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The extreme core allocations of the assignment game

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Abstract: Although assignment games are hardly ever convex, in this paper a characterization of their set of extreme points of the core is provided, which is also valid for the class of convex games. For each ordering in the player set, a payoff vector is defined where each player receives his marginal contribution to a certain reduced game played by his predecessors. We prove that the whole set of reduced marginal worth vectors, which for convex games coincide with the usual marginal worth vectors, is the set of extreme points of the core of the assignment game.
Key words: Core, assignment game, convex games.
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Resum: Tot i que els jocs d'assignació, un model de mercat a dues bandes amb utilitat transferible, no són en general jocs convexos, en aquest treball donem una caracterització dels punts extrems del seu core que també és certa per als jocs convexos. Per a cada ordenació del conjunt de jugadors, definim un vector de pagaments on cada jugador rep la seva contribució marginal en cert joc reduït jugat pels seus predecessors. Demostrem que el conjunt de vectors de contribució marginal reduïts, que per als jocs convexos coincideixen amb els vectors de contribució marginal usuals, coincideix amb el conjunt d'extrems del core del joc d'assignació.

1 Introduction

Assignment games were introduced by Shapley and Shubick (1972) as a model for a two–sided market with transferable utility. The player set consists of the union of two finite disjoint sets $M \cup M'$, where M is the set of buyers and M' is the set of sellers. We will denote by n the cardinality of $M \cup M$, $n = m + m'$, where m and m' are, respectively, the cardinalities of M and M'. The worth of any two–person coalition formed by a buyer $i \in M$ and a seller $j \in M'$ is $w(i,j) = a_{ij} \geq 0$. This real numbers can be arranged in a matrix and determine the worth of any other coalition $S \cup T$, where $S \subseteq M$ and $T \subseteq M'$, in the following way: $w(S \cup T) = \max\{\sum_{(i,j)\in\mu} a_{ij} \mid \mu \in \mathcal{M}(S,T)\}$, being $\mathcal{M}(S,T)$ the set of matchings between S and T. A matching (or assignment) between S and T is a subset μ of $S \times T$ such that each player belongs at most to one pair in μ. It will be assumed as usual that $\sum_{(i,j)\in\emptyset} a_{ij} = 0$ and thus, a coalition formed only by sellers or only by buyers, will have worth zero. Moreover, we say a buyer $i \in M$ is not assigned by μ if $(i,j) \notin \mu$ for all $j \in M'$ (and similarly for sellers).

Shapley and Shubik proved the balancedness of these games and also the existence of two special extreme core allocations: in one of them each seller achieves his maximum core payoff and in the other one each buyer does. Demange (1982) proves that this maximum payoff of a player in the core of the assignment games is his marginal contribution. The same result is stated without proof by Leonard (1983). The reader will find Demange's proof in the monography by Roth and Sotomayor (1990).

In 1987, Balinski and Gale give a characterization of the extreme points of the core of the assignment game in terms of the connectedness of a graph. From this characterization follows that in each extreme core point of an assignment game there is a player who receives a zero payoff. This last property was already stated by Thompson (1981) and has recently been used by Hammers *et. al.*(1999) to prove that every extreme point of the core of the assignment game is a marginal worth vector.

Assignment games are hardly ever convex games. Roughly speaking, an assignment game is convex if and only if the assignment matrix is diagonal up to a reordering of the player set. This will be proved later on in corollary 14 (see also Solymosi and Raghavan, 2000). Not being, in general, a convex game, the core of the assignment game has two important properties in common with the core of convex games:

Property 1 *In each extreme core allocation of the assignment game there is a player who is paid his marginal contribution.*

Property 2 *Moreover, all marginal contributions are attained in the core of the assignment game (see Demange, 1982 and Leonard, 1983).*

The main result of this paper will consist on a charactarization of the set of extreme core allocations of the assignment game which also holds for the class of convex games. To this end, some notations are needed.

Let (N, v) be a TU game, where as usual $N = \{1, 2, \ldots, n\}$ is its finite player set and $v : 2^N \longrightarrow \mathbb{R}$ its characteristic function satisfying $v(\emptyset) = 0$. A payoff vector will be $x \in \mathbb{R}^n$ and, for every coalition $S \subseteq N$ we shall write $x(S) := \sum_{i \in S} x_i$ the payoff to coalition S (where $x(\emptyset) = 0$). The core of the game (N, v) consists of those payoff vectors which allocate the worth of the grand coalition in such a way that every other coalition receives at least its worth by the characteristic function: $C(v) = \{x \in \mathbb{R}^n \mid x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } S \subset N\}$. A game (N, v) has a non-empty core if and only if it is balanced (see Bondareva, 1963 or Shapley, 1967), that is to say, the following balancedness conditions hold: $v(N) \geq \sum_{S \in \mathcal{C}} \alpha_S v(S)$, where $\mathcal{C}$ is a set of coalitions and the weights α_s satisfy that for all $i \in N$, $\sum_{i \in S \in \mathcal{C}} \alpha_i = 1$. It is straightforward to see that the core is a convex set and we say $x \in C(v)$ is an extreme point if $y, z \in C(v)$ and $x = \frac{1}{2}y + \frac{1}{2}z$ imply $y = z$. The subgame related to coalition S, $v_{|S}$, is the restriction of mapping v to the subcoalitions of S. A game is said to be superadditive when for all disjoint coalitions S and T, $v(S \cup T) \geq v(S) + v(T)$ holds. Notice that, from the definition, assignment games are always superadditive. Although a game may be balanced but not superadditive (see an example later), balanced games always satisfy superadditive inequalities involving the grand coalition. A well known class of balanced and superadditive games is the class of convex games. A game (N, v) is convex if and only if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all pair of coalitions S and T.

Marginal worth vectors (where given an ordering, each player receives his marginal contribution to the set of his predecessors), play an important role in the core of some games: the core of convex games is the convex hull of the whole set of marginal worth vectors. More formally, given a game (N, v) and an ordering $\theta = (i_1, i_2, \ldots, i_n)$ on N, the marginal worth vector $m_\theta^v \in \mathbb{R}^n$ is:

$$\begin{aligned}
(m_\theta^v)_{i_n} &= v(i_1, \ldots, i_{n-1}, i_n) - v(i_1, \ldots, i_{n-1}) \\
(m_\theta^v)_{i_{n-1}} &= v(i_1, \ldots, i_{n-1}) - v(i_1, \ldots, i_{n-2}) \\
\vdots \quad & \quad \vdots \\
(m_\theta^v)_{i_2} &= v(i_1, i_2) - v(i_1) \\
(m_\theta^v)_{i_1} &= v(i_1)
\end{aligned}$$

Following the same idea, given an ordering θ in N, we will introduce a new kind of marginal worth vectors where each player also receives her marginal contribution to her set of predecessors but a reduction of the game is performed in each step (Núñez and Rafels, 1998).

Definition 3 *Given a cooperative game* (N,v) *and a player* $i \in N$ *we denote by* $b_i^v = v(N) - v(N \setminus \{i\})$ *the marginal contribution of player* i *and define the* **i–marginal game** *of* (N,v) *as the game* $(N \setminus \{i\}, v^i)$ *where, for all* $\emptyset \neq S \subseteq N \setminus \{i\}$ *,*

$$v^i(S) = \max\{v(S \cup \{i\}) - b_i^v, v(S)\} \,,$$

This i–marginal game coincides with the reduced game *à la* Davis and Maschler (1965) on $N \setminus \{i\}$ at b_i^v .

Notice first that $v^i(N \setminus \{i\}) = v(N \setminus \{i\})$ and $C(v^i) \subseteq C(v_{|N\setminus\{i\}})$. Moreover, if (N,v) is convex, that is to say, $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ for all $i \in N$ and $S \subseteq T \subseteq N \setminus \{i\}$, then the i–marginal game v^i coincides with the subgame $v_{|N\setminus\{i\}}$.

Take now $\theta = (i_1, \ldots, i_n)$ an ordering in N , then the **reduced marginal worth vector** is $rm_\theta^v \in \mathbb{R}^n$:

$$\begin{array}{rcl}
(rm_\theta^v)_{i_n} & = & v(i_1, \ldots, i_{n-1}, i_n) - v(i_1, \ldots, i_{n-1}) = b_{i_n}^v \,, \\
(rm_\theta^v)_{i_{n-1}} & = & v^{i_n}(i_1, \ldots, i_{n-1}) - v^{i_n}(i_1, \ldots, i_{n-2}) = b_{i_{n-1}}^{v^{i_n}} \,, \\
\vdots & & \vdots \\
(rm_\theta^v)_{i_k} & = & v^{i_n i_{n-1} \cdots i_{k+1}}(i_1, \ldots, i_k) - v^{i_n i_{n-1} \cdots i_{k+1}}(i_1, \ldots, i_{k-1}) = b_{i_k}^{v^{i_n i_{n-1} \cdots i_{k+1}}} \,, \\
\vdots & & \vdots \\
(m_\theta^v)_{i_2} & = & v^{i_n i_{n-1} \cdots i_3}(i_1, i_2) - v^{i_n i_{n-1} \cdots i_3}(i_1) = b_{i_2}^{v^{i_n i_{n-1} \cdots i_3}} \,, \\
(m_\theta^v)_{i_1} & = & v^{i_n i_{n-1} \cdots i_2}(i_1) = b_{i_1}^{v^{i_n i_{n-1} \cdots i_2}} \,,
\end{array}$$

where, for all $k \in \{1, \ldots, n-1\}$, the game $v^{i_n i_{n-1} \cdots i_{k+1}}$ with player set $N \setminus \{i_n, i_{n-1}, \ldots, i_{k+1}\}$, is the i_{k+1}–marginal game of $v^{i_n i_{n-1} \cdots i_{k+2}}$.

