing uncertainty as it can include situations of risk and ignorance in the same formulation.

When using the D-S theory in decision making, the decision information must first be aggregated. A very common aggregation method is the ordered weighted averaging (OWA) operator developed by Yager (1988). Since it first appeared, the OWA operator has been used in a wide range of applications (see, among others, Calvo *et al.*, 2002; Yager and Kacprzyk, 1997). It provides a parameterized family of aggregation operators that includes the arithmetic mean, the maximum and the minimum as special cases (Yager, 1988). Recently, Chiclana *et al.* (2000) have developed the ordered weighted geometric (OWG) operator and it has subsequently been extensively analysed by a number of authors (see, among others, Herrera *et al.*, 2003; Merigó and Casanovas, 2006; Xu and Da, 2002). It combines the OWA operator with the geometric mean in the same aggregation thereby providing another parameterized family of aggregation operators that include the maximum and the minimum among others (Chiclana *et al.*, 2000).

In 1999, Yager and Filev introduced an extension of the OWA operator – the induced ordered weighted averaging (IOWA) operator – while, in 2003, Xu and Da introduced a geometric version of the IOWA operator, known as the induced ordered weighted geometric (IOWG) operator. Since their introduction, they have been examined in a number of studies (S.J. Chen and S.M. Chen, 2003; Chiclana *et al.*, 2004; Mitchell and Schaefer, 2000; Xu, 2005a; Xu, 2006a; Xu, 2006b; Xu, 2006c; Yager, 2002a; Yager, 2003a; Yager, 2004a). The main characteristic of the induced aggregation operators is that the reordering step is not conducted with the values of the arguments used for the OWA operators. In these cases, the reordering step is induced by means of another mechanism so that the order of the arguments depends upon the values of their associated inducing variables.

Yager (1992a) developed a more general formulation for decision making in the face of evidential knowledge by using the OWA operator. This problem has also been studied in (Merigó and Casanovas, 2006; Engemann *et al.*, 1996; Yager, 1996a; Yager, 2002b; Yager, 2004b; Yager, 2004c). In this paper, we suggest the use of induced aggregation operators in situations of decision making with D-S theory of evidence. The reason for doing this is because there are situations where we prefer to aggregate the variables with an inducing order instead of aggregating with the traditional OWA operator. For example, such a method is useful when the attitudinal character of the decision maker is particularly complex or when there are a number of external factors affecting the decision analysis. We also propose using different types of orderings in the aggregation of the D-S theory depending on the specific situation with which we are dealing. We study these problems in detail by conducting an extensive analysis of the induced aggregation operators in which we introduce different families of induced operators such as the step-IOWA operator, the window-IOWA operator, the olympic-IOWA operator, the E-Z IOWA operator and the median-IOWA operator, among others.

The remainder of this paper is organized as follows. In Section 2, we describe different types of aggregation operators. In Section 3, we briefly describe the Dempster-Shafer theory of evidence. In Section 4, we describe the process for using induced aggregation operators in decision making with D-S belief structures. In Section 5, we provide an illustrative example of the new approach. Finally, in Section 6 we summarize the main conclusions of the paper.

2. Aggregation operators

In this Section, we briefly describe the basic aggregation operators that are used in the paper.

2.1. OWA operator

The OWA operator, introduced by Yager (1988), provides a parameterized family of aggregation operators that include the arithmetic mean, the maximum and the minimum.

Definition 1. An OWA operator of dimension n is a mapping $OWA{:}R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$, then:

$$OWA(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j \tag{1}$$

where b_j is the jth largest of the a_i.

From a generalized perspective of the reordering step, we have to distinguish between the descending OWA (DOWA) operator and the ascending OWA (AOWA) operator (Yager, 1992b). The weights of these operators are related by $w_j = w^*_{n+1-j}$, where w_j is the *j*th weight of the DOWA (or OWA) operator and w^*_{n+1-j} the *j*th weight of the AOWA operator.

2.2. OWG operator

The OWG operator was introduced in Chiclana *et al.* (2000). It combines the OWA operator and the geometric mean in the same aggregation. The OWG operator provides a parameterized family that includes the minimum, the maximum and the geometric mean. In the following, we provide a definition of the OWG operator as introduced by Xu and Da (2002) where we can distinguish between descending and ascending orderings.

Definition 2. An OWG operator of dimension n is a mapping $OWG{:}{R^+}^n \to R^+$ that has an associated weighting vector W of dimension n such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$, then:

$$OWG(a_1, a_2, \ldots, a_n) = \prod_{j=1}^{n} b_j^{\,w_j} \tag{2}$$

where b_j is the *j*th largest of the a_i, and R^+ is the set of positive real numbers.

2.3. Induced OWA operator

The induced OWA (IOWA) operator was introduced in Yager and Filev (1999) and is an extension of the OWA operator. It differs in the fact that the reordering step is not carried out with the values of the arguments a_i. In this case, the reordering step is induced by another mechanism represented as u_i, where the ordered position of the arguments a_i depends upon the values of the inducing variables u_i.

Definition 3. An IOWA operator of dimension n is a mapping $IOWA$: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$, then:

$$IOWA(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle \ldots, \langle u_n,a_n\rangle) = \sum_{j=1}^{n} w_j b_j \tag{3}$$

where b_j is the a_i value of the OWA pair $\langle u_i,a_i\rangle$ having the jth largest u_i, u_i is the order inducing variable and a_i is the argument variable.

From a generalized perspective of the reordering step, we have to distinguish between the descending IOWA (DIOWA) operator and the ascending IOWA (AIOWA) operator. Note that these orderings are based on the inducing variable and their weighting vectors are related by $w_j = w^*_{n+1-j}$, where w_j is the jth weight of the DIOWA (or IOWA) operator and w^*_{n+1-j} the jth weight of the AIOWA operator. Note also that the elements b_j of the AIOWA operator are ordered in an increasing way such that $\langle \text{Min}\{u_i\},b_1\rangle \leq \ldots \leq \langle \text{Max}\{u_i\},b_n\rangle$.

