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RANK-TWO VECTOR BUNDLES ON NON-MINIMAL RULED SURFACES

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ABSTRACT. We continue previous work by various authors and study the birational geometry of moduli spaces of stable rank-two vector bundles on surfaces with Kodaira dimension $-\infty$. To this end, we express vector bundles as natural extensions by using two numerical invariants associated to vector bundles, similar to the invariants defined by Brînzănescu and Stoia in the case of minimal surfaces. We compute explicitly these natural extensions on blowups of general points on a minimal surface. In the case of rational surfaces, we prove that any irreducible component of a moduli space is either rational or stably rational.

1. INTRODUCTION

Following the inception of the GIT and its appearance in the works of Mumford, Takemoto, Maruyama, and Gieseker that set the foundations of modern vector bundle theory in the 1970s, several decisive results have suggested that the geometry of moduli spaces of vector bundles and the geometry of the base manifolds are interlaced. For example:

- A careful study of the geometry of the moduli spaces of vector bundles plays an essential role in Qin's proof of the Van Den Ven conjecture on the deformation invariance of the Kodaira dimension.
- Mukai proves that two-dimensional moduli spaces of vector bundles over K3 surfaces are also K3 surfaces. In arbitrary dimension, moduli spaces are holomorphically symplectic manifolds.
- A Beilinson spectral sequences analysis carried out by Horrocks, Barth, Hulek, and Maruyama shows that moduli spaces of vector bundles on the projective plane are either rational or unirational.

In [9], the following natural question is addressed. Is it true that the moduli spaces of rank-two vector bundles with large enough second Chern class on rational surfaces are themselves rational? Building on previous contributions in this direction [6], [7] this question is positively answered also in [9].

The first goal of our paper is to answer an improved version of this question, dropping the condition on the second Chern class. We prove in Theorems 4.4 and 4.5 that any non-empty irreducible component of an arbitrary moduli space of

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stable rank-two bundles on a rational surface is either rational or stably rational. Since, at least conjecturally, stably rational varieties ought to be rational, this is conclusive evidence that the moduli spaces are always rational without imposing any further condition on the second Chern class. (For a discussion on rationality, stable rationality, and differences between them see, for example, [3].)

The proof of our result relies on the use of natural numerical invariants associated to vector bundles, similar to the ones introduced by Brînzănescu and Stoia in the minimal case [4], [5]. The definition makes perfect sense for rank-two vector bundles on any surface X with Kodaira dimension $-\infty$. These invariants allow us to present any vector bundle on X as an extension of a certain type, and this construction comes with some advantages. Recall that Serre's method permits us to write any rank-two bundle on an arbitrary surface as an extension involving line bundles and some zero-dimensional subschemes. Conversely, non-trivial extensions are locally free if the zero-dimensional subschemes in question satisfy a certain condition, called *Cayley-Bacharach*. Unfortunately, this condition is locally closed and is neither closed nor open in general. Hence, if we want to use extension spaces corresponding to general bundles (in an irreducible component) to parametrise moduli spaces, we need to control the corresponding locus of zero-dimensional subschemes with the Cayley-Bacharach property. The ideal situation occurs, of course, if this locus coincides with the whole Hilbert scheme. The invariants we use place us precisely in this situation. Their definition and their basic properties form the content of section 3.

Along the way, we prove in section 4 that any irreducible component of a moduli space of rank-two vector bundles is dominated by a projective bundle over a Hilbert scheme, Theorem 4.1. The projective bundle in question is a space of extensions, and the essential fact is that all the zero-dimensional subschemes we work with satisfy the Cayley-Bacharach property; see the proof of Theorem 4.1. A similar result was previously obtained by Qin for minimal surfaces [19, Theorem C].

In section 5, we find explicit values for the numerical invariants of general stable vector bundles on surfaces X that are obtained as blowups of general points on a minimal surface S (Theorems 5.2 and 5.5). As a consequence, we obtain a refinement of Theorem 4.1 for these surfaces. If C is the curve over which S is ruled and F is the class of a general fibre lifted to X, we prove that the moduli spaces are birational to a projective bundle either over a product of two copies of the Jacobian of C with a symmetric product of C if $c_1 \cdot F$ is even or over just a product of two copies of the Jacobian of C if $c_1 \cdot F$ is odd, respectively; see Corollaries 5.3 and 5.6.

Notation. We will work over an algebraically closed field K of characteristic zero. Given a non-singular variety X we denote by K_X its canonical divisor and by $q(X)$ its irregularity. For any coherent sheaf E on X we are going to denote by $H^i(X,E)$ the cohomology groups; meanwhile $h^i(X,E)$ stands for their dimension. If E and E' are two coherent sheaves on X, the dimension of the space $\operatorname{Ext}^i_X(E,E')$ is denoted by $\operatorname{ext}^i_X(E,E')$. We denote by $\chi(X,E) := \sum_{i=0}^{\dim X}(-1)^i h^i(E)$ the Euler characteristic of E.

2. Background

We start collecting the main results that we will use concerning stable vector bundles on a smooth projective surface and their moduli spaces.

Definition 2.1. Let L be an ample divisor on a smooth projective surface X. A rank-two vector bundle V on X is L-semistable if for any rank-one subbundle E of V,

$$c_1(E) \cdot L \leq \frac{c_1(V) \cdot L}{2}.$$

If strict inequality holds, we say that V is L-stable. We say that V is simple if $\operatorname{Hom}_X(V, V) = K$. Notice that any L-stable vector bundle is simple.

We will denote by $\mathcal{M}_L(c_1, c_2)$ the moduli space of rank-two L-stable vector bundles V on a smooth projective surface X with $c_1(V) = c_1$ and $c_2(V) = c_2$.

Theorem 2.2. *Let X be a smooth projective surface, let L be an ample divisor on X, and let $c_1, c_2 \in H^*(S, \mathbb{Z})$ be Chern classes. For all $c_2 \gg 0$, $\mathcal{M}_L(c_1, c_2)$ is a smooth, irreducible, quasiprojective variety of dimension $4c_2 - c_1^2 - 3\chi(\mathcal{O}_X) + q(X)$.*

Proof. See [10], [24], [12], and [18].

One of the tools that we will use concerns prioritary sheaves. Prioritary sheaves were introduced on ruled surfaces by Walter in [23] as a generalization of semistable sheaves, and we recall its definition for the sake of completeness.

Definition 2.3. Let $\pi : S \longrightarrow C$ be a ruled surface and consider $F \in \operatorname{Num}(S)$ the numerical class of a fibre of π. A coherent sheaf E on S is said to be prioritary if it is torsion free and if $\operatorname{Ext}^2_S(E, E(-F)) = 0$.

Remark 2.4. If H is an ample divisor on a ruled surface S such that $H(K_S + F) < 0$, then any H-stable, torsion free sheaf is prioritary (see the proof of [23, Theorem 1]).

We denote by $\mathcal{S}pl(c_1, c_2)$ the moduli space of rank-two simple, prioritary, torsion free sheaves E on S with Chern classes c_1 and c_2. From [23, Proposition 2], we get the following result:

Theorem 2.5. *Let S be a smooth ruled surface, let L be an ample divisor on S, and let $c_1, c_2 \in H^*(S, \mathbb{Z})$ be Chern classes. Then, the moduli space $\mathcal{S}pl(c_1, c_2)$ is a smooth, irreducible, quasiprojective variety. Moreover, if $L \cdot (K_S + F) < 0$, then the moduli space $\mathcal{M}_L(c_1, c_2)$ is an open dense subset of $\mathcal{S}pl(c_1, c_2)$.*

We end the section gathering the relevant results on ruled surfaces that we will use throughout this paper.

