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FINITE GROUPS ACTING SYMPLECTICALLY ON $T^2 \times S^2$

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Abstract. For any symplectic form ω on $T^2 \times S^2$ we construct infinitely many nonisomorphic finite groups which admit effective smooth actions on $T^2 \times S^2$ that are trivial in cohomology but which do not admit any effective symplectic action on $(T^2 \times S^2, \omega)$. We also prove that for any ω there is another symplectic form ω' on $T^2 \times S^2$ and a finite group acting symplectically and effectively on $(T^2 \times S^2, \omega')$ which does not admit any effective symplectic action on $(T^2 \times S^2, \omega)$.

A basic ingredient in our arguments is the study of the Jordan property of the symplectomorphism groups of $T^2 \times S^2$. A group G is Jordan if there exists a constant C such that any finite subgroup Γ of G contains an abelian subgroup whose index in Γ is at most C. Csikós, Pyber and Szabó proved recently that the diffeomorphism group of $T^2 \times S^2$ is not Jordan. We prove that, in contrast, for any symplectic form ω on $T^2 \times S^2$ the group of symplectomorphisms $\mathrm{Symp}(T^2 \times S^2, \omega)$ is Jordan. We also give upper and lower bounds for the optimal value of the constant C in Jordan's property for $\mathrm{Symp}(T^2 \times S^2, \omega)$ depending on the cohomology class represented by ω. Our bounds are sharp for a large class of symplectic forms on $T^2 \times S^2$.

1. Introduction

1.1. In this paper we study effective symplectic finite group actions or, equivalently, finite subgroups of symplectomorphism groups. Despite the extraordinary development of symplectic geometry in the last three decades, the interactions between finite transformation groups and symplectic geometry seem to be so far a mostly unexplored terrain (with the remarkable exceptions of [5–7]).

The following notation will be useful in our discussion: for any group $\mathcal{G}$ we denote by $\mathcal{F}(\mathcal{G})$ the set of all isomorphism classes of finite subgroups of $\mathcal{G}$. Given a symplectic manifold (X, ω) we denote by $\mathrm{Diff}_{[\omega]}(X)$ the group of diffeomorphisms of X which preserve the de Rham cohomology class represented by ω. We have inclusions

$$\mathcal{F}(\mathrm{Symp}(X, \omega)) \subseteq \mathcal{F}(\mathrm{Diff}_{[\omega]}(X)) \subseteq \mathcal{F}(\mathrm{Diff}(X))$$

induced by the inclusions of the groups.

A basic question which apparently has not received attention is the following: given a symplectic manifold (X, ω), how big can the difference between $\mathcal{F}(\mathrm{Diff}_{[\omega]}(X))$ and $\mathcal{F}(\mathrm{Symp}(X, \omega))$ be? Similarly, one may want to compare $\mathcal{F}(\mathrm{Symp}(X, \omega))$ and $\mathcal{F}(\mathrm{Symp}(X, \omega'))$ for different symplectic structures ω, ω'.

If Σ is a closed, connected and orientable surface, then for any symplectic form ω on Σ we have $\mathcal{F}(\mathrm{Symp}(\Sigma, \omega)) = \mathcal{F}(\mathrm{Diff}_{[\omega]}(\Sigma)) = \mathcal{F}(\mathrm{Diff}^+(\Sigma))$, where Diff^+ refers to orientation preserving diffeomorphisms. To prove this claim, let us fix some

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symplectic form ω on Σ. Given a finite subgroup $\Gamma \subset \mathrm{Diff}^+(\Sigma)$ one may take, by the averaging trick, a Γ-invariant Riemannian metric g on Σ; the volume form ω_g associated to g and the orientation given by ω is Γ-invariant, and so is any constant multiple of ω_g. For some $\lambda \in \mathbb{R}_{>0}$ we have an equality of cohomology classes $[\lambda\omega_g] = [\omega]$, and by Moser's stability there is a diffeomorphism $\phi \in \mathrm{Diff}(\Sigma)$ such that $\phi^*(\lambda\omega_g) = \omega$ (see e.g. Exercise 3.21 in [20]). Conjugating the action of Γ by ϕ we obtain an action of Γ which fixes ω.

We will show in this paper that in higher dimensions the situation becomes much more interesting. We will study in detail $\mathcal{F}(\mathrm{Symp}(T^2 \times S^2, \omega))$ for every symplectic form ω on $T^2 \times S^2$, and we will prove that for every ω the difference

$$\mathcal{F}(\mathrm{Diff}_{[\omega]}(T^2 \times S^2)) \setminus \mathcal{F}(\mathrm{Symp}(T^2 \times S^2, \omega))$$

contains infinitely many elements. Hence, there is an infinite sequence of pairwise nonisomorphic finite groups $G_1, G_2, \dots$ such that each G_j acts smoothly and effectively on $T^2 \times S^2$ but, in contrast, there is no effective symplectic action of G_j on $(T^2 \times S^2, \omega)$. We will also prove that for any symplectic form ω there exists another symplectic form ω' such that

$$\mathcal{F}(\mathrm{Symp}(T^2 \times S^2, \omega')) \nsubseteq \mathcal{F}(\mathrm{Symp}(T^2 \times S^2, \omega)),$$

i.e., there exists some finite group G which admits an effective symplectic action on $(T^2 \times S^2, \omega')$ but no such action on $(T^2 \times S^2, \omega)$.

A related question which we do not answer in this paper is whether there exists some finite subgroup $G \subset \mathrm{Diff}^+(T^2 \times S^2)$ which does not admit effective symplectic actions on $(T^2 \times S^2, \omega)$ for any choice of ω (this question is closely related to the results in [5–7]).

By a theorem of Lalonde and McDuff (see Theorem 1.6 below) the symplectic forms on $T^2 \times S^2$ are classified up to isomorphism by the ratio λ between the volumes of the T^2 factor and the S^2 factor. The theorems proved in this paper imply that one can break the set $(0, \infty)$ of possible values of λ in infinitely many intervals of the form $(a, b]$ so that if two choices of λ belong to different intervals, then the corresponding symplectomorphism groups have different families of isomorphism classes of finite subgroups. From this perspective, our results are reminiscent of those of Abreu and McDuff [1] on the rational homotopy type of the symplectomorphism groups of $S^2 \times S^2$.

Note that the theorem of Lalonde and McDuff implies that $\mathcal{F}(\mathrm{Symp}(T^2 \times S^2, \omega))$ contains infinitely many elements for every ω. In fact, for any ω and any n there exists a subgroup of $\mathrm{Symp}(T^2 \times S^2, \omega)$ whose cardinal is n (see the remarks after Theorem 1.6). Hence, any argument ruling out the possibility that some finite group acts effectively and symplectically on $\mathrm{Symp}(T^2 \times S^2, \omega)$ must take into account more refined information than the cardinal of the group. The strategy we use in this paper to find obstructions for a finite group Γ to be isomorphic to a subgroup of $\mathrm{Symp}(T^2 \times S^2, \omega)$ is based on the notion of Jordan group, which we next explain.

1.2. **Jordan groups.** A group G is said to be Jordan [27] if there is some constant C such that any finite subgroup Γ of G contains an abelian subgroup whose index in Γ is at most C. The terminology comes from a classic theorem of Camille Jordan, which states that $\mathrm{GL}(n, \mathbb{C})$ is Jordan for every n (see [16] and [3, 9] for modern presentations). A number of papers have appeared in the last few years studying whether the automorphism groups of different geometric structures are Jordan or not: these include diffeomorphism groups, groups of birational transformations of algebraic varieties, or automorphism groups of algebraic varieties (see [28] for a survey).

Around twenty years ago, Étienne Ghys asked whether the diffeomorphism group of any smooth compact manifold is Jordan (see Question 13.1 in [13], and [22]). This question has been answered affirmatively in a number of cases (see the introduction and references in [22]). For example, if X is a smooth compact manifold with nonzero Euler characteristic, then $\mathrm{Diff}(X)$ is Jordan (see [22] for a proof in dimensions 2 and 4 and [23] for a proof in arbitrary dimensions using the classification of finite simple groups). However, Csikós, Pyber and Szabó [8] proved recently that the diffeomorphism group of $T^2 \times S^2$ is not Jordan, thus giving the first example of a compact manifold for which Ghys's question has a negative answer (see [24] for more examples). In contrast, in this paper we prove that for any symplectic form ω on $T^2 \times S^2$ the group of symplectomorphisms $\mathrm{Symp}(T^2 \times S^2, \omega)$ is Jordan. Furthermore, we relate the constant in Jordan property to the cohomology class represented by ω.

Consequently, from the perspective of Jordan property $\mathcal{F}(\mathrm{Symp}(T^2 \times S^2, \omega))$ is *qualitatively* smaller than $\mathcal{F}(\mathrm{Diff}_{[\omega]}(T^2 \times S^2))$ (the group actions defined in [8] are trivial in cohomology, so for any symplectic form ω they give finite subgroups of $\mathrm{Diff}_{[\omega]}(T^2 \times S^2)$).

To state our results with more precision we need to introduce some notation. Fix orientations on T^2 and S^2 and choose elements $t \in T^2$ and $s \in S^2$. Define for any symplectic form ω on $T^2 \times S^2$

$$\alpha(\omega) = \int_{T^2 \times \{s\}} \omega, \qquad \beta(\omega) = \int_{\{t\} \times S^2} \omega.$$

The numbers $\alpha(\omega)$ and $\beta(\omega)$ are independent of s and t by Stokes' theorem. Since ω is a symplectic form, both $\alpha(\omega)$ and $\beta(\omega)$ are nonzero. Define

$$\lambda(\omega) = \max\left\{ \left(2\mathbb{Z} \cap \left(-\infty, \left| \frac{2\alpha(\omega)}{\beta(\omega)} \right| \right) \right) \cup \{1\} \right\}.$$

In words, $\lambda(\omega)$ is the biggest even integer smaller than $|2\alpha(\omega)/\beta(\omega)|$ if $|\alpha(\omega)/\beta(\omega)| > 1$, and $\lambda(\omega) = 1$ otherwise.

Theorem 1.1. *Let ω be a symplectic form on $T^2 \times S^2$. Any finite subgroup $\Gamma \subset \mathrm{Symp}(T^2 \times S^2, \omega)$ contains an abelian subgroup $A \subseteq \Gamma$ such that*

$$[\Gamma : A] \leq \max\{144, 6\lambda(\omega)\}.$$

The next theorem shows that the bound in Theorem 1.1 is optimal if $6\lambda(\omega) \geq 144$.

Theorem 1.2. *Let ω be a symplectic form on $T^2 \times S^2$ such that $\lambda(\omega) \geq 8$. There exists a finite subgroup $\Gamma \subset \mathrm{Symp}(T^2 \times S^2, \omega)$ all of whose abelian subgroups $A \subseteq \Gamma$ satisfy $[\Gamma : A] \geq 6\lambda(\omega)$. Furthermore, the action of Γ on the cohomology of $T^2 \times S^2$ is trivial.*

Combining Theorems 1.1 and 1.2 we immediately obtain:

Corollary 1.3. *For any symplectic form ω on $T^2 \times S^2$ the difference*

$$\mathcal{F}(\mathrm{Diff}_{[\omega]}(T^2 \times S^2)) \setminus \mathcal{F}(\mathrm{Symp}(T^2 \times S^2, \omega))$$

contains infinitely many elements; more precisely, there are infinitely many nonisomorphic finite groups which admit smooth effective actions on $T^2 \times S^2$ that are trivial in cohomology but which are not isomorphic to any subgroup of $\mathrm{Symp}(T^2 \times S^2, \omega)$.

Furthermore, for any symplectic form ω on $T^2 \times S^2$ there exists another symplectic form ω' such that

$$\mathcal{F}(\mathrm{Symp}(T^2 \times S^2, \omega')) \nsubseteq \mathcal{F}(\mathrm{Symp}(T^2 \times S^2, \omega)).$$

If we restrict our attention to finite p-groups for primes $p > 3$, then our techniques give the following sharp result.