2.4. Induced OWG operator

The induced OWG (IOWG) operator was first introduced in Xu and Da (2003) and is an extension of the OWG operator. It involves combining the IOWA operator with the geometric mean. Unlike in the OWG operator, the reordering step in the IOWG is not carried out with the values of the arguments a_i. In this case, the reordering step is induced by another mechanism represented by u_i, where the ordered position of the arguments a_i depends upon the values of the inducing variable u_i.

Definition 4. An IOWG operator of dimension n is a mapping $IOWG$: $R^{+^n} \rightarrow R^+$ that has an associated weighting vector W of dimension n such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$, then:

$$IOWG(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle \ldots, \langle u_n,a_n\rangle) = \prod_{j=1}^{n} b_j^{\,w_j} \tag{4}$$

where b_j is the a_i value of the OWG pair $\langle u_i,a_i\rangle$ having the jth largest u_i, u_i is the order inducing variable and a_i is the argument variable.

From a generalized perspective of the reordering step, we can distinguish between the descending IOWG (DIOWG) operator and the ascending IOWG (AIOWG) operator. Note that these orderings are also based on the inducing variable such that the DIOWG operator is ordered as $\langle \text{Max}\{u_i\},b_1\rangle \leq \ldots \leq \langle \text{Min}\{u_i\},b_n\rangle$, and the AIOWG operator as $\langle \text{Min}\{u_i\},b_1\rangle \leq \ldots \leq \langle \text{Max}\{u_i\},b_n\rangle$. Note also that the weighting vectors are related by $w_j = w^*_{n+1-j}$, where w_j is the jth weight of the DIOWG (or IOWG) operator and w^*_{n+1-j} the jth weight of the AIOWG operator.

3. The Dempster-Shafer theory of evidence

The D-S theory of evidence was introduced by Dempster (1967; 1968) and Shafer (1976) and subsequently many new developments have been made (for example, Yager *et al.*, 1994; Srivastava and Mock, 2002). Formulations of this type provide a unifying framework for representing uncertainty as it can include cases of risk and ignorance as special occurrences. Obviously, the case of certainty is also included in this generalization as it can be seen as a particular situation of risk or ignorance. Note that the case of certainty could also appear in other particular situations of the D-S formulation. Apart from these traditional cases, the D-S framework allows other forms of information that a decision maker might have about the states of nature to be represented.

Definition 5. A D-S belief structure defined on a space X consists of a collection of n nonnull subsets of X, B_j for $j = 1,\ldots,n$, called focal elements and a mapping m, called the basic probability assignment, defined as, m: $2^X \rightarrow [0, 1]$ such that:

(1) $m(B_j) \in [0, 1]$.

(2) $\sum_{j=1}^{n} m(B_j) = 1$.

(3) $m(A) = 0, \ \forall A \neq B_j$.

As described above, the cases of risk and ignorance are included as special cases of belief structure in the D-S framework. In the case of risk, a belief structure is known as a Bayesian belief structure if it consists of n focal elements such that $B_j = \{x_j\}$, where each focal element is a singleton. Then, it is evident that we are in a situation of decision making under a risk environment as $m(B_j) = P_j = \text{Prob}\ \{x_j\}$.

The case of ignorance is found when the belief structure consists in only one focal element *B*, where *m(B)* essentially is the decision making under ignorance environment, as this focal element comprises all the states of nature. Thus, $m(B) = 1$. Other special cases of belief structures such as the consonant belief structure or the simple support function are studied in Shafer (1976).

Two important evidential functions associated with these belief structures are the measures of plausibility and belief. In the following, we provide a definition of these two measures as developed by Shafer (1976).

Definition 6. The plausibility measure Pl is defined as, Pl: $2^X \rightarrow [0, 1]$ such that:

$$\mathrm{Pl}(A) = \sum_{A \cap B_j \neq \varnothing} m(B_j) \tag{5}$$

Definition 7. The belief measure Bel is defined as Bel: $2^X \rightarrow [0, 1]$ such that:

$$\mathrm{Bel}(A) = \sum_{B_j \subseteq A} m(B_j) \tag{6}$$

Bel(A) represents the exact support to A and Pl(A) represents the possible support to A. With these two measures we can form the interval of support to A as [Bel(A),Pl(A)]. This interval can be seen as the lower and upper bounds of the probability to which A is supported such that $\mathrm{Bel}(A) \leq \mathrm{Prob}(A) \leq \mathrm{Pl}(A)$. From this we see that $\mathrm{Pl}(A) \geq \mathrm{Bel}(A)$ for all A. Another interesting feature about these two measures is that they are connected by $\mathrm{Bel}(A) = 1 - \mathrm{Pl}(\bar{A})$ or $\mathrm{Pl}(A) = 1 - \mathrm{Bel}(\bar{A})$, where $\bar{A}$ is the complement of A.

4. Induced OWA operators in decision making with Dempster-Shafer belief structures

4.1. Decision making approach

The problem of decision making with D-S belief structures has been studied by various authors (Merigó and Casanovas, 2006; Engemann *et al.*, 1996; Yager, 1996a; Yager, 2002b; Yager, 2004b; Yager, 2004c). In 1992a, Yager proposed a more generalized methodology by using the OWA operator.

A new method for decision making with D-S belief structures is possible by using the IOWA operator in the aggregation step instead of the OWA operator. The reason for using the IOWA operator in these cases is that the decision maker may, on occasions, have an attitudinal character that differs from the values of the arguments. Then, in order to aggregate the arguments, he prefers to use another mechanism in the reordering step which is closer accordance with his interests. Similar explanations for using the IOWA operator in such circumstances might be offered, but the principal idea is the possibility of using different reordering methods in the aggregation.

The procedure to follow for taking decisions with the IOWA operator in the D-S theory of evidence is similar to that used with OWA operators, with the difference that now the IOWA operator is used in the aggregation step. The procedure can be summarized as follows.

Assume we have a decision problem in which we have a collection of alternatives $\{A_1, \ldots, A_q\}$ with states of nature $\{S_1, \ldots, S_n\}$. a_{ih} is the payoff to the decision maker if he selects alternative A_i and the state of nature is S_h. The knowledge of the state of nature is captured in terms of a belief structure m with focal elements $B_1, \ldots, B_r$ and associated with each of these focal elements is a weight $m(B_k)$. The objective is to select the alternative which gives the best result to the decision maker. In order to do so, the following steps should be taken:

Step 1: Calculate the payoff matrix.