Let e and $m \geq 1$ be two integers. Let $p_1, \ldots, p_m$ be distinct points on a geometrically ruled surface S of invariant e over a smooth genus–g curve C, let $\pi : S \to C$ be the ruling, and let $\sigma : X \to S$ be a blowup of S in $p_1 = p_{11}, \ldots, p_m = p_{m1}$ and possibly other infinitely near points p_{ij} with $i = 1, \ldots, m$ and $j = 2, \ldots, k_i$. Put $\phi = \pi \circ \sigma$ and denote by $E_{11}, \ldots, E_{1k_1}$, $E_{21}, \ldots, E_{2k_2}, \ldots$, $E_{m1}, \ldots, E_{mk_m}$ the irreducible components of the exceptional divisor. In this notation, since $p_{i1} = p_i$, $E_{i1} = E_i$ is the first component of the blowup of S in p_i.

Denote by C_0 the minimal section of S so that $e = -C_0^2$, by F the fibre over a general point $p \in C$ of the ruling, and, if no confusion arises, the same notation for their pullbacks to X. Denote by $\widetilde{F_i}$ the strict transform of the fibre through p_i.

3. Numerical invariants associated to vector bundles

The main goal of this section will be to associate to any rank-two vector bundle V on X two different invariants that will be key ingredients in order to classify L-stable vector bundles later on. More precisely, in the minimal case ($m = 0$), Brînzănescu and Stoia introduced two numerical invariants associated to any rank-two vector bundle which are used to present the given bundle as a natural extension [5]. In the sequel, we will define similar invariants in the non-minimal case, and we will find the corresponding canonical extension. To do so, we fix V a rank-two vector bundle on X with $\det(V) = \mathcal{O}_X(\alpha C_0 + \beta F + G) \otimes \phi^* P$, where $P \in \mathrm{Pic}^0(C)$ and G is supported on the exceptional divisor. By twisting V with multiples of E_{ij} if necessary, we may assume that G is effective and $c_2(V) = c_2 \in \mathbb{Z}$. The first invariant will be given by the generic splitting type.

The invariant d_V. For a general point p in C the restriction of V to the corresponding fibre is of type $\mathcal{O}_F(d) \oplus \mathcal{O}_F(d')$ with $d \geq d'$ and $d + d' = \alpha$. The given d is the first invariant d_V. Note that this invariant is upper-continuous in flat families of rank-two bundles; indeed, $d_V \geq k$ if and only if $h^0(\mathcal{O}_{F_q}(-k)) \neq 0$ for any fibre F_q over $q \in C$.

Once d_V is determined, we define the second invariant.

The invariant r_V. The pushforward $\phi_* V(-d_V C_0)$ is either a line bundle (if $2d_V > \alpha$) or a rank-two bundle (if $2d_V = \alpha$) on C. Indeed, since the target of ϕ is a smooth curve, ϕ is flat, implying that $\phi_* V(-d_V C_0)$ is torsion free and hence locally free. Therefore, by Grauert's Theorem ([13, Corollary 12.9]) over an open subset of the target $rank(\phi_* V(-d_V C_0))$ equals one (if $2d_V > \alpha$) or two (if $2d_V = \alpha$). Define $r = r_V$ to be the maximum degree of a line subbundle of $\phi_* V(-d_V C_0)$; by a result of Nagata [16, Theorem 1], $2r \geq \deg(\phi_* V(-d_V C_0)) - g$. Alternatively, r_V is the maximum number for which there is a non-zero morphism $\mathcal{O}_X(d_V C_0 + rF) \otimes \phi^* M \to V$ with $M \in \mathrm{Pic}^0(C)$. If $C = \mathbb{P}^1$, then the invariant r_V has a simpler description:

$$r_V = \max \left\{ r \mid h^0((\phi_* V(-d_V C_0))(-r)) \neq 0 \right\}.$$

Note that if $2d_V = \alpha$ and the genus of the base curve is at least one, the maximal subbundle is not necessarily unique; see [14].

Lemma 3.1. *The invariant r_V is upper-semicontinuous in flat families of rank-two bundles with $d_V = 0$.*

Proof. Let $\{V_t\}_{t \in T}$ be a flat family of rank-two bundles with $d_{V_t} = 0$ for all t and let $r \in \mathbb{Z}$. We need to prove that the set

$$\{t \in T \mid r_{V_t} \geq r\} \subset T$$

is closed. Note that $r_{V_t} \geq r$ if and only if there exists a line bundle $\mathcal{L}$ of degree r on C such that $h^0(X, V_t \otimes \phi^* \mathcal{L}) \neq 0$ and hence

$$\{t \in T \mid r_{V_t} \geq r\} = \bigcup_{\mathcal{L} \in \mathrm{Pic}^r(C)} \{t \in T \mid h^0(V_t \otimes \phi^* \mathcal{L}) \neq 0\}.$$

The conclusion follows observing that the subset

$$\{(t, \mathcal{L}) \in T \times \mathrm{Pic}^r(C) \mid h^0(V_t \otimes \phi^* \mathcal{L}) \neq 0\} \subset T \times \mathrm{Pic}^r(C)$$

is closed and it maps, via the first projection $T \times \mathrm{Pic}^r(C) \to T$ which is a proper map, to the subset under question:

$$\bigcup_{\mathcal{L} \in \mathrm{Pic}^r(C)} \{t \in T | \ h^0(V_t \otimes \phi^*\mathcal{L}) \neq 0\} \subset T.$$

□

Let r be an integer such that $H^0(X, V(-d_V C_0 - rF) \otimes \phi^* M^{-1}) \neq 0$. A section σ in $H^0(X, V(-d_V C_0 - rF) \otimes \phi^* M^{-1})$ giving a morphism $\mathcal{O}_X(d_V C_0 + rF) \otimes \phi^* M \to V$ with $M \in \mathrm{Pic}^0(C)$ will vanish along a zero-dimensional lci subscheme Z plus possibly along an effective divisor D. In this case, V is presented as an extension

(1)
$$0 \to \mathcal{O}_X(d_V C_0 + rF + D) \otimes \phi^* M \to V \to \mathcal{I}_Z((\alpha - d_V)C_0 + (\beta - r)F + (G - D)) \otimes \phi^* N \to 0$$

with $M, N \in \mathrm{Pic}^0(C)$, $M \otimes N = P$.

By the definition of the invariants d_V and r_V, if $r = r_V$, then the divisor D must be supported along the exceptional divisor and strict transforms $\widetilde{F}_i$ and $h^0(X, \mathcal{O}_X(D - F_q)) = 0$ for any fibre F_q over $q \in C$. On the other hand, if $r_V > r$, then D must be supported along the exceptional divisor and strict transforms $\widetilde{F}_i$ and possibly copies of F.

