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Non-constant discounting in finite horizon: The free terminal time case

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Abstract: This paper derives the HJB (Hamilton-Jacobi-Bellman) equation for sophisticated agents in a finite horizon dynamic optimization problem with non-constant discounting in a continuous setting, by using a dynamic programming approach. A simple example is used in order to illustrate the applicability of this HJB equation, by suggesting a method for constructing the subgame perfect equilibrium solution to the problem. Conditions for the observational equivalence with an associated problem with constant discounting are analyzed. Special attention is paid to the case of free terminal time. Strotz's model (an eating cake problem of a nonrenewable resource with non-constant discounting) is revisited.
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Resumen: En este trabajo se deriva la ecuación de Hamilton-Jacobi-Bellman (HJB) para un agente sofisticado en un problema de optimización dinámica en tiempo continuo con horizonte finito, cuando la tasa de descuento de preferencia temporal es no constante, mediante la resolución de un problema de programación dinámica. Un sencillo ejemplo sirve para ilustrar la aplicabilidad de esta ecuación HJB. En particular, en el mismo se sugiere un método para construir el equilibrio perfecto en subjuegos solución del problema. El caso de tiempo final libre recibe una especial atención. Finalmente, se revisa el modelo de Strotz (un problema tipo “eating cake” de un recurso no renovable con descuento no constante).
Palabras clave: Descuento no constante, agentes naive y sofisticados, tiempo final libre, equivalencia observacional

1 Introduction

Variable rate of time preferences have received considerable attention in recent years. Effects of the so called hyperbolic discount functions (introduced by Phelps and Pollak (1968)) have been extensively studied in a discrete time context, within the field of behavioural economics. Laibson (1997) has made compelling observations about ways in which rates of time preference vary. However, this topic has received less attention in a continuous time setting. The main reason for this may be the complexity involved in the search for solutions in closed form in the non-constant discounting case. In fact, standard optimal control techniques cannot be used in this context, since they give rise to non-consistent policies.

The most relevant effect of non-constant discounting is that preferences change with time. In this sense, an agent making a decision in time t has different preferences compared with those in time t'. Therefore, we can consider him at different times as different agents. An agent making a decision in time t is usually called the t-agent. If the horizon planning is a finite interval $[0, T]$, we can understand the optimal control problem with non-constant discounting as a perfect information sequential game with a continuous number of players (the t-agents, for $t \in [0, T]$) making their decisions sequentially. A t-agent can act in two different ways: naive and sophisticated. Naive agents take decisions without taking into account that their preferences will change in the near future. Then, they will be continuously modifying their calculated choices for the future, and their decisions will be non-time consistent in general. In order to obtain a time consistent strategy, the t-agent should be sophisticated, in the sense of taking into account the preferences of all the t'-agents, for $t' \in (t, T]$. Therefore, the solution to the problem of the agent with non-constant discounting should be constructed by looking for the subgame perfect equilibria of the associated game with the infinite number of t-agents.

Historically, Ramsey (1928) believed that no discounting at all should be used, stating: "... we do not discount later enjoyments in comparison with earlier ones, a practice which is ethically indefensible and arises merely from the weakness of imagination". However, in the part of his analysis that allowed for time preference, Ramsey (1928) assumes an exponential discount factor, with constant discount rate, stating: "This is the only assumption we can make, without contradicting our fundamental hypothesis that successive generations are activated by the same system of preferences". The main property of non-constant discounting is implicit in this statement: it can create a time-consistency problem. In fact, Strotz (1956) illustrated how for a very simple model preferences are time consistent if, and only if, time preferences are exponentials with a constant discount rate. In order to avoid such time inconsistency, the agent could decide in a sophisticated way, making an analysis of what his actions would be in the future, as a consequence of his changing preferences. For instance, Pollak (1968) gave the right solution to the Strotz problem for both, naive and sophisticated agents under a logarithmic utility function.

Although the problem was first presented in a continuous time context (Strotz (1956)), almost all attention has been given to the discrete time setting introduced by Phelps and Pollak (1968). This is probably a consequence of the non-existence of a well-stated system of equations giving a general method for solving the problem, at least for sophisticated agents. There is no analogy to Pontryagin's maximum principle in this case if the discount factor is non constant. Therefore, each particular problem has been solved individually. This was the case of the Strotz model solved by Pollak in 1968, both for naive and sophisticated agents. Barro (1999) studied a modified version of the neoclassical growth model by including a variable rate of time preference. In his method for solving the problem, he employed a type of the well-known needle variations used in the proof of Pontryagin's maximum principle, but just in the initial period of time. In his analysis for a logarithmic utility function, he proved that the equilibrium features a constant effective rate of time preference and is observationally equivalent to the standard model. Unfortunately, the nice properties obtained for the logarithmic utility function - equivalence of solutions of naive and sophisticated agents (Pollak, 1968) and observational equivalence with the standard model (Barro, 1999) - seem to be a particularity of this utility function, and not the general rule.

For the case of naive agents, Pontryagin's maximum principle can be used in order to find the decision rule at time t of a t-agent. In fact, one should solve a standard optimal control problem for each time $t \in [0, T]$. Unfortunately, this method cannot be used if the agent is sophisticated. Instead, Markov subgame perfect equilibria must be found. This prompts the use of a dynamic programming approach, applying a kind of Bellman optimality principle.

In fact, Karp (2007) recently introduced a new method for analyzing the solutions in a non-constant-discount continuous time setting, deriving a dynamic programming equation (DPE) in an infinite time horizon problem. A different approach was adopted in Harris and Laibson (2005). In this paper we extend the results by Karp to a non-constant discounting problem in finite horizon. Special attention is paid to the free terminal time case. While the applicability of the Hamilton-Jacobi-Bellman equation for the case with non constant discounting seems to be, in principle, problematic, since it includes the solution as a data, it can be nevertheless be easily used in order to check the observational equivalence with a problem with constant discounting. Moreover, we suggest a guessing method for constructing the solution.

The paper is organized as follows. In Section 2 we describe the model. The dynamic programming equation is derived in Section 3, both in discrete and continuous settings. The observability equivalence problem is briefly analyzed, and a simple example illustrates how Pontryagin's maximum principle and the derived Hamilton-Jacobi-Bellman equation can be used in order to describe the solution for naive and sophisticated agents, respectively. Section 4 is devoted studying the problem with free terminal time, in several different contexts with some kind of commitment, or without commitment at all. In Section 5 we solve Strotz's model, an eating cake problem of a nonrenewable resource with non-constant discounting. It is shown how the coincidence of solutions for naive and sophisticated agents in a log-utility setting is lost if the time horizon is freely fixed by the agent. Finally, Section 6 contains the main conclusions of the paper.

2 The model: non constant discounting in a finite horizon planning

Let $x = (x^1, \ldots, x^n)$ be the vector of state variables, and $u = (u^1, \ldots, u^m)$ the vector of control (or decision) variables. In the conventional model, agent preferences in time t take the form

$$U_t = \int_t^T e^{-\rho(s-t)} L(x(s), u(s), s)\, ds + e^{-\rho(T-t)} F(x(T)) \ ,$$

where the state variables evolve according to the state (or control) equations $\dot{x}^i(s) = f^i(x(s), u(s), s)$, $x^i(t) = x^i_t$, for $i = 1, \ldots, n$. In order to maximize U_t, we must solve an optimal control problem and, since the discount rate is constant, the solution becomes time consistent.

Now, following Karp (2007), let us assume that the instantaneous discount rate is non-constant, but a function $r(s)$ at time s, $s \in [t, T]$. Function $r(s)$ is usually assumed to be non-increasing. Although this is the most reasonable choice in most decision problems, this assumption is not necessary for our model, and we will not make it. The discount factor at time t used to evaluate a payoff at time $t+\tau$, $\tau \geq 0$, is $\theta(\tau) = \exp\left(-\int_0^\tau r(s)\, ds\right)$.