Step 2: Calculate the belief function m about the states of nature.

Step 3: Calculate the attitudinal character of the decision maker by determining the values u_i. Note that in this case the measure $\alpha(W)$ is different from that adopted by Yager (1988) and is dependent upon the mechanism used in the reordering step. That is:

$$\alpha(W) = \sum_{j=1}^{n} w_j e_j \tag{7}$$

where e_j is the d_i value of the OWG pair $\langle u_i, d_i \rangle$ having the jth largest u_i, u_i is the order inducing variable and $d_i = (n - j) / (n - 1)$.

Step 4: Calculate the collection of weights, w, to be used in the IOWA aggregation for each different cardinality of focal elements. Note that it is possible to use different methods depending on the interests of the decision maker (Xu, 2005b).

Step 5: Determine the payoff collection, M_{ik}, if we select alternative A_i and the focal element B_k occurs, for all the values of i and k. Hence $M_{ik} = \{a_{ih} \mid S_h \in B_k\}$.

Step 6: Calculate the aggregated payoff, V_{ik} = IOWA(M_{ik}), using Eq. (3), for all the values of i and k.

Step 7: For each alternative, calculate the generalized expected value, C_i, where:

$$C_i = \sum_{k=1}^{r} V_{ik} m(B_k) \quad (8)$$

Step 8: Select the alternative with the largest C_i as the optimal. Note that in a situation of costs or similar, we should select the alternative with the lowest C_i.

From a generalized perspective of the reordering step, we can distinguish between ascending and descending orders in the IOWA aggregation. The reason for drawing this distinction is the reordering of the inducing variables, among which the highest value is sometimes the first result in the reordering step, but on other occasions the first result is the lowest value. This depends on the mechanism used for the reordering of the arguments.

The procedure to follow if we use the AIOWA operator in the aggregation step is the same than the procedure used for the IOWA or DIOWA operator with the following differences.

In *Step 3*, when calculating the inducing variables we should consider that in these cases, the lowest inducing variable is the first result in the reordering of the arguments.

In *Step 4*, when calculating the collection of weights, we should consider that the reordering will now be different so that we might associate each weight correctly with its corresponding position.

In *Step 6*, when calculating the aggregated payoff, we should use V_{ik} = AIOWA(M_{ik}), using Eq. (4), for all the values of i and k.

4.2. Using IOWA operators in belief structures

Analyzing the aggregation in *Steps 6* and *7* of Section 4.1., we can formulate the whole aggregation process in one equation. Then, the result obtained is that the focal weights are aggregating the results obtained by using the IOWA operator. We will call this process the BS-IOWA operator and it can be defined as follows.

Definition 8. A BS-IOWA operator is defined by

$$C_i = \sum_{k=1}^{r}\sum_{j_k=1}^{q_k} m(B_k) w_{j_k} b_{j_k} \tag{9}$$

where w_{j_k} is the weighting vector of the kth focal element such that $\sum_{j=1}^{n} w_{j_k} = 1$ and $w_{j_k} \in [0,1]$, b_{j_k} is the a_{i_k} value of the IOWA pair $\langle u_{i_k}, a_{i_k} \rangle$ having the j_kth largest u_{i_k}, u_{i_k} is the order inducing variable, a_{i_k} is the argument variable and $m(B_k)$ is the basic probability assignment. Note that q_k refers to the cardinality of each focal element and r is the total number of focal elements.

The BS-IOWA operator is commutative, monotonic, bounded and idempotent. We can prove these properties with the following theorems.

THEOREM 1 (Commutativity). Assume f is the BS-IOWA operator, then

$$f(\langle u_{1_1}, a_{1_1} \rangle, \ldots, \langle u_{q_r}, a_{q_r} \rangle) = f(\langle u^*_{1_1}, a^*_{1_1} \rangle, \ldots, \langle u^*_{q_r}, a^*_{q_r} \rangle) \tag{10}$$

where $(\langle u^*_{1_1}, a^*_{1_1}\rangle,...,\langle u^*_{q_r}, a^*_{q_r}\rangle)$ is any permutation of $(\langle u_{1_1}, a_{1_1}\rangle,...,\langle u_{q_r}, a_{q_r}\rangle)$ for each focal element *k*.

Proof. Let

$$f(\langle u_{1_1}, a_{1_1}\rangle,...,\langle u_{q_1}, a_{q_1}\rangle,...,\langle u_{q_r}, a_{q_r}\rangle) = \sum_{k=1}^{r}\sum_{j_k=1}^{q_k} m(B_k) w_{j_k} b_{j_k} \quad \text{and}$$

$$f(\langle u^*_{1_1}, a^*_{1_1}\rangle,...,\langle u^*_{q_1}, a^*_{q_1}\rangle,...,\langle u^*_{q_r}, a^*_{q_r}\rangle) = \sum_{k=1}^{r}\sum_{j_k=1}^{q_k} m(B_k) w_{j_k} b^*_{j_k}$$

Since $(\langle u^*_{1_1}, a^*_{1_1}\rangle,...,\langle u^*_{q_r}, a^*_{q_r}\rangle)$ is a permutation of $(\langle u_{1_1}, a_{1_1}\rangle,...,\langle u_{q_r}, a_{q_r}\rangle)$ for each focal element *k*, we have $b_{j_k} = b^*_{j_k}$, and then

$$f(\langle u_{1_1}, a_{1_1}\rangle,...,\langle u_{q_r}, a_{q_r}\rangle) = f(\langle u^*_{1_1}, a^*_{1_1}\rangle,...,\langle u^*_{q_r}, a^*_{q_r}\rangle) \qquad \blacksquare$$

THEOREM 2 (Monotonicity). Assume *f* is the BS-IOWA operator, if $a_{i_k} \geq \hat{a}_{i_k}$, $\forall i$, then

$$f(\langle u_{1_1}, a_{1_1}\rangle,...,\langle u_{q_r}, a_{q_r}\rangle) \geq f(\langle u_{1_1}, \hat{a}_{1_1}\rangle,...,\langle u_{q_r}, \hat{a}_{q_r}\rangle) \tag{11}$$