To an extension (1) one associates a natural numerical class

$$\zeta \equiv (2d_V - \alpha)C_0 + (2r - \beta)F + (2D - G);$$

see [22], [21]. It is clear from the definition that the length of the scheme Z from the extension (1) is computed as

$$\ell := \ell(Z) = c_2 + (\zeta^2 - c_1^2)/4 \geq 0. \tag{2}$$

Following Qin, [22], [21], [19], we denote by $E_\zeta(c_1, c_2)$ the family of non-trivial extensions of type (1); it is birational to a projective bundle over $X^{[\ell]} \times \mathrm{Pic}^0(C) \times \mathrm{Pic}^0(C)$. Since $2d_V \geq \alpha$ it follows that Z trivially satisfies the Cayley-Bacharach property with respect to $|K_X \otimes \mathcal{O}_X((\alpha - 2d_V)C_0 + (\beta - 2r)F + G - 2D) \otimes N \otimes M^{-1}|$, and hence for any Z, M, and N a general extension is a vector bundle. These extension families are crucial in the birational description of the moduli spaces.

Remark 3.2. If we replace V by a twist with a divisor supported on the exceptional divisor, the invariant d_V remains the same while the invariant r_V might change. Indeed, let $C = \mathbb{P}^1$ and assume $m = 1$ and $k_1 = 1$; i.e., the exceptional divisor has only one component E_1. If V is the trivial bundle, then the invariants d_V, r_V are zero. However the invariant $r_{V(-E_1)}$ of $V(-E_1)$ will be equal to -1 as $h^0(\mathcal{O}_X(-E_1)) = 0$ and $|\mathcal{O}_X(F - E_1)|$ consists of the strict transform $\widetilde{F}$ of a fibre. However, the canonical extension is

$$0 \to \mathcal{O}_X(-F + \widetilde{F}) \to \mathcal{O}_X(-E_1)^{\oplus 2} \to \mathcal{O}_X(-F + \widetilde{F}) \to 0;$$

i.e., it is indeed the canonical extension of the trivial bundle twisted by $\mathcal{O}_X(-E_1)$.

Remark 3.3. By definition, it is clear that any vector bundle in an extension (1) has $d_V = d$.

Moreover, if $2d_V > \alpha$, then any vector bundle V in the extension (1) will also have $r_V = r$. Besides, for any $M, N' \in \mathrm{Pic}^0(C)$ and $Z' \subset X$, there is no non-zero morphism from $\mathcal{O}_X(d_V C_0 + rF + D) \otimes \phi^* M$ to $\mathcal{O}_X((\alpha - d_V)C_0 + (\beta - r_V)F + (G - D)) \otimes \phi^* N' \otimes \mathcal{I}_{Z'}$ (as $d_V > \alpha - d_V$), and hence the divisor D, the line bundle

M, and the subscheme Z are also determined by V and $r_V = r$. In conclusion, if $2d_V > \alpha$, then the extension (1) is uniquely determined by V, and this phenomenon is mainly due to the fact that $\phi_* V(-d_V C_0)$ is a line bundle.

On the contrary, if $2d_V = \alpha$, it might happen that some bundles presented as an extension (1) have $r_V \neq r$ as is shown in the next example. The equality occurs if and only if the rank-two bundle $\phi_* V(-d_V C_0) \otimes \mathcal{O}_C(-rp) \otimes M^{-1}$ on C is normalized, i.e., $h^0(\phi_* V(-d_V C_0) \otimes \mathcal{O}_C(-rp) \otimes M^{-1}) \neq 0$ and $h^0(\phi_* V(-d_V C_0) \otimes \mathcal{O}_C(-(r+1)p) \otimes M^{-1}) = 0$.

Example 3.4. To construct an example in the simplest setup, let us assume that S is a ruled surface over $\mathbb{P}^1$ and that X is the blowup of S at one point. Let V be a rank-two vector bundle given by a non-trivial extension

$$0 \to \mathcal{O}_X(-nF) \to V \to \mathcal{I}_Z(nF + E_1) \to 0,$$

where Z is a 0-dimensional subscheme of length $2n+1$ such that 3 points lie on a fibre and the other ones lie in $2n-2$ different fibres. Notice that since $H^0(\mathcal{I}_Z((2n-1)F + E_1)) \neq 0$, we have $h^0 V((n-1)F) \neq 0$. Therefore, $r_V > r = -n$.

Next, we address the following question:

Question 3.5. We place ourselves in the case $\alpha = d_V = 0$. Let ζ be the numerical class $(2r-\beta)F + (2D - G)$ and let $E_\zeta(c_1, c_2)$ be the family of extensions

$$0 \to \mathcal{O}_X(rF + D) \otimes \phi^* M \to V \to \mathcal{I}_Z((\beta - r)F + (G - D)) \otimes \phi^* N \to 0 \tag{3}$$

with $M, N \in \mathrm{Pic}^0(C)$. When is $r_V = r$ for a general V in $E_\zeta(c_1, c_2)$?

We answer this question for $D = 0$ and we will see later that quite often $D = 0$ (see the proof of Theorem 5.2).

Proposition 3.6. *Let V_η be a vector bundle corresponding to a general extension $\eta \in E_\zeta(c_1, c_2)$ where $\zeta = (2r-\beta)F + 2D - G$ with $G = \sum_{i=1}^{\rho} E_i \geq 0$ and $D = \sum_{i=1}^{\rho} q_i E_i$ with $q_i \geq 0$.*

(a) *If $r_{V_\eta} = r$ for a general $\eta \in E_\zeta(c_1, c_2)$, then $2r \geq \beta - g - c_2$.*
(b) *If $D = 0$ and $2r \geq \beta - g - c_2$, then $r_{V_\eta} = r$.*

Proof. (a) By definition, for any η we have $r_{V_\eta} \geq r$. By semicontinuity, $r_{V_\eta} = r$ for a general η if and only if there exists a V with $r_V = r$; i.e., there exists V such that $(\phi_* V)(-rp) \otimes M^{-1}$ is normalized. By Nagata's Theorem ([16, Theorem 1]), we obtain

$$\deg((\phi_* V)(-rp)) \leq g.$$

On the other hand, the non-zero morphism $\mathcal{O}_X(rF) \otimes \phi^* M \to V$ gives rise to a canonical short exact sequence

$$0 \to \mathcal{O}_X(rF + \sum_{i=1}^{\rho} q_i E_i) \otimes \phi^* M \to V \to \mathcal{I}_Z((\beta - r)F + \sum_{i=1}^{\rho}(1-q_i)E_i)) \otimes \phi^* N \to 0$$

with $q_i \geq 0$, $N \in \mathrm{Pic}^0(C)$, and Z a zero-dimensional subscheme of length $\ell(Z) = c_2 + \sum_{i=1}^{\rho} q_i(1-q_i)$. Since C is a smooth curve, the pushforward of the above exact sequence gives a short exact sequence:

$$0 \to \mathcal{O}_C(rp) \otimes M \to \phi_* V \to \mathcal{O}_C((\beta - r)p) \otimes N \otimes \phi_*(\mathcal{I}_Z(\sum_{i=1}^{\rho}(1-q_i)E_i)) \to 0.$$

Hence, since $\phi_*\mathcal{O}_X(E_i) = \mathcal{O}_C$, we have

$$\deg((\phi_*V)(-rp)) = (\beta - 2r) + \deg(\phi_*(\mathcal{I}_Z(\sum_{i=1}^{\rho}(1-q_i)E_i)))$$

$$\leq (\beta - 2r) - (c_2 + \sum_{i=1}^{\rho} q_i(1-q_i)) + (\sum_{i,\ q_i\geq 2}(1-q_i))$$

$$= (\beta - 2r) - (c_2 + \sum_{i,\ q_i\geq 2} q_i(1-q_i)) + (\sum_{i,\ q_i\geq 2}(1-q_i)) = \beta - 2r - c_2 + \sum_{i,\ q_i\geq 2}(1-q_i)^2.$$