It is straightforward to see that reduced marginal worth vectors are always efficient, $rm_\theta^v(N) = v(N)$ and, if the game is convex the reduced marginal worth vector rm_θ^v coincides with the marginal worth vector m_θ^v . Moreover, if one reduced marginal worth vector belongs to the core, then it is an extreme point (Núñez and Rafels, 1998).

Section 2 relates every extreme point of the core of the assignment game with an extreme core point of one of its marginal games. Moreover, in this section we extend **property 1** to the successive reduced assignment games.

In section 3 we analyze the core of the successive reduced assignment game. One of the main difficulties is that the reduction of an asignment game is no more an assignment game, as was already pointed out by Owen (1992). However, we prove that all successive reductions $w^{k_n k_{n-1} \cdots k_r}$ of the assignment game are balanced (theorem 12). This comes from the fact that **property 2** also extends to those successive reductions of the assignment game.

Finally, in section 4, the set of extreme core allocations of the assignment game is characterized as the whole set of reduced marginal worth vectors (theorem 13). As a consequence of the above result, the case of the convex assignment game is analyzed.

2 The reduced assignment game

In the sequel we will denote by x_{-i} the restriction of x to coalition $N \setminus \{i\}$ and in general $x_{-i_1 i_2 \cdots i_k}$ will denote the restriction of x to coalition $N \setminus \{i_1, \ldots, i_k\}$. Given an arbitrary cooperative game (N, v), some relationships between the core elements of (N, v) and those of its marginal games $(N \setminus \{i\}, v^i)$ are already known. From the reduced game property (RGP) of the core elements (Peleg (1986)), if $x \in C(v)$ and $x_i = b_i^v$, then $x_{-i} \in C(v^i)$. In fact, this relationship is also valid for extreme core elements and, together with a sort of converse property, will play an important role throughout this paper. For the sake of comprehensiveness we state them in next proposition and the reader will find the proof in Núñez and Rafels (1998).

Proposition 4
1. *If $x \in Ext(C(v))$ and for some $i \in N$, $x_i = b_i^v$, then x_{-i} is an extreme point in the core of the i–marginal game, $x_{-i} \in Ext(C(v^i))$.*
2. *If $x_{-i} \in Ext(C(v^i))$ and the condition $v(i) \leq b_i^v$ holds, then $x = (x_{-i}; b_i^v) \in Ext(C(v))$, that is to say, the payoff which allocates to each player the same payoff that in x_{-i} and to player i his marginal contribution is an extreme core allocation.*

Notice that for balanced games, conditions $v(i) \leq b_i^v$, for all $i \in N$, always hold and thus

$$\bigcup_{k=1}^{n} Ext_{+k}(C(v^k)) \subseteq Ext(C(v)), \tag{1}$$

where $Ext_{+k}(C(v^k))$ denotes the set of $x \in \mathbb{R}^N$ such that $x_{-k} \in Ext(C(v^k))$ and $x_k = b_k^v$. In general, this inclusion is strict, but for some classes of games we get an identity.

Now we focus our attention in the extreme core allocations of the assignment game. Recall that, from Shapley and Shubick (1972), the core of the assignment game $(M \cup M', w)$ is nonempty and can be represented in terms of an optimal matching in $M \cup M'$. Let μ be one such optimal matching, then

$$C(w) = \left\{ (u,v) \in \mathbb{R}^{M \times M'} \;\middle|\; \begin{array}{l} u_i \geq 0, \text{ for all } i \in M\,;\, v_j \geq 0, \text{ for all } j \in M' \\ u_i + v_j = a_{ij} \text{ if } (i,j) \in \mu \\ u_i + v_j \geq a_{ij} \text{ if } (i,j) \notin \mu \\ u_i = 0 \text{ if } i \text{ not assigned by } \mu \\ v_j = 0 \text{ if } j \text{ not assigned by } \mu \end{array} \right\} \quad (2)$$

In the case of assignment games, inclusion (1) is in fact an equality.

Proposition 5 *Let $(M \cup M', w)$ be an assignment game, then*

$$Ext(C(w)) = \bigcup_{k=1}^{n} Ext_{+k}(C(w^k))\,.$$

PROOF: Take $x \in Ext(C(w))$, let us proof that there exists $k \in M \cup M'$ such that $x_k = b_k^w$. Recall first that, from Thompson (1981) and Balinski and Gale (1987), there exists $i' \in M \cup M'$ such that $x_{i'} = 0$. If this player i' is not assigned in any optimal matching of $M \cup M'$, then $b_{i'}^w = 0 = x_{i'}$. Otherwise, assume, without lost of generality, that $i' \in M$ and is assigned to $j' \in M'$ by an optimal matching μ of $M \cup M'$. Then, being x a core allocation, $x(M \cup (M' \setminus \{j'\})) \geq w(M \cup (M' \setminus \{j'\}))$ and, on the other hand, as $x_{i'} = 0$,

$$x(M \cup (M' \setminus \{j'\}) = \sum_{\substack{(i,j)\in\mu \\ (i,j)\neq(i',j')}} (x_i + x_j) = \sum_{\substack{(i,j)\in\mu \\ (i,j)\neq(i',j')}} a_{ij} \leq w(M \cup (M' \setminus \{j'\})\,,$$

as $\{(i,j) \in \mu \mid (i,j) \neq (i',j')\}$ is an assignment in $M \cup (M' \setminus \{j'\})$. Thus, $x(M \cup (M' \setminus \{j'\})) = w(M \cup (M' \setminus \{j'\}))$ and, by efficiency, $x_{j'} = b_{j'}^w$.

To sum up, we have just proved that for any $x \in Ext(C(w))$ there exists a player $k \in M \cup M'$ such that $x_k = b_k^w$. By the RGP of the extreme points of the core (part 1 of proposition 4), $x_{-k} \in Ext(C(w^k))$ and then $x \in Ext_{+k}(C(w^k))$. □

Notice that the above equality also holds for convex games.

Unfortunately, the reduction of an assignment game may not be another assignment game. This was already pointed out by Owen (1992) and remains true even for the particular reduced game which is the i–marginal game. Take for instance $M = \{1,2\}$, $M' = \{3,4,5\}$ and $w(i,j) = a_{ij}$ given by the matrix

	3	4	5
1	5	3	3
2	4	3	2

In the above example, the core allocations are those $(u,v) \in \mathbb{R}^5$, such that $u_1 + v_3 = 5$, $u_2 + v_4 = 3$, $v_5 = 0$ and $u_1 + v_4 \geq 3$, $u_1 + v_5 \geq 3$, $u_2 + v_3 \geq 4$, $u_2 + v_5 \geq 2$. This core is the convex hull of three extreme points which are (3,2,2,1,0), (4,3,1,0,0) and (3,3,2,0,0).

Let us now reduce the game on coalition $M' \setminus \{4\}$ at $b_4^w = w(M \cup M') - w(M \cup (M' \setminus \{4\})) = 1$. This 4–marginal game is $(M' \setminus \{4\}, w^4)$, where $w^4(S) = \max\{w(S \cup \{4\}) - b_4^w, w(S)\}$, for all $\emptyset \neq S \subseteq M' \setminus \{4\}$:

$$
\begin{array}{lll}
w^4(1) = 2 & w^4(12) = 2 & w^4(123) = 7 \\
w^4(2) = 2 & w^4(13) = 5 & w^4(125) = 5 \\
w^4(3) = 0 & w^4(15) = 3 & w^4(135) = 5 \\
w^4(5) = 0 & w^4(23) = 4 & w^4(235) = 4 \\
 & w^4(25) = 2 & \\
 & w^4(35) = 0 & w^4(1235) = 7
\end{array}
$$

Notice that the game w^4 is not superadditive ($2 = w^4(12) < w^4(1) + w^4(2) = 4$) and hence it cannot be an assignment game. Although not being an assignment game, w^4 still has a nonempty core. As $x = (3,2,2,1,0) \in C(w)$ and $x_4 = b_4^w = 1$, by part 1 of proposition 4, $x_{-4} = (3,2,2,0) \in C(w^4)$. In fact, as each player can be paid his marginal contribution in the core of the assignment game (Demange (1982) and Leonard (1981)), all marginal games of an assignment game are balanced. Formally, for any assignment game $(M \cup M', w)$ and any player $k \in M \cup M'$, we have $C(w^k) \neq \emptyset$. In section three, by extending the above property to all successive reduced games we will prove balancedness holds for them all.

On the other side, if the reduced games had been assignment games, by applying proposition 5 to the successive reduced games, we would obtain a natural method to express all extreme core allocations of the assignment game as reduced marginal worth vectors. In fact we will achieve the same result, in spite of them not being assignment games.

From now on, given an optimal matching μ in $(M \cup M', w)$ and an order $\theta = (k_1, k_2, \ldots, k_n)$ of the player set, for any $s \in \{1, 2, \ldots, n\}$, this notation will be used: let $I_s = M \cap \{k_n, k_{n-1}, \ldots, k_s\}$, $J_s = M' \cap \{k_n, k_{n-1}, \ldots, k_s\}$, $M_s = M \setminus I_s$, $M'_s = M' \setminus J_s$ and $\mu_s = \{(i,j) \in \mu \mid i \in M_s, j \in M'_s\}$. Notice that μ_s is the restriction to the player set $M_s \cup M'_s$ of the optimal matching μ fixed for the grand coalition $M \cup M'$.

The following is a technical lemma which will be used in proposition 7 to give a description of the core of the successive reduced assignment game

$w^{k_n k_{n-1}\cdots k_s}$ in terms of a fixed optimal matching for the player set $M \cup M'$ of the original game w, under some hypothesis.