Theorem 1.4. *Let $p > 3$ be a prime and let ω be a symplectic form on $T^2 \times S^2$. The group* $\mathrm{Symp}(T^2 \times S^2, \omega)$ *contains a nonabelian finite p-subgroup if and only if $2p \leq \lambda(\omega)$. Furthermore, if $2p \leq \lambda(\omega)$, then there exists a subgroup of* $\mathrm{Symp}(T^2 \times S^2, \omega)$ *which is isomorphic to the Heisenberg p-group*

$$\langle X, Y, Z \mid X^p = Y^p = Z^p = [X, Z] = [Y, Z] = 1,\ [X, Y] = Z \rangle.$$

Combining Theorem 1.1 with the main result in [22] we obtain the following.

Corollary 1.5. *Let (M, ω) be a symplectic 4-manifold diffeomorphic to the total space of an S^2-fibration over a compact Riemann surface or to the product of two compact Riemann surfaces. Then* $\mathrm{Symp}(M, \omega)$ *is Jordan.*

An important ingredient in the proofs of our theorems is a deep result of Lalonde and McDuff [17, Theorem 1.1] which has been mentioned above and which classifies symplectic structures on $T^2 \times S^2$ (in fact the main theorem in [17] applies to more general 4-manifolds, but we will only use the result for $T^2 \times S^2$). Fix symplectic forms ω_{T^2} and ω_{S^2} on T^2 and S^2 respectively, both with total volume 1.

Theorem 1.6 (Lalonde, McDuff)**.** *Let ω be a symplectic form on $T^2 \times S^2$. There exists a diffeomorphism ϕ of $T^2 \times S^2$ such that $\phi^*\omega = \alpha(\omega)\omega_{T^2} + \beta(\omega)\omega_{S^2}$.*

(Pullbacks are implicit in $\alpha(\omega)\omega_{T^2} + \beta(\omega)\omega_{S^2}$ and in similar expressions appearing in the rest of the paper.) An immediate consequence of Theorem 1.6 is that for any symplectic form ω on $T^2 \times S^2$ there exist arbitrarily large finite nonabelian subgroups of $\mathrm{Symp}(T^2 \times S^2, \omega)$: by Moser's stability, ω_{T^2} (resp. ω_{S^2}) is isomorphic to the volume form associated to a flat metric on T^2 (resp. a round metric on S^2); so we may take for example a subgroup of $\mathrm{Symp}(T^2 \times S^2, \omega)$ of the form $G_1 \times G_2$, where $G_1 \subset \mathrm{Symp}(T^2, \omega_{T^2})$ is an arbitrary large finite abelian group and $G_2 \subset \mathrm{Symp}(S^2, \omega_{S^2})$ is isomorphic to any finite nonabelian subgroup of $\mathrm{SO}(3, \mathbb{R})$.

By Theorem 1.6, to prove Theorem 1.1 it suffices to consider product symplectic forms $\alpha\omega_{T^2} + \beta\omega_{S^2}$. A standard technique in 4-dimensional symplectic geometry, based on pseudoholomorphic curves, allows us to prove that any symplectic finite group action on $T^2 \times S^2$ is equivalent to an action which preserves the fibration $T^2 \times S^2 \to T^2$ given by the projection to the first factor (Proposition 2.2). The proof of Theorem 1.1 then follows by combining results on finite group actions on T^2 and S^2 with a result on finite group actions on line bundles over T^2 (Proposition 2.10).

To prove Theorem 1.2 we observe that a slight modification of the construction in [8] can be made symplectic. (In particular, the groups in the statement of Theorem 1.2 can be taken to be finite Heisenberg groups.) This needs to be done carefully to estimate the cohomology class represented by the symplectic form.

Theorem 1.1 is proved in Section 2, Theorem 1.2 is proved in Section 3, Theorem 1.4 is proved in Section 4, and Corollary 1.5 is proved in Section 5.

1.3. **Notation and conventions.** All manifolds and group actions in this paper will be implicitly assumed to be smooth. As usual in the theory of finite transformation groups, in this paper $\mathbb{Z}_n$ denotes $\mathbb{Z}/n\mathbb{Z}$, not to be mistaken, when n is a prime p, with the p-adic integers. If p is a prime we denote by $\mathbb{F}_p$ the field of p elements. When we say that a group G can be generated by d elements we mean that there are elements $g_1, \dots, g_d \in G$, *not necessarily distinct*, which generate G. If a group G acts on a set X we denote the stabiliser of $x \in X$ by G_x, and for any subset $S \subset G$ we denote $X^S = \{x \in X \mid S \subseteq G_x\}$.

2. Proof of Theorem 1.1

2.1. We prove Theorem 1.1 modulo some results whose proofs are postponed to later paragraphs of this section. Denote throughout this section

$$X = T^2 \times S^2$$

and let

$$\Pi : X \to T^2$$

be the projection to the first factor. Take the product orientation on $T^2 \times S^2$, so that $\omega_{T^2} + \omega_{S^2}$ is compatible with the orientation.

Suppose that ω is a symplectic form on X and that $\Gamma \subset \mathrm{Symp}(X, \omega)$ is a finite group. Since both S^2 and T^2 admit orientation reversing diffeomorphisms we may assume, replacing ω by $\theta^*\omega$ for a suitable diffeomorphism θ of X, that

$$\alpha = \alpha(\omega) > 0 \qquad \text{and} \qquad \beta = \beta(\omega) > 0.$$

(We then conjugate the original action of Γ by θ, so that Γ acts by symplectomorphisms with respect to $\theta^*\omega$.) By Theorem 1.6 there is a diffeomorphism ξ of X such that $\xi^*\omega = \alpha\omega_{T^2} + \beta\omega_{S^2}$. Conjugating the action of Γ on X by ξ we may assume that

$$\Gamma \subset \mathrm{Symp}(X, \alpha\omega_{T^2} + \beta\omega_{S^2}).$$

Before continuing the proof, we introduce some useful terminology. Suppose that

$$q : E \to B$$

is a fibration of manifolds (by that we mean a locally trivial fibration in the category of smooth manifolds, so in particular q is a submersion). An action of a group Γ on E is said to be compatible with q if it sends fibers of q to fibers of q. This implies that there is an action of Γ on B such that if $x \in q^{-1}(b)$, then $\gamma \cdot x \in q^{-1}(\gamma \cdot b)$ for any $\gamma \in \Gamma$.

Let $\kappa_{S^2} \in H_2(X; \mathbb{Z})$ be the homology class represented by $\{t\} \times S^2$ for any $t \in T^2$, and let $\kappa_{T^2} \in H_2(X; \mathbb{Z})$ be the homology class represented by $T^2 \times \{s\}$ for any $s \in S^2$ (we use the chosen orientations of S^2 and T^2). By Proposition 2.2, there is an orientation preserving diffeomorphism $\phi : X \to X$ such that the action of Γ on X is compatible with the fibration $\Pi \circ \phi$ and such that $\phi_*\kappa_{S^2} = \kappa_{S^2}$, where ϕ_* is the map induced in homology by ϕ. Furthermore, there is a Γ-invariant almost complex structure J on X which is compatible with ω and with respect to which the fibers of $\Pi \circ \phi$ are J-complex.

Since ϕ is orientation preserving, it preserves the intersection pairing in $H_2(X; \mathbb{R}) \simeq \mathbb{R}^2$, which is hyperbolic. We have $\phi_*\kappa_{S^2} = \kappa_{S^2}$, and κ_{S^2} is isotropic. Hence the action of ϕ in $H_2(X; \mathbb{R})$ can be identified with an element of $\mathrm{O}(1,1)$ fixing a nonzero isotropic vector. The following lemma is an easy exercise in linear algebra.

Lemma 2.1. *If $A \in \mathrm{O}(1,1)$ fixes a nonzero isotropic vector, then A is the identity.*

We deduce that the action of ϕ on $H_2(X;\mathbb{R})$ is trivial. Hence ϕ acts trivially on $H^2(X;\mathbb{R})$, so in particular $\phi^*[\omega] = [\omega]$.

Replacing ω by $\phi^*\omega$ and conjugating both J and the action of Γ by ϕ, we put ourselves in the situation where the action of Γ is compatible with Π and J, and the fibers of Π are J-complex. The new symplectic form ω need no longer be a product symplectic form, but it is compatible with the almost complex structure J and its cohomology class has not changed:

$$[\omega] = \alpha[\omega_{T^2}] + \beta[\omega_{S^2}]. \tag{1}$$

Let $\Gamma_S \subseteq \Gamma$ be the subgroup whose elements act trivially on the base of the fibration Π. By Proposition 2.8 at least one of the following sets of conditions holds true:

(1) $\Gamma_S = \{1\}$.
(2) There exists a nontrivial element $\gamma \in \Gamma_S$ such that Γ preserves X^γ.
(3) There exists a nontrivial element $\gamma \in \Gamma_S$ and a subgroup $\Gamma_0 \subseteq \Gamma$ such that $[\Gamma : \Gamma_0] \leq 12$ and Γ_0 preserves X^γ; furthermore, there is some $h \in \Gamma_0 \cap \Gamma_S$ such that for any $t \in T^2$ the action of h on $\Pi^{-1}(t)$ exchanges the two points of $\Pi^{-1}(t) \cap X^\gamma$.

Suppose that $\Gamma_S = \{1\}$. Then the action of Γ on X comes from an effective action of Γ on T^2. By Lemma 2.5 there is an abelian subgroup $A \subseteq \Gamma$ such that $[\Gamma : A] \leq 6$. So in this case the proof of the theorem is finished.

Suppose for the rest of the proof that we are in the second or third situation given by Proposition 2.8. To facilitate a unified treatment, define $\Gamma_0 := \Gamma$ in case we are in the second situation. Let $\gamma \in \Gamma_S$ be the nontrivial element referred to by the proposition. For any $t \in T^2$ the intersection $X^\gamma \cap \Pi^{-1}(t)$ consists of two points (see the comments before Proposition 2.8). By Lemma 2.7 the restriction of Π to X^γ is a fibration of manifolds. Hence, $F := X^\gamma$ is a 2-dimensional manifold and the restriction

$$p : \Pi|_F : F \to T^2$$

is a degree two covering map. Furthermore, F is a J-complex submanifold of X.

By Proposition 2.9, F is a compact orientable surface which is either connected or has two connected components, and the normal bundle $N \to F$ has a structure of a complex line bundle satisfying $\deg N = 0$ if F is connected and $\deg N|_{F_1} + \deg N|_{F_2} = 0$ if F has two connected components F_1 and F_2. The degrees are defined using an orientation on F with respect to which the projection p is orientation preserving. Furthermore, by Lemma 2.6, the action of Γ_0 on the total space of N is effective.

We treat separately the cases F connected and F disconnected. In both cases we are going to apply Proposition 2.10 to the induced action of Γ_0 to N (or to its restriction $N|_{F_j}$). This can be done because, as the action of Γ preserves J and F is J-complex, the induced action of Γ on F is orientation preserving.

Suppose first of all that F is connected. Then $\deg N = 0$, so by Proposition 2.10 there is an abelian subgroup $A \subseteq \Gamma_0$ satisfying $[\Gamma_0 : A] \leq 6$. Since in any case $[\Gamma : \Gamma_0] \leq 12$, we have $[\Gamma : A] \leq 72$, so we are done.

Consider, for the rest of the proof, the case in which F has two connected components F_1 and F_2.