Therefore,

$$2r \geq \beta - g - c_2 + \sum_{i,\ q_i\geq 2}(1-q_i)^2 \geq \beta - g - c_2.$$

(b) Suppose that $2r - \beta \geq -g - c_2$. We denote $\ell := c_2$ and we will prove that there exists a V associated to an extension in $E_\zeta(c_1, c_2)$ for which $\phi_*V \otimes \mathcal{O}_C(-rp) \otimes M^{-1}$ is normalized. Let $Z = \{z_1, \dots, z_\ell\}$ be the reduced zero-dimensional subscheme of X obtained by intersection between C_0 and ℓ distinct fibres F_{q_i} of ϕ over general points $q_1, \dots, q_\ell \in C \setminus \{\phi(p_1), \dots, \phi(p_m)\}$. In this case, $\phi_*\mathcal{I}_Z = \phi_*(\mathcal{I}_Z(G)) \cong \mathcal{O}_C(-\sum_{i=1}^{l} q_i)$. We will choose also $M = N = \mathcal{O}_C$.

Claim. The map

$$\mathrm{Ext}^1_X(\mathcal{I}_Z((\beta - r)F + G), \mathcal{O}_X(rF)) \to \mathrm{Ext}^1_C(\mathcal{O}_C((\beta - 2r)p - \sum_{i=1}^{l} q_i), \mathcal{O}_C)$$

given by $V \mapsto (\phi_*V)(-rp)$ is surjective.

We prove the claim in several steps, factoring the given map in other surjective maps. First, we prove the surjectivity of the natural map

$$\mathrm{Ext}^1_X(\mathcal{I}_Z((\beta - r)F + G), \mathcal{O}_X(rF)) \to \mathrm{Ext}^1_X(\mathcal{I}_Z((\beta - r)F), \mathcal{O}_X(rF)).$$

Indeed, this map is dual, via Serre's duality to the map

$$H^1(X, \mathcal{I}_Z((\beta - 2r)F) \otimes K_X) \to H^1(X, \mathcal{I}_Z((\beta - 2r)F + G) \otimes K_X),$$

which is injective, as

$$H^0(G, \mathcal{I}_Z((\beta - 2r)F + G) \otimes K_X|_Z) = H^0(G, K_G) = 0$$

(use $\mathcal{I}_Z|_G \cong \mathcal{O}_G$ and $\mathcal{O}_X(F)|_G \cong \mathcal{O}_G$).

Second, we prove the surjectivity of the natural map

$$\mathrm{Ext}^1_X(\mathcal{I}_Z((\beta - r)F), \mathcal{O}_X(rF)) \to \mathrm{Ext}^1_X(\mathcal{O}_X((\beta - r)F - \sum_{i=1}^{l} F_i), \mathcal{O}_X(rF)).$$

Since $\mathcal{I}_{\{z_i\}\subset F_{q_i}} = \mathcal{O}_{F_{q_i}}(-1)$, we obtain a short exact sequence:

$$0 \to \mathcal{O}_X\left(-\sum_{i=1}^{\ell} F_{q_i}\right) \to \mathcal{I}_Z \to \bigoplus_{i=1}^{\ell} \mathcal{O}_{F_{q_i}}(-1) \to 0,$$

which yields, after tensorization with $K_X \otimes \mathcal{O}_X((\beta - 2r)F)$, to an injective map

$$H^1(X, K_X \otimes \mathcal{O}_X((\beta - 2r)F) \otimes \mathcal{O}_X(-\sum_{i=1}^{\ell} F_{q_i})) \to H^1(X, K_X \otimes \mathcal{O}_X((\beta - 2r)F) \otimes \mathcal{I}_Z).$$

Applying duality we obtain the surjective natural map

$$\mathrm{Ext}^1_X(\mathcal{I}_Z((\beta - r)F), \mathcal{O}_X(rF)) \to \mathrm{Ext}^1_X(\mathcal{O}_X((\beta - r)F - \sum_{i=1}^{\ell} F_{q_i}), \mathcal{O}_X(rF))$$

we were looking for.

Finally, the pushforward map

$$\mathrm{Ext}^1_X(\mathcal{O}_X((\beta - r)F - \sum_{i=1}^{\ell} F_{q_i}), \mathcal{O}_X(rF)) \to \mathrm{Ext}^1_C(\mathcal{O}_C((\beta - r)p - \sum_{i=1}^{\ell} q_i), \mathcal{O}_C(rp))$$

is surjective, as the pullback is a right-inverse, by projection formula. Hence, we have proved the claim.

Using [13, Exercise 2.5(c)], if $\beta - 2r - \ell \leq g$, then a general extension in $\mathrm{Ext}^1_C(\mathcal{O}_C((\beta - 2r)p - \sum_{i=1}^{\ell} q_i), \mathcal{O}_C)$ is normalized, which implies also, via the surjectivity of the map $V \to (\phi_* V)(-rp)$, that a general extension in

$$\mathrm{Ext}^1_X(\mathcal{I}_Z((\beta - r)F + G), \mathcal{O}_X(rF))$$

will have $r_V = r$. $\square$

4. The birational structure of moduli spaces

The main result of this section is the following birational structure characterisation of moduli spaces (compare to [21], [19], [7], [9]):

Theorem 4.1. *Let H be an ample divisor on X. Then any non-empty irreducible component $\mathcal{M}$ of the moduli space $\mathcal{M}_H(c_1, c_2)$ is dominated by a projective bundle over $C^{[\ell]} \times \mathrm{Pic}^0(C) \times \mathrm{Pic}^0(C)$ for a suitable positive integer ℓ.*

Proof. From the previous section, any vector bundle V in $\mathcal{M}$ is presented as an extension (1) for the corresponding d_V, r_V, and D. Given d_V, r_V, and D, the set of vector bundles that live in corresponding extensions (1) is a constructible set, as it is the image of a morphism from $E_\zeta(c_1, c_2)$ to $\mathcal{M}$. Note that the set of triples (d_V, r_V, D) is countable (D is supported on given fixed divisors), and hence $\mathcal{M}$ is a countable union of constructible subsets. This implies that $\mathcal{M}$ is also a finite union of this countable number of subsets and in particular one of the subsets must be dense (i.e., must contain an open subset); see for example [15]. In conclusion, a general vector bundle V in $\mathcal{M}$ is presented as an extension (1) with fixed d_V, r_V, and D. Since the general elements in the extension (1) are locally free, it follows that $\mathcal{M}$ is dominated by the space of extensions $E_\zeta(c_1, c_2)$.

In what concerns the structure of $E_\zeta(c_1, c_2)$ it is clear that it is birational to a projective bundle over $X^{[\ell]} \times \mathrm{Pic}^0(C) \times \mathrm{Pic}^0(C)$ via the map that associates to any extension the triple (Z, M, N), and the fibres of this map are the projective spaces

$$\mathbb{P}\mathrm{Ext}^1_X(\phi^* N \otimes \mathcal{I}_Z, \mathcal{O}_X((2d_V - \alpha)C_0 + (2r_V - \beta)F + (2D - G)) \otimes \phi^* M).$$

Since $X^{[\ell]}$ is also birational to a projective bundle over $C^{[\ell]}$ the conclusion follows. $\square$

Remark 4.2. The dominating extension family is not necessarily unique; see Theorem 5.2 and Example 5.4 in the next section.