Lemma 6 *Let $(M \cup M', w)$ be an assignment game, $\theta = (k_1, k_2, \ldots, k_n)$ an ordering in the player set and $s \in \{2, \ldots, n\}$. If $C(w^{k_n k_{n-1}\cdots k_r}) \neq \emptyset$ for all $r \in \{s, \ldots, n\}$, and we take*

$$\begin{array}{ll} \forall i \in M_s\,, & \alpha_i^s := \max_{k_l \in J_s}\{0, a_{ik_l} - b_{k_l}^{w^{k_n\cdots k_{l+1}}}\} \\ \forall j \in M'_s\,, & \beta_j^s := \max_{k_l \in I_s}\{0, a_{k_l i} - b_{k_l}^{w^{k_n\cdots k_{l+1}}}\}\,, \end{array} \tag{3}$$

then

1. *For all $i \in M_s$ not assigned by μ, $\alpha_i^s = 0$.*

1'. *For all $j \in M'_s$ not assigned by μ, $\beta_j^s = 0$.*

2. *For all $i \in M_s$ not assigned by μ_s, but assigned to $k_l \in J_s$ by μ, it holds $\alpha_i^s = a_{ik_l} - b_{k_l}^{w^{k_n k_{n-1}\cdots k_{l+1}}}$.*

2'. *For all $j \in M'_s$ not assigned by μ_s, but assigned to $k_l \in I_s$ by μ, it holds $\beta_j^s = a_{k_l j} - b_{k_l}^{w^{k_n k_{n-1}\cdots k_{l+1}}}$.*

PROOF: Notice first that, by hypothesis, there exists $x = (u, v) \in C(w^{k_n\cdots k_s})$ and, as $C(w^{k_n\cdots k_r}) \neq \emptyset$, for $s+1 \leq r \leq n$, by completing x with the corresponding marginal contributions, from part 2 of proposition 4, we get a core element of the assignment game, that is

$$(x; b_{k_n}^w, b_{k_{n-1}}^{w^{k_n k_{n-1}}}, \ldots, b_{k_s}^{w^{k_n k_{n-1}\cdots k_{s+1}}}) \in C(w)\,. \tag{4}$$

By the description (2) of the core of an assignment game, if μ is an optimal matching for $M \cup M'$, then,

(1) $u_i + v_j = a_{ij}$, for all $(i,j) \in \mu_s$,

(2) $u_i + b_{k_l}^{w^{k_n k_{n-1}\cdots k_{l+1}}} = a_{ik_l}$ if $(i, k_l) \in \mu$,

(3) $b_{k_l}^{w^{k_n k_{n-1}\cdots k_{l+1}}} + v_j = a_{k_l j}$ if $(k_l, j) \in \mu$,

(4) $u_i + v_j \geq a_{ij}$ for all $(i,j) \in M_s \times M'_s$, $(i,j) \notin \mu_s$,

(5) $u_i + b_{k_l}^{w^{k_n k_{n-1}\cdots k_{l+1}}} \geq a_{ik_l}$ for all $i \in M_s$ and $k_l \in J_s$.

(6) $b_{k_l}^{w^{k_n k_{n-1} \cdots k_{l+1}}} + v_j \geq a_{k_l j}$ for all $k_l \in I_s$ and $j \in M'_s$,

(7) $u_i \geq 0, v_j \geq 0$ forall $i \in M_s$ and $j \in M'_s$,

(8) $u_i = 0$ for all i not matched by μ,

(9) $v_j = 0$ for all j not matched by μ.

To prove *1*, if $i \in M_s$ is not matched by μ, then from (5) and (8), $0 = u_i \geq a_{ik_r} - b_{k_r}^{w^{k_n k_{n-1} \cdots k_{r+1}}}$, for all $k_r \in J_s$, and then $\alpha_i^s = 0$.

To prove *2*, if $(i, k_l) \in \mu$, then, from (2), (5) and (7) follows that

$$\begin{aligned} u_i + b_{k_l}^{w^{k_n k_{n-1} \cdots k_{l+1}}} &= a_{ik_l} \\ u_i + b_{k_r}^{w^{k_n k_{n-1} \cdots k_{r+1}}} &\geq a_{ik_r}, \qquad \text{for all } k_r \in J_s \\ u_i &\geq 0, \end{aligned}$$

and consequently $\alpha_i^s = a_{ik_l} - b_{k_l}^{w^{k_n k_{n-1} \cdots k_{l+1}}}$. The corresponding proofs for $j \in M'_s$ are left to the reader. □

The constants α_i^s and β_j^s will play an important role in the core of the reduced game $w^{k_n k_{n-1} \cdots k_s}$.

Proposition 7 *Let μ be an optimal matching for the game $(M \cup M', w)$, $\theta = (k_1, k_2, \ldots, k_n)$ an ordering in the player set, and $s \in \{2, \ldots, n\}$. If $C(w^{k_n k_{n-1} \cdots k_r}) \neq \emptyset$ for all $r \in \{s, \ldots, n\}$, then*

$$C(w^{k_n \cdots k_s}) = \left\{ (u, v) \in \mathbb{R}^{M_s \times M'_s} \;\middle|\; \begin{array}{l} u_i \geq \alpha_i^s, \textit{ for all } i \in M_s \\ v_j \geq \beta_j^s, \textit{ for all } j \in M'_s \\ u_i + v_j = a_{ij} \textit{ if } (i,j) \in \mu_s \\ u_i + v_j \geq a_{ij} \textit{ if } (i,j) \notin \mu_s \\ u_i = \alpha_i^s \textit{ if } i \textit{ not matched by } \mu_s \\ v_j = \beta_j^s \textit{ if } j \textit{ not matched by } \mu_s \end{array} \right\} \qquad (5)$$

where α_i^s, for all $i \in M_s$, and β_j^s, for all $j \in M'_s$, are defined in (3).

Proof: ($\subseteq$) We have just seen that if $(u, v) \in C(w^{k_n k_{n-1} \cdots k_s})$, the nine conditions listed in the proof of lemma 6 hold. From conditions (5) and (7), we get $u_i \geq \alpha_i^s$ for all $i \in M_s$ and from conditions (6) and (7) we get $v_j \geq \beta_i^s$ for all $j \in M'_s$. From (1) and (4) we get $u_i + v_j = a_{ij}$ if $(i, j) \in \mu_s$ and $u_i + v_j \geq a_{ij}$ if $(i, j) \notin \mu_s$. Moreover, by lemma 6 above, if i is not matched by μ_s, then $u_i = \alpha_i^s$ and if j is not matched by μ_s, $v_j = \beta_j^s$. Therefore, this first inclusion is proved.

($\supseteq$) Conversely, take $(u,v) \in \mathbb{R}^{M_s \times M'_s}$ satisfying all constraints defining the set in the right hand side of the equality we want to prove. By lemma 6, as $C(w^{k_n k_{n-1} \cdots k_r}) \neq \emptyset$, for all $r \in \{s, \dots, n\}$, $\alpha_i^s = 0$ if i not matched by μ and $\alpha_i^s = a_{ik_l} - b_{k_l}^{w^{k_n k_{n-1} \cdots k_{l+1}}}$ if $(i, k_l) \in \mu$ for some $k_l \in J_s$. Similarly, $\beta_i^s = 0$ if j not matched by μ and $\beta_j^s = a_{k_l j} - b_{k_l}^{w^{k_n k_{n-1} \cdots k_{l+1}}}$ if $(k_l, j) \in \mu$ for some $k_l \in I_s$. Now it is straightforward to see that $((u,v); b_{k_n}^w, b_{k_{n-1}}^{w^{k_n}}, \dots, b_{k_s}^{w^{k_n k_{n-1} \cdots k_{s+1}}}) \in C(w)$, as it fulfills all core constraints in description (2). Finally, by the reduced game property of the core elements, $(u,v) \in C(w^{k_n k_{n-1} \cdots k_s})$. □

It is well known that for any extreme point of the core of an assignment game there is a player with zero payoff (see Balinski and Gale (1987)). We now prove a similar property for the extreme core allocations of the reduced assignment game $w^{k_n k_{n-1} \cdots k_s}$. The result is that in every extreme core element there is a player who receives his lower bound in the representation of the core of proposition 7.

Lemma 8 *Let $(M \cup M', w)$ be an assignment game, $\theta = (k_1, k_2, \dots, k_n)$ an ordering in the player set and $s \in \{2, \dots, n\}$. If $C(w^{k_n k_{n-1} \cdots k_r}) \neq \emptyset$ for all $r \in \{s, \dots, n\}$, then for all $x = (u,v) \in Ext(C(w^{k_n k_{n-1} \cdots k_s}))$ there exists either $i \in M_s$ such that $x_i = \alpha_i^s$ or $j \in M'_s$ such that $x_j = \beta_j^s$.*

Proof: We shall consider two different cases. If there exists $i^* \in M_s$ such that i^* is not matched by μ_s, then for any $x \in C(w^{k_n k_{n-1} \cdots k_s})$, by proposition 7, $x_{i^*} = \alpha_{i^*}^s$. Similarly, if there exists $j^* \in M'_s$ not assigned in μ_s, then $x_{j^*} = \beta_{j^*}^s$.

