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The relationship of capitalization period length with market portfolio composition and betas

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Abstract:

Beta coefficients are not stable if we modify the observation periods of the returns. The market portfolio composition also varies, whereas changes in the *betas* are the same, whether they are calculated as regression coefficients or as a ratio of the risk premiums. *The instantaneous beta*, obtained when the capitalization frequency approaches infinity, may be a useful tool in portfolio selection.

JEL Classification: G11, G12

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Resumen:

Los coeficientes beta no son estables si se modifica la duración de los periodos en los que se mide la rentabilidad de los activos. La composición de la cartera de mercado también varía. Los cambios en las betas son los mismos si éstas han sido calculadas como coeficientes de regresión o como cocientes de primas de riesgo. La beta instantánea obtenida cuando la frecuencia de capitalización tiende a infinito puede ser utilizada como herramienta en la selección de carteras.

1. INTRODUCTION

The *beta* of an asset i in relation to market portfolio M admits two definitions: as a linear regression coefficient:

$$\beta_i^{(LR)} = \tag{1}$$

$$= \frac{\text{Covariance between returns on asset 'i' and returns on market M}}{\text{Variance in returns from market portfolio M}}$$

and as a quotient of the risk premiums:

$$\beta_i^{(E)} = \frac{\text{Expected return asset i - (Return risk-free asset)}}{\text{Expected return M - (Return risk-free asset)}} =$$

$$= \frac{\text{Risk premium asset i}}{\text{Risk premium M}} = \frac{E_i}{E_M} \tag{2}$$

The *Capital Asset Pricing Model* (CAPM) underlies the second way of calculating the *beta* coefficient above. The model assumes that betas of the different assets have been obtained according to (1) and uses a set of hypotheses to demonstrate that the risk premium on asset i is equal to the *beta* of asset i multiplied by the risk premium on the market portfolio M. Thus, mathematically:

$$\beta_i^{(LR)} = \beta_i^{(E)} \tag{3}$$

So, from this expression, the following one is obtained:

$$E_i = \beta_i^{(LR)} \cdot E_M \tag{4}$$

With the CAPM, it does not matter which of the two formulae (1) or (2) calculates the *beta*. Thus, two different calculation procedures exist for any single *beta*. As a consequence of (3), one usually speaks of **the** *beta* without worrying about which of the two procedures is used.

Whether (1) or (2) is employed, the historical data on the distribution of the return on the asset *i* need to be known. According to CAPM assumptions, for any asset *i*, these returns will refer to a single period, i.e. quarterly or monthly or so forth. The period length is constant. However, it has been proven that a change in the period length modifies the value of the *betas*.[1] Thus, for instance, in Meucci, A. (2005), the beta is defined depending on the period interval. Since the *betas* vary, we wondered whether, in theoretical terms, it was possible to find a functional relationship between the *betas* and the period. Furthermore, no empirical observations exist on market portfolio composition variation as a function of period length. We wondered if this variation did indeed exist and whether it was possible to find a functional relationship theoretically. Finally, we wondered about the limit of such a functional relationship when the period length approaches *0*.

In accordance with the CAMP assumptions and the stationarity and independence of the return distributions in different intervals, the purpose of this paper is as follows:

- First, we establish the functional dependence between market portfolio composition and period length (see theorem 1).
- Second, we demonstrate that there is a functional dependence between period length and *betas* (see theorem 2).
- Third, we introduce the concepts of *instantaneous beta* and *instantaneous market portfolio* and we obtain their formulae (see section "The CAPM when period length approaches 0").

Although this paper has a theoretical character, the aforementioned results have practical consequences for investors. For instance, in their daily practice they have to

[1] For an empirical study of this kind applied to shares in the Spanish stockmarket index, the Ibex35, see Fernandez, P. (2004).

consider the way the *betas* are calculated so as not to operate with non-homogeneous magnitudes.

This paper is divided into five parts. First of all we state the mathematical notations and prior assumptions. Next we obtain the market portfolio composition and, in the third part, betas are obtained as a function of dependence on the period length. In the fourth part, we develop the CAPM when period length approaches *0* and, specifically, we obtain instantaneous betas. Finally, we bring together the results and suggest some possible avenues for future development.