Then, the objective of the agent at time t (the t-agent) will be

$$\max_{\{u(s)\}} \int_t^T \theta(s-t) L(x(s), u(s), s)\, ds + \theta(T-t) F(x(T)) \ , \tag{1}$$

$$\dot{x}^i(s) = f^i(x(s), u(s), s) \ , \ x^i(t) = x_t^i \ , \ \text{for } i = 1, \ldots, n \ . \tag{2}$$

In Problem (1-2), we assume the usual regularity conditions, i.e., functions L, F and f^i are continuously differentiable in all their arguments.

Remark 1 *In Barro (1999), the discount factor was defined as* $e^{-[\rho\tau+\phi(\tau)]}$, *where* $\phi(\tau)$ *includes the aspects of time preference that cannot be described by the standard exponential factor* $e^{-\rho\tau}$. *Note that our expression of* $\theta(\tau)$ *is equivalent to the one in Barro (1999). Indeed, in the Barro's model, the instantaneous rate of time preference at the time distance* τ *is given by* $\rho + \phi'(\tau)$, *compared with* $r(\tau)$ *in the definition of Karp (2007). By writing* $r(\tau) = \rho + \phi'(\tau)$, *we can identify both discount factors.*

In this paper we will not assume any particular discount function. In the discrete time case, most papers work with the so-called hyperbolic discounting, first proposed by Phelps and Pollak (1968). The utility function is defined as

$$U_t = u_t + \beta(\delta u_{t+1} + \delta^2 u_{t+2} + \delta^3 u_{t+3} + \cdots) \ ,$$

where $0 < \beta \leq 1$, and u_k denotes the utility in period k. In fact, Laibson (1997) argues that β would be substantially less than one on an annual basis, perhaps between one-half and two-thirds.

As a natural extension of the discount function above to the continuous setting, Barro (1999) suggested the instantaneous discount rate $r(\tau) = \rho + be^{-\gamma\tau}$, where $b \geq 0$ and $\gamma > 0$.

Another natural choice for the discount factor comes from another application of Problem (1-2). Consider two agents with the same instantaneous utility and final functions, but different constant discount rates. Fro instance, this is the case of the model of renewable resources and economic sustainability studied (in an infinite horizon setting) by Li and Lofgren (2000), where agents are classified into conservationists (for which the instantaneous discount rate is zero) and non conservationists (which have a strictly positive instantaneous discount rate). If the agents have utility functions

$$U_t^1 = \int_t^T e^{-\rho_1(s-t)} L(x(s), u(s), s)\, ds + e^{-\rho_1(T-t)} F(x(T)) \ , \quad \text{and}$$

$$U_t^2 = \int_t^T e^{-\rho_2(s-t)} L(x(s), u(s), s)\, ds + e^{-\rho_2(T-t)} F(x(T)) \ ,$$

then the utility function of a Pareto optimum will be given by

$$U_t^P = \lambda U_t^1 + (1-\lambda) U_t^2 =$$

$$= \int_t^T \left(\lambda e^{-\rho_1(s-t)} + (1-\lambda) e^{-\rho_2(s-t)} \right) L(x(s), u(s), s)\, ds +$$
$$+ \left(\lambda e^{-\rho_1(T-t)} + (1-\lambda) e^{-\rho_2(T-t)} \right) F(x(T) \ ,$$

where $0 < \lambda < 1$. In this case, the discount factor is a linear combination of exponentials with constant but different discount rates,

$$\theta(s-t) = \left(\lambda e^{-\rho_1(s-t)} + (1-\lambda) e^{-\rho_2(s-t)} \right) = e^{f(s-t)} \ ,$$

where $f(\tau) = \ln \left(\lambda e^{-\rho_1 \tau} + (1-\lambda) e^{-\rho_2 \tau} \right)$.

3 Dynamic programming equation

We will first solve a discretized version of this model for a sophisticated agent, and then extend the solution to a continuous time setting.

3.1 Dynamic programming equation in discrete time

We will derive the dynamic programming equation in a discretized version of Problem (1-2), following a procedure similar to the one used in Karp (2007).

Let us divide the interval $[0, T]$ into n periods of constant length ϵ, in such a way that we identify $ds = \epsilon$, and $s = j\epsilon$, for $j = 0, 1, \ldots, n$. Then equation (2) becomes $x(s+\epsilon) - x(s) = f(x(s), u(s), s)\epsilon$. Denoting by $x(j\epsilon) = x_j$, $u(k\epsilon) = u_k$ $(j, k = 0, \ldots, n-1)$, the objective of the agent in period $t = j\epsilon$ will be

$$\max_{\{u_i\}} V_j = \sum_{i=0}^{n-j-1} \theta(i\epsilon) L(x_{(i+j)}, u_{(i+j)}, (i+j)\epsilon)\epsilon + \theta((n-j)\epsilon) F(x(T)) \ , \qquad (3)$$

$$x_{i+1} = x_i + f(x_i, u_i, i\epsilon)\epsilon \ , \ i = j, \ldots, n-1 \ , \ x_j \text{ given} \ . \qquad (4)$$

Let us state the dynamic programming algorithm for the discrete problem (3-4). In the final period, $t = n\epsilon = T$, we define $V_n^* = F(x(T))$, as usual. For $j = n-1$, the optimal value for (3) will be given by the solution to the problem

$$\begin{aligned} V^*_{(n-1)}(x_{(n-1)},(n-1)\epsilon) &= \max_{\{u_{(n-1)}\}} \left\{ L(x_{(n-1)},u_{(n-1)},(n-1)\epsilon)\epsilon + \theta_1 V^*_n \right\} \\ x_n &= x_{(n-1)} + f(x_{(n-1)},u_{(n-1)},(n-1)\epsilon)\epsilon \ . \end{aligned}$$

If

$$u^*_{(n-1)}(x_{(n-1)},(n-1)\epsilon) = \arg\max_{\{u_{(n-1)}\}} \left\{ L(x_{(n-1)},u_{(n-1)},(n-1)\epsilon)\epsilon + \right.$$
$$\left. +\theta_1 V^*_n(x_{(n-1)} + f(x_{(n-1)},u_{(n-1)},(n-1)\epsilon)\epsilon) \right\} \ ,$$

let us denote

$$H_{(n-1)}(x_{(n-1)},(n-1)\epsilon) = L(x_{(n-1)},u^*_{(n-1)}(x_{(n-1)},(n-1)\epsilon),(n-1)\epsilon) \ . \tag{5}$$

In general, for $j = 1,\ldots,n-1$, the optimal value in (3) can be written as

$$V^*_j(x_j,j\epsilon) = \max_{\{u_j\}} L(x_j,u_j,j\epsilon)\epsilon +$$
$$+ \sum_{k=1}^{n-j-1} \theta_k H_{(j+k)}(x_{(j+k)},(j+k)\epsilon)\epsilon + \theta_{n-j} V^*_n \tag{6}$$

with $x_{(j+1)} = x_j + f(x_j,u_j,j\epsilon)\epsilon$. Since

$$V^*_{j+1}(x_{(j+1)},(j+1)\epsilon) = \sum_{i=0}^{n-j-2} \theta_i H_{(j+i+1)}(x_{(j+i+1)},(j+i+1)\epsilon)\epsilon + \theta_{n-j-1} V^*_n \ , \tag{7}$$

then, solving $\theta_{n-j-1}V^*_n(x_n)$ in (7) and substituting in (6) we obtain:

Proposition 1 *For every initial state x_0, the equilibrium value $V^*(x_t)$ of problem (3-4) can be obtained as the solution of the following algorithm, which proceeds backward in time from period $n-1$ to period 0:*

$$V^*_n = F(x(T)) \ , \tag{8}$$

$$\theta_{n-j-1}V^*_j(x_j,j\epsilon) = \max_{\{u_j\}} \{\theta_{n-j-1}L(x_j,u_j,j\epsilon)\epsilon +$$
$$+ \sum_{k=1}^{n-j-1} \left(\theta_{n-j-1}\theta_k - \theta_{n-j}\theta_{k-1}\right) H_{(j+k)}(x_{(j+k)},(j+k)\epsilon)\epsilon + \tag{9}$$
$$+\theta_{n-j}V^*_{j+1}(x_{(j+1)},(j+1)\epsilon) \Big\} \ ,$$

$$x_{(j+1)} = x_j + f(x_j,u_j,j\epsilon)\epsilon \ , \tag{10}$$

for $j = 0,\ldots,n-1$.