Otherwise, all players in M_s are assigned to players in M'_s (and vice–versa). Assume $u_i > \alpha_i^s$ for all $i \in M_s$ and $v_j > \beta_j^s$ for all $j \in M'_s$. Then we can choose $\epsilon > 0$ such that if we define $\bar{x}$, $\bar{y} \in \mathbb{R}^{M_s \times M'_s}$,

$$\begin{aligned} &\bar{x}_i = u_i + \epsilon \text{ and } \bar{y}_i = u_i - \epsilon \ \text{ for all } i \in M_s\,, \\ &\bar{x}_j = v_j - \epsilon \text{ and } \bar{y}_j = v_j + \epsilon \ \text{ for all } j \in M'_s\,, \end{aligned}$$

then $\bar{x}$ and $\bar{y}$ belong to the core of the reduced assignment game. Notice that you can choose ϵ such that $\bar{x}_i \geq \alpha_i^s$ for all $i \in M_s$, $\bar{x}_j \geq \beta_j^s$ for all $j \in M'_s$. On the other hand if $(i,j) \in M_s \times M'_s$, then $\bar{x}_i + \bar{x}_j = u_i + v_j$. As $(u,v) \in C(w)$, $\bar{x}_i + \bar{x}_j \geq a_{ij}$ if $(i,j) \notin \mu_s$ and $\bar{x}_i + \bar{x}_j = a_{ij}$ if $(i,j) \in \mu_s$. The same argument follows for vector $\bar{y}$ and then, taking the same $\epsilon > 0$ for both vectors, we obtain $\bar{x}, \bar{y} \in C(w^{k_n k_{n-1} \cdots k_s})$ and $x = \frac{1}{2}\bar{x} + \frac{1}{2}\bar{y}$, which contradicts $x = (u,v)$ being an extreme point of $C(w^{k_n k_{n-1} \cdots k_s})$. □

The above property allows us to prove that in each extreme allocation of the core of a reduced assignment game there is a player receiving his marginal contribution. This will be the point to prove in section 4 that each extreme of the core of an assignment game is a reduced marginal worth vector.

Proposition 9 *Let* $(M \cup M', w)$ *be an assignment game,* $\theta = (k_1, k_2, \ldots, k_n)$ *an ordering in the player set and* $s \in \{2, \ldots, n\}$ *. If* $C(w^{k_n k_{n-1} \cdots k_r}) \neq \emptyset$ *for all* $r \in \{s, \ldots, n\}$ *, then, for all* $x \in Ext(C(w^{k_n k_{n-1} \cdots k_s}))$ *there exists* $k \in M_s \cup M'_s$ *such that* $x_k = b_k^{w^{k_n k_{n-1} \cdots k_s}}$ *.*

Proof: By the above lemma, we can assume, without lost of generality, that there exists $i^* \in M_s$ such that $x_{i^*} = \alpha^s_{i^*}$. Take μ an optimal matching in $M \cup M'$. We now consider two cases, depending on whether player i^* is matched by μ_s or not.

Case 1: Assume i^* not matched by μ_s.

As $x \in C(w^{k_n k_{n-1} \cdots k_s})$, then

$$x(M_s \cup M'_s) = w^{k_n k_{n-1} \cdots k_s}(M_s \cup M'_s)$$

and

$$x((M_s \setminus \{i^*\}) \cup M'_s) \geq w^{k_n k_{n-1} \cdots k_s}((M_s \setminus \{i^*\}) \cup M'_s) \,. \tag{6}$$

On the other hand,

$$x((M_s \setminus \{i^*\}) \cup M'_s) = \sum_{(i,j) \in \mu_s} (x_i + x_j) + \sum_{\substack{i \in M_s \setminus \{i^*\} \\ i \text{ not matched by } \mu_s}} x_i \;+ \sum_{\substack{j \in M'_s \\ j \text{ not matched by } \mu_s}} x_j \,.$$

By lemma 6 and the core description (7),

$$x((M_s \setminus \{i^*\}) \cup M'_s) = \sum_{(i,j) \in \mu_s} a_{ij} + \tilde{a}_{p_1 k_{l_1}} - b_{k_{l_1}}^{w^{k_n k_{n-1} \cdots k_{l_1+1}}} + \cdots + \tilde{a}_{p_q k_{l_q}} - b_{k_{l_q}}^{w^{k_n k_{n-1} \cdots k_{l_q+1}}}$$

where $p_1, p_2, \ldots, p_q$ are players in $(M_s \setminus \{i^*\}) \cup M'_s$ assigned to players $k_{l_1}, k_{l_2}, \ldots, k_{l_q}$ in $I_s \cup J_s$, and we assume $l_1 > l_2 > \cdots > l_q$, and

$$\tilde{a}_{p_r k_{l_r}} = \begin{cases} a_{p_r k_{l_r}} & \text{if } p_r \in M \\ a_{k_{l_r} p_r} & \text{if } p_r \in M' \,. \end{cases}$$

Now,

$$x((M_s \setminus \{i^*\}) \cup M'_s) = \textstyle\sum_{(i,j) \in \mu_s} a_{ij} + \sum_{r=1}^{q} (\tilde{a}_{p_r k_{l_r}} - b_{k_{l_r}}^{w^{k_n k_{n-1} \cdots k_{l_r+1}}}) \leq$$

$$w((M_s \setminus \{i^*\}) \cup M'_s \cup \{k_{l_1}, k_{l_2}, \ldots, k_{l_q}\}) - \textstyle\sum_{r=1}^{q} b_{k_{l_r}}^{w^{k_n \cdots k_{l_r+1}}} \leq$$

$$w^{k_n \cdots k_{l_1+1}}((M_s \setminus \{i^*\}) \cup M'_s \cup \{k_{l_1}, k_{l_2}, \ldots, k_{l_q}\}) - \textstyle\sum_{r=1}^{q} b_{k_{l_r}}^{w^{k_n \cdots k_{l_r+1}}} \leq$$

$$w^{k_n \cdots k_{l_1}}((M_s \setminus \{i^*\}) \cup M'_s \cup \{k_{l_2}, \ldots, k_{l_q}\}) - \textstyle\sum_{r=2}^{q} b_{k_{l_r}}^{w^{k_n \cdots k_{l_r+1}}} \leq$$

....

$$w^{k_n \cdots k_{l_q+1}}((M_s \setminus \{i^*\}) \cup M'_s \cup \{k_{l_q}\}) - b_{k_{l_q}}^{w^{k_n \cdots k_{l_q+1}}} \leq$$

$$w^{k_n \cdots k_{l_q}}((M_s \setminus \{i^*\}) \cup M'_s) \leq w^{k_n \cdots k_s}((M_s \setminus \{i^*\}) \cup M'_s)$$

where all these inequalities follow from the definition of marginal game, that is, $v^i(S) \geq v(S)$ and $v^i(S) \geq v(S \cup \{i\}) - b_i^v$, for all $S \neq \emptyset$ not containing player i. Therefore,

$$x((M_s \setminus \{i^*\}) \cup M'_s) \leq w^{k_n \cdots k_s}((M_s \setminus \{i^*\}) \cup M'_s) .$$

We have then obtained, from (6) and the above inequality, that

$$x((M_s \setminus \{i^*\}) \cup M'_s) = w^{k_n k_{n-1} \cdots k_s}((M_s \setminus \{i^*\}) \cup M'_s) ,$$

so, by efficiency,

$$x_i^* = w^{k_n k_{n-1} \cdots k_s}(M_s \cup M'_s) - w^{k_n k_{n-1} \cdots k_s}((M_s \setminus \{i^*\}) \cup M'_s) = b_{i^*}^{w^{k_n k_{n-1} \cdots k_s}}$$

Case 2: Player i^* is matched by μ_s to $j^* \in M'_s$.
By (3), $\alpha_{i^*}^s = \max_{k_l \in J_s}\{0, a_{i^* k_l} - b_{k_l}^{w^{k_n k_{n-1} \cdots k_{l+1}}}\}$ and thus there are two possibilities.
• If $x_{i^*} = \alpha_{i^*}^s = 0$, then on one side, being x a core element, $x(M_s \cup M'_s) = w^{k_n k_{n-1} \cdots k_s}(M_s \cup M'_s)$ and $x(M_s \cup (M'_s \setminus \{j^*\})) \geq w^{k_n k_{n-1} \cdots k_s}(M_s \cup (M'_s \setminus \{j^*\}))$.

On the other side, by using the same reasoning and notation as in Case 1, we obtain

$$x(M_s \cup (M'_s \setminus \{j^*\})) =$$

$$x((M_s \setminus \{i^*\}) \cup (M'_s \setminus \{j^*\})) =$$

$$\textstyle\sum_{(i,j) \in \mu_s,\, (i,j) \neq (i^*,j^*)} a_{ij} + \sum_{r=1}^{q} \left(\tilde{a}_{p_r k_{l_r}} - b_{k_{l_r}}^{w^{k_n k_{n-1} \cdots k_{l_r+1}}}\right) \leq$$

$$w((M_s \setminus \{i^*\}) \cup (M'_s \setminus \{j^*\}) \cup \{k_{l_1}, k_{l_1}, \ldots, k_{l_q}\}) - \textstyle\sum_{r=1}^{q} b_{k_{l_r}}^{w^{k_n \cdots k_{l_r+1}}} \leq \cdots \leq$$

$$w^{k_n \cdots k_s}((M_s \setminus \{i^*\}) \cup (M'_s \setminus \{j^*\})) \leq w^{k_n \cdots k_s}(M_s \cup (M'_s \setminus \{j^*\})) ,$$

where the last inequality holds by monotonicity of $w^{k_n k_{n-1} \cdots k_s}$.

We have then proved that

$$x(M_s \cup (M'_s \setminus \{j^*\})) = w^{k_n \cdots k_s}(M_s \cup (M'_s \setminus \{j^*\}))$$

and, by efficiency, $x_{j^*} = b_{j^*}^{w^{k_n k_{n-1} \cdots k_s}}$.
• If $x_{i^*} = \alpha_{i^*}^s = a_{i^* k_{l^*}} - b_{k_{l^*}}^{k_n k_{n-1} \cdots k_{l+1}}$ for some $k_{l^*} \in J_s$, there are again two possibilities. If there exists $i' \in M_s$ such that $(i', k_{l^*}) \in \mu$, then $x_{i'} = \alpha_{i'}^s$ and, as i' is not assigned in M'_s, by case 1 we have $x_{i'} = b_{i'}^{w^{k_n k_{n-1} \cdots k_s}}$, so there is a player who is paid his marginal contribution.