2. MATHEMATICAL NOTATIONS AND PRIOR ASSUMPTIONS

Given a group of N risky assets ($\mathrm{i} \in \mathrm{I} = \{1, 2, ..., N\}$) and a risk-free asset ***0***, for assets $\mathrm{i} \in \mathrm{I}$, let $\tilde{r}_i^{(p)}$ $(1 \leq i \leq N)$ be the random variable for the return on asset *i* in any period length *p*. Let p^0 be the capitalization period length, assumed to be constant and expressed in years.

We also assume that the distribution functions of the corresponding returns $\tilde{r}_i^{(p^0)} = \tilde{r}_i$ satisfy the following hypotheses:

(i) They are known.

(ii) For a concrete asset $i \in I$, they are identical for different time intervals of length p^0.

(iii) They have finite variance.

(iv) For unconnected time intervals of period length p^0, they are independent (both for a single asset and for various assets).

Hypothesis (i) supposes that, as in the CAPM model, all investors operating in the market have the same expectations; (ii) implies that these expectations are stationary, i.e. they do not vary from one period to another; (iii) requires the variance of returns to be finite, since otherwise (1) could not be calculated; (iv) implies that assets behave in accordance with the random - walk.[2]

Let: $A_i^{(p)} = E(1+\tilde{r}_i^{(p)}) = 1+E(\tilde{r}_i^{(p)})$ be the expected financial factor for a period length p, corresponding to asset i.

Let $A_i = A_i^{(p^0)}$

Let $r_0 = r_0^{(p_0)}$ be the effective return on a risk-free asset referring to a period length p^0.

Let $A_0 = A_0^{(p_0)} = 1+r_0^{(p_0)}$ be the certain financial factor.

Let $t=p/p^0$ (p>0 is the new period length under consideration)

It follows that for a period length p:

$$A_0^{(p)} = 1+r_o^{(p)} = (1+r_o)^{p/p^0} = (1+r_o)^t = A_0^t \tag{5}$$

Let $\left(E_i^{(p)}\right)_{Nx1} = \left(A_i^{(p)} - A_0^{(p)}\right)_{Nx1}$ be the column vector of the N risk premiums.

[2] In addition, note that (i) gives rise to two versions of the study, depending on whether the hypothesis is applicable or not to the market portfolio *M*. The first version will be the subject of another article. Here we discuss the second version. Hypothesis (i) will only be applicable to *M* if it admits the entrance of new assets and the exit of others. Otherwise, the return of *M* could not be stationary, since the assets with a higher expected return would be more likely to increase their weight in the portfolio than those assets with a lower expected return.

From (38) (see appendix), the vector column corresponding to the *mean risk premiums* for a period length p is:

$$\left(E_i^{(p)} \right)_{Nx1} = \left(A_i^t - A_0^t \right)_{Nx1} \tag{6}$$

Let $\left(\sigma_{ii'}^{(p)} \right)_{NxN}$ be the covariance matrix of the N risky assets, where:

$\sigma_{ii'}^{(p)} = \mathrm{cov}(\tilde{r}_i^{(p)}, \tilde{r}_{i'}^{(p)}) = \mathrm{cov}(1+\tilde{r}_i^{(p)}, 1+\tilde{r}_{i'}^{(p)})$ $(1 \le \mathrm{i} \le \mathrm{N},\ \ 1 \le \mathrm{i'} \le \mathrm{N})$ are the covariances of the returns on assets i and i' corresponding to one single period of length p.

Let $V_i^{(p)}$ be the variance of asset i for an interval with length p.

Let $\left(X_{M,i}^{(p)} \right)_{Nx1}$ be the market portfolio composition vector.