Equations (8-10) are the equilibrium dynamic programming equations in discrete time, and their solution is the Markov Perfect equilibrium (MPE) solution to problem (3-4).

3.2 The Hamilton-Jacobi-Bellman equation

In this paper we are interested in deriving a dynamic programming equation in continuous time. In this section we will obtain a modified Hamilton-Jacobi-Bellman equation for the problem with non-constant discounting, which reduces to the usual HJB equation in case where the discount rate is constant. The idea of our derivation consists in taking the limit $\epsilon \to 0$ of the dynamic programming equation of the discrete stage equilibrium problem described in Proposition 1.

Let $W(x,t)$ represent the value function of the t-agent, with initial condition $x(t)=x$. We assume that $W(x,t)$ is continuously differentiable in all its arguments. Since $s=j\epsilon$ and $x(s+\epsilon)=x(s)+f(x(s),u(s),s)\epsilon$, then $W(x(t),t)=V_j(x_j,j\epsilon)$ and

$$W(x(t+\epsilon),t+\epsilon)=W(x(t),t)+\nabla_x W(x(t),t)\cdot f(x(t),u(t),t)\epsilon+\nabla_t W(x(t),t)\epsilon+o(\epsilon)\ ,$$

where $\lim_{\epsilon\to 0}\dfrac{o(\epsilon)}{\epsilon}=0$. Since $\theta_k=\exp\left(-\int_0^{k\epsilon} r(s)ds\right)$, then

$$\theta_{n-j}=\theta_{n-j-1}\left[1-r((n-j)\epsilon)\epsilon\right]+o(\epsilon)=\theta_{n-j-1}\left[1-r(T-t)\epsilon\right]+o(\epsilon)\ ,$$

$$\theta_{k-1}=\theta_k\left[1+r(k\epsilon)\epsilon\right]+o(\epsilon)$$

and substituting in (9) we obtain

$$W(x(t),t)=\max_{\{u(t)\}}\{L(x(t),u(t),t)\epsilon+W(x(t),t)+\nabla_x W(x(t),t)\cdot f(x(t),u(t),t)\epsilon+$$
$$+\nabla_t W(x(t),t)\epsilon-r(T-t)W(x(t),t)\epsilon-K(x(t),t)\epsilon\}+o(\epsilon)\ , \qquad (11)$$

where $K(x(t),t)$ is given by

$$K(x(t),t)=\sum_{k=1}^{n-j-1}\theta(k\epsilon)\left[r(k\epsilon)-r(T-t)\right]H_{(t+k\epsilon)}(x(t+k\epsilon),t+k\epsilon)\epsilon+o(\epsilon^2)\ . \qquad (12)$$

Dividing equations (11) and (12) by ϵ, and taking the limit $\epsilon\to 0$, $n\to\infty$ (with $T=n\epsilon$), we get

$$r(T-t)W(x(t),t)+K(x(t),t)-\nabla_t W(x(t),t)=$$
$$=\max_{\{u\}}\{L(x(t),u,t)+\nabla_x W(x(t),t)\cdot f(x(t),u,t)\}\ ,$$

where

$$K(x(t),t)=\int_0^{T-t}\theta(s)\left[r(s)-r(T-t)\right]H(x(t+s),t+s)ds=$$
$$=\int_t^T\theta(s-t)\left[r(s)-r(T-t)\right]H(x(s),s)ds\ ,$$

and $H(x(s), s) = L(x(s), u^*(x(s), s), s)$.

Finally, note that for the equilibrium rule $u^* = u^*(x(s), s)$, $s \in [t, T]$, since $\dot{x}(t+s) = f(x(t+s), u(t+s), t+s)$, we can write $x(t+s)$ as a function of $x(t)$ and s ($x(t+s) = x(x(t), s)$). Therefore, $H(x(t+s), t+s) = H(x(t), s)$.

Hence, we have proved:

Theorem 1 *Let $W(x,t)$ be the value function of the t-agent. If $W(x,t)$ is continuously differentiable in (x,t), it satisfies the dynamic programming equation*

$$r(T-t)W(x,t) + K(x,t) - \nabla_t W(x,t) = \tag{13}$$

$$= \max_{\{u\}} \left\{ L(x,u,t) + \nabla_x W(x,t) \cdot f(x,u,t) \right\} \ ,$$

$$W(x,T) = F(x) \ , \tag{14}$$

where

$$K(x,t) = \int_t^T \theta(s-t) \left[r(s-t) - r(T-t) \right] H(x,s) ds \ . \tag{15}$$

Assume also that $u^*(x,t)$ attains the maximum in Equation (13). If, for each pair (x,t), the state trajectory $x^*(s)$ solution to $\dot{x}(s) = f(x(s), u^*(x(s), s), s)$ with initial condition $x(t) = x$ is unique, and the control trajectory $u^*(x(t), t)$ is piecewise continuous in t, then $W(x,t)$ is called the Markov Perfect equilibrium (MPE), and the control trajectory $u^*(t)$, $t \in [0,T]$ is the equilibrium rule.

Remark 2 *Note that in the case of constant discount rate $r(t) = r$, for every $t \in [0,T]$, we recover the usual Hamilton-Jacobi-Bellman equation.*

Remark 3 *It is clear from condition (14) and the differentiability assumption of the value function $W(x,t)$ that, if the terminal state is free, we recover the well-known transversality condition*

$$\nabla_x W(x^*, T) = \nabla_x F(x^*(T))$$

In particular, if there is no final function ($F(x) = 0$), then $\nabla_x W(x,T) = 0$.

Remark 4 *Assume now that, in addition to the initial state $x(0) = x_0$, the final state $x(T) = x_T$ is given. In this case, the terminal condition $W(T,x) = F(x)$ makes no sense ($x(T)$ is fixed). Instead, we have the extra condition $x(T) = x_T$ in order to integrate the differential equations.*

The standard HJB equation for Problem (1-2) with $t = 0$ is

$$r(t)W(x,t) - \nabla_t W(x,t) = \max_{\{u\}} \{L(x,u,t) + \nabla_x W(x,t) \cdot f(x,u,t)\} \ . \tag{16}$$

In our framework of changing preferences, the solution to this equation corresponds to the so-called commitment solution, in the sense that it is optimal as long as the agent can precommit his future behaviour at time $t = 0$. Note that, when comparing equations (16) and (13), there are two differences. First, a new term $K(x,t)$ appears in equation (13), and second the term $r(t)W(x,t)$ in (16) changes to $r(T-t)W(x,t)$.

In principle, the modified Hamilton-Jacobi Bellman equation given by (13-15) appears not to be very useful, insofar as it includes implicitly the equilibrium rule (that is, the solution to the problem) in the definition of $K(x,t)$ (via $H(x,s)$). In this paper we will illustrate with several examples how equations (13-15) can be used in order to solve particular problems, and suggest a guessing method for searching for the solution, as well as analyzing nice properties such as the observational equivalence with a problem with non-constant discounting (this is a very important property in an economic context).

Anyway, there is a case in which equations (13-15) are greatly simplified: when the "utility function" $L(x,u,t) = 0$, i.e., the Mayer problem in non constant discounting. This situation occurs in some economic problems, in which one is concerned about maximizing a final value after, for instance, a saved quantity subsequent to an investment or trading period of time. In this case, function $K(x(t),t)$ vanishes and the HJB equation describing the MPE becomes:

Corollary 1 *If the value function of a t-agent for the problem*

$$\max_{\{u(s)\}} \theta(T-t)F(x(T)) \ , \tag{17}$$

$$\dot{x}(s) = f(x(s),u(s),s) \ , \quad x(t) = x_t \tag{18}$$

is continuously differentiable, then it verifies the partial differential equation

$$r(T-t)W(x,t) - \nabla_t W(x,t) = \max_{\{u\}} \{\nabla_x W(x,t) \cdot f(x,u,t)\} \ ,$$

with $W(x,T) = F(x)$.