Suppose that there is some $h \in \Gamma_0 \cap \Gamma_S$ such that for any $t \in T^2$ the action of h on $\Pi^{-1}(t)$ exchanges the two points of $\Pi^{-1}(t) \cap X^\gamma$. Then h exchanges the two connected components F_1 and F_2, and since the action of h is compatible with J, we get an isomorphism of complex line bundles $N|_{F_1} \simeq N|_{F_2}$. In view of the equality $\deg N|_{F_1} + \deg N|_{F_2} = 0$ we obtain $\deg N|_{F_1} = \deg N|_{F_2} = 0$. Let $\Gamma_1 \subseteq \Gamma_0$ be the subgroup preserving the connected components F_1, F_2. By Lemma 2.6 the action of Γ_1 on $N|_{F_1}$ is effective. By Proposition 2.10 there is an abelian subgroup $A \subseteq \Gamma_1$ such that $[\Gamma_1 : A] \leq 6$. Combining all the estimates on indices we get

$$[\Gamma : A] = [\Gamma : \Gamma_0][\Gamma_0 : \Gamma_1][\Gamma_1 : A] \leq 12 \cdot 2 \cdot 6 = 144,$$

so the proof is complete in this case.

Consider, to finish, the case in which no element of Γ_0 exchanges the connected components F_1, F_2. In that case we have $\Gamma_0 = \Gamma$. We are going to bound the absolute value of the degrees of $\deg N|_{F_j}$ in terms of the numbers α, β. Let $[F_j] \in H_2(X; \mathbb{Z})$ be the homology class represented by F_j using the orientation on F_j which is compatible with p. Since p restricts to a diffeomorphism $F_j \to T^2$ for $j = 1, 2$, we have

$$[F_j] = \kappa_{T^2} + \lambda_j \kappa_{S^2}$$

for some integer λ_j. Let $T^{\mathrm{ver}} = \operatorname{Ker} d\Pi \subset TX$ denote the vertical tangent bundle of the fibration Π. We have $T^{\mathrm{ver}} = T^2 \times TS^2$, so $c_1(T^{\mathrm{ver}}) = 2[\omega_{S^2}]$ (the factor of 2 is the Euler characteristic $\chi(S^2)$; recall that ω_{S^2} has total volume 1). Since F intersects each fiber of Π transversely in two points, N can be identified with the restriction of T^{ver} to F, so we have

$$\deg N|_{F_j} = \langle c_1(T^{\mathrm{ver}}), [F_j] \rangle = \langle 2[\omega_{S^2}], \kappa_{T^2} + \lambda_j \kappa_{S^2} \rangle = 2\lambda_j.$$

Hence,

$$\lambda_j = \frac{\deg N|_{F_j}}{2}.$$

In particular, the degree $\deg N|_{F_j}$ is an even integer. Since both F_1 and F_2 are J-complex submanifolds and J is compatible with ω, we have, using (1) and the fact that the total volumes of ω_{T^2} and ω_{S^2} are 1,

$$0 < \langle [\omega], [F_j] \rangle = \langle \alpha[\omega_{T^2}] + \beta[\omega_{S^2}], \kappa_{T^2} + \lambda_j \kappa_{S^2} \rangle = \alpha + \beta\lambda_j = \alpha + \beta \frac{\deg N|_{F_j}}{2}.$$

Consequently

$$\deg N|_{F_j} > -\frac{2\alpha}{\beta}$$

for $j = 1, 2$. Since $\deg N|_{F_1} + \deg N|_{F_2} = 0$, this implies that

$$|\deg N|_{F_j}| < \frac{2\alpha}{\beta},$$

and since $\deg N|_{F_j}$ is an even integer it follows that $|\deg N|_{F_j}| \leq \lambda(\omega)$.

By assumption Γ_0 preserves F_1, so by Lemma 2.6 the action of Γ_0 on $N|_{F_1}$ is effective. By Proposition 2.10 there is an abelian subgroup $A \subseteq \Gamma_0$ such that

$$[\Gamma_0 : A] \leq 6 \max\{1, |\deg N|_{F_1}|\} \leq 6 \cdot \lambda(\omega).$$

Since $\Gamma_0 = \Gamma$, the proof of Theorem 1.1 is complete.

2.2. **Construction of a Γ-invariant S^2-bundle structure.** Recall that $\kappa_{S^2} \in H_2(X;\mathbb{Z})$ denotes the homology class represented by $\{t\} \times S^2$ for any $t \in T^2$.

Proposition 2.2. *Let α, β be positive real numbers and consider the symplectic form $\omega = \alpha\omega_{T^2} + \beta\omega_{S^2}$. Suppose that a finite group Γ acts symplectically on (X, ω). There exists an orientation preserving diffeomorphism $\phi : X \to X$ such that the action of Γ is compatible with the fibration $\Pi \circ \phi$, and a Γ-invariant almost complex structure J on X such that the fibers of $\Pi \circ \phi$ are J-complex. Finally we have $\phi_*\kappa_{S^2} = \kappa_{S^2}$.*

Proof. The proof uses pseudoholomorphic curves and is a slight generalisation of [19, Proposition 4.1] and the note afterwards. We sketch the main ideas for completeness, giving precise references when necessary (the reader not familiar with pseudoholomorphic curve theory may look at the beautiful survey [18] for an introduction targeted to results on 4-dimensional ruled symplectic manifolds). Let $\mathcal{J}$ denote the Fréchet space of $\mathcal{C}^\infty$ almost complex structures on X which are compatible with ω. The idea is that upon fixing any $J \in \mathcal{J}$ the J-holomorphic spheres cohomologous to κ_{S^2} will fit into a fibration.

Fix a complex structure J_{S^2} on S^2 compatible with the orientation. Let, for any $J \in \mathcal{J}$,

$$\mathcal{M}(J) = \{u : S^2 \to X \mid \overline{\partial}_J u = 0,\ u_*[S^2] = \kappa_{S^2}\}.$$

Here $\overline{\partial}_J u = \frac{1}{2}(du \circ J_{S^2} - J \circ du)$ and $[S^2] \in H_2(S^2;\mathbb{Z})$ denotes the fundamental class defined by the orientation. The group $G \simeq \mathrm{PSL}(2,\mathbb{C})$ of complex automorphisms of S^2 acts on $\mathcal{M}(J)$ by precomposition. The compact open topology on the set of maps from S^2 to X induces a topology on $\mathcal{M}(J)$ with respect to which the action of G is continuous and proper. The Gromov compactness theorem implies that $\mathcal{M}(J)/G$ is compact because one cannot write $\kappa_{S^2} = A_1 + A_2$ in such a way that both A_1 and A_2 belong to the image of the Hurewicz homomorphism $\pi_2(X) \to H_2(X;\mathbb{Z})$, and also $\langle \omega, A_j \rangle > 0$ for $j = 1, 2$ (hence, no bubbling can occur).

Since $\langle c_1(TX), \kappa_{S^2} \rangle = 2 > 1$, the main result in [15] (see also [18, §3.3.2]) implies that $\mathcal{M}(J)$ has a natural structure of a smooth oriented manifold of dimension $2(\langle c_1(TX), \kappa_{S^2}\rangle + 1) = 6$, and the action of G on $\mathcal{M}(J)$ is smooth. By the adjunction formula (see [18, Exercise 3.5]) each $u \in \mathcal{M}(J)$ is an embedding. In particular, the action of G on $\mathcal{M}(J)$ is free and $\mathcal{M}(J)/G$ has a natural structure of a smooth oriented compact surface.

The natural evaluation map $\psi_J : \mathcal{M}(J) \times_G S^2 \to X$ that sends the class of $(u, s) \in \mathcal{M}(J) \times S^2$ to $u(s)$ is an orientation preserving diffeomorphism (see [19, Proposition 4.1] and the note afterwards, and also [18, §4.3] — the latter refers only to fibrations over S^2, but everything works identically for fibrations over general Riemann surfaces). The fact that the evaluation map is orientation preserving is not explicitly mentioned either in [19, Proposition 4.1] or in [18, §4.3], but it is an immediate consequence of the fact that the evaluation map has degree 1. Using the multiplicativity of Euler characteristics in fibrations, it follows that $\chi(\mathcal{M}(J)/G) = 0$, so that $\mathcal{M}(J)/G$ is diffeomorphic to T^2. Hence the projection $f : \mathcal{M}(J) \times_G S^2 \to \mathcal{M}(J)/G$ is a fibration over T^2 with fibers diffeomorphic to S^2, and its total space is orientable.

It is well known that over a given surface there exist two oriented S^2-fibrations up to isomorphism, the trivial one and a twisted one (see e.g. [20, Lemma 6.25]), and their total spaces are not diffeomorphic. Therefore $\mathcal{M}(J) \times_G S^2$ must be the trivial fibration over T^2, so there exist

$$\xi\colon \mathcal{M}(J) \times_G S^2 \to X, \qquad \eta\colon \mathcal{M}(J)/G \to T^2$$

such that $\Pi \circ \xi = \eta \circ f$.

We emphasize that the preceding results hold true for *every* $J \in \mathcal{J}$.

Now let $\mathcal{J}_\Gamma \subset \mathcal{J}$ be the subset of Γ-invariant almost complex structures (see [20, Proposition 5.49] and the comments before it). For any $J \in \mathcal{J}_\Gamma$ the diffeomorphism

$$\phi := \xi \circ \psi_J^{-1} : X \to X$$

and the almost complex structure J satisfy the properties of the theorem. Indeed, the fact that $\pi_2(T^2) = 1$ implies that any diffeomorphism of $T^2 \times S^2$ sends κ_{S^2} to $\pm\kappa_{S^2}$. Since Γ preserves $\alpha\omega_{T_2} + \beta\omega_{S^2}$, it follows that Γ preserves κ_{S^2}. Consequently Γ acts on $\mathcal{M}(J)$; this induces an action on $\mathcal{M}(J) \times_G S^2$ preserving the fibers of η and with respect to which ξ is Γ-equivariant. □

2.3. Lemmas on finite groups acting on the sphere and the torus.

Lemma 2.3. *If H is a nontrivial finite cyclic group acting effectively and orientation preservingly on S^2, then $(S^2)^H$ consists of two points.*

Given two groups $H' \subseteq H$ we denote by $\Sigma_H(H')$ the collection of all subgroups of H which are equal to the image of H' by some automorphism of H, i.e.

$$\Sigma_H(H') = \{\phi(H') \mid \phi \in \operatorname{Aut}(H)\}.$$

For example, H' is a characteristic subgroup of H if and only if $\Sigma_H(H') = \{H'\}$.

Lemma 2.4. *Any nontrivial finite group H acting effectively and orientation preservingly on S^2 has a nontrivial cyclic subgroup $H' \subseteq H$ such that at least one of these sets of conditions is satisfied:*

(1) $|\Sigma_H(H')| \leq 1$,
(2) $|\Sigma_H(H')| \leq 12$ *and there is some $h \in H$ in the normalizer of H' which exchanges the two points in $(S^2)^{H'}$.*

Furthermore, if $p > 2$ is a prime and H is a finite p-group acting effectively and orientation preservingly on S^2, then H is cyclic.

Lemma 2.5. *Any finite group H acting effectively and orientation preservingly on T^2 has an abelian subgroup $H' \subseteq H$ such that: $[H : H'] \leq 6$, the action of H' on T^2 is free, H' is isomorphic to a subgroup of $S^1 \times S^1$, and the induced action of H' on $H^1(T^2;\mathbb{Z})$ is trivial. Furhermore, if $p > 3$ is a prime and H is a finite p-group acting effectively and orientation preservingly on T^2, then the subgroup H' can be chosen to be H itself.*

To prove the preceding lemmas, we use the following argument. If a finite group H acts by orientation preserving diffeomorphisms on a surface Σ, then one may take an invariant Riemannian metric on Σ and consider the induced conformal structure. The surface Σ then becomes a Riemann surface, and the action of H on Σ is by Riemann surface automorphisms. At this point we may use results on automorphisms of Riemann surfaces to understand the action of H.