Proof. Let

$$f(\langle u_{1_1}, a_{1_1}\rangle,...,\langle u_{q_1}, a_{q_1}\rangle,...,\langle u_{q_r}, a_{q_r}\rangle) = \sum_{k=1}^{r}\sum_{j_k=1}^{q_k} m(B_k) w_{j_k} b_{j_k} \quad \text{and}$$

$$f(\langle u_{1_1}, \hat{a}_{1_1}\rangle, \ldots, \langle u_{q_1}, \hat{a}_{q_1}\rangle, \ldots, \langle u_{q_r}, \hat{a}_{q_r}\rangle) = \sum_{k=1}^{r} \sum_{j_k=1}^{q_k} m(B_k) w_{j_k} \hat{b}_{j_k}$$

Since $a_{i_k} \geq \hat{a}_{i_k}$, $\forall i$, it follows that $b_{j_k} \geq \hat{b}_{j_k}$, and then

$$f(\langle u_{1_1}, a_{1_1}\rangle, \ldots, \langle u_{q_r}, a_{q_r}\rangle) \geq f(\langle u_{1_1}, \hat{a}_{1_1}\rangle, \ldots, \langle u_{q_r}, \hat{a}_{q_r}\rangle) \quad \blacksquare$$

THEOREM 3 (Boundedness). Assume *f* is the BS-IOWA operator, then

$$\min\{a_i\} \leq f(\langle u_{1_1}, a_{1_1}\rangle, \ldots, \langle u_{q_1}, a_{q_1}\rangle, \ldots, \langle u_{q_r}, a_{q_r}\rangle) \leq \max\{a_i\} \tag{12}$$

Proof. Let $\max\{a_i\} = b$ and $\min\{a_i\} = a$, then

$$f(\langle u_{1_1}, a_{1_1}\rangle, \ldots, \langle u_{q_r}, a_{q_r}\rangle) = \sum_{k=1}^{r} \sum_{j_k=1}^{q_k} m(B_k) w_{j_k} b_{j_k} \leq \sum_{k=1}^{r} \sum_{j_k=1}^{q_k} m(B_k) w_{j_k} b = b \sum_{k=1}^{r} \sum_{j_k=1}^{q_k} m(B_k) w_{j_k}$$

$$f(\langle u_{1_1}, a_{1_1}\rangle, \ldots, \langle u_{q_r}, a_{q_r}\rangle) = \sum_{k=1}^{r} \sum_{j_k=1}^{q_k} m(B_k) w_{j_k} b_{j_k} \leq \sum_{k=1}^{r} \sum_{j_k=1}^{q_k} m(B_k) w_{j_k} a = a \sum_{k=1}^{r} \sum_{j_k=1}^{q_k} m(B_k) w_{j_k}$$

Since $\sum_{j_k=1}^{q_k} w_{j_k} = 1$ for each focal element and $\sum_{k=1}^{r} m(B_k) = 1$, we get

$$f(\langle u_{1_1}, a_{1_1}\rangle, \ldots, \langle u_{q_r}, a_{q_r}\rangle) = b$$

$$f(\langle u_{1_1}, a_{1_1}\rangle, \ldots, \langle u_{q_r}, a_{q_r}\rangle) = a$$

Therefore,

$$\min\{a_i\} \leq f(\langle u_{1_1}, a_{1_1}\rangle, \ldots, \langle u_{q_1}, a_{q_1}\rangle, \ldots, \langle u_{q_r}, a_{q_r}\rangle) \leq \max\{a_i\} \qquad \blacksquare$$

THEOREM 4 (Idempotency). Assume f is the BS-IOWA operator, if $a_i = a \ \forall i \in N$, then

$$f(\langle u_{1_1}, a_{1_1}\rangle, \ldots, \langle u_{q_1}, a_{q_1}\rangle, \ldots, \langle u_{q_r}, a_{q_r}\rangle) = a \tag{13}$$

Proof. Since $a_i = a \ \forall i \in N$, we have

$$f(\langle u_{1_1}, a_{1_1}\rangle, \ldots, \langle u_{q_r}, a_{q_r}\rangle) = \sum_{k=1}^{r} \sum_{j_k=1}^{q_k} m(B_k) w_{j_k} b_{j_k} \leq \sum_{k=1}^{r} \sum_{j_k=1}^{q_k} m(B_k) w_{j_k} a = a \sum_{k=1}^{r} \sum_{j_k=1}^{q_k} m(B_k) w_{j_k}$$

Since $\sum_{j_k=1}^{q_k} w_{j_k} = 1$ for each focal element and $\sum_{k=1}^{r} m(B_k) = 1$, we get

$$f(\langle u_{1_1}, a_{1_1}\rangle, \ldots, \langle u_{q_1}, a_{q_1}\rangle, \ldots, \langle u_{q_r}, a_{q_r}\rangle) = a \qquad \blacksquare$$

A further interesting feature is the distinction drawn between descending and ascending orders by using $w_j = w^*_{n+1-j}$, where w_j is the jth weight of the BS-DIOWA (or BS-IOWA) operator and w^*_{n+1-j} the jth weight of the BS-AIOWA operator. Then, we obtain the BS-DIOWA and the BS-AIOWA operator. Obviously, these operators also fulfil the properties discussed in Theorems (1) - (4).

If there is a tie in the reordering step of the BS-DIOWA or the BS-AIOWA operator, we can follow the policy explained in Yager and Filev (1999) where the tied arguments are substituted by their arithmetic mean. Note also that it is possible to conduct the reordering of the arguments with words or lexicographic orders that combine words with numbers in the aggregation (Zadeh, 1996).

4.3. Families of BS-IOWA operators

Following a similar methodology as that adopted for the OWA operator, we are able to develop different types of aggregation operators by choosing a different manifestation of the weighting vector in the BS-IOWA operator. Note that it is possible to obtain these results both with the BS-DIOWA or the BS-AIOWA operators by using $w_j = w^*_{n+1-j}$, where w_j is the jth weight of the BS-DIOWA (or BS-IOWA) operator and w^*_{n+1-j} the jth weight of the BS-AIOWA operator.