In contrast, let us now assume that there is no final function. Then equations (13-15) can be written as follows:

Corollary 2 *If, in Problem (1-2), there is no final function ($F(x(T)) = 0$), then the HJB equation can be written as*

$$\bar{K}(x,t) - \nabla_t W(x,t) = \max_{\{u\}} \{L(x,u,t) + \nabla_x W(x,t) \cdot f(x,u,t)\} \ ,$$

where

$$\bar{K}(x,t) = \int_t^T \theta(s-t) r(s-t) H(x,s) ds$$

and $W(x,T) = 0$.

Proof: Along the equilibrium path,

$$W(x,t) = \int_t^T \theta(s-t) H(x,s) ds + \theta(T-t) F(x(T) \ .$$

Therefore

$$K(x,t) = \int_t^T \theta(s-t) r(s-t) H(x,s) ds - r(T-t) \int_t^T \theta(s-t) H(x,s) ds =$$

$$= \bar{K} - r(T-t) W(x,t) - \theta(T-t) r(T-t) F(x(T)) \ .$$

If $F(x(T)) = 0$, then the result follows by substituting the expression above in (13). □

3.3 Observational equivalence with a standard model under constant discounting

Barro (1999) noted that, in an infinite horizon context, the neoclassical growth model with variable rate of time preference and log utility is observationally equivalent (for a sophisticated agent) to the standard model, in the sense that there exists a constant rate for which the optimal control rule coincides with that under non-constant discounting. His proof consisted in constructing explicitly the solution to the problem, guessing for a particular form of the solution.

It is clear that equations (13-15) can be easily used in order to solve the problem of observational equivalence, once the solution of the problem for an arbitrary constant discount rate has been obtained. It suffices to check if the solution of the problem with constant rate satisfies the Hamilton-Jacobi-Bellman equations (13-15). This provides a first and straightforward application of the dynamic programming equation derived above. In fact, we will use this method in the examples discussed in this paper.

In general, let us consider the standard model with constant discounting

$$\max_{\{u(s)\}} \int_t^T e^{-\rho(s-t)} L(x(s), u(s), s)\, ds \ , \ \dot{x}(s) = f(x(s), u(s), s) \ , \ x(t) = x_t \ . \qquad (19)$$

Problem (1-2) will be observationally equivalent to Problem (19) if, and only if, the control (or equilibrium) rules obtained as solution to the HJB equations

$$r(T-t)W + K - \nabla_t W = \max_u \{L + \nabla_x W \cdot f\} \ , \ W(x_T, T) = F(x_T, T) \qquad (20)$$

and

$$\rho V - \nabla_t V = \max_u \{L + \nabla_x V \cdot f\} \ , \ V(x_T, T) = F(x_T, T) \ , \qquad (21)$$

coincide. If the hessian matrix $\left(\dfrac{\partial^2 (L + \nabla_x W \cdot f)}{\partial u^a \partial u^b}\right)$, $a, b = 1, \ldots, m$, is non singular, we can solve the equilibrium rule $u^* = u^*(x, \nabla_x W(x,t), t)$ from the right hand term of (20). In a similar way, if the hessian matrix $\left(\dfrac{\partial^2 (L + \nabla_x V \cdot f)}{\partial u^a \partial u^b}\right)$, $a, b = 1, \ldots, m$, is non singular, we can solve the feedback control law $u^* = u^*(x, \nabla_x V(x,t), t)$ from the right hand term of (21). It is clear that, if Problems (1-2) and (19) have the same solution, then $u^*(x,t) = u^*(x, \nabla_x W(x,t), t) = u^*(x, \nabla_x V(x,t), t)$. This is naturally satisfied if $\nabla_x W = \nabla_x V$, in whose case the right hand terms of equations (20) and (21) coincide.

Now, if $\nabla_x W = \nabla_x V$, then $W(x,t) = V(x,t) + \gamma(t)$ and therefore

$$\rho V(x,t) - \nabla_t V(x,t) = r(T-t)W(x,t) - \nabla_t W(x,t) + K(x,t) =$$

$$= r(T-t)(V(x,t) + \gamma(t)) - \nabla_t V(x,t) - \gamma'(t) + K(x,t) \ .$$

Hence, if there exists a function $\gamma'(t)$ and a constant ρ such that

$$K(x,t) = \gamma'(t) - r(T-t)\gamma(t) + (\rho - r(T-t))V(x,t) \ , \qquad (22)$$

Problem (1-2) will be observationally equivalent to Problem (19). For instance, if $V(x,t) = W(x,t)$, $\gamma(t) = 0$ and condition (22) becomes $K = (\rho - r(T-t))W$.

In particular, if the functions $f_i(x,u,t)$, $i = 1, \ldots n$, defining the control equations are affine in the control variables, i.e., $f_i(x,u,t) = \sum_{a=1}^m g_i^a(x,t) u_a + h_i(x,t)$ (this is a very common situation in economics models), $n = m$ (the number of state and control variables coincide) and $\mathrm{rank}(h_i^a) = n$, then necessarily $\nabla_x W = \nabla_x V$. This is the case, for instance, of the (neoclassical growth) Ramsey model, where there are just one state and one control variables.

Remark 5 *In the limit $T \to \infty$, if the Problem (1-2) is autonomous (L and f are time-independent), an standard argument shows that $W(x,t) = W(x)$. Then, from equations (13) and (15) we recover the dynamic programming equation describing the MPE obtained in Karp (2007). In this case, if $\bar{r} = \lim_{t\to\infty} r(t)$ (or, following Karp, $\bar{r}$ is the long term discount rate), condition (22) becomes $K(x) = -r\gamma + (\rho - \bar{r})W(x)$ or, equivalently, $\nabla_x K(x) = (\rho - \bar{r})\nabla_x V(x)$. If $n = m = 1$, the effective discount rate is $\rho = \bar{r} + \frac{K'(x)}{V'(x)}$. For instance, if $f(x,u) = h(x) - u$, then $V'(x) = L_u(x,u)$ and condition (22) becomes $\frac{K'(x)}{L_u(x,u)} =$ constant, which is precisely the condition derived by Karp for the Ramsey model (the assumption $L_x = 0$ is no longer necessary).*

In the discussion above, we follow the definition of observational equivalence introduced in Barro (1999). However, it seems interesting, both from a mathematical and economical point of view, to study if Problem (1-2) is observationally equivalent to a problem such as 19, but for a different Lagrangian L (that is, a different instantaneous utility function). This problem is closely related to the inverse problem of variational calculus, a long-standing problem in mathematical physics, first posed in 1887 by H. Helmholtz, which has received renewed interest in the last thirty years, when substantial progress has been recorded.

3.4 A simple example

Let us illustrate with a simple example how equations (13-15) can be used in order to solve particular problems with non-constant discounting. This example will provide some insight into how the non-constant discount factor modifies, via the new term $K(x,t)$, the solution of a similar problem with a constant discount factor. In fact, we suggest a guessing method in order to find a solution. Moreover, we obtain impossibility results for the observational equivalence in this example, both for sophisticated and for naive agents.

Let us consider the following problem:

$$\min_{\{u\}} \int_t^T \frac{1}{2} u^2 \theta(s-t)\, ds \ , \tag{23}$$

$$\dot{x}(s) = u(s) \ , \quad x(t) = x_t > 0 \ , \quad x(T) = 0 \ . \tag{24}$$

In the case of constant discounting, $\theta(\tau) = e^{-r\tau}$, the optimal solution to the optimal control problem is

$$u^*(t) = -r\frac{e^{rt}}{e^{rT} - e^{rt}} x^*(t) \ , \quad x^*(t) = x_0 \frac{e^{rT} - e^{rt}}{e^{rT} - 1} \ . \tag{25}$$

Note that the feedback control law can be expressed as a linear function in the state variable x.