3. MARKET PORTFOLIO COMPOSITION AS A FUNCTION OF PERIOD LENGTH

In the proof of the CAPM equation, the relative weights in the market portfolio of the N assets are proportional to the vector resulting from the multiplication of the inverse of the covariance matrix by the column vector of the risk premiums.[3] Division of the resulting vector by the sum of its components gives a vector with the sum of its components equal to 1. This vector provides the relative weights of the different assets in the market portfolio. If we apply the CAPM to a period length p, the result is:[4]

[3] For this proof, see Jaquillat (1989: 153-156).
[4] Comments on expression (7):

$$\left(X_{M,i}^{(p)}\right)_{Nx1} = \frac{\left(\sigma_{ij}^{(p)}\right)_{NxN}^{-1} \cdot \left(E_i^{(p)}\right)_{Nx1}}{(1)'_{1xN} \cdot \left(\sigma_{ij}^{(p)}\right)_{NxN}^{-1} \cdot \left(E_i^{(p)}\right)_{Nx1}} \qquad (7)$$

Expression (7) shows that the market portfolio composition depends on the covariance matrix and on the vector of the risk premiums. However, both the matrix and the vector depend on p (see (38) and (39) and bear in mind that $t=p/p^0$). As a consequence, the market portfolio composition depends on the length p of the period that we are considering.

<u>Theorem 1:</u> The market portfolio composition is determined by the following expression as a function of p:

$$X_{M,i}^{(p)} = \frac{\left|SE_i^{(p)}\right|}{\sum_{j=1}^{N} \left|SE_i^{(p)}\right|} \qquad (i=1,2,......N) \qquad (8)$$

where:

$$\left|SE_i^{(p)}\right| = \begin{vmatrix} \sigma_{11}^{(p)} & . & E_1^{(p)} & . & \sigma_{1n}^{(p)} \\ \sigma_{21}^{(p)} & . & E_2^{(p)} & . & \sigma_{2n}^{(p)} \\ ..... & . & .... & . & ..... \\ \sigma_{n1}^{(p)} & . & E_n^{(p)} & . & \sigma_{nn}^{(p)} \end{vmatrix} \qquad (9)$$

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a) The result of operating the numerator is an Nx1 vector whose components are proportional to the relative weights of the N assets in the market portfolio.
b) $(1)'_{Nx1}$ represents a row vector whose components are all equal to 1.
c) The result of operating the denominator is a 1x 1 matrix (in fact it is a real number). It is possible to prove that this number is the sum of the components of the vector obtained in the numerator.
d) As a consequence, the final result is a vector whose components add up to 1.

is the determinant of the matrix resulting from the replacement, in the covariance matrix, of the vector of the risk premiums by the ith column.[5]

Proof: Let us consider the system of N+1 equations in N unknowns:

$$\left.\begin{aligned} &E_i^{(p)} = \frac{(e_i)'_{1xN} \cdot (\sigma_{ij}^{(p)})_{NxN} \cdot (X_{M,j}^{(p)})_{Nx1}}{(X_{M,i}^{(p)})'_{1xN} \cdot (\sigma_{ij}^{(p)})_{NxN} \cdot (X_{M,j}^{(p)})_{Nx1}} \cdot \left(\sum_{i=1}^{N} X_{M,i}^{(p)} \cdot E_i^{(p)} \right) \\ &\qquad (i = 1,2 \ldots\ldots N) \\ &\sum_{i=1}^{N} X_{M,i}^{(p)} = 1 \end{aligned}\right\} \qquad (10)$$

In the above expression, (e_i) is the ith vector of the canonical base of R^n (i.e., it is a vector that has all its components equal to zero except the ith component, which is equal to 1).

The first N equations are a consequence of the CAPM equation (the fraction is the *beta* and the parenthesis is the market risk premium). The last equation establishes that the sum of the weight of the N assets must be 1.

Starting from the previous equation system, by dividing equations 2 to N by the first equation, the following linear system is obtained:

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[5]Comment on proposition:

a) What happens if application of formula (8), which provides the market portfolio composition, gives a negative weight $(X_i < 0)$? Concrete examples demonstrate that this is mathematically possible, but while an individual portfolio can have some negative weights, the market portfolio must have, by definition, no negative weights.

b) A possible solution to the problem outlined in point a) is to discard the assets that provide negative weights (thus, $X_i=0$ for the assets with initially negative weights) and to repeat of the calculation of the formulas of theorem 1, without the rows and columns for the assets with initially negative weights. The aim is to find the best possible approach for choosing the most appropriate solution from the non-negative solutions that satisfy systems (10) and (11).