As can be seen in Definition 8, each focal element uses a different weighting vector in the aggregation step with the IOWA operator. Therefore, it is possible to use different families of IOWA operators in the same BS-IOWA process. For example, assuming that we have three focal elements, we could use the maximum criteria for the first, the minimum criteria for the second and the average criteria for the third. For this reason, in order to conduct the analysis, we will consider different families of IOWA operators individually for each focal element. Note that the nomenclature used in this subsection is not the same as that adopted in earlier subsections.

For example, it is possible to obtain the maximum, the minimum, the average, the Hurwicz criteria, the weighted average (WA) and the OWA operator. These families are obtained in accordance with Yager and Filev (1994). In other words, the maximum is obtained if $w_p = 1$ and $w_j = 0$, for all $j \neq p$, and $u_p = \mathrm{Max}_i\{a_i\}$; the minimum, if $w_p = 1$ and $w_j = 0$, for all $j \neq p$, and $u_p = \mathrm{Min}_i\{a_i\}$; the average criteria, when $w_j = 1/n$, for all a_i; the Hurwicz criteria, when $w_p = \alpha$, $w_q = 1 - \alpha$, $w_j = 0$, for all $j \neq p,q$, and $u_p = \mathrm{Max_i}\{a_i\}$, $u_q = \mathrm{Min_i}\{a_i\}$; the weighted average, if $u_i > u_{i+1}$, for all i; and the OWA operator if the ordered position of u_i is the same as that of b_j such that b_j is the jth largest of a_i.

Other families of aggregation operators could be used in the IOWA operator by using a different manifestation of the weighting vector. For example, when $w_p = 1$ and $w_j = 0$ for all $j \neq p$ we are using the step-IOWA operator (Yager, 1993). Note that if $u_p = \mathrm{Max_i}\{a_i\}$, the step-IOWA becomes the maximum and if $u_p = \mathrm{Min_i}\{a_i\}$, the step-IOWA becomes the minimum.

When $w_j = 1/m$ for $k \leq j \leq k + m - 1$ and $w_j = 0$ for $j > k + m$ and $j < k$, we are using the window-IOWA operator that is based on the window-OWA operator (Yager, 1993). Note that k and m must be positive integers such that $k + m - 1 \leq n$. Also note that if $m = k = 1$, and the initial position of the highest u_i is also the initial position of the highest a_i, then, the window-IOWA becomes the maximum. If $m = 1$, $k = n$, and the initial position of the lowest u_i is also the initial position of the lowest a_i, then, the window-IOWA becomes the minimum. And if $m = n$ and $k = 1$, the window-IOWA becomes the average criteria.

If $w_1 = w_n = 0$, and for all others $w_j = 1/(n - 2)$, we are using the olympic-IOWA operator that is based on the olympic-OWA (Yager, 1996b). Note that if $n = 3$ or $n = 4$, the olympic-IOWA becomes the IOWA-median and if $m = n - 2$ and $k = 2$, the window-IOWA becomes the olympic-IOWA. Also note that the olympic-IOWA becomes the olympic average if $w_p = w_q = 0$, such that $u_p = \mathrm{Max_i}\{a_i\}$ and $u_q = \mathrm{Min_i}\{a_i\}$, and for all others $w_j = 1/(n - 2)$.

Another type of aggregation that could be used is the E-Z IOWA weights that it is based on the E-Z OWA weights (Yager, 2003b). In this case, we should distinguish between two classes. In the first class, we assign $w_j = (1/k)$ for $j = 1$ to k and $w_j = 0$ for $j > k$, and in the second class, we assign $w_j = 0$ for $j = 1$ to $n - k$ and $w_j = (1/k)$ for $j = n - k + 1$ to n. Note that the E-Z IOWA weights becomes the E-Z OWA weights if the ordered position of u_i is the same as that of the ordered position of b_j such that b_j is the jth largest of a_i, from $j = 1$ to k for the first class, or $j = n - k + 1$ to n for the second class. Note also for the first class that the maximum is obtained if $k = 1$ and $b_1 = \text{Max}\{a_i\}$, and the average criteria if $k = n$. In the second class, the minimum is obtained if $k = 1$ and $b_n = \text{Min}\{a_i\}$, and the average if $k = n$.

It should also be noted that the median and the weighted median can be used as induced aggregation operators. For the IOWA-median, if n is odd we assign $w_{(n+1)/2} = 1$ and $w_j = 0$ for all others, and this affects the argument a_i with the $[(n + 1)/2]$th largest u_i. If n is even we assign for example, $w_{n/2} = w_{(n/2)+1} = 0.5$, and this affects the arguments with the $(n/2)$th and $[(n/2) + 1]$th largest u_i. Note that if the ordered position of u_i is the same than the ordered position of b_j such that b_j is the jth largest of a_i, then, the IOWA-median is transformed in the OWA-median.

For the weighted IOWA-median, we follow a similar procedure to that described by Yager (1994). We select the argument a_i that has the kth largest inducing variable u_i, such that the sum of the weights from 1 to k is equal or higher than 0.5 and the sum of the weights from 1 to $k - 1$ is less than 0.5. Note that if the ordered position of u_i is the same than the ordered position of b_j such that b_j is the jth largest of a_i, then, the weighted IOWA-median becomes the weighted OWA-median.

A further type of IOWA aggregation is the S-IOWA operator based on the S-OWA operator (Yager and Filev, 1994). For this type, we have to distinguish between three cases, the “orlike”, the “andlike” and the generalized S-IOWA operator. The “orlike” S-IOWA operator is obtained when $w_p = (1/n)(1 - \alpha) + \alpha$, $u_p = \text{Max}\{a_i\}$, and $w_j = (1/n)(1 - \alpha)$ for all $j \neq p$ with $\alpha \in [0, 1]$. Note that if $\alpha = 0$, we obtain the average and if $\alpha = 1$, the maximum. The “andlike” S-IOWA operator is obtained when $w_q = (1/n)(1 - \beta) + \beta$, $u_q = \text{Min}\{a_i\}$, and $w_j = (1/n)(1 - \beta)$ for all $j \neq q$ with $\beta \in [0, 1]$. In this case, if $\beta = 0$ we obtain the average and if $\beta = 1$, the minimum. Finally, the generalized S-IOWA operator is found when $w_p = (1/n)(1 - (\alpha + \beta) + \alpha$, with $u_p = \text{Max}\{a_i\}$; $w_q = (1/n)(1 - (\alpha + \beta) + \beta$, with $u_q = \text{Min}\{a_i\}$; and $w_j = (1/n)(1 - (\alpha + \beta)$ for all $j \neq p,q$ where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$. Note that if $\alpha = 0$, the generalized S-IOWA operator becomes the “andlike” S-IOWA operator and if $\beta = 0$, the “orlike” S-IOWA operator.