2.3.1. *Proof of Lemmas* 2.3 *and* 2.4. Lemma 2.3 follows from Riemann's uniformization theorem and the identification of the automorphisms of $\mathrm{Aut}(\mathbb{C}P^1)$ with $\mathrm{PSL}(2,\mathbb{C})$. For Lemma 2.4 we use the classification of the finite subgroups of $\mathrm{PSL}(2,\mathbb{C})$. These coincide, up to conjugation, with those of $\mathrm{SO}(3,\mathbb{R})$, because $\mathrm{SO}(3,\mathbb{R}) \subset \mathrm{PSL}(2,\mathbb{C})$ is a maximal compact subgroup. Each finite subgroup of $\mathrm{SO}(3,\mathbb{R})$ is isomorphic to one of these: a cyclic group C_n, a dihedral group D_{2n} ($n \geq 3$), or the group G_{12} (resp. G_{24}, G_{60}) of orientation preserving isometries of a regular tetrahedron (resp. cube, icosahedron), the subindex denoting the number of elements (see e.g. [10, Lect. 1]).

We prove Lemma 2.4 treating each case separately. If $H \simeq C_n$ then we set $H' := H$, so $|\Sigma_H(H')| = 1$. If $H \simeq D_{2n}$, then we define $H' \subset H$ to be the subgroup generated by all the elements of H of order bigger than 2; the subgroup H' is a nontrivial characteristic cyclic subgroup of H, so $|\Sigma_H(H')| = 1$. If $H \simeq G_{12}$, then taking $H' \subset H$ to be any cyclic subgroup of order 2 we have $|\Sigma_H(H')| = 3$; H' can be identified with the orientation preserving isometries of a regular tetrahedron fixing the midpoints of two opposite edges, and there is some orientation preserving isometry h that exchanges the two midpoints. If $H \simeq G_{24}$, then taking $H' \subset H$ to be any cyclic subgroup of order 4 we have $|\Sigma_H(H')| = 3$; H' can be identified with the orientation preserving isometries of a cube fixing the centers of two opposite faces, and there is some orientation preserving isometry h that exchanges the centers of the two faces. Finally, if $H \simeq G_{60}$, then taking $H' \subset H$ to be any cyclic subgroup of order 5 we have $|\Sigma_H(H')| = 12$; H' can be identified with the orientation preserving isometries of a regular icosahedron fixing two opposite vertices, and there is some orientation preserving isometry h that exchanges the two opposite vertices. The statement of p-groups follows from the classification of finite subgroups of $\mathrm{SO}(3,\mathbb{R})$.

2.3.2. *Proof of Lemma* 2.5. We may identify T^2 with an elliptic curve $T = \mathbb{C}/\Lambda$, where $\Lambda \subset \mathbb{R}^2 \simeq \mathbb{C}$ is a full rank lattice, in such a way that H acts on T by complex automorphisms. Let $\mathrm{Aut}_0(T) \subset \mathrm{Aut}(T)$ denote the subgroup of automorphisms fixing the identity element e. We have $\mathrm{Aut}(T) = T \cdot \mathrm{Aut}_0(T)$. It is well known that $\mathrm{Aut}_0(T)$ coincides with the group of discrete symmetries of the lattice Λ which are induced by complex linear automorphisms of $\mathbb{C}$, so $\mathrm{Aut}_0(T)$ is a cyclic group of order 2, 3, 4 or 6. Hence $[\mathrm{Aut}(T) : T] \leq 6$. It follows that $H' := H \cap T$ satisfies $[H : H'] \leq 6$. Since T is isomorphic to $S^1 \times S^1$ as a Lie group, H' is isomorphic to an abelian subgroup of $S^1 \times S^1$. Since the action of T on itself is trivial in $H^1(T;\mathbb{Z})$, so is the action of H'. The statement on p-groups follows from the observation that the only primes dividing an element of $\{2,3,4,6\}$ are 2 and 3.

2.4. **Lemmas on finite group actions and invariant submanifolds.**

Lemma 2.6. *Let E be a compact and connected manifold. Suppose that a finite group H acts effectively on E and that $F \subset E$ is an H-invariant submanifold. Let $N \to F$ be the normal bundle. The action of H on E induces, linearising in the normal directions of F, an effective action of H on N by bundle automorphisms.*

Lemma 2.7. *Let $q : E \to B$ be a fibration of compact manifolds. Suppose that a finite group H acts on E compatibly with q, preserving an almost complex structure J on E and preserving all fibers of q. Then for any subset $U \subseteq H$ the fixed point set E^U is a J-complex submanifold and the restriction of q to E^U is a fibration of manifolds.*

The proofs of these lemmas are standard, so we just sketch the main ideas. Suppose that a finite group H acts on a compact manifold E. Let g be an H-invariant Riemannian metric on E. Let $x \in E$ be any point, and let $H_x \subseteq H$ be its isotropy group. The action of H_x on E induces a linear action on T_xE, and the exponential map $\exp_x^g : T_xE \to E$ is H_x-equivariant. So, near x, E^{H_x} is a submanifold whose tangent space at x can be identified with the linear subspace $(T_xE)^{H_x} \subseteq T_xE$. Repeating the same argument at each point of E^{H_x} it follows that E^{H_x} is a closed submanifold of E. The same argument implies that E^H is a closed submanifold of E.

If the action of H on E is effective and E is connected, then for any nontrivial subgroup $H' \subseteq H$ the fixed point set $E^{H'}$ has dimension smaller than that of E. This implies that for any $x \in E^{H'}$ the linear action of H' on T_xE identifies H' with a subgroup of $\mathrm{Aut}(T_xE)$, and hence is effective. Lemma 2.6 follows immediately from this observation.

The proof of Lemma 2.7 follows easily from the previous arguments. Replacing H by the subgroup generated by U it suffices to consider the case $U = H$. For the last statement, note that by Ehresmann's theorem [11] it suffices to check that the restriction of q to E^H is a submersion.

2.5. **Finite groups of automorphisms of spherical fibrations over T^2.** Let J be an almost complex structure on X with respect to which the fibers of

$$\Pi : X \to T^2$$

are J-complex. The following observation is implicitly used in the next proposition. If a finite group G acts on X preserving the fibers of Π and respecting the almost complex structure J, then for any nontrivial $g \in G$ and any $t \in T^2$ the fixed point set $(\Pi^{-1}(t))^g$ consists of two points. This is a consequence of Lemma 2.3 and the fact that, since the action of G preserves J and the fibers of Π are J-complex, the restriction of the action of G to any fiber of Π is orientation preserving.

Proposition 2.8. *Suppose that a finite group Γ acts effectively on X respecting J, and suppose that the action is compatible with the fibration Π. Let $\Gamma_S \subseteq \Gamma$ be the subgroup whose elements act trivially on the base of the fibration Π. At least one of the following sets of conditions holds true:*

(1) $\Gamma_S = \{1\}$.
(2) *There exists a nontrivial element $\gamma \in \Gamma_S$ such that Γ preserves X^γ.*
(3) *There exists a nontrivial element $\gamma \in \Gamma_S$ and a subgroup $\Gamma_0 \subseteq \Gamma$ such that $[\Gamma : \Gamma_0] \leq 12$ and Γ_0 preserves X^γ; furthermore, there is some $h \in \Gamma_0 \cap \Gamma_S$ such that for any $t \in T^2$ the action of h on $\Pi^{-1}(t)$ exchanges the two points of $\Pi^{-1}(t) \cap X^\gamma$.*

Proof. Let Γ be a finite group acting effectively on X and preserving both J and Π. As mentioned before, since the fibers of Π are J-complex, the induced action of Γ on each fiber of Π is orientation preserving. Let $\Gamma_S \subseteq \Gamma$ be the normal subgroup whose elements preserve the fibers of Π. If $\Gamma_S = \{1\}$, then the proposition holds trivially. So assume for the rest of the proof that $\Gamma_S \neq \{1\}$.

Let $S \subset X$ be any of the fibers of Π. We claim that the action of Γ_S on S is effective. Indeed, if for some element $\eta \in \Gamma_S$ we had $S^\eta = S$, then, since by Lemma 2.7 the projection $\Pi : X^\eta \to T^2$ is a fibration, we would deduce that the fibers of

$\Pi : X^\eta \to T^2$ are 2-dimensional closed submanifolds of the fibers of $\Pi : X \to T^2$, hence $X^\eta = X$, contradicting the assumption that Γ acts effectively on X.

Since the action of Γ_S on S is effective and orientation preserving, we may apply Lemma 2.4 and deduce that there is a nontrivial cyclic subgroup $\Gamma'_S \subseteq \Gamma_S$ for which at least one of the following two sets of conditions holds true:

(1) $|\Sigma_{\Gamma_S}(\Gamma'_S)| = 1$;
(2) $|\Sigma_{\Gamma_S}(\Gamma'_S)| \leq 12$ and there is some $h \in \Gamma_S$ which normalizes Γ'_S and which exchanges the two points in $S^{\Gamma'_S}$.

In the first case we take γ to be a generator of Γ'_S. Then $X^\gamma = X^{\Gamma'_S}$ and, since Γ'_S is a characteristic subgroup of a normal subgroup Γ_S of Γ, Γ'_S is normal in Γ. This implies that $X^{\Gamma'_S}$ (and hence also X^γ) is preserved by Γ.

In the second case we again take the generator $\gamma \in \Gamma'_S$ and we define

$$\Gamma_0 = \{g \in \Gamma \mid g\Gamma'_S g^{-1} = \Gamma'_S\}.$$

Since Γ_S is normal in Γ, Γ_0 satisfies $[\Gamma : \Gamma_0] \leq |\Sigma_{\Gamma_S}(\Gamma'_S)| \leq 12$. Furthermore Γ_0 preserves $X^\gamma = X^{\Gamma'_S}$ because Γ'_S is normal in Γ_0. We claim that for any $t \in T^2$ the action of h on $\Pi^{-1}(t)$ exchanges the two points of $\Pi^{-1}(t) \cap X^\gamma$. Clearly $h \in \Gamma_0$, because by assumption h normalizes Γ'_S, so the action of h preserves X^γ. Since $h \in \Gamma_S$, the action of h also preserves all the fibers of Π. Applying Lemma 2.7 to the action of the subgroup $G \subseteq \Gamma_S$ generated by h and the elements of Γ'_S, it follows that the restriction of Π to X^G is a fibration of manifolds. Since $X^G \cap S = \emptyset$, we deduce that $X^G = \emptyset$, and this means that for any $t \in T^2$ the action of h exchanges the two points in $\Pi^{-1}(t) \cap X^\gamma$. □

Proposition 2.9. *Suppose that $F \subset X$ is a J-complex closed submanifold intersecting transversely each fiber of Π and such that the restriction of Π to F is a 2-sheeted (unramified) covering $F \to T^2$. Let $N \to F$ be the normal bundle of the inclusion $F \hookrightarrow X$, endowed with the structure of a complex line bundle inherited by J. Then either F is connected or it has two connected components F_1, F_2. In the first case, F is diffeomorphic to T^2 and $\deg N = 0$; in the second case, F_j is diffeomorphic to T^2 for $j = 1, 2$ and $\deg N|_{F_1} + \deg N|_{F_2} = 0$.*

The hypotheses of the proposition imply that F is a compact orientable surface, and to give a sense to the degree of N, we orient F in such a way that p is orientation preserving.

Proof. Clearly, either F is connected or has two connected components. A computation with the Euler characteristic shows that in the first case F is a torus. In the second case the restriction of p to each connected component of F is a diffeomorphism, so F is the disjoint union of two tori.