$$\left.\begin{array}{l}\dfrac{E_i^{(p)}}{E_1^{(p)}}=\dfrac{\sum\limits_{j=1}^{N}\sigma_{ij}^{(p)}.X_{M,j}^{(p)}}{\sum\limits_{j=1}^{N}\sigma_{1j}^{(p)}.X_{M,j}^{(p)}} \qquad\qquad (i=2......N)\\ \sum\limits_{i=1}^{N}X_{M,i}^{(p)}=1\end{array}\right\} \tag{11}$$

Solution of this system gives the solutions determined by expressions (8). It can easily be confirmed that these solutions satisfy equations systems (10) and (11), if one bears in mind the following result:

$$\sum_{j=1}^{N}\sigma_{ij}^{(p)}.\left|SE_j^{(p)}\right|=E_i^{(p)}.\left|\sigma_{ij}^{(p)}\right|_{NxN} \tag{12}$$ □

4. THE BETA AS A FUNCTION OF PERIOD LENGTH

Theorem 2:

a) *Betas* of the assets also depend on *p*.

b) For any *p*, the *betas* of the diverse assets are proportional to the respective risk premiums.

This can be expressed as follows:

$$\left(\beta_i^{(p)}\right)'_{1xN}=\frac{(1)'_{1xN}\cdot\left(\sigma_{ij}^{(p)}\right)^{-1}_{NxN}\cdot\left(E_i^{(p)}\right)_{Nx1}}{\left(E_i^{(p)}\right)'_{1xN}\cdot\left(\sigma_{ij}^{(p)}\right)^{-1}_{NxN}\cdot\left(E_i^{(p)}\right)_{Nx1}}\left(E_i^{(p)}\right)'_{1xN} \tag{13}$$

Proof:

We begin with the expression:

$$\beta_i^{(p)} = \frac{\sigma_{iM}^{(p)}}{V_M^{(p)}} =$$

$$= \frac{\left(e_i\right)_{1xN} \cdot \left(\sigma_{ij}^{(p)}\right)_{NxN} \dfrac{\left(\sigma_{ij}^{(p)}\right)_{NxN}^{-1} \cdot \left(E_i^{(p)}\right)_{Nx1}}{(1)_{1xN}^{'} \cdot \left(\sigma_{ij}^{(p)}\right)_{NxN}^{-1} \cdot \left(E_i^{(p)}\right)_{Nx1}}}{\dfrac{\left(\left(E_i^{(p)}\right)_{Nx1}\right)^{'} \cdot \left(\left(\sigma_{ij}^{(p)}\right)_{NxN}^{-1}\right)^{'}}{(1)_{1xN}^{'} \cdot \left(\sigma_{ij}^{(p)}\right)_{NxN}^{-1} \cdot \left(E_i^{(p)}\right)_{Nx1}} \cdot \left(\sigma_{ij}^{(p)}\right)_{NxN} \cdot \dfrac{\left(\sigma_{ij}^{(p)}\right)_{NxN}^{-1} \cdot \left(E_i^{(p)}\right)_{Nx1}}{(1)_{1xN}^{'} \cdot \left(\sigma_{ij}^{(p)}\right)_{NxN}^{-1} \cdot \left(E_i^{(p)}\right)_{Nx1}}} \tag{14}$$

Simplification of the previous expression gives:

$$\beta_i^{(p)} = \frac{(1)_{1xN}^{'} \cdot \left(\sigma_{ij}^{(p)}\right)_{NxN}^{-1} \cdot \left(E_i^{(p)}\right)_{Nx1}}{\left(E_i^{(p)}\right)_{1xN}^{'} \cdot \left(\sigma_{ij}^{(p)}\right)_{NxN}^{-1} \cdot \left(E_i^{(p)}\right)_{Nx1}} E_i^{(p)} \tag{15}$$

Expression (15) shows that the *betas* of the diverse assets are proportional to the respective risk premium. Expressing (15) vectorially results in (13). □

<u>Corollary 1:</u> The *beta* of the market portfolio M is equal to 1 for any real $p>0$.