Other families of IOWA operators that could be developed include those that depend on the aggregated objects. For example, we could develop the BADD-IOWA operator as follows.

$$w_j = \frac{b_j^{\alpha}}{\sum_{j=1}^{n} b_j^{\alpha}} \tag{14}$$

with $\alpha \in (-\infty, \infty)$ and b_j is the value a_i of the OWA pair with the jth largest u_i. Note that $\Sigma_j w_j = 1$ and $w_j \in [0,1]$. Note also that if $\alpha = 0$, we obtain the average and if $\alpha = \infty$, the maximum. Other families of IOWA operators that depend on the aggregated objects could be developed by using $(1 - b_j)^{\alpha}$, $(1/b_j)^{\alpha}$, etc., instead of b_j^{α}. These families were developed for the OWA operator in Yager (1993).

A further useful method for obtaining the weighting vector is the functional method known as basic unit interval monotonic function (BUM) (Yager, 1996b). Let f be a function f: [0, 1] $\rightarrow$ [0, 1] such that $f(0) = f(1)$ y $f(x) \geq f(y)$ for $x > y$. Using this BUM function we obtain the IOWA weights for j = 1 to n as:

$$w_j = f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right) \tag{15}$$

Using this method, it is easy to see that $\Sigma_j w_j = 1$ and $w_j \in [0,1]$.

A further type of IOWA operator that could be used in the aggregation is the centered-IOWA operator. Following the same methodology that Yager used for the OWA operator (Yager, 2007), we can define the centered-IOWA operator as an aggregation that is symmetric, strongly decaying and inclusive. It is symmetric if $w_j = w_{j+n-1}$. It is strongly decaying when $i < j \leq (n + 1)/2$, then $w_i < w_j$ and when $i > j \geq (n + 1)/2$, then $w_i < w_j$. It is inclusive if $w_j > 0$. Note that it is possible to consider a relaxation of the second condition by using $w_i \leq w_j$ instead of $w_i < w_j$. This situation is known as softly decaying centered-IOWA operator. An example of this particular situation is the average criteria. Another particular situation appears if we remove the third condition. This case is known as non-inclusive centered-IOWA operator. An example of this case is the IOWA-median. Note that the attitudinal character of the centered-IOWA is not equal to 0.5 because it depends on the inducing variables.

A special type of centered-IOWA operators are the Gaussian-IOWA weights. Note that it is based on the Gaussian-OWA weights developed by Xu in (2005b). In order to define the weighting vector, first we have to consider a Gaussian distribution $\eta(\mu, \sigma)$ where:

$$\mu_n = \frac{1}{n} \sum_{j=1}^{n} j = \frac{n+1}{2} \tag{16}$$

$$\sigma_n = \sqrt{\frac{1}{n} \sum_{j=1}^{n} (j - \mu_n)^2} \tag{17}$$

Assuming that:

$$\eta(j) = \frac{1}{\sqrt{2\Pi\sigma_n}} e^{-(j-\mu_n)^2 / 2\sigma_n^2} \tag{18}$$

we define the IOWA weights as:

$$w_j = \frac{\eta_j}{\sum_{j=}^{n} \eta(j)} = \frac{e^{-(j-\mu_n)^2 / 2\sigma_n^2}}{\sum_{j=1}^{n} e^{-(j-\mu_n)^2 / 2\sigma_n^2}} \tag{19}$$

Note that $\Sigma_j w_j = 1$ and $w_j \in [0,1]$.

Finally, if we assume that all the focal elements use the same weighting vector, then, we can refer to these families as the BS-maximum, the BS-minimum, the BS-average, the BS-WA, the BS-step-IOWA operator, the BS-window-IOWA, the BS-median-IOWA, the BS-olympic-IOWA, the BS-centered-IOWA, the BS-Gaussian-IOWA, the BS-S-IOWA, the BS-EZ-IOWA, etc.

5. Induced OWG operators in decision making with Dempster-Shafer belief structures

An alternative for decision making with D-S theory is made possible by using the IOWG operator in the aggregation step. The reason for using IOWG operators arises because there are situations in which it is better to reorder the arguments with a different mechanism rather than using that of their values. In this mechanism, we introduce an inducing variable for each argument from which we can develop the reordering step. Then, it is possible to aggregate with a different method to that used with the OWG operator (Merigó and Casanovas, 2006). Note that in the 21st Century, it seems more useful to use the IOWA operator but mathematically it is also interesting to consider the geometric version, especially for future research related with decision making with preference relations.

The procedure to follow when taking decisions with the IOWG operator is very similar to the previous method commented when using IOWA operators in the D-S belief structure. The difference is that in this case, the arguments are aggregated with the IOWG operator. Assuming the same variables as with the IOWA operator explained in Section 4.1, we could summarize the procedure with the following steps:

Step 1: Calculate the payoff matrix.

Step 2: Calculate the belief function m about the states of nature.

Step 3: Calculate the attitudinal character of the decision maker by determining the values u_i.

Step 4: Calculate the collection of weights, w, to be used in the IOWG aggregation for each different cardinality of focal elements. Note that it is possible to use different methods depending on the interests of the decision maker (Xu, 2005b).

Step 5: Determine the payoff collection, M_{ik}, if we select alternative A_i and the focal element B_k occurs, for all the values of i and k. Hence $M_{ik} = \{C_{ij} \mid S_j \in B_k\}$.

Step 6: Calculate the aggregated payoff, V_{ik} = IOWG(M_{ik}), using Eq. (4), for all the values of i and k.

Step 7: For each alternative, calculate the generalized expected value, C_i, where:

$$C_i = \sum_{k=1}^{r} V_{ik}\, m(B_k) \tag{20}$$

Step 8: Select the alternative with the largest C_i as the optimal. Note that in a situation of costs or similar, we should select the alternative with the lowest C_i.