To prove the formulas on the degree of N, recall that on a real vector bundle of rank two a choice of complex structure is equivalent up to homotopy to a choice of orientation. Via this equivalence, the first Chern class is equal to the Euler class. There is a natural (up to homotopy) isomorphism between N and the vertical tangent bundle of Π. Endowing the latter with the orientation induced by J, this isomorphism is orientation preserving. As a fibration of smooth oriented manifolds, we can identify $\Pi : X \to T^2$ with the total space of $P \times_{\mathrm{PSL}(2,\mathbb{C})} \mathbb{C}P^1$, where P is the trivial principal $\mathrm{PSL}(2, \mathbb{C})$-bundle. But P admits a reduction of the structure group to $\mathrm{SO}(3, \mathbb{R})$ with respect to which F is invariant under the antipodal map

$X \to X$, because for any two distinct points $p, q \in \mathbb{C}P^1$ the space

$$\{f : \mathbb{C}P^1 \to S^2 \text{ conformal isomorphism } \mid f(p) \text{ and } f(q) \text{ are antipodal}\}/\operatorname{SO}(3,\mathbb{R})$$

is contractible (S^2 is the round sphere in $\mathbb{R}^3$). This implies the formulas on $\deg N$. □

2.6. **Finite groups of automorphisms of a complex line bundle over T^2.**

Proposition 2.10. *Let $L \to T^2$ be a complex line bundle. Assume that a finite group Γ acts effectively on L by vector bundle automorphisms and that the induced action on T^2 is orientation preserving. Then there is an abelian subgroup $\Gamma_{\mathrm{ab}} \subseteq \Gamma$ satisfying*

$$[\Gamma : \Gamma_{\mathrm{ab}}] \leq 6 \cdot \max\{1, |\deg L|\}.$$

Suppose in addition that Γ acts trivially on $H^1(T^2;\mathbb{Z})$ and that the induced action of Γ on T^2 factors through a free action of an abelian quotient of Γ which can be generated by 2 elements. Then there is an abelian subgroup $\Gamma_{\mathrm{ab}} \subseteq \Gamma$ satisfying

$$[\Gamma : \Gamma_{\mathrm{ab}}] \leq \max\{1, |\deg L|\}.$$

Proof. Let $\Gamma_0 \subseteq \Gamma$ denote the subgroup consisting of those elements which preserve the fibers of L. There is an exact sequence $0 \to \Gamma_0 \to \Gamma \to \Gamma_B \to 0$, where Γ_B acts effectively and orientation preservingly on T^2. By Lemma 2.5 there is an abelian subgroup $\Gamma_B' \subseteq \Gamma_B$ such that $[\Gamma_B : \Gamma_B'] \leq 6$, Γ_B' acts freely on T^2 and trivially on $H^1(T^2;\mathbb{Z})$, and Γ_B' can be identified with a subgroup of $S^1 \times S^1$. The latter implies that Γ_B' can be generated by two elements. So if we replace Γ by $\eta^{-1}(\Gamma_B')$, where $\eta : \Gamma \to \Gamma_B$ is the quotient map, then we are in the situation of the second statement. Consequently, the second statement implies the first.

Let us prove the second statement. Assume that a finite group Γ acts effectively on a line bundle $L \to T^2$ and that the induced action of Γ on T^2 is orientation preserving and factors through a free action of an abelian quotient of Γ which can be generated by 2 elements. We also assume that Γ acts trivially on $H^1(T^2;\mathbb{Z})$. If Γ is abelian, then we set $\Gamma_{\mathrm{ab}} = \Gamma$ and we are done. So we assume for the rest of the proof that Γ is not abelian.

Let, as before, $\Gamma_0 \subseteq \Gamma$ denote the subgroup whose elements act trivially on the base T^2 so that $\Gamma_B = \Gamma/\Gamma_0$ acts freely on T^2 and Γ_B is abelian and can be generated by two elements. Let $\eta : \Gamma \to \Gamma_B$ be the quotient morphism. We have an exact sequence of groups

$$1 \to \Gamma_0 \to \Gamma \xrightarrow{\eta} \Gamma_B \to 1.$$

The subgroup $\Gamma_0 \subset \Gamma$ is central because its elements act by homothecies on the fibers of L and the action of Γ on L is linear. Furthermore, the action of Γ on L defines a monomorphism $\Gamma_0 \hookrightarrow S^1$, since the elements of Γ_0 act on L as multiplication by a complex number of modulus one. This implies that Γ_0 is cyclic.

Define a map

$$Q : \Gamma_B \times \Gamma_B \to \Gamma_0$$

as follows. Given elements $a, b \in \Gamma_B$ take lifts $\alpha, \beta \in \Gamma$ and set

$$Q(a,b) := [\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}.$$

The term $\alpha\beta\alpha^{-1}\beta^{-1}$ belongs to Γ_0 because Γ_B is abelian, so $\eta(\alpha\beta\alpha^{-1}\beta^{-1}) = 1$. It is straightforward to check that $[\alpha, \beta]$ only depends on a and b, so Q is well defined.

Lemma 2.11. *The map Q has the following properties:*

1. *For all $a, b, c \in \Gamma_B$ we have $Q(ab, c) = Q(a, c)Q(b, c)$, $Q(a, bc) = Q(a, b)Q(a, c)$ and $Q(a, a) = Q(1, a) = Q(a, 1) = 1$;*
2. *for any $a, b \in \Gamma_B$ the order of $Q(a, b) \in \Gamma$ divides* $\mathrm{GCD}(\mathrm{ord}_B(a), \mathrm{ord}_B(b))$, *where* ord_B *refers to the order of elements in Γ_B;*
3. *if p, q are different primes, $a \in \Gamma_B$ is a p-element and $b \in \Gamma_B$ is a q-element, then $Q(a, b) = 1$;*
4. *if a, b are both p-elements, then the order of $Q(a, b)$ is at most* $\max\{\mathrm{ord}_B(a), \mathrm{ord}_B(b)\}$.

Proof. Suppose that $\alpha, \beta, \gamma \in \Gamma$ satisfy $\eta(\alpha) = a$, $\eta(\beta) = b$ and $\eta(\gamma) = c$. We have

$$\begin{aligned} Q(ab, c) &= (\alpha\beta)\gamma(\alpha\beta)^{-1}\gamma^{-1} = \alpha\beta\gamma\beta^{-1}\alpha^{-1}\gamma^{-1} = \alpha(\beta\gamma\beta^{-1}\gamma^{-1})\gamma\alpha^{-1}\gamma^{-1} \\ &= \alpha\gamma\alpha^{-1}\gamma^{-1}(\beta\gamma\beta^{-1}\gamma^{-1}) \qquad \text{because } \beta\gamma\beta^{-1}\gamma^{-1} = [\beta, \gamma] \text{ is central} \\ &= Q(a, c)Q(b, c). \end{aligned}$$

The proof of $Q(a, bc) = Q(a, b)Q(a, c)$ is identical, and $Q(a, a) = Q(1, a) = Q(a, 1) = 1$ is immediate, so (1) is proved. Using (1) we get $Q(a, b)^{\mathrm{ord}_B(a)} = Q(a^{\mathrm{ord}_B(a)}, b) = Q(1, b) = 1$ and similarly $Q(a, b)^{\mathrm{ord}_B(b)} = 1$, which gives (2). Finally, (3) and (4) follow from (2). □

Let $\Gamma_c \subseteq \Gamma_0$ be the subgroup generated by the elements $Q(a, b) \in \Gamma_0$ as a, b run through all elements of Γ_B. Clearly $\Gamma_c = [\Gamma, \Gamma]$, so $\Gamma_c \neq \{1\}$ by assumption.

Before concluding the proof of Proposition 2.10 we prove three lemmas.

Let $d_c = |\Gamma_c|$.

Lemma 2.12. $|\Gamma_B|$ *divides the product* $d_c \deg L$.

Proof. Consider the line bundle $\Lambda = L^{\otimes d_c}$. The action of Γ on L induces an action on Λ defined by $\gamma \cdot (v_1 \otimes \cdots \otimes v_{d_c}) = (\gamma \cdot v_1) \otimes \cdots \otimes (\gamma \cdot v_{d_c})$, and the subgroup of Γ defined as $\Gamma_\Lambda^* = \{\gamma \in \Gamma \mid \gamma \text{ acts trivially on } \Lambda\}$ coincides with the set elements of Γ_0 whose order divides d_c. Since Γ_0 is cyclic and $|\Gamma_c| = d_c$, we have $\Gamma_\Lambda^* = \Gamma_c$. The quotient $\Gamma_\Lambda := \Gamma/\Gamma_\Lambda^* = \Gamma/\Gamma_c = \Gamma/[\Gamma, \Gamma]$ acts effectively on Λ, and defining $\Gamma_{\Lambda,0} := \Gamma_0/\Gamma_c$ there is an exact sequence

$$1 \to \Gamma_{\Lambda,0} \to \Gamma_\Lambda \to \Gamma_B \to 1.$$

The action of Γ_Λ on Λ gives a monomorphism $i : \Gamma_{\Lambda,0} \hookrightarrow S^1$. Since Γ_Λ is finite and abelian, there is a homomorphism $\rho : \Gamma_\Lambda \to S^1$ which extends i. Denote by

$$\phi : \Gamma_\Lambda \times \Lambda \to \Lambda$$

the map corresponding to the action of Γ on Λ so that $\phi(\gamma, \lambda) = \gamma \cdot \lambda$. Define a map

$$\psi : \Gamma_\Lambda \times \Lambda \to \Lambda$$

by $\psi(\gamma, \lambda) = \rho(\gamma)^{-1}\phi(\gamma, \lambda)$. The map ψ defines a new action of Γ on Λ, with respect to which $\Gamma_{\Lambda,0}$ acts trivially. Hence this new action factors through an action of Γ_B on Λ lifting the action on T^2. Since the action of Γ_B on T^2 is free, so is the action of Γ_B on Λ. Consequently, the bundle Λ descends to a line bundle on the quotient T^2/Γ_B. Equivalently, there is a line bundle $\Lambda' \to T^2/\Gamma_B$ satisfying $\Lambda \simeq q^*\Lambda'$, where $q : T^2 \to T^2/\Gamma_B$ is the quotient map. Since q has degree $|\Gamma_B|$, it follows that $\deg \Lambda$ is divisible by $|\Gamma_B|$. Finally, $\deg \Lambda = d_c \deg L$, so the proof is complete. □

Lemma 2.13. *We have* $\deg L \neq 0$.

Proof. Suppose that $\deg L = 0$. We are going to prove that the action of Γ on L factors through an abelian group. This is a contradiction because by assumption Γ is not abelian and the action of Γ on L is effective.

Since $\deg L = 0$, there is a nowhere vanishing smooth section $\sigma : T^2 \to L$. For any $\gamma \in \Gamma$ there is a unique smooth map $\phi_\gamma : T^2 \to \mathbb{C}^*$ defined by the property that $\gamma \cdot \sigma(p) = \phi_\gamma(p) \cdot \sigma(\gamma \cdot p)$ for every $p \in T^2$. For any $\gamma, \gamma' \in \Gamma$ the equality $\gamma' \cdot (\gamma \cdot \sigma(p)) = (\gamma'\gamma) \cdot \sigma(p)$ gives the cocycle condition

$$\phi_{\gamma'\gamma}(p) = \phi_{\gamma'}(\gamma \cdot p)\phi_\gamma(p).$$

Denoting by $\rho_\gamma : T^2 \to T^2$ the map $\rho_\gamma(p) = \gamma \cdot p$, we can rewrite the cocycle condition as $\phi_{\gamma'\gamma} = (\phi'_\gamma \circ \rho_\gamma)\phi_\gamma$. Associating to each map $T^2 \to \mathbb{C}^*$ its homotopy class and using the canonical identification $[T^2, \mathbb{C}^*] \simeq H^1(T^2; \mathbb{Z})$, each ϕ_γ corresponds to a cohomology class $\Phi_\gamma \in H^1(T^2; \mathbb{Z})$, and the cocycle condition implies $\Phi_{\gamma'\gamma} = \rho_\gamma^*\Phi_{\gamma'} + \Phi_\gamma$. Since the action of Γ on $H^1(T^2; \mathbb{Z})$ is trivial, we have $\rho_\gamma^*\Phi_{\gamma'} = \Phi_{\gamma'}$, so we have

$$\Phi_{\gamma'\gamma} = \Phi_{\gamma'} + \Phi_\gamma$$

for every γ, γ'. Now, $H^1(T^2; \mathbb{Z})$ is torsion free and Γ is finite, so $\Phi_\gamma = 0$ for every $\gamma \in \Gamma$. So each γ_γ is null homotopic, and this implies that we can choose for every γ a smooth map $\psi_\gamma : T^2 \to \mathbb{C}$ such that $\phi_\gamma = \exp(\psi_\gamma)$.