Solution for a naive agent:

If the discount factor is non constant, and the agent is naive, the t-agent will solve Problem (23-24) as a standard optimal control problem. The solution is

$$u(s) = -\frac{[\theta(s-t)]^{-1}}{\int_t^T [\theta(\tau - t)]^{-1} \, d\tau} x_t \, . \tag{26}$$

Since the s-agent will not be time consistent with this solution for $s > t$ (he will change the decision rule continuously), the feedback control law (26) will be satisfied only for $s = t$ in general, and therefore

$$u^*(t) = -\frac{x_t}{\int_t^T [\theta(s-t)]^{-1} \, ds} \, . \tag{27}$$

Note that the solution for the problem of a naive agent is non-observationally equivalent to a standard problem with constant discounting: the feedback control rules (25) and (27) coincide if, and only if,

$$\frac{1}{r}\left[e^{r(T-t)} - 1\right] = \int_t^T [\theta(s-t)]^{-1} \, ds = \int_0^{T-t} [\theta(\tau)]^{-1} \, d\tau$$

for every $t \in [0, T]$, i.e.,

$$\frac{1}{r}\left[e^{rs} - 1\right] = \int_0^s [\theta(\tau)]^{-1} \, d\tau \, ,$$

and differentiating with respect to s, we obtain $\theta(s) = e^{-rs}$.

Solution for a sophisticated agent:

If the decision maker is sophisticated, Equation (13) becomes

$$r(T-t)W(x,t) + K(x,t) - W_t(x,t) = \max_{\{u\}} \left\{ -\frac{1}{2}u^2 + W_x(x,t)u \right\} \, .$$

By maximizing the right hand term of the expression above, we obtain the control law $u^*(x,t) = W_x(x,t)$, and the HJB equation becomes

$$r(T-t)W(x,t) + K(x,t) - W_t(x,t) = \frac{1}{2}W_x^2(x,t) \, . \tag{28}$$

From (25), it seems natural to assume that the non constancy of the discount factor will affect only the time dependence of the control law. Therefore, let us check if there exists a function $\lambda(t)$ such that the feedback control law $u(x,t) = \lambda(t)x$ is a solution of the nonlinear partial differential equation (28) with the final condition $W(0,T) = 0$.

Since $W_x(x,t) = u = \lambda(t)x$, then $W(x,t) = \frac{1}{2}\lambda(t)x^2 + \mu(t)$ and equation (28) becomes

$$K(x,t) = -r(T-t)\left(\frac{1}{2}\lambda(t)x^2 + \mu(t)\right) + \frac{1}{2}\lambda'(t)x^2 + \mu'(t) + \frac{1}{2}\lambda^2(t)x^2 \; , \tag{29}$$

where

$$K(x,t) = \int_t^T \theta(s-t)[r(s-t) - r(T-t)]\left[-\frac{1}{2}(u^*(x_t,s))^2\right] ds \; .$$

The solution to the state equation $\dot{x}(s) = u = \lambda(s)x(s)$ with the initial condition $x(t) = x_t$ is $x(s) = \exp(\int_t^s \lambda(\tau)d\tau)$, hence $u^*(x_t,s) = \lambda(s)x_t e^{\int_t^s \lambda(\tau)d\tau}$ and

$$K(x,t) = -\frac{1}{2}x_t^2 \int_t^T \theta(s-t)[r(s-t) - r(T-t)]\lambda^2(s)e^{2\int_t^s \lambda(\tau)d\tau} d\tau \; . \tag{30}$$

The dependence on x in equations (29-30) implies

$$\mu'(t) - r(T-t)\mu(t) = 0 \quad \text{and}$$

$$\lambda'(t) - r(T-t)\lambda(t) + \lambda^2(t) = \tag{31}$$

$$= -\int_t^T \theta(s-t)[r(s-t) - r(T-t)]\lambda^2(s)e^{2\int_t^s \lambda(\tau)d\tau} ds \; .$$

From the solution of the two equations above we obtain the equilibrium value function $W(x,t)$ and, therefore, the control law $u^*(x,t) = \lambda(t)x$. Then, $x(t) = x_0 \exp(\int_0^t \lambda(s)ds)$ and $u^* = x_0\lambda(t)\exp(\int_0^t \lambda(s)ds)$.

Unfortunately, it is clear from this example that searching for an analytic solution in closed form becomes extremely difficult in general, since an integro-differential equation should be solved. However, we have illustrated that the general form of the solution in the standard case is preserved, and this gives us some hints about the searching for a numerical solution.

Let us study if there exists observational equivalence with a problem with constant discount factor ρ. From (25) we know that we must simply check if $\lambda(t) = -\rho\dfrac{e^{\rho t}}{e^{\rho T} - e^{\rho t}}$ is a solution of equation (31). After integrating and simplifying, we obtain

$$\left[1 - \frac{r(T-t)}{\rho}\right] e^{\rho t}(e^{\rho T} - e^{\rho t}) =$$

$$= [2\rho - r(T-t)] \int_t^T \theta(s-t)e^{2\rho s}\, ds + e^{2\rho t} - \theta(T-t)e^{2\rho T} \; .$$

Although we cannot solve the integral in the right hand term for an arbitrary discount factor $\theta(\tau)$, it becomes clear that the equation above will not be satisfied in general for every $t \in [0, T]$. Therefore, Problem (23-24) will not be in general observationally equivalent to a similar problem with constant discounting.

4 Free Terminal Time

In the previous section, and in particular in Theorem 1, we have assumed implicitly that we are dealing with a standard problem in which the terminal time T is fixed. Let now us analyze how the transversality conditions in an optimal control problem in which the terminal time T is freely fixed by the agent are extended to the general context of non constant discounting. We study the general case in which the final function can also depend on T, i.e., $F = F(x(T), T)$. An interesting question emerges in this context: who decides when to stop? Recall that, since preferences are changing due to the non constancy of the discount factor, the agent can be understood as a continuum number of t agents.

It is clear that, if there is no commitment, the terminal time T is decided by the final agent (the T-agent in this case), i.e., who decides whether or not to stop, independently of if the decision maker is sophisticated or naive.

We can also consider the commitment situation in which the 0-agent can fix by contract the terminal time. In this case, the objective of the 0-agent will be to choose T as the maximizer of his value function.

In this paper, we study the general framework in which there exists a time $t \in [0, T]$ in which the t-agent can fix the terminal time T. If $t < T$, we are in a commitment situation. If $t = T$, we assume no commitment at all. Depending on the economic applications we were interested in, the first or the second settings will be more appropriate.

4.1 No commitment in the terminal time T

It is clear that, if there is no commitment, naive agents will solve the free terminal time problem by assuming that, at each moment t, they decide the value of T; that is, the decision maker at time t will solve a standard optimal control problem where the terminal time is fixed in such a way that it is optimal from the viewpoint of the t-agent. Therefore, in general, not only the decisions measured by the control variables will be time inconsistent, but also the calculated terminal times.

If the decision maker is sophisticated, we must analyze the game theoretic framework in which the stopping time is decided by the final agent, who is the one to decide whether or not to stop. In this section we analyze section the case when the state variables at time T are free. With fixed final states, in Section 5 we will propose and illustrate with an example a method for solving the free terminal time problem.

Let T^* be the stopping time. Since T^* is fixed by the T^*-agent, he will compare the value $F(x(T^*), T^*)$ obtained if he decides to stop in T^* with the value obtained if he decides to continue his (consumption, investment) activity. Moreover, necessarily all the previous t-agents ($t \leq T^*$) will prefer to continue until $T = T^*$.