Analyzing the aggregation in *Step 6* and *Step 7*, we can formulate the whole aggregation process in one equation as follows. We will call this the BS-IOWG operator.

Definition 9. A BS-IOWG operator is defined by

$$C_i = \sum_{k=1}^{r} \prod_{j_k=1}^{q_k} m(B_k)\, b_{j_k}^{w_{j_k}} \tag{21}$$

where w_{j_k} is the weighting vector of the kth focal element such that $\sum_{j=1}^{n} w_{j_k} = 1$ and $w_{j_k} \in [0,1]$, b_{j_k} is the a_{i_k} value of the IOWG pair $\langle u_{i_k}, a_{i_k} \rangle$ having the j_kth largest u_{i_k}, u_{i_k} is the order inducing variable, a_{i_k} is the argument variable and $m(B_k)$ is the basic probability assignment. Note that q_k refers to the cardinality of each focal element and *r* is the total number of focal elements. Also note that the IOWG operator can only aggregate positive numbers.

The BS-IOWG operator is commutative, monotonic, bounded and idempotent. Their demonstration of these properties is straightforward by looking to theorems (1) – (4) developed for the BS-IOWA operator.

From a generalized perspective of the reordering step it is possible to distinguish between the DIOWG and the AIOWG operators. Their use in D-S framework is straightforward by using $w_j = w^*_{n+1-j}$, where w_j is the *j*th weight of the BS-DIOWG (or BS-IOWG) operator and w^*_{n+1-j} the *j*th weight of the BS-AIOWG operator. The reason for using BS-AIOWG operators arises because it is sometimes better to use an ascending order in the inducing variable. For example, we could use it in situations where the lowest inducing variable is the best result and we want to start the reordering step from this result. Note that it is possible to use words in the inducing variables and if there is a tie in the reordering step of the BS-DIOWG or the BS-AIOWG operator, we should also follow the policy explained in Yager and Filev (1999).

Adopting the same methodology than Section 4.3, we can develop a wide range of families of BS-IOWG operators. As each focal element uses a different weighting vector, the analysis should be conducted individually. Then we could analyze among others, the maximum, the minimum, the geometric mean (GM), the weighted geometric mean (WGM), the step-IOWG operator, the window-IOWG, the IOWG median, the olympic-IOWG, the centered-IOWG, the S-IOWG, etc. If we assume that all the focal elements use the same weighting vector, then, we can refer to these families as the BS-maximum, the BS-minimum, the BS-GM, the BS-WGM, the BS-step-IOWG operator, the BS-window-IOWG, the BS-IOWG median, the BS-olympic-IOWG, the BS-centered-IOWG, the BS-S-IOWG, etc.

6. Illustrative example

In the following, we present an illustrative example using the methodologies described above. We analyze a decision making problem with the D-S belief structure. We use different types of aggregation operators to solve the problem such as the arithmetic mean (AM), the WA, the OWA, the AOWA, the IOWA and the AIOWA operator. We also use different types of geometric aggregation operators such as the GM, the WGM, the OWG, the AOWG, the IOWG and the AIOWG operator. Note that in all cases we assume a situation in which the highest value is the best result. Then, for each situation, we select the alternative with the highest result.

Step 1: Assume a decision maker has five possible investment opportunities and he wants to select the alternative that adapts best to his interests.

1) A_1 is a car company.
2) A_2 is a pharmaceutical company.
3) A_3 is a computer company.
4) A_4 is a chemical company.
5) A_5 is a TV company.

The possible results, depending on the future state of nature, are represented in Table 1. Note that the results are income values.

Table 1. Payoff matrix

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8
A_1	20	40	50	80	30	60	80	50
A_2	30	30	60	70	40	70	50	40
A_3	50	60	20	40	30	50	80	70
A_4	40	50	30	60	50	60	60	60
A_5	60	40	50	30	70	70	60	30

The states of nature represent different economic situations affecting the companies. These situations are evaluated by the world growth rate: S_1 = strong recession, S_2 = weak recession, S_3 = growth rate near 0, S_4 = very low growth rate, S_5 = low growth rate, S_6 = medium growth rate, S_7 = high growth rate, S_8 = very high growth rate.

Step 2: The decision maker has brought together a group of experts in order to solve the problem. After careful analysis, the experts have obtained some probabilistic information about the state of nature that will occur in the future. This information is represented by the following belief function m about the states of nature.

Focal element

$B_1 = \{S_1, S_5, S_6, S_7\} = 0.4$

$B_2 = \{S_1, S_3, S_8\} = 0.3$

$B_3 = \{S_2, S_3, S_4\} = 0.3$

Step 3: Assume the following attitudinal character for the decision maker when using induced aggregation operators, represented in Table 2.

Table 2. Inducing variables

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8
A_1	25	16	24	18	20	13	19	14
A_2	18	34	22	12	24	16	20	26
A_3	13	21	28	22	19	25	16	26
A_4	20	24	14	31	27	25	19	18
A_5	25	16	23	30	15	21	18	26

Step 4: Assume we have used one of the existing methods for determining the weights (Yager, 1993; Xu, 2005) and we have obtained the following results for the different number of arguments.

Weighting vector

$W_2 = (0.7, 0.3)$

$W_3 = (0.4, 0.4, 0.2)$

$W_4 = (0.3, 0.3, 0.2, 0.2)$

Step 5: Calculate the payoff collection, M_{ik}, if we select alternative A_i and the focal element B_k occurs, for all the values of i and k.

A_1: $M_{11} = \langle 20, 30, 60, 80 \rangle$; $M_{12} = \langle 20, 50, 50 \rangle$; $M_{13} = \langle 40, 50, 80 \rangle$.

A_2: $M_{21} = \langle 30, 40, 70, 50 \rangle$; $M_{22} = \langle 30, 60, 40 \rangle$; $M_{23} = \langle 30, 60, 70 \rangle$.

A_3: $M_{31} = \langle 50, 30, 50, 80 \rangle$; $M_{32} = \langle 50, 20, 70 \rangle$; $M_{33} = \langle 60, 20, 40 \rangle$.