Remark 4.3. A similar argument shows that for any surface S, any irreducible component $\mathcal{M}$ of a moduli space of stable rank-two vector bundles with fixed Chern classes is dominated by a locally closed subset of a projective bundle over $S^{[\ell]} \times \mathrm{Pic}^0(S) \times \mathrm{Pic}^0(S)$. Indeed, any bundle can be presented as an element of the countable family of extensions

$$0 \to M_0 \otimes M \to V \to N_0 \otimes N \otimes I_Z \to 0,$$

where M_0 and N_0 are fixed line bundles, $M, N \in \mathrm{Pic}^0(S)$, and Z is a zero-dimensional subscheme. A general vector bundle will belong to a single given extension family, and the locally closed subvariety is obtained from the condition that Z satisfies the Cayley-Bacharach property with respect to the corresponding adjoint bundle. However, it seems that a nice description of this locus in the widest generality is out of reach.

To obtain a full birational description of the irreducible components $\mathcal{M}$, we need first to describe the general fibres of the map $\psi : E_\zeta(c_1, c_2) \to \mathcal{M}$. If $2d_V > \alpha$, then, from Remark 3.3, the map ψ is birational. In particular, we obtain (see also [9, Theorem A]):

Theorem 4.4. *If $c_1 \cdot F$ is odd, then any non-empty irreducible component $\mathcal{M}$ of the moduli space $\mathcal{M}_H(c_1, c_2)$ is birational to a projective bundle over $C^{[\ell]} \times \mathrm{Pic}^0(C) \times \mathrm{Pic}^0(C)$ for a suitable positive integer ℓ. In particular, for $C = \mathbb{P}^1$ the moduli space is rational.*

Proof. Since $c_1 \cdot F$ is odd, we necessarily have $2d_V > \alpha$. □

In the case $2d_V = \alpha$ and $C = \mathbb{P}^1$, we can prove the following.

Theorem 4.5. *If $C = \mathbb{P}^1$ and $c_1 \cdot F$ is even, then any non-empty irreducible component $\mathcal{M}$ of the moduli space $\mathcal{M}_H(c_1, c_2)$ is stably rational.*

Proof. We use the notation from the proof of Theorem 4.1 and the previous section. If $2d_V > \alpha$, then we can apply the discussion above to conclude that $\mathcal{M}$ is rational. We are hence in the situation where $2d_V = \alpha$. The fibre of ψ over V in this case is the open subset of $\mathbb{P}H^0(V(-d_V C_0 - rF - D))$ corresponding to sections that vanish along a zero-dimensional subscheme only. Indeed, any section of

$$H^0(V(-d_V C_0 - rF - D))$$

vanishing along a zero-dimensional subscheme gives a presentation of V as an extension in $E_\zeta(c_1, c_2)$, and conversely, any presentation corresponds to a section. Hence $E_\zeta(c_1, c_2)$ is birationally a projective bundle over $\mathcal{M}$. On the other hand, $E_\zeta(c_1, c_2)$ is rational, which implies that $\mathcal{M}$ must be stably rational. □

5. Computation of the extension spaces

In the previous section we have proved that general elements in a given irreducible component of a moduli space of stable rank-two bundles are presented as an extension in some particular extension space. If $c_1 \cdot F = 1$, this extension space is unique. However, in the case $c_1 \cdot F = 0$, this extension space is not unique anymore; see Remark 3.3. One can raise some very natural questions related to this situation. Can one determine effectively this extension space in the case $c_1 \cdot F = 1$? If $c_1 \cdot F = 0$, what is the most natural extension space that can cover (an irreducible component of) a moduli space? We answer these questions in the case when X is the blowup of general points on the minimal surface S; i.e., there are no infinitely near blown-up points.

So, throughout this section, $X \to S$ will be the blowup of S at m general points $p_1, \cdots, p_m$. Since we are dealing with rank-two stable vector bundles V on X we can assume that $c_1(V) = \alpha C_0 + \beta F + \sum_{i=1}^m \gamma_i E_i$ with $\alpha, \beta, \gamma_i \in \{0,1\}$.

Given any smooth surface Y, H an ample divisor on Y, and $\sigma : \tilde{Y} \to Y$ the blowup of Y at one point, for any $n \gg 0$, the divisor

$$H_n := n\sigma^* H - E_1$$

is ample. Moreover, by [17, Theorem 1], for $n \gg 0$, there exists an open immersion

$$\mathcal{M}_{Y,H}(c_1, c_2) \subset \mathcal{M}_{\tilde{Y},H_n}(\sigma^* c_1, c_2)$$

between smooth irreducible moduli spaces of the same dimension. Therefore, while describing the moduli space $\mathcal{M}_L(c_1, c_2)$ of L-stable rank-two vector bundles on X, we can assume without loss of generality that

$$c_1 = \alpha C_0 + \beta F + \sum_{i=1}^{\rho} E_i$$

with $\alpha, \beta \in \{0,1\}$ and $\rho = m$.

From now on, and until the end of the paper, we will work under this assumption and we will analyse separately the cases $\alpha = 1$ and $\alpha = 0$.

5.1. **The case** $c_1 \cdot F = 0$. Let $c_1 = \eta F + \sum_{i=1}^m E_i$ with $\eta \in \{0,1\}$ and $c_2 = 2n + \varepsilon \gg 0$ with $\varepsilon \in \{0,1\}$. Let L be a (c_1, c_2)–suitable ample divisor; i.e., L belongs to a chamber of type (c_1, c_2) whose closure contains the ray spanned by F, [11, Definition 1, p. 142]. In order to find the most natural extension space that dominates $\mathcal{M}_L(c_1, c_2)$, we need to identify first the invariants d and r of general bundles.

Lemma 5.1. *For any V in $\mathcal{M}_L(c_1, c_2)$ we have $d_V = 0$.*

Proof. The result is proved in its most general settings in [11, Theorem 5]. For the convenience of the reader, we present here the proof in our case.

Suppose $d = d_V > 0$ for some V and put $r = r_V$. Then V lies in an extension space $E_\zeta(c_1, c_2)$ with $\zeta = 2dC_0 + (2r - \eta)F + (2D - \sum_{i=1}^m E_i)$. Note that $\zeta \cdot F = 2d > 0$ and $\zeta \cdot L < 0$, by the L-stability of V.

We claim that $c_1^2 - 4c_2 \le \zeta^2 < 0$. In this case, ζ would define a wall separating F and L, which would be in contradiction with the (c_1, c_2)–suitability of L. The inequality $c_2 + (\zeta^2 - c_1^2)/4 \ge 0$ is automatic and follows from (2). To verify $\zeta^2 < 0$, we consider the numerical class $\xi := (L \cdot F)\zeta - (L \cdot \zeta)F$. It is orthogonal to L, and hence, by the Hodge Index Theorem, it follows that $\xi^2 \le 0$ with equality if and only if $\xi = 0$. We compute $\xi^2 = (L \cdot F)^2(\zeta^2) - 2(L \cdot F)(\zeta \cdot L)(\zeta \cdot F) \le 0$. Since $L \cdot F > 0$, $\zeta \cdot L < 0$, and $\zeta \cdot F > 0$, it follows that $\zeta^2 < 0$, and the claim is proved. □

Theorem 5.2. *A general vector bundle V in any irreducible component of $\mathcal{M}_L(c_1, c_2)$ lies in an extension of type*

$$0 \to \mathcal{O}_X(r_0 F) \otimes \phi^* M \to V \to \mathcal{I}_Z((\eta - r_0)F + \sum_{i=1}^m E_i) \otimes \phi^* N \to 0 \tag{4}$$

with $h^0(V(-r_0 F) \otimes \phi^* M^{-1}) = 1$ and $r_V = r_0 := \left\lceil \frac{\eta - c_2 - g}{2} \right\rceil = \left\lceil \frac{\eta - 2n - \varepsilon - g}{2} \right\rceil$.

