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Option Valuation As an Expectation in The Complex Domain: The Black-Scholes Case

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Abstract: It is very well known that the first succesful valuation of a stock option was done by solving a deterministic partial differential equation (PDE) of the parabolic type with some complementary conditions specific for the option. In this approach, the randomness in the option value process is eliminated through a no-arbitrage argument. An alternative approach is to construct a replicating portfolio for the option. From this viewpoint the payoff function for the option is a random process which, under a new probabilistic measure, turns out to be of a special type, a martingale. Accordingly, the value of the replicating portfolio (equivalently, of the option) is calculated as an expectation, with respect to this new measure, of the discounted value of the payoff function. Since the expectation is, by definition, an integral, its calculation can be made simpler by resorting to powerful methods already available in the theory of analytic functions. In this paper we use precisely two of those techniques to find the well-known value of a European call.
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Resum: La primera valoració, generalment acceptada, d'una opció sobre una acció es feu solucionant una equació diferencial, en derivades parcials, determinista, del tipus parabòlic introduint algunes condicions de contorn específiques de l'opció. En aquest plantejament, la aleatorietat inherent en la valoració de l'opció s'elimina introduint l'hipòtesi d'inexistència de possibilitats d'arbitratge. Un plantejament alternatiu és construir una cartera que repliqui els resultats de l'opció. Des d'aquest punt de vista la funci ó de pagament per l'opció es un procés aleatori que, sota una nova mesura de probabilitat, converteix el procés en una martingala. El valor de la cartera replicant, equivalent al valor de l'opció, es calcula com a esperança del valor descomptat dels pagaments, respecte aquesta nova mesura. Com que l'esperança matemàtica és, per definició, una integral, el seu càlcul es pot fer de forma més simple utilitzant mètodes procedents de la teoria de funcions analí tiques. En aquest article s'utilitzen dues d'aquestes tècniques per trobar el valor d'una call europea.

1 Introduction

The valuation of a stock option in the classic case of the Black-Scholes model involves the calculation of a suitably defined expectation. This expectation is, in disguise, the integral of the function of a random variable ξ with respect to the distribution function of ξ. In the case of the European call the resulting integral can be solved by standard methods. However, the theory of functions of a complex variable provides us with an assortment of techniques that can prove very useful in the solution of integrals. In general, because contour integration is equivalent to calculating residues, integration in the complex plane can be reduced to the much simpler task of differentiation. And not even that, provided one has at hand a good table of integral transforms. In the first of our approaches to the valuation of a European call we employ the Laplace transform with respect to one of the parameters of the integrand. Needless to say, this stands in contrast to the common but inaccurate belief that transformations of this type are done only with respect to "variables", in the ordinary sense After integration in the complex plane we revert to the original parameter to obtain the desired solution. In the second approach, we rely again on an indirect method of integration found in the theory of the Fourier transform, namely, on the Parseval formula. Besides the Introduction, there are three more sections in this paper. Section 2 deals briefly with the standard or classic solution. This section was written mainly to provide some context and for comparison purposes. Section 3 provides the solution employing the Laplace transform. Section 4, the last, contains the solution found by means of the Parseval formula.

2 The Classic Solution

In the Black-Scholes model, the value at time t of an option is given by the following expectation with respect to the risk-neutral measure:

$$V_t = E\left\{e^{-r(T-t)}h\,|F_t\right\}. \tag{1}$$

Here r is the riskless interest rate, T is the expiration time of the option and h is the random process (driven by a Brownian process W_t) which defines the option. F_t is supposed to model the flow of relevant information and is called the filtration. (The financial and technical underpinnings of the Black-Scholes model can be found in almost any modern book on the mathematics of finance. See, e.g., [5]). It is further assumed that under the real world measure the stock S_t follows a geometric Brownian motion with drift parameter μ and volatility σ. Since for a European call with strike price K

$$h = f(S_T) = \max(S_T - K, 0) = (S_T - K)_+,$$

then, omitting the details, after making the substitution $\tau = T - t$, one can write (1) thus

$$V_t = E\left\{e^{-r\tau}\left(S_t e^{(r-\sigma^2/2)\tau+\sigma W_\tau} - K\right)_+\right\}. \tag{2}$$