Otherwise, there is no $i \in M_s$ assigned to k_{l^*}. Then, as $x \in C(w^{k_n k_{n-1} \cdots k_s})$, $x(M_s \cup (M'_s \setminus \{j^*\})) \geq w^{k_n k_{n-1} \cdots k_s}(M_s \cup (M'_s \setminus \{j^*\}))$. On the other hand, by an argument similar to the one above,

$x(M_s \cup (M'_s \setminus \{j^*\})) =$

$x((M_s \setminus \{i^*\}) \cup (M'_s \setminus \{j^*\})) + x_{i^*} =$

$\sum_{(i,j) \in \mu_s, (i,j) \neq (i^*,j^*)} a_{ij} + \sum_{r=1}^{q} (\tilde{a}_{p_r k_{l_r}} - b_{k_{l_r}}^{w^{k_n k_{n-1} \cdots k_{l_r+1}}}) + a_{i^* k_{l^*}} - b_{k_{l^*}}^{w^{k_n \cdots k_{l^*+1}}} \leq$

$w(M_s \cup (M'_s \setminus \{j^*\}) \cup \{k_{l_1}, k_{l_2}, \ldots, k_{l_q}\} \cup \{k_{l^*}\}) - \sum_{r=1}^{q} b_{k_{l_r}}^{w^{k_n \cdots k_{l_r+1}}} - b_{k_{l^*}}^{w^{k_n \cdots k_{l^*+1}}} \leq \cdots \leq$

$w^{k_n \cdots k_s}(M_s \cup (M'_s \setminus \{j^*\}))$

We have then proved that

$$x(M_s \cup (M'_s \setminus \{j^*\})) = w^{k_n \cdots k_s}(M_s \cup M'_s \setminus \{j^*\}))$$

and, by efficiency, we get $x_{j^*} = b_{j^*}^{w^{k_n k_{n-1} \cdots k_s}}$. □

3 Structure of the core of the reduced assignment game

In this section we study the structure of the successive marginal game $w^{k_n k_{n-1} \cdots k_s}$, trying to extend to this reduced game some well known properties of the original assignment game.

Recall first that the core of an assignment game has a particular structure with two special extreme points. One of them gives each seller her maximum possible payoff in the core (and so each buyer gets then his minimum possible payoff inside the core), while the other extreme gives each buyer his maximum possible payoff in the core. This was proved by Shapley and Shubik (1972). Now next lemma shows this property is preserved in the core of the reduced assignment game. For this purpose, when $C(w^{k_n k_{n-1} \cdots k_s}) \neq \emptyset$, we write

$$\begin{array}{l} u_i^* = \max\{u_i \mid (u,v) \in C(w^{k_n k_{n-1} \cdots k_s})\} \\ v_j^* = \max\{v_j \mid (u,v) \in C(w^{k_n k_{n-1} \cdots k_s})\} \\ u_{*i} = \min\{u_i \mid (u,v) \in C(w^{k_n k_{n-1} \cdots k_s})\} \\ v_{*j} = \min\{v_j \mid (u,v) \in C(w^{k_n k_{n-1} \cdots k_s})\} \end{array} \tag{7}$$

Lemma 10 *Let $(M \cup M', w)$ be an assignment game, $\theta = (k_1, k_2, \ldots, k_n)$ an ordering in the player set and $s \in \{2, \ldots, n\}$. If $C(w^{k_n k_{n-1} \cdots k_r}) \neq \emptyset$ for* *all* $r \in \{s, \ldots, n\}$ *, then* (u^*, v_*) *and* (u_*, v^*) *are extreme core allocations of* $w^{k_n k_{n-1} \cdots k_r}$.

PROOF: Notice first that if $(u, v), (u', v') \in C(w^{k_n k_{n-1} \cdots k_s})$ and you define for all $i \in M_s$

$$\underline{u}_i = \min\{u_i, u'_i\} \quad \text{and} \quad \overline{u}_i = \max\{u_i, u'_i\}$$

and for all $j \in M'_s$

$$\underline{v}_j = \min\{v_j, v'_j\} \quad \text{and} \quad \overline{v}_j = \max\{v_j, v'_j\}$$

it is easy to prove that $(\overline{u}, \underline{v}), (\underline{u}, \overline{v}) \in C(w^{k_n k_{n-1} \cdots k_s})$. We will prove it only for the point $(\underline{u}, \overline{v})$.

Let us fix an optimal matching μ of $M \cup M'$. As (u, v) and (u', v') are in the core, by proposition 7, $\underline{u}_i \geq \alpha_i^s$ and $\overline{v}_j \geq \beta_j^s$, for all $i \in M_s$, $j \in M'_s$. Moreover, either $\underline{u}_i + \overline{v}_j = u_i + \overline{v}_j \geq u_i + v_j \geq a_{ij}$ or $\underline{u}_i + \overline{v}_j = u'_i + \overline{v}_j \geq u'_i + v'_j \geq a_{ij}$.

If $(i, j) \in \mu_s$, then $u_i + v_j = a_{ij}$ and $u'_i + v'_j = a_{ij}$. Notice that if $u_i \geq u'_i$, then $v_j \leq v'_j$ and as a consequence $\underline{u}_i + \overline{v}_j = u'_i + v'_j = a_{ij}$. And if $u_i \leq u'_i$, then $v_j \geq v'_j$ and $\underline{u}_i + \overline{v}_j = u_i + v_j = a_{ij}$.

And last, if there exists $i \in M_s$ not assigned in μ_s , then, again by proposition 7, $u_i = u'_i = \alpha_i^s$, which implies $\underline{u}_i = \overline{u}_i = \alpha_i^s$.

Therefore, $(\underline{u}, \overline{v}) \in C(w^{k_n k_{n-1} \cdots k_s})$. Now, from the classical argument done by Shapley and Shubick (1972), follows that $(u_*, v^*) \in C(w^{k_n k_{n-1} \cdots k_s})$, and in fact it is straightforward to see that it is an extreme core allocation. □

Following the proof of Roth and Sotomayor (1990), we will now see that the maximum payoff of a player in the core of the reduced assignment game is his marginal contribution. In fact, what we get is that, apart from the technical details, the reduced assignment game also satisfies **property 2**: all marginal contributions are attained in the core of the reduced assignment game.

Proposition 11 *Let* $(M \cup M', w)$ *be an assignment game,* $\theta = (k_1, k_2, \ldots, k_n)$ *an ordering in the player set and* $s \in \{2, \ldots, n\}$ *. If* $C(w^{k_n k_{n-1} \cdots k_r}) \neq \emptyset$ *for all* $r \in \{s, \ldots, n\}$ *, then*

1. *For all* $i' \in M_s$ *,* $u^*_{i'} = b_{i'}^{w^{k_n k_{n-1} \cdots k_s}}$ *;*

2. *For all* $j' \in M'_s$ *,* $v^*_{j'} = b_{j'}^{w^{k_n k_{n-1} \cdots k_s}}$ *,*

where u^*_i *and* v^*_j *have been defined in (7).*

Proof: We shall only prove the statement for all player $i' \in M_s$, as by a similar argument the reader will obtain the result for all players $j' \in M'_s$.

From lemma 10 we know there exists $(u^*, v_*) \in Ext(C(w^{k_n k_{n-1} \cdots k_s}))$ such that for all $i \in M_s$ and all $j \in M'_s$,

$$u^*_i = \max\{u_i \mid (u,v) \in C(w^{k_n k_{n-1} \cdots k_s})\} \quad \text{and} \quad v_{*j} = \min\{v_j \mid (u,v) \in C(w^{k_n k_{n-1} \cdots k_s})\}.$$

Take then $i' \in M_s$ and μ an optimal matching in $M \cup M'$. We now consider different cases:

Case 1: If i' not assigned in $M_s \cup M'_s$ by μ_s, then from proposition 7 $u^*_{i'} = \alpha^s_{i'}$ and, by the proof of case 1 of proposition 9, $\alpha^s_{i'} = b^{w^{k_n k_{n-1} \cdots k_s}}_{i'}$.

Case 2: If i' is assigned to $j_1 \in M'_s$ by μ_s and $v_{*j_1} = \beta^s_{j_1}$, then, by applying to player j_1 the case 2 of proposition 9, it is straightforward to check that $u^*_{i'} = b^{w^{k_n k_{n-1} \cdots k_s}}_{i'}$.

Case 3: Otherwise i' is matched by μ_s to $j_1 \in M'_s$ and $v_{*j_1} > \beta^s_{j_1}$.

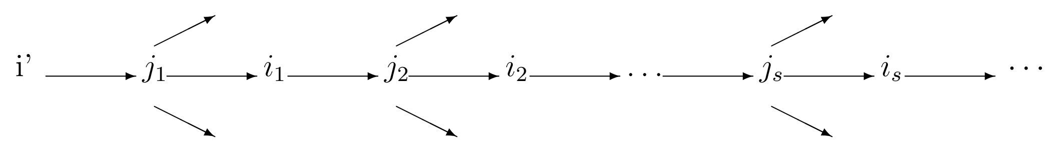
Let x be the allocation of the assignment game obtained by completing (u^*, v_*) with the corresponding marginal contributions:

$$x = (x_k)_{k \in M \cup M'} = ((u^*, v_*); b^w_{k_n}, b^{w^{k_n}}_{k_{n-1}}, \cdots, b^{w^{k_n k_{n-1} \cdots k_{s+1}}}_{k_s}).$$

By the balancedness hypothesis and repeatedly applying proposition 4, x is a core allocation, in fact an extreme point of $C(w)$.

Following the proof of lemma 8.15 in Roth and Sotomayor (1990), construct an oriented graph with vertices $M \cup M'$ and two kind of arcs: given $(i,j) \in M \times M'$, if $(i,j) \in \mu$, then $i \longrightarrow j$ and if $x_i + x_j = a_{ij}$ but $(i,j) \notin \mu$, then $j \longrightarrow i$.