<u>Proof:</u> For any p, the sum of the previous *betas*, weighted by the assets, is equal to 1. In effect, (7) and (13) give:

$$\sum_{i=1}^{N} X_{M,i}^{(p)} \beta_i^{(p)} = \left(\beta_i^{(p)}\right)_{1xN}^{'} \cdot \left(X_{M,i}^{(p)}\right)_{Nx1} =$$

$$= \frac{(1)_{1xN}^{'} \cdot \left(\sigma_{ij}^{(p)}\right)_{NxN}^{-1} \cdot \left(E_i^{(p)}\right)_{Nx1}}{\left(E_i^{(p)}\right)_{1xN}^{'} \cdot \left(\sigma_{ij}^{(p)}\right)_{NxN}^{-1} \cdot \left(E_i^{(p)}\right)_{Nx1}} \cdot \left(E_i^{(p)}\right)_{1xN}^{'} \cdot$$

$$\cdot \frac{\left(\sigma_{ij}^{(p)}\right)_{NxN}^{-1} \cdot \left(E_i^{(p)}\right)_{Nx1}}{(1)_{1xN}^{'} \cdot \left(\sigma_{ij}^{(p)}\right)_{NxN}^{-1} \cdot \left(E_i^{(p)}\right)_{Nx1}} = 1 = \beta_M^{(p)} \tag{16}$$

Corollary 2: For any real p we have

$$\beta_i^{(p)} = \frac{E_i^{(p)}}{E_M^{(p)}} \tag{17}$$

Proof: From (7), the risk premium on the market portfolio is obtained by weighting the risk premium of the N assets:

$$E_M^{(p)} = \sum_{i=1}^{N} X_{M.i}^{(p)} \cdot E_i^{(p)} = \left(E_i^{(p)}\right)_{1xN}^{'} \cdot \left(X_{M,i}^{(p)}\right)_{Nx1} =$$

$$= \left(E_i^{(p)}\right)_{1xN}^{'} \frac{\left(\sigma_{ij}^{(p)}\right)_{NxN}^{-1} \cdot \left(E_i^{(p)}\right)_{Nx1}}{(1)_{1xN}^{'} \cdot \left(\sigma_{ij}^{(p)}\right)_{NxN}^{-1} \cdot \left(E_i^{(p)}\right)_{Nx1}} \Rightarrow$$

$$\Rightarrow \frac{1}{E_M^{(p)}} = \frac{(1)_{1xN}^{'} \cdot \left(\sigma_{ij}^{(p)}\right)_{NxN}^{-1} \cdot \left(E_i^{(p)}\right)_{Nx1}}{\left(E_i^{(p)}\right)_{1xN}^{'} \cdot \left(\sigma_{ij}^{(p)}\right)_{NxN}^{-1} \cdot \left(E_i^{(p)}\right)_{Nx1}} \tag{18}$$

By applying (18) to (15), we obtain (17) □

Corollary 3: For any real positive p the *beta* obtained as a regression coefficient coincides with the *beta* obtained as a quotient of the risk premiums.

Proof: It is possible to calculate the *beta* as a regression coefficient directly as a quotient of the risk premiums:

$$\beta_i^{(p)} = \frac{E_i^{(p)}}{E_M^{(p)}} = \frac{E_i^{(p)}}{\sum_{i=1}^{N} X_{M,i}^{(p)} \cdot E_i^{(p)}} = \frac{E_i^{(p)}}{\left(E_i^{(p)}\right)'_{1xN} \cdot \left(X_{M,i}^{(p)}\right)_{Nx1}} \tag{19}$$