This expectation is simple enough to be found directly. Traditionally it is calculated as follows: Since W_τ is a normal random variable of mean zero and variance τ, (2) can be written in the form

$$V_\tau = E\left\{e^{-r\tau}\left(S_t e^{(r-\sigma^2/2)\tau+\sigma\sqrt{\tau}Z} - K\right)_+\right\}, \tag{3}$$

where Z is a standard normal with density $\phi_Z(z) = e^{-z^2/2}/\sqrt{2\pi}$. Calling the function under the expectation $g(z)$, then by a standard result of probability theory,

$$V_t = \int_{-\infty}^{\infty} g(z)\phi_Z(z)dz.$$

For later use we will write the solution of the above integral. It is

$$S_t\Phi\left(\frac{\log\frac{S_t}{K} + (r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right) - Ke^{-r\tau}\Phi\left(\frac{\log\frac{S_t}{K} + (r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right), \tag{4}$$

$\Phi(\cdot)$ being the cdf of a standard normal random variable.

3 The Laplace Transform

We will make (3) our starting point. After factoring out the constant K, this expression can obviously be written as

$$\begin{aligned} V_t &= E\left\{Ke^{-r\tau}\left(e^{\log\frac{S_\tau}{K}+(r-\sigma^2/2)\tau+\sigma\sqrt{\tau}Z}-1\right)_+\right\} \\ &= E\left\{Ke^{-r\tau}\left(e^X-1\right)_+\right\}, \end{aligned} \tag{5}$$

where $X=\log\frac{S_\tau}{K}+(r-\sigma^2/2)\tau+\sigma\sqrt{\tau}Z$ is a normal random variable (recall that $Z\sim N(0,1)$) with density function, say, $f_X(\cdot)$, and mean μ_X and variance σ_X^2 given by

$$\begin{aligned} \mu_X &= \log\frac{S_\tau}{K}+(r-\sigma^2/2)\tau \\ \sigma_X^2 &= \sigma^2\tau. \end{aligned} \tag{6}$$

Since $(e^x-1)_+\neq 0$ only for $x>0$ (when $e^x>1$) then, from (5) we obtain

$$\begin{aligned} V_t &= \int_{-\infty}^{\infty} Ke^{-r\tau}\left(e^x-1\right)_+ f_X(x)dx \\ &= \int_0^{\infty} Ke^{-r\tau}\left(e^x-1\right)\frac{1}{\sqrt{2\pi}\sigma_X}e^{-(x-\mu_X)^2/2\sigma_X^2}dx. \end{aligned} \tag{7}$$

We will now exploit the fact that $\sigma_X^2\geq 0$. Setting

$$y=\sigma_X^2 \tag{8}$$

in the last integral above we obtain

$$V_t=\int_0^{\infty} Ke^{-r\tau}\left(e^x-1\right)\frac{1}{\sqrt{2\pi}\sqrt{y}}e^{-(x-\mu_X)^2/2y}dx. \tag{9}$$

We now take the Laplace transform[1] of (9) with respect to y according to the formula (see[3] and [4] for the transform pairs used here)

$$y^{-1/2}e^{-a/(4y)} \risingdotseq \sqrt{\pi}p^{-1/2}e^{-\sqrt{ap}},\ \text{Re } a\geq 0,\ \text{Re } p\geq 0,$$

with $a=2(x-\mu_X)^2$. The transformed integral, $\widetilde{V}_t$, after simplifications becomes

$$\widetilde{V}_t=\frac{K}{\sqrt{2}}p^{-1/2}e^{-r\tau+\mu_X\sqrt{2p}}\int_0^{\infty}\left(e^x-1\right)e^{-\sqrt{2p}x}dx.$$

[1] If the Laplace transform of $f(y)$ is $F(p)$ then we write $f(y)\risingdotseq F(p)=\int_0^\infty e^{-py}f(y)dy$.