Proof. By semicontinuity, for a general vector bundle V in an irreducible component of $\mathcal{M}_L(c_1, c_2)$, we have $r_V = r_1$ and (using an argument as in the proof of Theorem 4.1) V sits in an exact sequence
(5)

$$0 \to \mathcal{O}_X(r_1F + \sum_{i=1}^m \ell_i E_i) \otimes \phi^* M \to V \to \mathcal{I}_Z((\eta - r_1)F + \sum_{i=1}^m (1-\ell_i)E_i) \otimes \phi^* N \to 0,$$

where $\ell_i \geq 0$ and Z is a zero-dimensional subscheme of length $\ell(Z) = 2n + \varepsilon + \sum_{i=1}^m \ell_i(1 - \ell_i)$.

We prove that

$$r_1 = r_0, \ \ell_i = 0 \text{ for all } i, \text{ and } h^0(V(-r_0F) \otimes \phi^* M^{-1}) = 1.$$

To this end, we compute the dimension of the family $\mathcal{F}$ of vector bundles given by extensions of type (5). Note that

$$\dim(\mathcal{F}) = \mathrm{ext}^1_X + 2\dim(\mathrm{Pic}^0(C)) + 2\ell(Z) - h^0(V(-r_1F - \sum_{i=1}^m \ell_i E_i) \otimes \phi^* M^{-1}),$$

where

$$\begin{aligned}\mathrm{ext}^1_X &= \dim \mathrm{Ext}^1_X(\mathcal{I}_Z((\eta - r_1)F + \sum_{i=1}^m (1-\ell_i)E_i) \otimes \phi^* N, \mathcal{O}_X(r_1F + \sum_{i=1}^m \ell_i E_i) \otimes \phi^* M) \\ &= h^1(\mathcal{I}_Z((\eta - 2r_1)F + \sum_{i=1}^m (1-2\ell_i)E_i) \otimes \phi^*(M^{-1} \otimes N) \otimes K_X)\end{aligned}$$

for a general choice of Z, M, and N. Note that

$$h^0(\mathcal{I}_Z((\eta - 2r_1)F + \sum_{i=1}^m (1-2\ell_i)E_i) \otimes \phi^*(M^{-1} \otimes N) \otimes K_X) = 0,$$

as the coefficient of C_0 in the expression of the corresponding line bundle equals -2, and

$$\begin{aligned}&h^2(\mathcal{I}_Z((\eta - 2r_1)F + \sum_{i=1}^m (1-2\ell_i)E_i) \otimes \phi^*(M^{-1} \otimes N) \otimes K_X) \\ &= h^2(\mathcal{O}_X((\eta - 2r_1)F + \sum_{i=1}^m (1-2\ell_i)E_i) \otimes \phi^*(M^{-1} \otimes N) \otimes K_X) \\ &= h^0(\mathcal{O}_X((2r_1 - \eta)F + \sum_{i=1}^m (2\ell_i - 1)E_i) \otimes \phi^*(M \otimes N^{-1})).\end{aligned}$$

We claim that the latter h^0 vanishes. Indeed, if it was different from zero, then we necessarily would have

$$2r_1 - \eta \geq \#\{E_i | \ \ell_i = 0\}$$

since F is numerically equivalent to each $E_i + \widetilde{F}_i$. Equivalently

$$r_1 \geq (\eta - r_1) + \#\{E_i | \ \ell_i = 0\},$$

which implies that

$$r_1F \cdot L \geq (\eta - r_1)F \cdot L + \sum (1-\ell_i)F \cdot L \geq (\eta - r_1)F \cdot L + \sum (1-\ell_i)E_i \cdot L,$$

taking into account that $F \cdot L \geq E_i \cdot L$. This shows that the sequence (5) destabilises V, which is a contradiction.

In particular, it follows that

$$\begin{aligned}\operatorname{ext}^1 &= -\chi(X, I_Z((\eta-2r_1)F+\sum_{i=1}^{m}(1-2\ell_i)E_i)\otimes\phi^*(M^{-1}\otimes N)\otimes K_X)\\ &= -\chi(X,\mathcal{O}_X((2r_1-\eta)F+\sum_{i=1}^{m}(2\ell_i-1)E_i)\otimes\phi^*(M\otimes N^{-1}))+\ell(Z).\end{aligned}$$

We compute from the Riemann-Roch Theorem

$$\begin{aligned}&\chi(X,\mathcal{O}_X((2r_1-\eta)F+\sum_{i=1}^{m}(2\ell_i-1)E_i)\otimes\phi^*(M\otimes N^{-1}))\\ &\quad = 1-g+2r_1-\eta-\sum_{i=1}^{m}(1-2\ell_i)(1-\ell_i).\end{aligned}$$

Hence,

$$\begin{aligned}\dim\mathcal{F} &= -2r_1+\eta+3g-1+\sum_{i=1}^{m}(1-\ell_i^2)+3(2n+\varepsilon)-h^0(V(-r_1F-\sum_{i=1}^{m}\ell_iE_i)\otimes\phi^*M^{-1})\\ &= -2r_1+(\eta+3g-1)+(m-\sum_{i=1}^{m}\ell_i^2)+3(2n+\varepsilon)-h^0(V(-r_1F-\sum_{i=1}^{m}\ell_iE_i)\otimes\phi^*M^{-1}).\end{aligned}$$

Since

$$-2r_1\le -1-\eta+c_2+g,\ m-\sum_{i=1}^{m}\ell_i^2\le m,\ \text{and}\ h^0(V(-r_1F-\sum_{i=1}^{m}\ell_iE_i)\otimes\phi^*M^{-1})\ge 1,$$

it follows that

$$\dim\mathcal{F}\le(-1-\eta+c_2+g)+(\eta+3g-1)+m+3(2n+\varepsilon)-1=4(2n+\varepsilon)+4g-3+m.$$

Moreover, if either $-2r_1<-1-\eta+c_2+g$ or there exists i such that $\ell_i\neq 0$ or $h^0(V(-r_1F-\sum_{i=1}^{m}\ell_iE_i)\otimes\phi^*M^{-1})\ge 2$, then

$$\dim\mathcal{F}<(-1-\eta+c_2+g)+(\eta+3g-1)+m+3(2n+\varepsilon)-1=4(2n+\varepsilon)+4g-3+m.$$