Application of (7) gives:

$$\beta_i^{(p)} = \frac{E_i^{(p)}}{\left(E_i^{(p)}\right)'_{1xN} \dfrac{\left(\sigma_{ij}^{(p)}\right)^{-1}_{NxN} \cdot \left(E_i^{(p)}\right)_{Nx1}}{(1)'_{1xN} \cdot \left(\sigma_{ij}^{(p)}\right)^{-1}_{NxN} \cdot \left(E_i^{(p)}\right)_{Nx1}}} =$$

$$= \frac{(1)'_{1xN} \cdot \left(\sigma_{ij}^{(p)}\right)^{-1}_{NxN} \cdot \left(E_i^{(p)}\right)_{Nx1}}{\left(E_i^{(p)}\right)'_{1xN} \cdot \left(\sigma_{ij}^{(p)}\right)^{-1}_{NxN} \cdot \left(E_i^{(p)}\right)_{Nx1}} \cdot E_i^{(p)} \tag{20}$$

which is identical to (15) □

5. THE CAPM WHEN PERIOD LENGTH APPROACHES 0

If we calculate the following limits:

$$\lim_{p\to 0} \beta_i^{(p)} = \lim_{p\to 0} \frac{(1)'_{1xN} \cdot \left(\sigma_{ij}^{(p)}\right)^{-1}_{NxN} \cdot \left(E_i^{(p)}\right)_{Nx1}}{\left(E_i^{(p)}\right)'_{1xN} \cdot \left(\sigma_{ij}^{(p)}\right)^{-1}_{NxN} \cdot \left(E_i^{(p)}\right)_{Nx1}} E_i^{(p)} \tag{21}$$

$$\lim_{p\to 0} \left(X_{M,i}^{(p)}\right)_{Nx1} = \lim_{p\to 0} \frac{\left(\sigma_{ij}^{(p)}\right)^{-1}_{NxN} \cdot \left(E_i^{(p)}\right)_{Nx1}}{(1)'_{1xN} \cdot \left(\sigma_{ij}^{(p)}\right)^{-1}_{NxN} \cdot \left(E_i^{(p)}\right)_{Nx1}} \tag{22}$$

we obtain *the instantaneous beta* $\left(\beta_i^{(INST)}\right)$ and *the instantaneous market portfolio composition* $\left(X_i^{(INST)}\right)$, respectively.

When calculating the limit when *p* approaches *0* in (19) and applying L'Hôpital's rule, we obtain:

$$\beta_i^{(INST)} = \frac{\rho_i - \rho_0}{\sum_{j=1}^{N} X_{M,j}^{(INST)} \cdot \left(\rho_j - \rho_0\right)} = \frac{\rho_i - \rho_0}{\rho_M - \rho_0} \tag{23}$$

where:

$$\rho_i = Ln(A_i) \quad ; \quad \rho_o = Ln(A_0) \quad ; \quad \rho_M = \sum_{j=1}^{N} X_{M,j}^{(INST)} \cdot \rho_j \tag{24}$$

In addition, we obtain the instantaneous market portfolio composition by calculating the limit, when p approaches 0, through applying both L'Hôpital's rule N times and the derivation rules for determinants on (8):

$$X_i^{(INS)} = \frac{\left|SE_i^{(INS)}\right|}{\sum_{j=1}^{N} \left|SE_i^{(INS)}\right|} \qquad (i = 1,2,......N) \tag{25}$$

where:

$$\left|SE_i^{(INS)}\right| = \begin{vmatrix} Ln\left(1+\frac{\sigma_{11}}{A_1^2}\right) & \dots & Ln\left(\frac{A_1}{A_0}\right) & \dots & Ln\left(1+\frac{\sigma_{1N}}{A_1 \cdot A_N}\right) \\ Ln\left(1+\frac{\sigma_{21}}{A_2 \cdot A_1}\right) & \dots & Ln\left(\frac{A_2}{A_0}\right) & \dots & Ln\left(1+\frac{\sigma_{2N}}{A_2 \cdot A_N}\right) \\ \dots\dots\dots\dots & \dots & \dots\dots\dots & \dots & \dots\dots\dots\dots \\ Ln\left(1+\frac{\sigma_{N1}}{A_N \cdot A_1}\right) & \dots & Ln\left(\frac{A_N}{A_0}\right) & \dots & Ln\left(1+\frac{\sigma_{NN}}{A_N^2}\right) \end{vmatrix} \quad (26)$$

Using the instantaneous market portfolio and instantaneous betas has the advantage that the value of the above mentioned magnitudes does not depend on the initial period p^0. Thus, the limits mentioned above enable us to unify criteria when dealing with the variables related to the CAPM.