Now let $\gamma, \gamma' \in \Gamma$ be arbitrary elements and let $\zeta = [\gamma^{-1}, \gamma'^{-1}]$ so that $\gamma\gamma' = \gamma'\gamma\zeta$. We are going to prove that ζ acts trivially on L. First note that since the action of Γ on T^2 factors through an abelian quotient, ζ acts trivially on T^2, so the cocycle condition implies that

$$(\phi_\gamma \circ \rho_{\gamma'})\phi_{\gamma'} = \phi_{\gamma\gamma'} = \phi_{\gamma'\gamma\zeta} = (\phi_{\gamma'\gamma} \circ \rho_\zeta)\phi_\zeta = \phi_{\gamma'\gamma}\phi_\zeta = (\phi_{\gamma'} \circ \rho_\gamma)\phi_\gamma\phi_\zeta.$$

It follows that the smooth map $\chi : T^2 \to \mathbb{C}$ defined by the equality

$$\psi_\gamma \circ \rho_{\gamma'} + \psi_\gamma = \psi_{\gamma'} \circ \rho_\gamma + \psi_\gamma + \chi \tag{2}$$

satisfies $\exp \chi = \phi_\zeta$ (note that χ need not be equal to ψ_ζ; what is true is that the difference $\chi - \psi_\zeta$ is a constant integral multiple of $2\pi\mathbf{i}$). Let δ be the order of ζ in Γ. Since ζ acts trivially on T^2 the cocycle condition for ϕ_ζ implies that $\phi_\zeta^\delta = 1$. Hence the condition $\exp \chi = \phi_\zeta$ implies that $\chi(p) \in \delta^{-1}2\pi\mathbf{i}\mathbb{Z}$ for every $p \in T^2$. Since χ is smooth, we conclude that χ is constant. Fix any point $p \in T^2$. It follows from (2) that

$$\sum_{\eta\in\Gamma} \psi_\gamma(\gamma'\eta \cdot p) + \psi_\gamma(\eta \cdot p) = \sum_{\eta\in\Gamma} \psi_{\gamma'}(\gamma\eta \cdot p) + \psi_\gamma(\eta \cdot p) + \chi(\eta \cdot p).$$

Clearly $\sum_{\eta\in\Gamma} \psi_\gamma(\gamma'\eta \cdot p) = \sum_{\nu\in\Gamma} \psi_\gamma(\nu \cdot p)$ and $\sum_{\eta\in\Gamma} \psi_{\gamma'}(\gamma\eta \cdot p) = \sum_{\nu\in\Gamma} \psi_{\gamma'}(\nu \cdot p)$, so the terms involving ψ's in the equality above cancel each other, and it follows that $\sum_{\eta\in\Gamma} \chi(\eta \cdot p) = 0$. Since χ is constant, this implies that $\chi = 0$, which implies $\phi_\zeta = 1$, so ζ acts trivially on L. □

Lemma 2.14. *We have* $d_c^2 \leq |\Gamma_B|$.

Proof. We first prove that Γ_c can be generated by an element of the form $Q(a, b)$ for some $a, b \in \Gamma_B$. Begin with a generator of Γ_c of the form

$$h = Q(a_1, b_1) \cdots Q(a_r, b_r).$$

Since Γ_B is abelian we can write $a_i = \prod_p a_{ip}$, $b_i = \prod_p b_{ip}$, where each product is over the set of primes and a_{ip}, b_{ip} are p-elements of Γ_B. In the next arguments we repeatedly use Lemma 2.11. We have

$$Q(a_i, b_i) = \prod_{p,q} Q(a_{ip}, b_{iq}) = \prod_p Q(a_{ip}, b_{ip}),$$

and hence, if we denote by $\operatorname{ord}\gamma$ the order of any $\gamma \in \Gamma$,

$$\operatorname{ord} h = \operatorname{ord} \prod_i \prod_p Q(a_{ip}, b_{ip}) = \operatorname{ord} \prod_p \prod_i Q(a_{ip}, b_{ip}) \leq \prod_p \max_i \operatorname{ord} Q(a_{ip}, b_{ip}).$$

Choose for any p an index $i(p)$ such that $Q(a_{i(p)p}, b_{i(p)p}) = \max_i \operatorname{ord} Q(a_{ip}, b_{ip})$. Let $a = \prod_p a_{i(p)p}$ and $b = \prod_p b_{i(p)p}$. We have

$$d_c = \operatorname{ord} h \leq \prod_p \max_i \operatorname{ord} Q(a_{ip}, b_{ip}) = \operatorname{ord} Q(a, b).$$

This implies that $Q(a, b)$ is a generator of Γ_c. We claim that the set

$$S = \{a^i b^j \in \Gamma_B \mid 0 \leq i < d_c,\ 0 \leq j < d_c\}$$

contains d_c^2 elements. Otherwise there would exist $0 \leq k < d_c$ and $0 \leq l < d_c$ such that $a^k b^l = 1$, hence $b^{-l} = a^k$. This would imply $Q(a, b)^k = Q(a^k, b) = Q(b^{-l}, b) = Q(b, b)^{-l} = 1$. Hence $\operatorname{ord} Q(a, b) < d_c$, a contradiction with our previous computation. It follows that Γ_B contains at least d_c^2 elements, so the lemma is proved. □

We are now ready to finish the proof of Proposition 2.10. By Lemma 2.13 we have $\deg L \neq 0$. By Lemma 2.12, the nonvanishing of $\deg L$ implies that $|\Gamma_B| \leq |d_c \deg L|$. Using this inequality and Lemma 2.14 we have

$$|\Gamma_B|^2 \leq d_c^2 (\deg L)^2 \leq |\Gamma_B| (\deg L)^2.$$

Dividing both sides by $|\Gamma_B|$ we get

$$|\Gamma_B| \leq (\deg L)^2.$$

Since Γ_B can be generated by two elements, there are three possibilities: Γ_B is trivial, Γ_B is nontrivial cyclic, or Γ_B is isomorphic to $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$ where n_1, n_2 are natural numbers bigger than one. In each of the three cases there exists a cyclic subgroup $\Gamma_{\text{cyc}} \subseteq \Gamma_B$ such that $[\Gamma_B : \Gamma_{\text{cyc}}] \leq |\Gamma_B|^{1/2} \leq |\deg L|$. Define

$$\Gamma_{\text{ab}} := \eta^{-1}(\Gamma_{\text{cyc}}).$$

By (1) in Lemma 2.11, Γ_{ab} is abelian. Finally, $[\Gamma : \Gamma_{\text{ab}}] \leq |\deg L|$, so we are done. □

3. Proof of Theorem 1.2

The first three subsections of this section are devoted to introducing the preliminaries of the proof of Theorem 1.2, which is given in Subsection 3.5.

3.1. **The group Γ_n.** Let I be an ideal of a commutative ring R with unit. Consider the group

$$T(R,I) = \left\{ A(x,y,z) := \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{Mat}_{3\times 3}(R) \mid x,y,z \in I \right\}$$

with the group structure given by matrix multiplication. For any natural number n, $T(\mathbb{Z}, n\mathbb{Z})$ is a normal subgroup of $T(\mathbb{Z},\mathbb{Z})$, so we may define the quotient group

$$\Gamma_n := T(\mathbb{Z},\mathbb{Z})/T(\mathbb{Z},n\mathbb{Z}).$$

The map

$$\eta : \Gamma_n \to V := \mathbb{Z}_n \times \mathbb{Z}_n$$

which sends the class of $A(x,y,z)$ to $([x],[y])$ is a surjective morphism of groups. The kernel of η can be identified with $\Gamma_n^0 = \{[A(0,0,z)] \mid z \in \mathbb{Z}\}$, which is the center of Γ_n. The map $\psi : \Gamma_n^0 \to \mathbb{Z}_n$ that sends $[A(0,0,z)]$ to $[z]$ is an isomorphism of groups. Hence Γ_n sits in an exact sequence of groups,

$$0 \to \mathbb{Z}_n \to \Gamma_n \xrightarrow{\eta} \mathbb{Z}_n \times \mathbb{Z}_n \to 0.$$

The group Γ_n is sometimes called a finite Heisenberg group. When n is a prime p, Γ_n is isomorphic to the group in the statement of Theorem 1.4.

Lemma 3.1. *For any abelian subgroup $A \subseteq \Gamma_n$ we have $[\Gamma_n : A] \geq n$.*

Proof. This is proved in Section 3 of [29] (note that $\Gamma_n \simeq \mathfrak{G}_K^1$ taking $N = n$ in [29]). ☐

3.2. **The circle bundle $M_n \to T_n^2$.** Fix a natural number n. Let

$$T_n^2 := \mathbb{R}^2/n\mathbb{Z}^2$$

with its natural smooth structure. The group $\mathbb{Z}_n \times \mathbb{Z}_n$ acts on T_n^2 in the obvious way: $([a],[b]) \cdot [(x,y)] = [(a+x, b+y)]$.

Define

$$M_n := T(\mathbb{Z}, n\mathbb{Z})\backslash T(\mathbb{R},\mathbb{R}).$$

Endow $T(\mathbb{R},\mathbb{R})$ with the structure of a differential manifold with respect to which $\mathbb{R}^3 \ni (x,y,z) \mapsto A(x,y,z) \in T(\mathbb{R},\mathbb{R})$ is a diffeomorphism. Since the action of $T(\mathbb{Z},n\mathbb{Z})$ on $T(\mathbb{R},\mathbb{R})$ is smooth and properly discontinuous, M_n has a natural structure of a differential manifold. The group Γ_n acts smoothly and effectively on M_n on the left via product of matrices. On the other hand, the projection $T(\mathbb{R},\mathbb{R}) \ni A(x,y,z) \mapsto (x,y) \in \mathbb{R}^2$ descends to a projection

$$\rho : M_n \to T_n^2$$

which is a principal circle bundle. The structure of a principal bundle is induced by right multiplication on $T(\mathbb{R},\mathbb{R})$ by central elements. More concretely,

$$e^{2\pi \mathbf{i} t} \cdot [A(x,y,z)] = [A(x,y,z)A(0,0,nt)]. \tag{3}$$

The action of Γ_n on M_n is by principal bundle automorphisms, lifting the action of Γ_n on T_n^2 defined through the map $\eta : \Gamma_n \to \mathbb{Z}_n \times \mathbb{Z}_n$ and the action of $\mathbb{Z}_n \times \mathbb{Z}_n$ on T_n^2 defined above.

We identify the tangent space $T_{\mathrm{Id}}T(\mathbb{R},\mathbb{R})$ with the set of 3×3 upper diagonal real matrices with zeroes in the diagonal, namely

$$T_{\mathrm{Id}}(\mathbb{R},\mathbb{R}) = \{\alpha(x,y,z) = A(x,y,z) - A(0,0,0) \mid x,y,z \in \mathbb{R}\}. \tag{4}$$

Let

$$e_x = (1,0,0), \qquad e_y = (\cos 2\pi/6, \sin 2\pi/6, 0), \qquad e_z = (0,0,1)$$

and consider the isomorphism of vector spaces

$$f : T_{\mathrm{Id}}(\mathbb{R},\mathbb{R}) \to \mathbb{R}^3, \qquad f(\alpha(x,y,z)) = xe_x + ye_y + ze_z.$$

Consider the left invariant Riemannian metric $\widetilde{g}$ on $T(\mathbb{R},\mathbb{R})$ whose restriction to $T_{\mathrm{Id}}T(\mathbb{R},\mathbb{R})$ is the pairing

$$\langle \alpha, \alpha' \rangle := \langle f(\alpha), f(\alpha') \rangle_{\mathbb{R}^3},$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^3}$ denotes the Euclidean pairing in $\mathbb{R}^3$. We use this choice of metric because the $\mathbb{Z}$-span of the vectors e_x, e_y is a lattice in the plane $\{(a,b,c) \mid c = 0\}$ with rotational $\mathbb{Z}_6$-symmetry; this will be crucial in Subsection 3.3.