On the other hand, the expected dimension of $\mathcal{M}_L(c_1,c_2)$ is precisely $4(2n+\varepsilon)+4g-3+m$, and this shows that if either $-2r_1<-1-\eta+c_2+g$ or there exists i such that $\ell_i\neq 0$ or $h^0(V(-r_1F-\sum_{i=1}^{m}\ell_iE_i)\otimes\phi^*M^{-1})\ge 2$, then the family $\mathcal{F}$ cannot dominate a component of the moduli space. In particular, $\ell_i=0$ for all i, $h^0(V(-r_1F-\sum_{i=1}^{m}\ell_iE_i)\otimes\phi^*M^{-1})=1$, and $2r_1\le 1+\eta-c_2-g$. Since $2r_1\ge\eta-c_2-g$ from Proposition 3.6, it follows that $r_1=r_0$. □

Corollary 5.3. *Let $c_1=\eta F+\sum_{i=1}^{m}E_i$ with $\eta\in\{0,1\}$ and $c_2=2n+\varepsilon\gg 0$ with $\varepsilon\in\{0,1\}$ and let L be a (c_1,c_2)–suitable ample divisor. Then, the moduli space $\mathcal{M}_L(c_1,c_2)$ is smooth and irreducible, and there exists a generically finite rational map from a projective bundle over* $\operatorname{Pic}^0(C)\times\operatorname{Pic}^0(C)\times C^{[2n+\varepsilon]}$ *to $\mathcal{M}_L(c_1,c_2)$.*

Proof. We apply the previous theorem. The generic finiteness follows from the dimension computation of the family of extensions. □

Example 5.4. In this example we show that the dominating families of moduli spaces are not necessarily unique.

Let S be a geometrically ruled surface of invariant $e>0$ over $\mathbb{P}^1$, let $n\gg 0$ be an integer, and let $c_1=F$, $c_2=2n$. Fix an ample (c_1,c_2)–suitable divisor

$L = C_0 + mF$ on S, with $m \gg 0$. By Theorem 5.2 a general vector bundle V in $\mathcal{M}_L(F, 2n)$ lies in an extension of type

$$0 \to \mathcal{O}_S(-(n-1)F) \to V \to \mathcal{I}_Z(nF) \to 0, \tag{6}$$

where Z is a 0-dimensional subscheme of length $2n$. Let us construct another extension family dominating $\mathcal{M}_L(F, 2n)$. To this end, we consider the irreducible family $\mathcal{F}_n$ of rank 2 vector bundles V on S given by a non-trivial extension

$$0 \to \mathcal{O}_S(-D) \to V \to \mathcal{I}_Z(D+F) \to 0, \tag{7}$$

where $D = nF$ and Z is a sufficiently general 0-dimensional subscheme of length $2n$.

First of all notice that $h^0(V(D)) = 3$. In fact, it follows from the exact cohomology sequence associated to the exact sequence (7) and the fact that from the generality of Z we have $h^0(\mathcal{I}_Z(2D+F)) = 2$. Now, we are going to compute the dimension of $\mathcal{F}_n$. By definition we have

$$\begin{aligned} \dim \mathcal{F}_n \quad &= \#\text{moduli}(Z) + \text{ext}^1(\mathcal{I}_Z(D+F), \mathcal{O}_S(-D)) - h^0(V(D)) \\ &= 2\ell(Z) + \text{ext}^1(\mathcal{I}_Z, \mathcal{O}_S(-2D-F)) - h^0(V(D)). \end{aligned}$$

By Serre's duality and applying the Riemann-Roch Theorem we get

$$\text{ext}^1_X(\mathcal{I}_Z, \mathcal{O}_S(-2D-F)) = -\chi(X, \mathcal{O}_S(-2D-F)) + \ell(Z) = 4n.$$

Therefore,

$$\dim \mathcal{F}_n = 2(2n) + 4n - 3 = 4(2n) - 3.$$

It is easy to check that for any $V \in \mathcal{F}_n$, $c_1(V) = F$ and $c_2(V) = 2n$. Let us see that V is L-stable; i.e., for any rank-one subbundle $\mathcal{O}_S(A)$ of V we have $c_1(\mathcal{O}_S(A)) \cdot L < \frac{1}{2}$ or, equivalently,

$$c_1(\mathcal{O}_S(A)) \cdot L < \frac{c_1(V) \cdot L}{2}.$$

Indeed, since V sits in an extension of type (7) we have

$$\begin{array}{ll} (1) & \mathcal{O}_S(A) \hookrightarrow \mathcal{O}_S(-nF) \quad \text{or} \\ (2) & \mathcal{O}_S(A) \hookrightarrow \mathcal{I}_Z((n+1)F). \end{array}$$

In the first case, $-A - nF$ is an effective divisor. Since L is an ample divisor we have $(-A - nF) \cdot L \geq 0$ and

$$c_1(\mathcal{O}_S(A)) \cdot L = A \cdot L \leq -nF \cdot L = -n < \frac{1}{2} = \frac{c_1(V) \cdot L}{2}.$$

If $\mathcal{O}_S(A) \hookrightarrow \mathcal{O}_S((n+1)F) \otimes \mathcal{I}_Z$, then $(n+1)F - A$ is an effective divisor. On the other hand, using the generality of Z we have

$$H^0(\mathcal{O}_S(A + (n-2)F)) \subset H^0(\mathcal{I}_Z((2n-1)F)) = 0.$$

So $A + (n-2)F$ is not an effective divisor, and writing $A = \alpha C_0 + \beta F$, we have either $\beta + n - 2 < 0$ or $\alpha < 0$.

Assume that $\beta + n - 2 < 0$, in particular $\beta < 0$. Since $(n+1)F - A$ is an effective divisor, it follows that $\alpha \leq 0$ and, using $m \gg e$, we have

$$\begin{aligned} c_1(\mathcal{O}_S(A)) \cdot L = A \cdot L \quad &= -\alpha e + \alpha m + \beta \\ &= \alpha(m-e) + \beta \\ &< \tfrac{1}{2} = \tfrac{c_1(V) \cdot L}{2}. \end{aligned}$$

Assume that $\alpha < 0$ and $\beta + n - 2 \geq 0$. Using again the fact that $(n+1)F - A$ is an effective divisor and hence $\beta \leq n+1$, we obtain

$$\begin{aligned} c_1(\mathcal{O}_S(A)) \cdot L = A \cdot L &= -\alpha e + \alpha m + \beta \\ &\leq -\alpha e + \alpha m + n + 1 \\ &= \alpha(m-e) + n + 1 \\ &< \tfrac{1}{2} = \tfrac{c_1(V)L}{2}, \end{aligned}$$

as $m \gg n + \frac{3}{2} + e$, which proves the L-stability of V. In conclusion, we have a dominant morphism

$$\phi : \mathcal{F}_n \longrightarrow \mathcal{M}_L(F, 2n).$$

5.2. **The case $c_1 \cdot F = 1$.** Let $c_1 = C_0 + \beta F + \sum_{i=1}^{m} E_i$ and let L be a fixed ample divisor. The most natural extension space that dominates $\mathcal{M}_L(c_1, c_2)$, for large c_2, is given by the following result:

Theorem 5.5. *Let L be an ample divisor with $L \cdot (K_X + F) < 0$. Then, for $c_2 \gg 0$, a general vector bundle V in $\mathcal{M}_L(c_1, c_2)$ lies in an extension of type*

$$(8) \qquad 0 \to \mathcal{O}_X(C_0 - (c_2 - \beta)F) \otimes \phi^* M \to V \to \mathcal{O}_X(c_2 F + \sum_{i=1}^{m} E_i) \otimes \phi^* N \to 0$$

with $M, N \in \operatorname{Pic}^0(C)$. In particular, $d_V = 1$ and $r_V = \beta - c_2$.