Let T be the set of $i \in M$ that can be reached from i' through an oriented path. We will assume $i' \in T$. Let S be the set of $j \in M'$ that can be reached from i' through an oriented path. Notice that, under the assumptions of case 3, neither T nor S are the empty set, as $i' \in T$ and $j_1 \in S$. Moreover, all oriented paths starting at i' pass through j_1.



We first prove that

$$\text{for all } i \notin T, \text{ and all } j \in S, \ x_i + x_j > a_{ij}. \tag{8}$$

Assume on the contrary that there exists $i \notin T$ and $j \in S$ such that $x_i + x_j = a_{ij}$ (recall $x \in C(w)$). If $(i,j) \in \mu$, then, being $j \in S$ there exists $\tilde{\imath} \in T$ such that $\tilde{\imath} \longrightarrow j$, but then $(\tilde{\imath}, j) \in \mu$ and contradicts μ being a matching. On the other hand, if $(i,j) \notin \mu$, then $j \longrightarrow i$ and as $j \in S$, we get $i \in T$ in contradiction with the hypothesis.

We now claim that "there exists an oriented path c starting at i' and ending either at $i_d \in M$ not matched by μ to a player in M'_s, or at $j_{d+1} \in S$ such that $x_{j_{d+1}} = 0$, in such a way that, in both cases, all $j \in M'$ in the path belong to M'_s". The proof of this claim will consider several cases.

If there exists some $l \in \{s, \ldots, n\}$ such that $k_l \in S$, by definition of set S it is known to exist a path $c = (i', j_1, i_1, j_2, \ldots, j_d, i_d, k_l)$ connecting i' with k_l. Take then the path $c = (i', j_1, i_1, j_2, \ldots, j_d, i_d)$. If $j_t \in M'_s$ for all $t \in \{1, 2, \ldots, d\}$, this is the path claimed. Otherwise take $t^* = \min\{t \in \{1, 2, \ldots, d\} \mid j_t \notin M'_s\}$ and notice that $t^* > 1$ as $j_1 \in M'_s$. Take then $c = (i', j_1, \ldots, j_{t^*-1}, i_{t^*-1}\}$.
Assume $k_l \notin S$ for all $l \in \{s, \ldots, n\}$.
• If there exists $i_d \in T$ not matched by μ, then there is a path $c = (i', j_1, \ldots, j_d, i_d)$ with the properties claimed.
• Otherwise, all $i \in T$ are matched by μ to some player in M'_s. We will prove that there exists $k \in S$ such that $x_k = 0$.
Assume $x_k > 0$ for all $k \in S$. We then can choose $\epsilon > 0$ such that the payoff $x' \in \mathbb{R}^{M \cup M'}$ belongs to $C(w)$, where

$$\begin{array}{ll} x'_k = x_k + \epsilon & k \in T \\ x'_k = x_k - \epsilon & k \in S \\ x'_k = x_k & k \notin S \cup T\,. \end{array}$$

You only have to take $0 < \epsilon < x_k$ for all $k \in S$ and $\epsilon < x_i + x_j - a_{ij}$ for all $j \in S$ and $i \notin T$, which is possible by claim (8). Then $x'_k \geq 0$ for all $k \in M \cup M'$ and $x'_i + x'_j \geq a_{ij}$ for $i \in M \setminus T$ and $j \in S$. Moreover, if $i \in T$ and $j \notin S$, $x'_i + x'_j = x_i + x_j + \epsilon \geq a_{ij}$. On the other hand, for $i \in T$ and $j \in S$, or $i \in M \setminus T$ and $j \in M' \setminus S$, $x'_i + x'_j = x_i + x_j$. Notice that only in the two cases $i \in T$ and $j \in S$, or $i \in M \setminus T$ and $j \in M' \setminus S$, it is possible to have $(i,j) \in \mu$ and thus all core constraints are satisfied.

- If there exists some $k_l \in T$, take $l^* = \max\{l \in \{s, \ldots, n\} \mid k_l \in T\}$. If $l^* < n$, then for all $l^* < l \leq n$, $x'_{k_l} = x_{k_l} = b_{k_l}^{w^{k_n k_{n-1} \cdots k_{l+1}}}$, and that implies, by RGP of the core, $x'_{-k_n k_{n-1} \cdots k_{l^*+1}} \in C(w^{k_n k_{n-1} \cdots k_{l^*+1}})$. But then
$$x'_{k_{l^*}} = x_{k_{l^*}} + \epsilon > x_{k_{l^*}} = b_{k_{l^*}}^{w^{k_n k_{n-1} \cdots k_{l^*+1}}}$$

which contradicts the fact that every player payoff in the core is bounded above by his marginal contribution. Notice that if $l^* = n$, then $x'_{k_n} = x_{k_n} + \epsilon = b^w_{k_n} + \epsilon > b^w_{k_n}$ and the same contradiction is reached in $C(w)$.

- Otherwise, $k_l \notin S \cup T$ for all $l \in \{s, \dots, n\}$ and then $x'_{k_l} = x_{k_l} = b^{w^{k_n k_{n-1} \cdots k_{l+1}}}_{k_l}$ for all $l \in \{s, s+1, \dots, n\}$. Now, as $x' \in C(w)$, by the RGP of the core, $x'_{-k_n k_{n-1} \cdots k_s} \in C(w^{k_n k_{n-1} \cdots k_s})$. But $x'_k < x_k = v_{*k}$ for all $k \in S$, in contradiction with the definition of (u^*, v_*).

Once proved the claim, take an oriented path starting from i' and ending either at $i_d \in M$ not assigned in M'_s or at $j_{d+1} \in S$ with $x_{j_{d+1}} = 0$, and such that all $j \in S$ being in the path belong to M'_s. Let this path be $c = (i', j_1, i_1, j_2, \dots, j_d, i_d)$ or $c = (i', j_1, i_1, j_2, \dots, j_d, i_d, j_{d+1})$.

Define a matching μ' in $M_s \cup M'_s$ in the following way:

$$\mu' = \left\{ (i_t, j_t) \;\middle|\; \begin{array}{l} t \in \{1, \dots, d\} \\ i_t \in M_s \end{array} \right\} \cup \left\{ (i,j) \in M_s \times M'_s \;\middle|\; \begin{array}{l} (i,j) \in \mu_s \text{ and} \\ i \notin \{i', i_1, \dots, i_d\} \end{array} \right\}$$

Then, on one hand, being $(u^*, v_*) \in C(w^{k_n k_{n-1} \cdots k_s})$,

$$\sum_{i \in M_s,\, i \neq i'} u^*_i + \sum_{j \in M'_s} v_{*j} \geq w^{k_n k_{n-1} \cdots k_s}((M_s \setminus \{i'\}) \cup M'_s).$$

On the other hand, $\{A_1, A_2, A_3\}$ is a partition of $M_s \setminus \{i'\}$ where

$$\begin{aligned} A_1 &= \{i_1, i_2, \dots, i_d\} \cap M_s \\ A_2 &= \{i \in M_s \setminus \{i', i_1, \dots, i_d\} \mid i \text{ matched by } \mu_s\} \\ A_3 &= \{i \in M_s \setminus \{i', i_1, \dots, i_d\} \mid i \text{ not matched by } \mu_s\} \end{aligned}$$

and $\{B_1, B_2, B_3, B_4\}$ is a partition of M'_s if $c = (i', i_1, \dots, i_d)$ and of $M' \setminus \{j_{d+1}\}$ otherwise, where

$$\begin{aligned} B_1 &= \{j_t \in \{j_1, \dots, j_d\} \mid i_t \in M_s\} \\ B_2 &= \{j_t \in \{j_1, \dots, j_d\} \mid i_t \notin M_s\} \\ B_3 &= \begin{cases} \{j \notin \{j_1, \dots, j_d\} \mid j \text{ matched by } \mu_s\} & \text{if } c = (i', j_1, i_1, \dots, i_d) \\ \{j \notin \{j_1, \dots, j_d, j_{d+1}\} \mid j \text{ matched by } \mu_s\} & \text{if } c = (i', j_1, i_1, \dots, i_d, j_{d+1}) \end{cases} \\ B_4 &= \begin{cases} \{j \notin \{j_1, \dots, j_d\} \mid j \text{ not matched by } \mu_s\} & \text{if } c = (i', j_1, i_1, \dots, i_d) \\ \{j \notin \{j_1, \dots, j_d, j_{d+1}\} \mid j \text{ not matched by } \mu_s\} & \text{if } c = (i', j_1, i_1, \dots, i_d, j_{d+1}). \end{cases} \end{aligned}$$

As $u^*_{i_t} + v_{*j_t} = a_{i_t j_t}$ and the players in A_2 are matched by μ_s to the players in B_3, we can write

$$\sum_{i\in M_s,\, i\neq i'} u_i^* + \sum_{j\in M_s'} v_{*j} =$$

$$\sum_{\substack{i_t\in A_1\\ j_t\in B_1}} (u_{i_t}^* + v_{*j_t}) + \sum_{\substack{i\in A_2\\ j\in B_3\\ (i,j)\in\mu_s}} (u_i^* + v_{*j}) + \sum_{i\in A_3} u_i^* + \sum_{j_t\in B_2} v_{*j_t} + \sum_{j\in B_4} v_{*j} + (v_{*j_{d+1}}) =$$

$$\sum_{(i,j)\in\mu'} a_{ij} + \sum_{r=1}^{q} \left(\tilde{a}_{p_r k_{l_r}} - b_{k_{l_r}}^{w^{k_n k_{n-1}\cdots k_{l_r+1}}}\right),$$

where the term in brackets is only considered when $c = (i', j_1, \ldots, i_d, j_{d+1})$ (then $v_{*j_{d+1}} = 0$) and $p_1, p_2, \ldots, p_q$ are either players in $M_s \cup M_s'$ not in the chosen path c and matched by μ to players k_{l_r} in $I_s \cup J_s$ for some $l_r \in \{s, \ldots, n\}$, or players j_t in the path such that $i_t = k_l \in I_s$, for some $s \leq l \leq n$. In the first case, if $p \in M_s$ and $(p, k_l) \in \mu$, then by lemma 6 and proposition 7, $u_p^* = a_{pk_l} - b_{k_l}^{w^{k_n\cdots k_{l+1}}}$ and if $p \in M_s'$ such that $(k_l, p) \in \mu$, $v_{*p} = a_{k_l p} - b_{k_l}^{w^{k_n\cdots k_{l+1}}}$. In the second case, as $j_t \longrightarrow i_t = k_l$, we get $a_{i_t j_t} = x_{i_t} + x_{j_t} = b_{k_l}^{w^{k_n k_{n-1}\cdots k_{l+1}}} + v_{*j_t}$.