By invariance, the metric $\widetilde{g}$ descends to a metric g_n on M_n. The metric g_n on M_n is also S^1-invariant, since the action of S^1 on M_n is defined via multiplication by central elements of $T(\mathbb{R},\mathbb{R})$, i.e. $A(x,y,z)A(0,0,nt) = A(0,0,nt)A(x,y,z)$.

3.3. **Introducing an extra $\mathbb{Z}_6$-symmetry.** Define the following smooth map:

$$h : T(\mathbb{R},\mathbb{R}) \to T(\mathbb{R},\mathbb{R}), \qquad h(A(x,y,z)) = A\left(-y, x+y, z - xy - \frac{1}{2}y^2\right).$$

A simple but tedious computation proves that $h^6 = \mathrm{Id}$ (so in particular h is a diffeomorphism) and that h is an morphism (hence an isomorphism) of groups:

$$h(A(x,y,z))h(A(x',y',z')) = h(A(x,y,z)A(x',y',z')).$$

The definition of h may seem a bit awkward, especially for the presence of quadratic terms. See the appendix for a geometric interpretation of h in which these quadratic terms come up from an easy computation with iterated integrals.

The identity element $A(0,0,0)$ is fixed by h and the action on $T_{\mathrm{Id}}T(\mathbb{R},\mathbb{R})$ induced by h is the linear map which, in terms of (4), takes the form

$$\alpha(x,y,z) \mapsto \alpha(-y, x+y, z).$$

It follows that h fixes the Riemannian metric $\widetilde{g}$ defined in the previous subsection.

Suppose for the rest of this subsection that n is an *even* natural number. Then h preserves $T(\mathbb{Z}, n\mathbb{Z})$, so h gives rise to a diffeomorphism h_n of M_n which is a g_n-isometry. Furthermore, since h acts trivially on the subgroup $\{A(0,0,z) \mid z \in \mathbb{R}\} \subset T(\mathbb{R},\mathbb{R})$, the action of h_n commutes with the S^1-action on M_n, so h_n acts by principal bundle automorphisms on $M_n \to T^2_n$.

Let $\widehat{\Gamma}_n \subset \mathrm{Diff}(M_n)$ be the subgroup generated by (the action on M_n of the elements of) Γ_n and h_n. Combining our previous observations on the action of Γ_n and h, we deduce that $\widehat{\Gamma}_n$ acts on M_n by S^1-principal bundle automorphisms and by g_n-isometries.

Lemma 3.2. *If $n \geq 8$, then any abelian subgroup $A \subseteq \widehat{\Gamma}_n$ satisfies $[\widehat{\Gamma}_n : A] \geq 6n$.*

Proof. Let $B_n \subset \mathrm{Diff}(T^2_n)$ be the subgroup generated by the diffeomorphisms $\chi, t_a, t_b \in \mathrm{Diff}(T^2_n)$ defined as

$$\chi([x],[y]) = ([-y],[x+y]), \quad t_a([x],[y]) = ([x+1],[y]), \quad t_b([x],[y]) = ([x],[y+1]).$$

Since $\chi^{-1}t_a\chi = t_a t_b^{-1}$ and $\chi^{-1}t_b\chi = t_a$ (we omit the symbol $\circ$ in the compositions), the subgroup $\langle t_a, t_b\rangle$, which is isomorphic to $\mathbb{Z}_n \times \mathbb{Z}_n$, is a normal subgroup of B_n. Hence, there is an exact sequence

$$0 \to \mathbb{Z}_n \times \mathbb{Z}_n \to B_n \xrightarrow{\zeta} \mathbb{Z}_6 \to 0,$$

where $\zeta(\chi) \in \mathbb{Z}_6$ is a generator and the element $(u, v) \in \mathbb{Z}_n \times \mathbb{Z}_n$ is mapped to $t_a^u t_b^v$. Furthermore, the action of $\mathbb{Z}_6$ on $\mathbb{Z}_n \times \mathbb{Z}_n$ given by conjugation in B_n is $\chi \cdot (u, v) = (u + v, -u)$.

Suppose that $A \subseteq B_n$ is an abelian subgroup and that $\zeta(A) \neq 0$. There are three possibilities for the image $\zeta(A)$. Suppose first that $\zeta(A) = \mathbb{Z}_6$. Then for any $(u, v) \in A \cap \operatorname{Ker}\zeta \subseteq \mathbb{Z}_n \times \mathbb{Z}_n$ we have $\chi \cdot (u, v) = (u+v, -u) = (u, v)$, which implies $(u, v) = (0, 0)$, i.e.,

$$\zeta(A) = \langle\chi\rangle \quad \Longrightarrow \quad A \cap \operatorname{Ker}\zeta = 0.$$

Next suppose that $\zeta(A) = \langle\chi^2\rangle \subset \mathbb{Z}_6$. Then for any $(u, v) \in A \cap \operatorname{Ker}\zeta \subseteq \mathbb{Z}_n \times \mathbb{Z}_n$ we have $\chi^2 \cdot (u, v) = (v, -u - v) = (u, v)$, which implies $(u, v) = (0, 0)$ if n is not divisible by 3 and $(u, v) \in \{(0, 0), (n/3, n/3)\}$ if n is divisible by 3. In any case,

$$\zeta(A) = \langle\chi^2\rangle \quad \Longrightarrow \quad A \cap \operatorname{Ker}\zeta \subseteq K_2 := \{(0, 0), (n/3, n/3)\},$$

where we agree that the second term only appears if n is divisible by 3. Finally, suppose that $\zeta(A) = \langle\chi^3\rangle$. Then for any $(u, v) \in A \cap \operatorname{Ker}\zeta \subseteq \mathbb{Z}_n \times \mathbb{Z}_n$ we have $\chi^3 \cdot (u, v) = (-u, -v) = (u, v)$, which implies $(u, v) \in \{0, n/2\} \times \{0, n/2\}$; hence

$$\zeta(A) = \langle\chi^3\rangle \quad \Longrightarrow \quad A \cap \operatorname{Ker}\zeta \subseteq K_3 := \{0, n/2\} \times \{0, n/2\}.$$

It is immediate from the definitions that there is a morphism of groups $\widehat{\eta} : \widehat{\Gamma} \to B_n$ with the property that each $\phi \in \widehat{\Gamma}_n$, seen as a diffeomorphism of M_n, lifts $\widehat{\eta}(\phi)$. Setting $\theta = \zeta \circ \widehat{\eta}$ we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_n \arrow[r] \arrow[d, "\eta"] & \widehat{\Gamma}_n \arrow[r, "\theta"] \arrow[d, "\widehat{\eta}"] & \mathbb{Z}_6 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{Z}_n^2 \arrow[r] & B_n \arrow[r, "\zeta"] & \mathbb{Z}_6 \arrow[r] & 0.
\end{tikzcd}$$

Suppose that $\widehat{A} \subseteq \widehat{\Gamma}_n$ is abelian. Then $\widehat{\eta}(\widehat{A}) \subseteq B_n$ is also abelian. We are going to bound $[\widehat{\Gamma}_n : \widehat{A}]$, treating different cases separately. If $\zeta(\widehat{\eta}(\widehat{A})) = 0$, then $\widehat{A} \subseteq \operatorname{Ker}\theta$, so $\widehat{A}$ can be identified with an abelian subgroup of Γ_n. By Lemma 3.1 we have

$$[\widehat{\Gamma}_n : \widehat{A}] = 6[\Gamma_n : \widehat{A}] \geq 6n.$$

If $\zeta(\widehat{\eta}(\widehat{A})) = \mathbb{Z}_6$, then, by our previous comment, $\widehat{\eta}(\widehat{A}) \cap \operatorname{Ker}\zeta = 0$, which implies that $\widehat{A} \cap \operatorname{Ker}\theta \subseteq \operatorname{Ker}\eta$. This implies that $|\widehat{A}| \leq 6|\operatorname{Ker}\eta| = 6n$, so

$$[\widehat{\Gamma}_n : \widehat{A}] \geq \frac{6n^3}{6n} = n^2 \geq 6n.$$

If $\zeta(\widehat{\eta}(\widehat{A})) = \langle\chi^2\rangle$, then $\widehat{A} \cap \operatorname{Ker}\theta \subseteq \eta^{-1}(K_2)$, so $|\widehat{A}| \leq |\langle\chi^2\rangle| \cdot |\eta^{-1}(K_2)| = 6 \cdot |\operatorname{Ker}\eta| = 6n$, which gives

$$[\widehat{\Gamma}_n : \widehat{A}] \geq \frac{6n^3}{6n} = n^2 \geq 6n.$$

If $\zeta(\widehat{\eta}(\widehat{A})) = \langle\chi^3\rangle$, then $\widehat{A} \cap \operatorname{Ker}\theta \subseteq \eta^{-1}(K_3)$, so $|\widehat{A}| \leq |\langle\chi^3\rangle| \cdot |\eta^{-1}(K_3)| = 8 \cdot |\operatorname{Ker}\eta| = 8n$, which gives

$$[\widehat{\Gamma}_n : \widehat{A}] \geq \frac{6n^3}{8n} = \frac{6n^2}{8} \geq 6n,$$

and so the proof of the lemma is complete. □

3.4. **A $\widehat{\Gamma}_n$-invariant symplectic form on $M_n \times_{S^1} S^2$.** Suppose, as in the previous subsection, that n is an even natural number.

Let us identify T^2 with T^2_1 and consider the diffeomorphism

$$\phi : T^2 \to T^2_n, \qquad \phi(([x],[y])) = ([nx],[ny]).$$

Let $(x, y) \in \mathbb{R}^2$ denote the canonical coordinates. These coordinates define translation invariant vector fields ∂_x, ∂_y on $\mathbb{R}^2$, which induce by projection vector fields on each T^2_n; we denote these vector fields on T^2_n with the same symbols ∂_x, ∂_y. We denote the dual forms on T^2_n by dx, dy.

Lemma 3.3. *There exists a $\widehat{\Gamma}_n$-invariant connection A on $M_n \to T^2_n$ whose curvature F_A satisfies*

$$\phi^* F_A = 2\pi \mathbf{i} n \, dx \wedge dy.$$

Proof. Define a connection A on $M_n \to T^2_n$ by the prescription that its horizontal distribution is g_n-orthogonal to the tangent spaces of the S^1-orbits. Since the action of $\widehat{\Gamma}_n$ on M_n is by principal bundle automorphisms and g_n-isometries, A is $\widehat{\Gamma}_n$-invariant. To compute the curvature of A we work on $T(\mathbb{R}, \mathbb{R})$. Consider the matrices

$$m_x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad m_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and let X, Y be the left invariant vector fields on $T(\mathbb{R}, \mathbb{R})$ whose restrictions to $T_{\mathrm{Id}} T(\mathbb{R}, \mathbb{R})$ are given by m_x, m_y respectively. The vector fields X, Y descend to S^1-invariant horizontal vector fields X', Y' on M_n whose projections to T^2_n satisfy $D\rho(X') = \partial_x$ and $D\rho(Y') = \partial_y$. On the other hand, $[X, Y]$ is the left invariant vector field whose restriction to $T_{\mathrm{Id}} T(\mathbb{R}, \mathbb{R})$ is equal to $[m_x, m_y]$. The latter can easily be identified with the restriction of $2\pi n^{-1} \mathcal{X}$ to $T_{\mathrm{Id}} T(\mathbb{R}, \mathbb{R})$, where $\mathcal{X}$ is the vector field on $T(\mathbb{R}, \mathbb{R})$ induced by the infinitesimal action of $\mathbf{i} \in \operatorname{Lie} S^1$ that results from deriving the action (3). It follows that $F_A = 2\pi \mathbf{i} n^{-1} \, dx \wedge dy$. Since $\phi^* dx = n \, dx$ and $\phi^* dy = n \, dy$, the result follows. □

Incidentally, note that Lemma 3.3 implies by Chern–Weil theory that $\deg M_n = n$, which, combined with Lemmas 3.1 and 3.2, implies that the first (resp. second) statement of Proposition 2.8 is sharp for line bundles L of even degree satisfying $|\deg L| \geq 8$ (resp. for any L).