Proof. The proof will follow after different steps.

Step 1. We show that the dimension of the irreducible family $\mathcal{F}$ of non-trivial extensions of type (8) equals the dimension of $\mathcal{M}_L(c_1, c_2)$.

Observe that $h^0(\mathcal{O}_X(-C_0 + (2c_2 - \beta)F + \sum_{i=1}^{m} E_i) \otimes \phi^*(N \otimes M^{-1})) = 0$, which implies that $h^0(V(-C_0 + (c_2 - \beta F)) \otimes \phi^* M^{-1}) = 1$ for any extension. Moreover

$$\operatorname{ext}^1_X(\mathcal{O}_X(c_2 F + \sum_{i=1}^{m} E_i) \otimes \phi^* N, \mathcal{O}_X(C_0 - (c_2 - \beta)F) \otimes \phi^* M)$$

$$= h^1(X, \mathcal{O}_X(C_0 - (2c_2 - \beta)F - \sum_{i=1}^{m} E_i) \otimes \phi^*(M \otimes N^{-1})).$$

Note that for $c_2 \gg 0$ we have

$$h^0(X, \mathcal{O}_X(C_0 - (2c_2 - \beta)F - \sum_{i=1}^{m} E_i) \otimes \phi^*(M \otimes N^{-1})) = 0,$$

and, by duality,

$$h^2(X, \mathcal{O}_X(C_0 - (2c_2 - \beta)F - \sum_{i=1}^{m} E_i) \otimes \phi^*(M \otimes N^{-1})) = 0,$$

and hence

$$\operatorname{ext}^1_X(\mathcal{O}_X(c_2 F + \sum_{i=1}^{m} E_i) \otimes \phi^* N, \mathcal{O}_X(C_0 - (c_2 - \beta)F) \otimes \phi^* M)$$

$$= -\chi(X, \mathcal{O}_X(C_0 - (2c_2 - \beta)F - \sum_{i=1}^{m} E_i) \otimes \phi^*(M \otimes N^{-1})).$$

From the Riemann-Roch Theorem we have

$$-\chi(X, \mathcal{O}_X(C_0 - (2c_2 - \beta)F - \sum_{i=1}^{m} E_i) \otimes \phi^*(M \otimes N^{-1}))$$
$$= 4c_2 - 2\beta + \rho + 2g + e - 2.$$

It follows that

$$\begin{aligned}\dim \mathcal{F} &= \operatorname{ext}^1_X(\mathcal{O}_X(c_2F + \sum_{i=1}^{m} E_i) \otimes \phi^*N, \mathcal{O}_X(C_0 - (c_2 - \beta)F) \otimes \phi^*M) \\ &\quad + 2\dim(\operatorname{Pic}^0(C)) - 1 \\ &= 4c_2 - 2\beta + \rho + 4g - 3 + e.\end{aligned}$$

Note that the dimension of $\mathcal{M}_L(c_1, c_2)$ equals the expected dimension

$$4c_2 - c_1^2 - 3\chi(X, \mathcal{O}_X) + g = 4c_2 + e - 2\beta + \rho - 3 + 4g;$$

i.e. it equals the dimension of $\mathcal{F}$.

Step 2. We prove that any bundle V in an extension (8) is simple. To this end, we consider the exact sequence

$$\begin{aligned}0 &\to \operatorname{Hom}(\mathcal{O}_X(c_2F + \sum_{i=1}^{m} E_i) \otimes \phi^*N, V) \\ &\to \operatorname{Hom}(V, V) \to \operatorname{Hom}(\mathcal{O}_X(C_0 - (c_2 - \beta)F) \otimes \phi^*M, V).\end{aligned}$$

The simplicity of V follows from the following two facts: $h^0(V(-C_0+(c_2-\beta F))\otimes \phi^*M^{-1}) = 1$, which we have already used, and $h^0(V(-c_2F - \sum_{i=1}^{m} E_i) \otimes \phi^*N^{-1}) = 0$. The latter vanishing is a direct consequence of the non-triviality of the extension (8) and of the vanishing of $h^0(X, \mathcal{O}_X(C_0 - (2c_2 - \beta)F - \sum_{i=1}^{m} E_i) \otimes \phi^*(M \otimes N^{-1}))$.

Step 3. We prove that any bundle V in an extension (8) is prioritary. To this end, consider the exact sequence

$$\operatorname{Ext}^2_X(\mathcal{O}_X(c_2F + \sum_{i=1}^{m} E_i) \otimes \phi^*N, V(-F)) \to \operatorname{Ext}^2_X(V, V(-F))$$
$$\to \operatorname{Ext}^2_X(\mathcal{O}_X(C_0 - (c_2 - \beta)F) \otimes \phi^*M, V(-F)),$$

and we easily check that

$$h^2(V(-(c_2+1)F - \sum_{i=1}^{m} E_i) \otimes \phi^*N^{-1}) = h^2(V(-C_0 + (c_2 - \beta - 1)F) \otimes \phi^*M^{-1}) = 0.$$

From the previous steps we obtain a natural map from the extension family $\mathcal{F}$ to the moduli space $\mathcal{S}pl(c_1, c_2)$ of simple prioritary bundles with Chern classes c_1 and c_2. This map is injective. Indeed, if a bundle V is presented as two extensions

$$0 \to \mathcal{O}_X(C_0 - (c_2 - \beta)F) \otimes \phi^*M \xrightarrow{f} V \xrightarrow{g} \mathcal{O}_X(c_2F + \sum_{i=1}^{m} E_i) \otimes \phi^*N \to 0$$

and

$$0 \to \mathcal{O}_X(C_0 - (c_2 - \beta)F) \otimes \phi^*M' \xrightarrow{f'} V \xrightarrow{g'} \mathcal{O}_X(c_2F + \sum_{i=1}^{m} E_i) \otimes \phi^*N' \to 0,$$

since $g \circ f'$ and $g' \circ f$ are necessarily zero, we obtain an isomorphism between $\mathcal{O}_X(C_0 - (c_2 - \beta)F) \otimes \phi^* M$ and $\mathcal{O}_X(C_0 - (c_2 - \beta)F) \otimes \phi^* M'$. The injectivity follows then from the fact that $h^0(V(-C_0 + (c_2 - \beta F)) \otimes \phi^* M^{-1}) = 1$. On the other hand, since $L \cdot (K_X + F) < 0$ the moduli space $\mathcal{S}pl(c_1, c_2)$ is irreducible (Theorem 2.5) and contains $\mathcal{M}_L(c_1, c_2)$ as an open subscheme. In particular, since $\dim \mathcal{F} = \dim(\mathcal{S}pl(c_1, c_2))$, it follows that $\mathcal{M}_L(c_1, c_2)$ and the image of $\mathcal{F}$ in $\mathcal{S}pl(c_1, c_2)$ contain a common non-empty open subset. $\square$

As a consequence of this analysis we obtain the following refinement of Theorem 4.1:

Corollary 5.6. *For any ample divisor L on X with $L \cdot (K_X + F) < 0$, $c_1 = C_0 + \beta F + \sum_{i=1}^{m} E_i$, and $c_2 \gg 0$, the moduli space $\mathcal{M}_L(c_1, c_2)$ is irreducible and birational to a projective bundle over* $\mathrm{Pic}^0(C) \times \mathrm{Pic}^0(C)$.

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