Moreover, if $j \in B_4$, j cannot be matched by μ to a player i_t, for $t \in \{1, \ldots, d\}$. The reason is that if $t < d$, i_t is matched by μ to j_{t+1}, while i_d is either not assigned to a player in M_s' or assigned to j_{d+1} when $c = (i', i_1, \ldots, i_d, j_{d+1})$. We finally assume $l_1 > l_2 > \cdots > l_q$, and define

$$\tilde{a}_{p_r k_{l_r}} = \begin{cases} a_{p_r k_{l_r}} & \text{if } p_r \in M \\ a_{k_{l_r} p_r} & \text{if } p_r \in M'. \end{cases}$$

Now,

$$\sum_{i\in M_s,\, i\neq i'} u_i^* + \sum_{j\in M_s'} v_{*j} =$$

$$\sum_{(i,j)\in\mu'} a_{ij} + \sum_{r=1}^{q} \left(\tilde{a}_{p_r k_{l_r}} - b_{k_{l_r}}^{w^{k_n k_{n-1}\cdots k_{l_r+1}}}\right) \leq$$

$$w((M_s \setminus \{i'\}) \cup M_s' \cup \{k_{l_1}, k_{l_2}, \ldots, k_{l_q}\}) - \sum_{r=1}^{q} b_{k_{l_r}}^{w^{k_n\cdots k_{l_r+1}}} \leq$$

$$w^{k_n\cdots k_{l_1+1}}((M_s \setminus \{i'\}) \cup M_s' \cup \{k_{l_1}, k_{l_2}, \ldots, k_{l_q}\}) - \sum_{r=1}^{q} b_{k_{l_r}}^{w^{k_n\cdots k_{l_r+1}}} \leq$$

$$w^{k_n\cdots k_{l_1}}((M_s \setminus \{i'\}) \cup M_s' \cup \{k_{l_2}, \ldots, k_{l_q}\}) - \sum_{r=2}^{q} b_{k_{l_r}}^{w^{k_n\cdots k_{l_r+1}}} \leq$$

$$\ldots\ldots\ldots$$

$$w^{k_n\cdots k_{l_q+1}}((M_s \setminus \{i'\}) \cup M_s' \cup \{k_{l_q}\}) - b_{k_{l_q}}^{w^{k_n\cdots k_{l_q+1}}} \leq$$

$$w^{k_n\cdots k_{l_q}}((M_s \setminus \{i'\})) \cup M_s') \leq w^{k_n\cdots k_s}((M_s \setminus \{i'\}) \cup M_s')$$

where all these inequalities follow from the definition of marginal game.

We thus have

$$\sum_{i\in M_s,\, i\neq i'} u_i^* + \sum_{j\in M'_s} v_{*j} = w^{k_n k_{n-1}\cdots k_s}((M_s\setminus\{i'\})\cup M'_s)$$

which, by efficiency, means that $u_{i'}^* = b_{i'}^{w^{k_n k_{n-1}\cdots k_s}}$. □

From the above proposition, each player can attain his marginal contribution in the core of the successive reduced game $w^{k_n k_{n-1}\cdots k_s}$, provided all the previous reduced games are balanced. Finally, next theorem states that given an assignment game all successive reduced games are balanced.

Theorem 12 *Let $(M\cup M', w)$ be an assignment game and an arbitrary ordering $\theta=(k_1,k_2,\ldots,k_{n-1},k_n)$ in the player set. Then, for all $l\in\{2,\ldots,n\}$, $C(w^{k_n k_{n-1}\cdots k_l})\neq\emptyset$.*

Proof: Since w is an assignment game, it is well known that $C(w)\neq\emptyset$ (Shapley and Shubick (1972)). Take $\theta=(k_1,\ldots,k_n)$ an arbitrary ordering in $M\cup M'$. By property 2, given k_n there exists $x\in C(w)$, such that $x_{k_n}=b^w_{k_n}$. Now by the reduced game property of core elements, $x_{-k_n}\in C(w^{k_n})$, which proves the marginal game $C(w^{k_n})$ is balanced.

Assume iteratively that given $s\in\{2,\ldots,n\}$, $C(w^{k_n k_{n-1}\cdots k_r})\neq\emptyset$ for all $r\in\{s,\ldots,n\}$. By lemma 10 and proposition 11 there exists $x=(u^*,v_*)$ and $y=(u_*,v^*)$, both in $C(w^{k_n k_{n-1}\cdots k_s})$, such that if $k_{s-1}\in M_s$ then $x_{k_{s-1}}=b_{k_{s-1}}^{w^{k_n k_{n-1}\cdots k_s}}$ and if $k_{s-1}\in M'_s$ then $y_{k_{s-1}}=b_{k_{s-1}}^{w^{k_n k_{n-1}\cdots k_s}}$. Anyway, there is a core element z where player k_{s-1} gets his marginal contribution in the game $w^{k_n\cdots k_s}$. Then, again by the reduced game property for core allocations, $z_{-k_{s-1}}\in C(w^{k_n k_{n-1}\cdots k_s k_{s-1}})$ and thus the game $w^{k_n\cdots k_s k_{s-1}}$ is also balanced. □

After the above theorem follows that the marginal game, w^i , although may be not superadditive (see the example in page 6), always satisfies superadditive inequalities involving the grand coalition, which are part of balancedness inequalities (Bondareva, 1963 and Shapley, 1967). In general, the successive reduced game $w^{k_n\cdots k_s}$, being balanced, will also preserve this property.

4 The extreme core points of the assignment game

Once proved that all the successive reduced games $w^{k_n k_{n-1}\cdots k_l}$ are balanced, by proposition 7 $C(w^{k_n k_{n-1}\cdots k_l})$ can be expressed in terms of any optimal matching of the gran coalition and, by proposition 9, in each extreme point of

$C(w^{k_n k_{n-1} \cdots k_l})$ there is a player who is paid his marginal contribution in the successive reduced game $w^{k_n k_{n-1} \cdots k_l}$. Moreover, by proposition 11, for each player of the game $w^{k_n k_{n-1} \cdots k_l}$ there is an extreme core point where he is paid his marginal contribution.

Recall from proposition 4 that in order to lift a core allocation of the i–marginal game to a core element of the original game by completing with the marginal contribution of player i, we need the original game v to satisfy condition $v(i) \leq b_i^v$. The balancedness of v and the successively reduced games that has just been proved, guarantees the above inequality holds in every step.

We can now state and proof the main result of this paper: the set of extreme allocations in the core of the assignment game coincides with the whole set of reduced marginal worth vectors.

Theorem 13 *Let $(M \cup M', w)$ be an assignment game, then*

$$ExtC(w) = \{rm_\theta^w\}_{\theta \in \mathcal{S}_n}$$

where $\mathcal{S}_n$ is the set of orderings over the player set $M \cup M'$.

Proof: Take $x \in Ext(C(w))$, by proposition 9 there exists $k_n \in M \cup M'$ such that $x_{k_n} = b_{k_n}^w$. Now, by RGP for the extreme core points, $x_{-k_n} \in Ext(C(w^{k_n}))$. Again by proposition 9 there exists $k_{n-1} \in (M \setminus \{k_n\}) \cup M'$ such that $x_{k_{n-1}} = b_{k_{n-1}}^{w^{k_n}}$. By repeating the process, in a finite number of steps we get an ordering $\theta = (k_1, k_2, \ldots, k_{n-1}, k_n) \in \mathcal{S}_n$ such that $x = rm_\theta^w$.

Conversely, take $x = rm_\theta^w$ for some $\theta = (k_1, k_2, \ldots, k_n)$, which means that $x_{k_{l-1}} = b_{k_{l-1}}^{w^{k_n k_{n-1} \cdots k_l}}$ for all $l \in \{2, \ldots, n\}$ and $x_{k_n} = b_{k_n}^w$. In the one–player game $w^{k_n k_{n-1} \ldots k_2}$, $x_{k_1} = b_{k_1}^{w^{k_n k_{n-1} \cdots k_2}} = w^{k_n k_{n-1} \cdots k_2}(k_1) \in Ext(C(w^{k_n k_{n-1} \cdots k_2}))$. From theorem 12, $C(w^{k_n k_{n-1} \cdots k_3}) \neq \emptyset$, and then, by proposition 4, we get $(x_{k_1}; b_{k_2}^{w^{k_n k_{n-1} \cdots k_3}}) \in Ext(C(w^{k_n k_{n-1} \cdots k_3}))$. Repeatedly applying theorem 12 and proposition 4, we get $x = rm_\theta^w \in Ext(C(w))$. □

After the above characterization of the extreme core allocations of an assignment game, we show several characterizations of convex assignment games.