Let $W^T(x(T^*), T^*)$ denote the valuation for the T^*-agent of the problem with terminal time T (recall that $W(x, t)$ represents the valuation from the perspective of the t-agent). Then T^* will be optimal for the T^*-agent if

$$F(x(T^*), T^*) = \max_{T' \geq T^*} W^{T'}(x(T^*), T^*) \ , \tag{32}$$

where T' represents the optimal stopping times if the T^* decides to continue. In a similar way, a condition assuring that previous T-agents prefer to continue until T^* is that

$$F(x(T), T) \leq W^{T^*}(x(T), T) \ , \quad \text{for every } T \leq T^* \ . \tag{33}$$

Using these ideas, we can state the following result:

Proposition 2 *In Problem (1-2) with T free, if the agent is sophisticated and there is no commitment in the terminal time, then the following condition is satisfied in the terminal time T^*:*

$$\left[L + \frac{\partial F}{\partial x} \cdot f \right] \Big|_{T^*} = \left[r(0) F - \frac{\partial F}{\partial T} \right] \Big|_{T^*} \ .$$

Proof: First of all, the terminal time T^* is optimum from the perspective of the T^*-agent. If the optimum for the $(T^* + \epsilon)$-agent, $\epsilon > 0$, is to stop, then from the optimality of T^* we get $F(x(T^*), T^*) \geq W^{T^*+\epsilon}(x(T^*), T^*)$. In this case,

$$F(x(T^*), T^*) \geq \int_{T^*}^{T^*+\epsilon} \theta(s - T^*) L(x, u, s) ds + \theta(\epsilon) F(x(T^* + \epsilon), T^* + \epsilon) =$$

$$= F(x(T^*), T^*) + \epsilon \left[L - r(0) F + \left(\frac{\partial F}{\partial x} \cdot \dot{x} + \frac{\partial F}{\partial T} \right) \right] \Big|_{T^*} + o(\epsilon) \ , \tag{34}$$

where we have used that $\theta(0) = 1$ and $\theta'(0) = -r(0)$.

Otherwise, if the $(T^*+\epsilon)$-agent prefers to continue until a new stopping time $T' \geq T^*+\epsilon$ (which will be optimum for a T'-agent), the optimality of T^* for the T^*-agent implies that $F(x(T^*),T^*) \geq W^{T'}(x(T^*),T^*)$, where $W^{T'}(x(T^*+\epsilon),T^*+\epsilon) \geq F(x(T^*+\epsilon),T^*+\epsilon)$. Therefore,

$$F(x(T^*),T^*) \geq W^{T'}(x(T^*),T^*) = L(x(T^*),T^*)\epsilon + W^{T'}(x(T^*+\epsilon),T^*+\epsilon) + o(\epsilon) \geq$$

$$\geq L(x(T^*),T^*)\epsilon + F(x(T^*+\epsilon),T^*+\epsilon) + o(\epsilon) =$$

$$= F(x(T^*),T^*) + \epsilon \left[L - r(0)F + \left(\frac{\partial F}{\partial x} \cdot \dot{x} + \frac{\partial F}{\partial T} \right) \right] \Bigg|_{T^*} + o(\epsilon)\ . \quad (35)$$

Therefore, from (32) and taking the limit $\epsilon \to 0^+$ in (34-35) we obtain

$$\left[L - r(0)F + \frac{\partial F}{\partial x} \cdot f + \frac{\partial F}{\partial T} \right] \Bigg|_{T^*{}_+} \leq 0\ . \quad (36)$$

In a similar way, a $(T^*-\epsilon)$-agent, $\epsilon > 0$, will decide to continue until T^* if

$$F(x(T^*-\epsilon),T^*-\epsilon) \leq \int_{T^*-\epsilon}^{T^*} \theta(T^*-s)L(x,u,s)ds + \theta(\epsilon)F(x(T^*),T^*)$$

and in the limit case $\epsilon \to 0^+$ we obtain

$$\left[L - r(0)F + \frac{\partial F}{\partial x} \cdot f + \frac{\partial F}{\partial T} \right] \Bigg|_{T^*{}_-} \geq 0\ . \quad (37)$$

The result follows from (36) and (37) (recall that L and F are assumed to be of class C^1 in all their arguments). $\square$

Note that, from condition $W(x(T),T) = F(x(T),T)$, we recover the more familiar expression

$$\left[L + \frac{\partial W}{\partial x} \cdot f \right] \Bigg|_{T^*} = \left[r(0)F - \frac{\partial F}{\partial T} \right] \Bigg|_{T^*}\ .$$

In the analysis above we have implicitly assumed that the terminal state $x(T)$ is free.

4.2 Commitment in the terminal time

Let us now consider the opposite situation to that studied above: the 0-agent, and not the T-agent, decides when to stop.

If the decision maker is naive, then all we are required to do is to calculate the terminal time by using the standard transversality condition for the optimal control problem solved by the 0-agent, and next solve the problem for the t-agents, $t > 0$, as a problem with fixed terminal time.

For sophisticated agents we have:

Proposition 3 *Let us consider Problem (1-2) for a sophisticated agent, with the terminal time T free. If the agent can fix the terminal time at time $t = 0$, then a necessary condition for the optimality of T^* from the perspective of the 0-agent is*

$$\left[L + \frac{\partial F}{\partial x} \cdot f\right]\bigg|_{T^*} = \left[r(T)F - \frac{\partial F}{\partial T}\right]\bigg|_{T^*} .$$

Proof: We can adapt the proof given in Hartl and Sethi (1983) for an ordinary optimal control problem to our generalized setting with non constant discounting.

For a given fixed terminal time $\bar{T}$, let $(x(t, \bar{T}), u(t, \bar{T}))$ be the equilibrium pair. We define

$$\tilde{W}(T, \bar{T}) = \int_0^T \theta(t) L(x(t, \bar{T}), u(t, \bar{T}), t)\, dt + \theta(T) F(x(T, \bar{T}), T) \text{ if } T \in [0, \bar{T}] \text{ , and}$$

$$\tilde{W}(T, \bar{T}) = \int_0^{\bar{T}} \theta(t) L(x(t, \bar{T}), u(t, \bar{T}), t)\, dt + \theta(T) F(x(\bar{T}, \bar{T}), T) +$$
$$+ \int_{\bar{T}}^{T} \theta(t) L(x(t, \bar{T}), u(\bar{T}, \bar{T}), t)\, dt + \theta(\bar{T}) F(x(T, \bar{T}), \bar{T}) \text{ if } T > \bar{T} .$$

Let T^* be optimum from the perspective of the 0-agent. The decision rule $u(t, T^*)$ can be assumed to be continuous in $t = T^*$, and hence $x(t, T^*)$ is of class C^1 with respect to t in $t = T^*$. Then, $\tilde{W}(T, T^*)$ is continuously differentiable with respect to T in $T = T^*$. It is clear then that a necessary condition for the optimality of T^* for the 0-agent is that

$$0 = \frac{\partial \tilde{W}(T, T^*)}{\partial T}\bigg|_{T=T^*} = \theta(T)L + \theta(T)\frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial T} + \theta'(T)F + \theta(T)\frac{\partial F}{\partial T}\bigg|_{T=T^*}$$

and, from $\theta'(T^*) = -r(T^*)\theta(T^*)$ the result follows. □

4.3 A general result

Finally. let us consider the general case in which there is a unique t, $t \in [0, T]$, such that the t-agent can decide the terminal time. Then we have the following result for sophisticated agents.

Theorem 2 *In Problem (1-2), with T is free, if the decision maker is sophisticated and there exists a moment $t \in [0, T]$ such that the t-agent can fix the terminal time, then if T^* is optimum from the perspective of the t-agent, the following condition must be satisfied:*

$$\left[L + \frac{\partial F}{\partial x} \cdot f\right]\bigg|_{T^*} = \left[r(T - t)F - \frac{\partial F}{\partial T}\right]\bigg|_{T^*} .$$

Proof: If $t = T$, we are in the no commitment situation and the result follows from Proposition 2. If $t = 0$, we recover the result proved in Proposition 3. It remains to prove the theorem for $t \in (0, T)$. However, in this case it is easy to see that the proof of Proposition 3 can be adapted in such a way that Theorem 2 follows. $\square$

5 Strotz's model revisited

The model used by Strotz (1956) in order to illustrate the main features of non constant discounting is essentially the Ramsey model with no production function in finite horizon, and can be understood as a problem of consumption of a non renewable resource. Pollak (1968) solved this model by constructing the solution beginning from a discrete setting and taking the limit to the continuum. Actually, he solved the problem from the perspective of naive agents, and then showed that naive and sophisticated agents have the same solution for this particular problem. In this section, we will derive the solution for naive and sophisticated agents, by applying the guessing method suggested in Section 3.4 to equations (13-15). If the terminal time is fixed, we will recover the results by Pollak. If a linear production function is included in the model, we show that the main properties of the solutions are preserved. However, this is not the case if the terminal time is freely chosen by the decision maker.