6. CONCLUSIONS

When the CAPM is applied to a group of N assets assumed to have stationary and independent return distributions, for different periods, the results obtained depend on the period length p. In short, when p varies, the following variables also change:

- The market portfolio composition vector.
- The risk premium on the N assets.
- The risk premium market return.
- Betas of each of the N assets.

Therefore, we have as many CAPM models as positive values of p.

The market portfolio composition vector and the vector of the *betas* of the diverse assets vary, inasmuch as the period length varies in which the returns on the assets are measured. In particular, expressions (7) and (13) show these results, which provide, as a function of p, the market portfolio composition and the vector of the betas respectively. In both expressions, the inverse of the covariance matrix and the vector of the risk premiums intervene.

The limit when *p* approaches *0* is relevant. In this case, we will obtain *instantaneous betas* and the *instantaneous market portfolio composition*.

The following requires further study:

a) The vectorial function:

$$\begin{aligned} &R^{+} \to R^{N} \\ &p \to \left(\beta_i^{(p)} \right)_{Nx1} \end{aligned} \tag{27}$$

as well as each of the components of this vectorial function (they are real functions of a real variable).

b) The vectorial function:

$$\begin{aligned} &R^{+} \to R^{N} \\ &p \to \left(X_{M,i}^{(p)} \right)_{Nx1} \end{aligned} \tag{28}$$

as well as each of the components of this vectorial function (they are real functions of a real variable).

c) It is particularly important to study the intervals in which the components increase and decrease, as well as the limits at zero and infinity relating to each component of the two previous vector functions, using the risk premiums and the covariances between the various assets. It would be interesting to analyse whether the evolution of the variables linked to the assets with a *beta* below 1 is qualitatively different from that of the assets with a *beta* larger than 1. It would also be useful to analyse the behavior of possible negative *betas*.

The greatest difficulty in achieving these three objectives lies in obtaining, for any p, the general expression of the inverse of the covariance matrix and, in particular, its limit when p approaches 0.

APPENDIX

Set out below are the statistical properties of the means, variances and covariances when modifying the period length p in which returns are measured. In the properties A1) and A2) of this appendix we do not suppose that the returns on assets are stationary.

A) Properties for period length $K{\cdot}p^0$ (K positive integer).

1) $$E(1+\tilde{r}_i^{(K\cdot p^0)}) = \prod_{j=1}^{K} (1+E(\tilde{r}_{i,j})) \tag{29}$$

($\tilde{r}_{i,j}^{(p_0)}$ is the return on asset i in the period j of length p^0)

Proof: Since the random variables are independent of each other, following Cramer (1962:23) the expectation of the product is a product of expectations, as a result of which the property is demonstrated. □

1 ') corollary: If the random variables corresponding to every period j of length p^0 have the same distribution, $A_{i,j}=A_i$, then we have the following result:

$$E(1+\tilde{r}_i^{(K\cdot p^0)}) = \prod_{j=1}^{K} A_i = A_i^K \tag{30}$$

2) $$\sigma_{ii'}^{(K\cdot p^0)} = \prod_{j=1}^{K}\left(\sigma_{ii',j}^{(p^0)} + A_{i,j}^{(p^0)}.A_{i',j}^{(p^0)}\right) - \prod_{j=1}^{K} A_{i,j}^{(p^0)}.A_{i',j}^{(p^0)} \tag{31}$$

($\sigma_{ii',j}^{(p_0)}$ is the covariance between the returns on assets i and i' in the period j with length p^0)

Proof:

$$\sigma_{ii'}^{(K\cdot p^0)} = E\left[(1+\tilde{r}_i^{(K\cdot p^0)} - \prod_{j=1}^{K} A_{i,j}^{(p^0)})\cdot(1+\tilde{r}_{i'}^{(K\cdot p^0)} - \prod_{j=1}^{K} A_{i',j}^{(p^0)})\right] =$$

$$= E\left[\left(1+\tilde{r}_i^{(K\cdot p^0)}\right)\cdot\left(1+\tilde{r}_{i'}^{(K\cdot p^0)}\right)\right] - \prod_{j=1}^{K} A_{i,j}^{(p^0)}\cdot \prod_{j=1}^{K} A_{i',j}^{(p^0)} =$$