Note that, unlike the integral in (9), the above integral can be computed easily, the result being

$$\widetilde{V}_t = \frac{K}{2}e^{-r\tau}\left(\frac{1}{\sqrt{p}(\sqrt{p}-1/\sqrt{2})} - \frac{1}{p}\right)e^{\mu_X\sqrt{2p}}. \tag{10}$$

To revert back to the original function V_t we use the transform pairs

$$\begin{aligned} e^{hb+h^2y}\operatorname{erf}\mathrm{c}(\frac{b+2hy}{2\sqrt{y}}) &\risingdotseq \frac{e^{-b\sqrt{p}}}{\sqrt{p}(\sqrt{p}+h)},\ b>0 \\ \operatorname{erf}\mathrm{c}(\frac{a}{\sqrt{y}}) &\risingdotseq \frac{1}{p}e^{-2a\sqrt{p}},\ \mathrm{Re}\,a>0,\ \mathrm{Re}\,p>0 \end{aligned}$$

on both summands on the right hand side of (10) with $h=-1/\sqrt{2}$, $b=-\sqrt{2}\mu_X$ and $a=-\mu_X/\sqrt{2}$. Here $\operatorname{erfc}(\cdot)$ stands for the complementary error function. Using (8) in the result and simplifying yields

$$V_t = \frac{S_t}{2}\operatorname{erf}\mathrm{c}\left(-\frac{\mu_X+\sigma_X^2}{\sqrt{2}\sigma_X}\right) - \frac{K}{2}e^{-r\tau}\operatorname{erf}\mathrm{c}\left(-\frac{\mu_X}{\sqrt{2}\sigma_X}\right). \tag{11}$$

That this is the same as solution (4) can be seen by using (6) and the fact that $\Phi(x)=\frac{1}{2}\operatorname{erfc}(-x/\sqrt{2})$.

4 The Parseval Formula

On account of (5) and (6) we can write

$$\begin{aligned} V_t &= E\left\{Ke^{-r\tau}\left(e^{\mu_X+\sigma_X z}-1\right)_+\right\} \\ &= Ke^{-r\tau}\int_{-\infty}^{\infty}\left(e^{\mu_X+\sigma_X z}-1\right)\theta(z+\frac{\mu_X}{\sigma_X})\phi_Z(z)dz, \end{aligned} \tag{12}$$

where $\theta(\cdot)$ is Heaviside's unit step function and $\phi_Z(\cdot)$ is the density function of the standard normal random variable Z. According to Parseval's formula (see [1]), if the functions $f(z)$ and $g(z)$ have, respectively, $F(s)$ and $G(s)$ as their Fourier transforms[2], then

$$\int_{-\infty}^{\infty} f(z)g(z)dz = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)G^*(s)ds, \tag{13}$$

[2]If the Fourier transform of $f(x)$ is $F(s)$ then we write $f(x) \risingdotseq F(s) = \int_{-\infty}^{\infty} e^{isx}f(x)dx$.

where the asterisk stands for complex conjugation. To compute the integral in (12) with the help of the above formula, we will regard the integrand appearing there as the product of the functions $f(z) = e^{\mu_X + \sigma_X z} - 1$ and $g(z) = \theta(z + \frac{\mu_X}{\sigma_X})\phi_Z(z)$. The function $f(z)$ does not possess an ordinary Fourier transform. However, in the sense of distributions (see [2]) we have

$$e^{\mu_X + \sigma_X z} - 1 \risingdotseq 2\pi[e^{\mu_X}\delta(s - i\sigma_X) - \delta(s)],$$

$\delta(\cdot)$ being Dirac's delta function. As regards the function $g(z)$, rather simple and direct calculations involving the definition of Fourier transform lead to

$$\theta(z + \frac{\mu_X}{\sigma_X})\phi_Z(z) \risingdotseq e^{-s^2/2}\Phi(\frac{\mu_X}{\sigma_X} + is),$$

where as before $\Phi(\cdot)$ is the cdf of a standard normal rv. Substitution of these two transforms in the right hand side of (13) and simplification of the constant factor imply that (12) is

$$V_t = Ke^{-r\tau}\int_{-\infty}^{\infty}[e^{\mu_X}\delta(s - i\sigma_X) - \delta(s)]e^{-s^2/2}\Phi(\frac{\mu_X}{\sigma_X} - is)ds.$$

By virtue of the fundamental property of the δ -function, it readily follows that

$$V_t = Ke^{-r\tau}\left\{e^{\mu_X + \sigma_X^2/2}\Phi(\frac{\mu_X}{\sigma_X} + \sigma_X) - \Phi(\frac{\mu_X}{\sigma_X})\right\}$$

Using (6) this can be easily given the form as in (4).

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