5.1 Strotz model with log-utility and fixed terminal time

The model studied by Strotz is the following:

$$\max_c \int_t^T \theta(s-t) \ln c \, ds \ , \quad \text{for} \quad t \in [0, T] \ , \tag{38}$$

$$\dot{k} = -c \ , \quad k(t) = k_t \ , \quad k(T) = 0 \ . \tag{39}$$

If the discount factor is constant, $\theta(s-t) = e^{-r(s-t)}$, the solution to the problem is $k^*(t) = k_0 \frac{e^{-rt} - e^{-rT}}{1 - e^{-rT}}$, $c^*(t) = k_0 r \frac{e^{-rt}}{1 - e^{-rT}}$. Note that we can write $c^* = rk^* + \mu$, where $\mu = k_0 r \frac{e^{-rT}}{1 - e^{-rT}}$, or $c^* = \lambda(t) k^*$, where $\lambda(t) = \frac{re^{-rt}}{e^{-rt} - e^{-rT}}$.

The value function can be written as $V(k, t) = \frac{1}{r} \ln (rk + \mu) + Ae^{rt} - \frac{1}{r}$, where A is the solution to condition $V(0, T) = \frac{1}{r} \ln \mu + Ae^{rT} - \frac{1}{r} = 0$.

Alternatively, $V(k,t) = \left(Ae^{rt} + \frac{1}{r}\right)\ln k + g(t)$, where $g(t)$ is the solution to the differential equation $g' - rg = 1 + \ln\left(Ae^{rt} + \frac{1}{r}\right)$.

Solution for a naive agent:

Naive t-agents will solve problem (38-39) using Pontryagin's maximum principle. The feedback control law is now

$$c(s) = \frac{\theta(s-t)}{\int_t^T \theta(\tau - t)\, d\tau} k_t \ .$$

Since the t-agent will no be time consistent for $s > t$, the actual consumption rule is obtained for the case $s = t$ from the equation above, and therefore

$$c^*(t) = \frac{k_t}{\int_t^T \theta(\tau - t)\, d\tau} \ , \tag{40}$$

recovering easily in this way the solution obtained by Pollak (1968), and showing that in this case there will be non observational equivalence for every nonconstant discount function $\theta(\tau)$. The optimal pair $(k^*(t), c^*(t))$ is

$$k^*(t) = k_0 e^{-\int_0^t \left[\int_\tau^T \theta(s-\tau)\, ds\right]^{-1} d\tau} \quad \text{and} \quad c^*(t) = k_0 \frac{e^{-\int_0^t \left[\int_\tau^T \theta(s-\tau)\, ds\right]^{-1} d\tau}}{\int_t^T \theta(s-t)\, ds} \ . \tag{41}$$

Solution for a sophisticated agent:

A sophisticated agent will solve equations (13-15). From equation (13) we obtain $c^* = (W_k)^{-1}$. Therefore,

$$r(T-t)W + K - W_t = \ln c^* + W_k(-c^*) = -\ln W_k - 1 \ . \tag{42}$$

From the insight obtained in the example discussed in Section 3.4, and the result obtained in the case of constant discounting, let us conjecture that the feedback control law is given by $c(t) = \lambda(t)k$, for some function $\lambda(t)$.

Since $k(s) = k_t e^{-\int_t^s \lambda(\tau)\, d\tau}$ is the solution to $\dot{k}(s) = -c(s)$ with the initial condition $k(t) = k_t$, then

$$K = \int_t^T \theta(s-t)\left[r(s-t) - r(T-t)\right] \ln\left(\lambda(s) k_t e^{-\int_t^s \lambda(\tau)\, d\tau}\right) ds \ .$$

Let $\Lambda(t) = (\lambda(t))^{-1}$. Then $W_k = \frac{\Lambda(t)}{k}$ so $W(k,t) = \Lambda(t)\ln k + f(t)$ and substituting in (42) we obtain

$$r(T-t)[\Lambda(t)\ln k + f(t)] - \Lambda'(t)\ln k - f'(t) + 1 + \ln\Lambda(t) - \ln k =$$

$$= \int_t^T \theta(s-t)[r(T-t)-r(s-t)] \ln \left(\lambda(s) e^{-\int_t^s \lambda(\tau)\, d\tau} \right) ds +$$

$$+ \int_t^T \theta(s-t)[r(T-t)-r(s-t)] \ln k_t \, ds \; .$$

Since the equation above must be satisfied for every k, then

$$\Lambda' - r(T-t)\Lambda + 1 = \int_t^T \theta(s-t)[r(s-t)-r(T-t)]\, ds \; .$$

Finally,

$$\int_t^T \theta(s-t) r(s-t)\, ds = -\theta(s-t)|_t^T = -\theta(T-t)+1$$

and by simplifying we obtain

$$\Lambda' - r(T-t)\Lambda = -\theta(T-t) - r(T-t) \int_t^T \theta(s-t)\, ds \; .$$

The general solution of this linear first order differential equation is

$$\Lambda(t) = A e^{\int_0^t r(T-s) ds} + \int_t^T \theta(s-t)\, ds$$

where, since $\Lambda(T) = A e^{\int_0^T r(T-s) ds} = A e^{\int_0^T r(\tau) d\tau} = \dfrac{A}{\theta(T)}$, then $A = \theta(T)\Lambda(T)$. Finally, the terminal condition $0 = k(T) = \Lambda(T)c(T)$ implies that $\Lambda(T) = 0$ ($c(T)$ is necessarily strictly positive) and, therefore, $A = 0$. In conclusion, $\Lambda(t) = \int_t^T \theta(s-t)\, ds$ and

$$c^*(t) = \frac{k_t}{\int_t^T \theta(s-t)\, ds} \; ,$$

which coincides with the solution obtained for a naive agent, in agreement with Pollak (1968).

5.2 The Strotz model with isoelastic utility function

Let us briefly analyze the Strotz model with isoelastic utility function, $u(c) = (c^{1-\sigma} - 1)/(1-\sigma)$, $\sigma > 0$, instead of $u(c) = \ln c$.

If the discount factor is constant, it is easy to show that

$$c^*(t) = \frac{r}{\sigma} \frac{e^{-\frac{r}{\sigma}t}}{e^{-\frac{r}{\sigma}t} - e^{-\frac{r}{\sigma}T}} k_t \; . \tag{43}$$

In the case of non-constant discounting, if the agent is naive, and proceeding as in the case of the logarithmic instantaneous utility function, the consumption rule becomes

$$c^*(t) = \frac{k_t}{\int_t^T [\theta(\tau - t)]^{\frac{1}{\sigma}}\, d\tau} \; . \tag{44}$$

Once again, it can be checked that no constant discount rate r exists such that solutions (43) and (44) coincide.

If the agent is sophisticated, $u'(c) = c^{-\sigma} = W_k$, so $c^* = W_k^{-\frac{1}{\sigma}}$. The HJB becomes

$$r(T-t)W + K - W_t = \frac{W_k^{1-\frac{1}{\sigma}} - 1}{1-\sigma} - W_k^{1-\frac{1}{\sigma}} \; . \tag{45}$$

Since the feedback control law in the case of constant discounting is linear in k_t, we guess for a control feedback law $c = \lambda(t)k$. Now we have

$$K = \int_t^T \theta(s-t)[r(s-t) - r(T-t)] \frac{\left(\lambda(s) k_t e^{-\int_t^s \lambda(\tau)\, d\tau}\right)^{1-\sigma} - 1}{1-\sigma}\, ds \; .$$

By substituting in (45) and simplifying, we obtain that $\lambda(t)$ will be the solution to the integro-differential equation

$$\frac{r(T-t)}{1-\sigma}\lambda^{-\sigma} + \frac{\sigma}{1-\sigma}\frac{\lambda'}{\lambda^{1+\sigma}} - \frac{\sigma}{1-\sigma}\lambda^{1-\sigma} =$$

$$= \frac{1}{1-\sigma}\int_t^T \theta(s-t)[r(T-t) - r(s-t)](\lambda(s))^{1-\sigma} e^{-(1-\sigma)\int_t^s \lambda(\tau)\, d\tau}\, ds \; .$$

It can be shown that, in general, the solutions obtained in equations (43) and (44) for constant discounting and a naive agent are non consistent with the integro-differential equation above. Therefore, naive and sophisticated agents will make different decisions and they will be, in general, non observationally equivalent to a problem with constant discount rate.