$$= E\left[\prod_{j=1}^{K}(1+\tilde{r}_{i,j}^{(p^0)})\cdot(1+\tilde{r}_{i',j}^{(p^0)})\right] - \prod_{j=1}^{K} A_{i,j}^{(p^0)}\cdot \prod_{j=1}^{K} A_{i',j}^{(p^0)} =$$

$$= \prod_{j=1}^{K} E\left[(1+\tilde{r}_{i,j}^{(p^0)})\cdot(1+\tilde{r}_{i',j}^{(p^0)})\right] - \prod_{j=1}^{K} A_{i,j}^{(p^0)}\cdot \prod_{j=1}^{K} A_{i',j}^{(p^0)} =$$

$$= \prod_{j=1}^{K}\left[\sigma_{ii',j}^{(p^0)} + A_{i,j}^{(p^0)}\cdot A_{i',j}^{(p^0)}\right] - \prod_{j=1}^{K} A_{i,j}^{(p^0)}\cdot A_{i',j}^{(p^0)}$$ □

2 ') corollary: If the random variables are stationary - i.e. if for every period of length p they have the same distribution - we have:

$$\sigma_{ii'}^{(K\cdot p^0)} = \prod_{t=1}^{K}(\sigma_{ii',t} + A_{i,t}\cdot A_{i',t}) - \prod_{t=1}^{K} A_{i,t}\cdot A_{i',t} = \left(\sigma_{ii'} + A_i\cdot A_{i'}\right)^K - \left(A_i\cdot A_{i'}\right)^K \tag{32}$$

3) $$\mathrm{V}_{\mathrm{i}}^{(K\cdot p^0)} = (\mathrm{V}_{\mathrm{i}} + \mathrm{A}_{\mathrm{i}}^2)^{\mathrm{K}} - \mathrm{A}_i^{2K} \tag{33}$$

Proof: This is the result of property 2 ', making i=i ' □

B) Properties for period length *p^0/m*   (*m* positive integer).

In this section it is assumed that the distributions of returns on the various assets are stationary for periods of length p^0/m.

1) $$E(1+\tilde{r}_i^{(p^0/m)})=A_i^{1/m} \tag{34}$$

Proof: On applying property 1 of section A), taking $K=m$, we have:

$$\left[\mathrm{E}(1+\tilde{r}_i^{(p^0/m)})\right]^m=\mathrm{E}(1+\tilde{r}_i)=A_i \Rightarrow \mathrm{E}(1+\tilde{r}_i^{(p^0/m)})=A_i^{1/m} \quad \square$$

2) $$\sigma_{ii'}^{(p^0/m)}=(\sigma_{ii'}^{(p^0)}+A_i\cdot A_{i'})^{1/m}-(A_i A_{i'})^{1/m} \tag{35}$$

Proof: On applying property 2 ' of section A), making $K=m$, and property 1B), we have:

$$\sigma_{ii'}^{(p^0)}=(\sigma_{ii'}^{(p^0/m)}+A_i^{1/m}\cdot A_{i'}^{1/m})^m-(A_i^{1/m}\cdot A_{i'}^{1/m})^m \tag{36}$$

On solving $\sigma_{ii'}^{(p^0/m)}$ we have the equality that we sought to demonstrate. $\square$

3) $$V_i^{(p^0/m)}=(V_i^{(p^0)}+A_i^2)^{1/m}-A_i^{2/m} \tag{37}$$

Proof: This results directly from property 2) above, taking $i'=i$. $\square$

C) Properties for period length *p* (*p*=*t·p*^{0}, t positive real).

In this section we assume that the distributions are stationary. By combining the properties of A) and B) in this appendix, we find that, for any positive real $t=p/p^0$, the following occurs:

1) $$E(1+\tilde{r}_i^{(p)})=A_i^t \tag{38}$$

2) $$\sigma_{ii'}^{(p)}=(\sigma_{ii'}+A_i\cdot A_{i'})^t-(A_i\cdot A_{i'})^t \tag{39}$$

3) $$V_i^{(p)}=(V_i+A_i^2)^t-A_i^{2t} \tag{40}$$

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