For a convex game (N, v), as we have pointed out in the introduction of this paper, and it is straightforward to see, for each permutation θ of the player set, $rm_\theta^v = m_\theta^v$. In fact this coincidence characterizes convexity in the class of assignment games, that is to say, if for an assignment game $(M \cup M', w)$, $rm_\theta^w = m_\theta^w$ holds for every ordering θ in $M \cup M'$, then, from theorem 13, all marginal worth vectors belong to the core and thus w is convex. However, there is an easier characterization of convexity in assignment games in terms of the assignment matrix, the proof of which is straightforward and can be found in Solymosi (2000). Next proposition gathers these two characterizations.

Proposition 14 *Let* $(M \cup M', w)$ *be an assignment game defined by matrix* $A \in \mathcal{M}_{m\times m'}(\mathbb{R})$ *. The following statements are equivalent:*

1. w *is convex.*
2. $rm^w_\theta = m^w_\theta$ *for each ordering* θ *in* $M \cup M'$ *.*
3. A *is diagonal (i.e., for all* $i \in M$ *there exists at most one* $j \in M'$ *such that* $a_{ij} > 0$ *and for all* $j \in M'$ *there exists at most one* $i \in M$ *such that* $a_{ij} > 0$ *).*

From the above proposition follows easily that the set of convex assignment games coincides with the set of convex neighbour games defined in Hammers *et.al.* (1999).

Also by proposition 14, the worth of a coalition in a convex assignment game is then the addition of the worth of all possible two–person subcoalitions, which means it is a 2-game (van der Nouweland *et. al.* (1996)) and then the Shapley value and the nucleolus coincide and are easy to compute: for all $i \in M$, if $a_{ij} > 0$, then $\phi_i(w) = \frac{a_{ij}}{2}$.

The Shapley value (Shapley, 1953) is a well known point solution concept for TU-games, and thus also for assignment games. This value consists on the average of all marginal worth vectors and then always belongs to the core if the game is convex. Moreover, as all subgames of a convex game are also convex, the Shapley value of each subgame belongs to the core of the corresponding subgame.

It is easy to find examples of assignment games such that the Shapley value is not a core allocation, but there are also non-convex assignment games where the Shapley value belongs to the core. Take for instance the 2×2 assignment game w defined by the matrix $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$. The Shapley value is $\phi(w) = (1.083, 1.417, 0.917, 1.583)$ and is a core allocation, although this assignment game is not convex. However, when computing the Shapley value of the subgame $w_{|S}$ where $S = \{123\}$, we get $\phi(w_{|S}) = (0.667, 0.167, 1.167)$. Notice on one hand that this allocation does not belong to the core of the subgame $w_{|S}$ and, on the other hand, that the payoff of player 3, acording to the Shapley value, decreases when player 4 joins coalition $\{1, 2, 3\}$, as $1.167 > 0.9171$.

We look for assignment games where the Shapley value has the property of being population monotonic, that is to say, each player payoff increases as the coalition to which he belongs grows larger. Population monotonicity was defined by Sprumont (1990) and, formally, the extended Shapley value $(\phi(w_{|S}))_{S \subseteq N}$ is a Population Monotonic Allocation Scheme (PMAS) if and only if for all $i \in N$ and all pair of coalitions S and T such that $i \in S \subseteq T$, it holds $\phi_i(w_{|S}) \leq \phi_i(w_{|T})$. It is well known that for convex games the extended Shapley value is a PMAS. We now prove that in fact this is another characterization of convexity in the class of assignment games.

Proposition 15 *Let $(M \cup M', w)$ be an assignment game and, for all coalition $S \subseteq M \cup M'$, $\phi(w_{|S})$ the Shapley value of subgame $w_{|S}$. The following statements are equivalent:*

1. *w is convex.*
2. *$(\phi(w_{|S}))_{S \subseteq M \cup M'}$ is a PMAS.*

PROOF: We only have to prove $2 \Rightarrow 1$. From the definition of PMAS follows straightforward that if $(\phi(w_{|S})_{S \subseteq M \cup M'}$ is a PMAS, then $\phi(w_{|S}) \in C(w_{|S})$. Assume now that w is not convex. Then, without lost of generality, we may assume there is some $i \in M$ such that for two different $k_1, k_2 \in M'$, $a_{ik_1} > a_{ik_2} > 0$. Take coalition $S = \{i, k_1, k_2\}$. Then

$$\phi_i(w_{|S}) + \phi_{k_1}(w_{|S}) = a_{ik_1} - \frac{a_{ik_2}}{2} < a_{ik_1} = w_{|S}(ik_1)$$

which implies $\phi(w_{|S}) \notin C(w_{|S})$ and contradicts the extended Shapley value being a PMAS. □

Looking at the prove of the above proposition, the next corollary follows easily.

Corollary 16 *An assignment game $(M \cup M', w)$ is convex if and only if, for every three person coalition S, $\phi(w_{|S}) \in C(w_{|S})$*

Notice that the game in the exemple above is a non-convex assignment game and this is why the Shapley value is not a PMAS and the Shapley value of a three-person subgame is not a core allocation.

In fact convex assignment games are the only assignment games such that the Shapley value of every subgame belongs to the core of the corresponding subgame. Which are the assignment games such that the Shapley value is a core allocation remains an open question.

Let us remark to finish this section that most non–convex assignment games lack a PMAS, since Sprumont (1990) showed that if the assignment matrix has a 2×2 submatrix with all entries positive, the game has no PMAS.

5 Concluding remarks

In Owen (1992) an assignment game with reservation prices is considered. Given a $m \times m'$ matrix $A = (a_{ij})$ with non-negative entries, and non-negative reservation prices $p_1, \ldots, p_m$ for the sellers and $q_1, \ldots, q_{m'}$ for the buyers, the game

$$w(S \cup T) = \max_{\mu \in \mathcal{M}(S,T)} \left(\sum_{(i,j) \in \mu} a_{ij} + \sum_{i \in M \text{ not assigned by } \mu} p_i + \sum_{j \in M' \text{ not assigned by } \mu} q_j \right)$$

is defined. The interpretacion could be that each agent has a non-negative profit by remaining unmatched.

It is straightforward to prove that this game is strategically equivalent to the assignment game $(M \cup M', w')$ with matrix $A' = (a'_{ij})$ where $a'_{ij} = \max\{a_{ij} - p_i - q_j, 0\}$, that is to say, if $x = (p, q) \in \mathbb{R}^{m+m'}$, then for all $S \subseteq M$ and $T \subseteq M'$, we have $w'(S \cup T) = w(S \cup T) - x(S \cup T)$.

The coincidence between the set of extreme points of the core and the set of reduced marginal worth vectors is preserved by strategic equivalence and thus, once proved this coincidence for assignment games, it also holds for assignment games with reservation prices.

For recent results concerning assignment games the reader is referred to Solymosi (1999) and Holzman (2000) where it is proved that the core of an assignment game coincides with the bargaining set *à la* Davis and Maschler and *à la* Mas-Colell; and to Solymosi and Raghavan (2000), where the assignment games with stable core are characterized.

Interesting results on the extreme points of the core of the assignment game can be found in Thompson (1981) and Balinski and Gale (1987). Both state that the core of the assignment game has at most $\binom{2m}{m}$ extreme points, where $m = |M| \leq |M'|$, and prove that this upper bound is attained. In Balinski and Gale (1987) a lower bound for the number of extreme points of the assignment game, under nondegeneracy conditions, is proved to be $2m$ for square games and $m + 1$ for rectangular games. Games for which this lower bounds are attained are provided. A case of degeneracy in an assignment game is the Böhm-Bawerk's horse market: each seller j values his horse in h_j and, in absence of product diferentation, each buyer i values any horse in c_i. The resulting assignment matrix is now defined by $a_{ij} = \max\{0, h_j - c_i\}$ and, as can be seen in Shapley and Shubik (1972), the core consists on a line segment with two extreme allocations.

Finally, it is easy to check that the core of a convex assignment game $(M \cup M', w)$ has at most 2^m extreme points (assuming $m \leq m'$). By proposition 14, once ordered the player set in such a way that $a_{ij} = 0$ for $i \neq j$, it is straightforward to see that

$$C(w) = \left\{ (u,v) \in \mathbb{R}^{m+m'} \;\middle|\; \begin{array}{l} 0 \leq u_i \leq a_{ii} \text{ and } v_i = a_{ii} - u_i \text{ for all } 1 \leq i \leq m \\ v_i = 0 \text{ for } i > m \end{array} \right\}$$

and thus, in each extreme core allocation, u_i must be either 0 or a_{ii}. If $a_{ii} > 0$ for all $i \in M$, then the number of extreme core allocations is precisely 2^m.

We must also point out that the characterization of the set of extreme points of the core we have just obtained for bilateral assignment games does not hold neither for transport games (Thompson, 1981) nor for multiple partners assignment games (Sotomayor, 1999). Take the following exemple: $M = \{1, 2\}$, $M' = \{3, 4\}$, the matrix $A = \begin{pmatrix} 1 & 13 \\ 1 & 1 \end{pmatrix}$, $p = (p_1, p_2) = (2, 1)$ the capacities of agents 1 and 2, and $q = (q_3, q_4) = (1, 1)$ the capacities of agents 3 and 4. The transport game (which in this case is also a multiple partners assignment game, as no player is matched more than once with the same partner) is $v(1) = v(2) = v(3) = v(4) = v(12) = v(34) = 0$, $v(13) = v(23) = v(24) = v(123) = v(234) = 1$, $v(14) = v(124) = 13$ and $v(134) = v(1234) = 14$. The core of the game is $C(v) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4_+ \mid x_1 \leq 12, x_2 = 0, x_3 = 1, 1 \leq x_4 \leq 12, x_1 + x_4 = 13\}$ and player 1 never achieves his marginal contribution $b^v_1 = 13$ in the core.

Finally, let us say that we could consider a new value or point solution for TU games defined as the average of the whole set of reduced marginal worth vectors. This value will coincide with the Shapley value for convex games and will belong to the core for assignment games. We left the study of this solution for a next work.

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