5.3 An extension: linear production function

If a linear production function is included in the model, the main results obtained for a logarithmic utility function do not change. Let us solve Problem (38) where equation (39) is replaced by

$$\dot{k} = ak - c \; , \quad k(t) = k_t \; , \quad k(T) = 0 \tag{46}$$

for a positive.

If the discount factor is constant, $\theta(s-t) = e^{-r(s-t)}$, the feedback control rule is again $c^*(t) = \dfrac{rk_t}{1-e^{-r(T-t)}}$.

The same happens for naive and sophisticated agents, where the control law is given by (40). Naive and sophisticated agents with non-constant discount rate take the same decisions with time, and they do not coincide with those of decision makers with a constant discount rate.

5.4 The free terminal time case

Finally, let us solve Problem (38-39) with free terminal time. If $a \neq 0$ in (46), the problem is solved in a similar way.

First of all, let us solve the problem for a constant discount rate. In this case, we must impose the standard transversality condition $H|_{t=T} = 0$. Equivalently, note that the value function $V(k_0, 0)$ is given, as a function of T, by

$$V(k_0, 0) = \int_0^T e^{-rt} \left[\ln (k_0 r) - rt - \ln \left(1 - e^{-rT}\right) \right] \, dt \; .$$

Therefore, in the free terminal time case, $V'(T) = 0$ implies that T is the solution to the nonlinear equation $\ln\left(1 - e^{-rT}\right) + rT = \ln(rk_0) - 1$. Note that $c(T) = e$.

Solution for a naive agent:

A naive t-agent will solve Problem (38-39) by applying the control rule (40),

$$c^*(t) = \frac{k_t}{\int_t^{T_t} \theta(\tau - t)\, d\tau} \;, \tag{47}$$

where the terminal time T_t is calculated by solving the standard optimal control problem (38-39) with free terminal time. By applying the conditions of the Pontryagin's maximum principle to the Hamiltonian function $H_t = \theta(T_t - t) \ln c - pc$ with initial condition $k(t) = k_t$, where p is the co-state variable, and using the well-known transversality condition $H_t|_{T_t} = \theta(T_t - t) \ln c(T_t) - p(T_t) c(T_t) = 0$, after several calculations we obtain the condition characterizing the value of T_t

$$\theta(T_t - t) k_t = e \int_t^{T_t} \theta(s-t)\, ds \; ,$$

and from the expression of k_t given by (41) we obtain

$$k_0 \theta(T_t - t) e^{-\left(1 + \int_0^t \left[\int_s^{T_t} \theta(\tau - s)\, d\tau\right]^{-1} ds\right)} = \int_t^{T_t} \theta(s-t)\, ds \; . \tag{48}$$

For instance, let us solve the problem for the discount function $\theta(\tau) = 1 - \mu\tau$, for $\mu \geq 0$. In this case, the instantaneous discount factor is given by $r(s) = \mu/(1-\mu s)$, which is an increasing function in s (we assume that, along the horizon planning, $1 - \mu s > 0$). Therefore, this will not be a reasonable discount factor for most of applications. However, we choose it in our calculations for simplicity's sake. By substituting in (48) and simplifying we easily obtain

$$2(1 - \mu(T_t - t))k_t = e(T_t - t)(2 - \mu(T_t - t)) \ .$$

For example, the 0-agent will choose T_0 as the solution to

$$2(1 - \mu T_0)k_0 = eT_0(2 - \mu T_0) \ . \tag{49}$$

In the limit $\mu = 0$ we obtain $T_0 = k_0/e$. If $\mu > 0$,

$$T_0 = \frac{\mu k_0 + e - \sqrt{\mu^2 k_0^2 + e^2}}{e\mu} \ .$$

In general,

$$T_t - t = \frac{\mu k_t + e - \sqrt{\mu^2 k_t^2 + e^2}}{e\mu} \ .$$

Note that $\theta(T_t - t) > 0$, for every $t \in [0, T_T]$. Moreover, since $\dfrac{\partial(T_t - t)}{\partial k_t} > 0$ and $k(t)$ is a decreasing function on t, then $T_t - t$ is a decreasing function on t.

Solution for a sophisticated agent:

Finally, let us solve the problem for a sophisticated decision maker. Since we have the final condition $k(T) = 0$, the problem does not verify the conditions of Theorem 2.

Let T^* be the stopping time. Then, T^* is optimal time from the viewpoint of the T^*-agent. Let $T_\epsilon \in (T^* - \epsilon, T^* + \epsilon)$, $\epsilon > 0$. In the limit $\epsilon \to 0^+$, the agent discounts at the constant instantaneous discount rate $r(0)$. Therefore, if $r(t)$ is a continuous function, we can assume that the terminal-agent discounts in a neighborhood of T^* at $r(0)$, and it is optimal for him to stop at time T^*. Hence, it seems natural to analyze the critical consumption level c for which a decision maker with constant discount rate $r(0)$ decides to stop. This consumption level will coincide with the value of $c(T^*)$ for our problem with non-constant discounting, since in the limit $\epsilon \to 0^+$ the T_ϵ^*-agent discount rate is precisely $r(0)$.

From the discussion above, we propose the following method for solving the problem for a sophisticated agent under no commitment if the terminal time T is free.

1. We first solve the optimal control problem with free terminal time for the constant discount rate $r(0)$. If T is the optimal terminal time for this problem (which is calculated by using the standard transversality condition), let $c(T) = \bar{c}$ be the critical consumption level.

2. The problem with non-constant discount rate is solved by searching for the solution to the HJB equation (13-15) with the additional final condition $c(T) = \bar{c}$.

In Strotz's model, we have already obtained that, for any constant discount rate, $\bar{c} = c(T) = e$. Therefore, following the previous method, we impose the condition $c^*(T^*) = \bar{c} = e$, where $c^*(t)$ is given by equation (40).

For instance, for the discount function $\theta(\tau) = 1 - \mu\tau$, $0 \leq \mu \leq 2e/k_0$, from the condition above we obtain

$$T = \frac{2k_0}{2e + \mu k_0} \; . \tag{50}$$

The conditions on the parameter μ assure us that $\theta(T) \geq 0$.

Note that the value of T obtained for a sophisticated agent under no commitment in the terminal time (50) does not coincide with that of a naive agent if commitment is not allowed (see, for instance, equation (49) for the 0-agent), unless $\mu = 0$. Therefore, the solutions for naive and sophisticated agents do not coincide in the case of free terminal time.

6 Concluding remarks

In this paper we have obtained the Hamilton-Jacobi-Bellman equation for sophisticated agents in a finite horizon dynamic optimization problem with non-constant discounting. In our derivation we have adapted the results in Karp (2007) for autonomous infinite horizon problems. Conditions for the observational equivalence with an associated problem with constant discount rate (which is solved by using the standard techniques of optimal control theory) have been analyzed.

Special attention has been paid to the case of free terminal case for sophisticated agents. If the final state is free, we have summarized in Theorem 2 the different transversality conditions corresponding to several contexts, depending on if there exists some kind of commitment (there exists a moment t such that the t-agent can decide the terminal time) or there is no commitment at all (and hence the decision about when to stop is adopted by the final agent). If the final state is fixed, we have suggested, from economic insight, a procedure for solving the free terminal time case for some particular problems.

Several examples have been given in order to illustrate the applicability of the derived HJB equation. In particular, we have reproduced the results by Pollak (1968) for the Strotz's model. Some extensions of this model have been also considered. The most interesting one is, probably, the free terminal case. In all the examples we have compared the results for both, naive and sophisticated agents. The solution for naive agents has been obtained by adapting the Pontryagin's maximum principle to our particular problem, whereas for the results for sophisticated agents we have used the a guessing method combined with the HJB equation derived in the paper.

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