A_4: $M_{41} = \langle 40, 50, 60, 60 \rangle$; $M_{42} = \langle 40, 30, 60 \rangle$; $M_{43} = \langle 50, 30, 60 \rangle$.

A_5: $M_{51} = \langle 60, 70, 70, 60 \rangle$; $M_{52} = \langle 60, 50, 30 \rangle$; $M_{53} = \langle 40, 50, 30 \rangle$.

From the sixth step, we can distinguish between four different types of aggregation operators: the IOWA operator, the AIOWA operator, the IOWG operator and the AIOWG operator.

Step 6: Calculate the aggregated payoff, V_{ik}, using Eq. (3) for the IOWA operator and using Eq. (4) for the IOWG operator. The results are shown in Tables 3 and 4.

Table 3. Aggregated payoff

	AM	*WA*	*OWA*	*AOWA*	*IOWA*	*AIOWA*
V_{11}	47.5	43	52	43	43	52
V_{12}	40	38	44	38	38	44
V_{13}	56.6	50	58	50	60	58
V_{21}	47.5	45	50	45	47	48
V_{22}	43.3	44	46	40	46	44
V_{23}	53.3	50	58	50	50	58
V_{31}	52.5	50	55	50	50	55
V_{32}	46.6	42	52	42	46	52
V_{33}	40	40	44	36	36	44
V_{41}	52.5	51	54	51	53	52
V_{42}	43.3	40	46	40	46	44
V_{43}	46.6	44	50	44	50	44
V_{51}	65	65	66	64	65	65
V_{52}	46.6	50	50	44	46	50
V_{53}	40	42	42	38	40	42

Table 4. Aggregated payoff for the geometric operators

	GM	*WGM*	*OWG*	*AOWG*	*IOWG*	*AIOWG*
V_{11}	41.19	37.12	45.70	37.12	37.12	45.70
V_{12}	36.84	34.65	41.62	34.65	34.65	41.62
V_{13}	50.13	46.89	55.55	46.89	57.70	55.18
V_{21}	45.27	42.91	47.75	42.91	45.15	45.38
V_{22}	41.60	41.92	44.41	38.66	44.41	41.92
V_{23}	50.13	46.89	55.55	46.89	46.89	55.55
V_{31}	49.49	47.12	51.97	47.12	47.12	51.97
V_{32}	41.21	37.06	47.62	37.06	39.65	47.62
V_{33}	36.34	35.65	40.95	32.87	32.87	40.95
V_{41}	51.80	50.30	53.34	50.30	52.38	51.22
V_{42}	41.60	38.66	44.41	38.66	44.41	41.92
V_{43}	44.81	42.27	48.55	42.27	48.55	42.27
V_{51}	64.80	64.80	65.81	63.81	64.80	64.80
V_{52}	44.81	48.55	48.55	42.27	43.84	48.55
V_{53}	39.14	41.28	41.28	37.27	38.98	41.28

Step 7: For each alternative, calculate the generalized expected value, C_i, using Eq. (8) for the IOWA operator and Eq. (20) for the IOWG operator.

Table 5. Generalized expected value

	AM	*WA*	*OWA*	*AOWA*	*IOWA*	*AIOWA*
A_1	48	43.6	51.4	43.6	46.6	51.4
A_2	48	46.2	51.2	45	47.6	49.8
A_3	47	44.6	50.8	43.4	44.6	50.8
A_4	48	45.6	50.4	45.6	50	47.2
A_5	52	53.6	54	50.2	51.8	53.6

Table 6. Generalized expected value for the geometric operators

	GM	*WGM*	*OWG*	*AOWG*	*IOWG*	*AIOWG*
A_1	42.56	39.31	47.43	39.31	42.55	47.32
A_2	45.62	43.80	49.08	42.82	45.45	47.39
A_3	43.06	40.66	47.35	39.82	40.60	47.35
A_4	46.64	44.39	49.22	44.39	48.84	45.74
A_5	51.10	52.86	53.27	49.39	50.76	52.86

Step 8: Select the best alternative for each aggregation operator. As we can see, in this problem, the best alternative is A_5.

If we establish an order for the investments, a typical situation if we want to select more than one alternative, we can see that each aggregation gives us a different order. Note that $\succ$ means *preferred to*. The results are shown in Table 7.

Table 7. Ordering of the investments

	Ordering		*Ordering*
AM	$A_5 \succ A_1 = A_2 = A_4 \succ A_3$	*GM*	$A_5 \succ A_4 \succ A_2 \succ A_3 \succ A_1$
WA	$A_5 \succ A_2 \succ A_4 \succ A_3 \succ A_1$	*WGA*	$A_5 \succ A_4 \succ A_2 \succ A_3 \succ A_1$
OWA	$A_5 \succ A_1 \succ A_2 \succ A_3 \succ A_4$	*OWG*	$A_5 \succ A_4 \succ A_2 \succ A_1 \succ A_3$
AOWA	$A_5 \succ A_4 \succ A_2 \succ A_1 \succ A_3$	*AOWG*	$A_5 \succ A_4 \succ A_2 \succ A_3 \succ A_1$
IOWA	$A_5 \succ A_4 \succ A_2 \succ A_1 \succ A_3$	*IOWG*	$A_5 \succ A_4 \succ A_2 \succ A_1 \succ A_3$
AIOWA	$A_5 \succ A_1 \succ A_3 \succ A_2 \succ A_4$	*AIOWG*	$A_5 \succ A_2 \succ A_3 \succ A_1 \succ A_4$

7. Conclusions

In this paper, we have proposed the use of induced aggregation operators in decision making with D-S belief structure. First, we have reviewed some basic aggregation operators and we have forwarded a number of suggestions concerning certain new theoretical aspects, such as the distinction between ascending and descending orders and different families of induced aggregation operators. This analysis has been conducted for both the IOWA and the IOWG operators. We have studied the D-S belief structure and its application in decision making. We have outlined the process that should be followed when using induced aggregation operators in the D-S theory of evidence, and here again we have studied some of its main properties, including the distinction between ascending and descending orders and different families of induced aggregation operators. We have also examined the process that should be adopted with IOWG operators in decision making with D-S theory. Finally, an illustrative example has been provided in which we have reported the results obtained when using the OWA, the OWG, the IOWA and the IOWG operators in decision making with D-S belief structure.

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