Define

$$P_n = \phi^* M_n, \qquad A_n = \phi^* A,$$

so that P_n is a principal circle bundle over T^2 carrying an effective action of Γ_n and A_n is a Γ_n invariant connection on P_n whose curvature is equal to $F_{A_n} = 2\pi \mathbf{i} n \, dx \wedge dy$.

Let us identify S^2 with the unit sphere centered at 0 in $\mathbb{R}^3$, and consider the action of S^1 on S^2 given by rotations around the z-axis:

$$e^{2\pi \mathbf{i} t} \cdot (x, y, z) = (x \cos t - y \sin t, x \sin t + y \cos t, z). \tag{5}$$

Let ω_{FS} be the volume form associated to restriction of the Euclidean metric on S^2 and the orientation specified by the ordered basis (∂_x, ∂_y) of $T_{(0,0,1)}S^2$. We may look at ω_{FS} as a symplectic form on S^2, with respect to which the action of S^1 given by rotation is Hamiltonian. The moment map $\mu_{\mathrm{FS}} : S^2 \to \mathbf{i}\mathbb{R}$ is

$$\mu_{\mathrm{FS}}(x, y, z) = \mathbf{i}z,$$

so $\mu_{\mathrm{FS}}(S^2) = \mathbf{i}[-1, 1]$. We have

$$\int_{S^2} \omega_{\mathrm{FS}} = 4\pi. \tag{6}$$

Consider the associated bundle $P_n \times_{S^1} S^2$ and the projection

$$\Pi_n : P_n \times_{S^1} S^2 \to T^2.$$

We are next going to construct a Γ_n-invariant symplectic form on $P_n \times_{S^1} S^2$ using the minimal coupling construction (see e.g. [20, §6.1]). In order to keep track of the cohomology class represented by the symplectic form we will give the construction in some detail.

Let $D\Pi_n$ denote the vertical tangent bundle of the fibration Π_n. Each fiber of Π_n can be identified, in a way unique up to the action of S^1, with S^2. Since ω_{FS} is S^1-invariant it defines, via these identifications, a section ω_0^{ver} of $\Lambda^2(\operatorname{Ker} D\Pi_n)^*$. On its turn, the connection A_n induces a left inverse of the inclusion $\operatorname{Ker} D\Pi_n \hookrightarrow T(P_n \times_{S^1} S^2)$, which when combined with ω_0^{ver} leads to a 2-form

$$\widetilde{\omega}_0 \in \Omega^2(P_n \times_{S^1} S^2)$$

whose restriction to each fiber coincides with ω_{FS}. The form $\widetilde{\omega}_0$ is not closed (unless A_n is flat), but the following 2-form is closed (see e.g. [2, Theorem 7.34]):

$$\omega_0 = \widetilde{\omega}_0 + \mu_{\mathrm{FS}} \cdot \Pi_n^* F_{A_n}. \tag{7}$$

Lemma 3.4. *For any real number $\delta > 2\pi n$*

$$\omega_\delta = \omega_0 + \delta \Pi_n^*(dx \wedge dy)$$

is a $\widehat{\Gamma}_n$-invariant symplectic form on $P_n \times_{S^1} S^2$.

Proof. It is clear that ω_δ is closed (this holds regardless of the value of δ), so we prove that ω_δ is nondegenerate if $\delta > 2\pi n$. The vertical and horizontal distributions in $T(P_n \times_{S^1} S^2)$ are ω_δ-orthogonal, so it suffices to prove that the restrictions of ω_δ to both distributions are nondegenerate. The restriction to the vertical distribution coincides with ω_0^{ver}, which is nondegenerate because it coincides on each fiber with ω_{FS}. To prove that the restriction to the horizontal distribution is nondegenerate if $\delta > 2\pi n$, use $F_{A_n} = 2\pi \mathbf{i} n \, dx \wedge dy$ and $|\mu(u)| \leq 1$ for every $u \in S^2$. Finally, to prove that ω_δ is $\widehat{\Gamma}_n$-invariant observe that ω_0 is $\widehat{\Gamma}_n$-invariant (this is a consequence of the invariance of the connection A_n) and that $dx \wedge dy$ is invariant under the action of B_n (see the proof of Lemma 3.2) on T^2 given by conjugating the action on T_n^2 via the diffeomorphism $\phi : T^2 \to T_n^2$. □

3.5. **Completion of the proof.** The action (5) factors through a morphism

$$S^1 \to \mathrm{SO}(3, \mathbb{R})$$

(via the standard action of $\mathrm{SO}(3, \mathbb{R})$ on S^2) which represents an element of order 2 in $\pi_1(\mathrm{SO}(3, \mathbb{R})) \simeq \mathbb{Z}_2$. Hence for every even natural number n there is a diffeomorphism $\psi_n : T^2 \times S^2 \to P_n \times_{S^1} S^2$ satisfying $\Pi_n \circ \psi_n = \Pi$ (recall that $\Pi : T^2 \times S^2 \to T^2$ is the projection).

Lemma 3.5. *For any $n \in 2\mathbb{N}$ we have $[\psi_n^* \omega_\delta] = \delta[\omega_{T^2}] + 4\pi[\omega_{S^2}]$.*

Proof. It suffices to prove that $[\psi_n^*\omega_0] = 4\pi[\omega_{S^2}]$. Let $\sigma_0, \sigma_1 \subset P_n \times_{S^1} S^2$ be the submanifolds corresponding to the fixed points $(0, 0, 1)$, $(0, 0, -1)$ respectively of the action of S^1 on S^2, i.e.

$$\sigma_0 = P_n \times_{S^1} \{(0, 0, 1)\}, \qquad \sigma_1 = P_n \times_{S^1} \{(0, 0, -1)\},$$

and let $S_j = \psi_n^{-1}(\sigma_j)$. Orient σ_j and S_j so that their projections to T^2, which are diffeomorphisms, are orientation preserving. Since S_1, S_2 are disjoint, a simple computation using the intersection product on $H_*(T^2 \times S^2)$ proves that the homology classes represented by S_j are

$$[S_0] = [T^2] + k[S^2], \qquad [S_1] = [T^2] - k[S^2]$$

for some integer k. It follows that for any $s \in S^2$

$$\int_{T^2\times\{s\}} \psi_n^*\omega_0 = \frac{1}{2}\left(\int_{S_0} \psi_n^*\omega_0 + \int_{S_1} \psi_n^*\omega_0\right) = \frac{1}{2}\left(\int_{\sigma_0} \omega_0 + \int_{\sigma_1} \omega_0\right).$$

Since $\mu_{\mathrm{FS}}([1:0]) + \mu_{\mathrm{FS}}([0:1]) = 0$, it follows from the definition of ω_0 (7) that

$$\frac{1}{2}\left(\int_{\sigma_0} \omega_0 + \int_{\sigma_1} \omega_0\right) = 0.$$

Consequently $[\psi_n^*\omega_0] = \beta[\omega_{S^2}]$ for some real number β. But β coincides with the total volume of ω_{FS}, which by (6) is equal to 4π. □

We are now ready to prove Theorem 1.2. Let ω be an arbitrary symplectic form on $T^2 \times S^2$. Let $\alpha = \alpha(\omega)$ and $\beta = \beta(\omega)$, let $n = \lambda(\omega)$ and let $\xi = \alpha/\beta$. Suppose that n is an even natural number satisfying $n \geq 8$. It follows, combining Lemma 3.4 and Lemma 3.5, that there exists a $\widehat{\Gamma}_n$-invariant symplectic form $\omega_{4\pi\xi}$ on $P_n \times_{S^1} S^2$ satisfying

$$\frac{\beta}{4\pi}[\psi_n^*\omega_{4\pi\xi}] = [\omega].$$

By Lalonde and McDuff's Theorem 1.6 there is a diffeomorphism ϕ of $T^2 \times S^2$ such that

$$\frac{\beta}{4\pi}\psi_n^*\omega_{4\pi\xi} = \phi^*\omega.$$

Since two symplectic forms that differ by multiplication by a constant have identical symplectomorphism groups, it follows that there is a subgroup of $\mathrm{Symp}(T^2 \times S^2, \omega)$ which is isomorphic to $\widehat{\Gamma}_n$. Applying Lemma 3.2, the proof of the first statement of Theorem 1.2 is complete. It only remains to prove that the action of Γ on the cohomology of $T^2 \times S^2$ is trivial. This follows from the next lemma.

Lemma 3.6. *For any symplectic form ω on $T^2 \times S^2$ and any symplectomorphism ϕ of ω, the action of ϕ on $H^*(T^2 \times S^2; \mathbb{Z})$ is trivial.*

Proof. It suffices to prove that the action of ϕ on $H_2(T^2 \times S^2; \mathbb{R})$ is trivial. Let $\kappa_{S^2}, \kappa_{T^2} \in H_2(T^2 \times S^2; \mathbb{Z})$ be as before the classes represented by $\{t\} \times S^2$ and $T^2 \times \{s\}$ respectively, for any $t \in T^2$ and $s \in S^2$. Since $\pi_2(T^2)$ is trivial, $(\Pi \circ \phi)_*\kappa_{S^2} = 0$, so $\phi_*\kappa_{S^2} = \lambda\kappa_{S^2}$ for some $\lambda \in \mathbb{Z}$. Since $[\omega] = \alpha(\omega)[\omega_{T^2}] + \beta(\omega)[\omega_{S^2}]$ with $\beta(\omega) \neq 0$ and ω_{T^2} pairs trivially with κ_{S^2}, we have $\lambda = 1$. The proof is finished using Lemma 2.1 and the arguments preceding it. □

4. Proof of Theorem 1.4

We first prove that if ω is a symplectic form on $T^2\times S^2$, $p>3$ is a prime such that $2p>\lambda(\omega)$, and $\Gamma\subset \mathrm{Symp}(T^2\times S^2,\omega)$ is a finite p-group, then Γ is abelian. This follows from the same arguments as in the proof of Theorem 1.1. The difference with the general situation considered in Theorem 1.1 is that when applying Lemmas 2.4 and 2.5 to a p-group H with $p>3$, the subgroup H' whose existence is claimed turns out to be H itself in both lemmas. When we apply Proposition 2.8 during the proof of Theorem 1.1 there are three possible outcomes, which in the context of a finite p-group Γ (with $p>3$ and $2p>\lambda(\omega)$) simplify as follows. If $\Gamma_S=\{1\}$, then the abelian subgroup $A\subseteq\Gamma$ which is constructed turns out to be Γ itself, so Γ is abelian. In the two other cases, the group Γ_0 coincides with Γ, and similarly Γ_1 is also equal to Γ. The proof that Γ is abelian is completed by observing that, in Proposition 2.10, if Γ is a p-group ($p>3$), $\deg L$ is even, and $2p>\deg L$, then $\Gamma_{\mathrm{ab}}=\Gamma$. To justify this, first note that it suffices to consider the second statement (again because in Lemma 2.5 for a p-group H, $p>3$, the subgroup H' coincides with H). The fact that $\deg L$ is even and $2p>\deg L$ implies that p does not divide $\deg L$. This implies, using Lemma 2.12, that $|\Gamma_B|$ divides $d_c=|[\Gamma,\Gamma]|$. In particular $|\Gamma_B|\leq|[\Gamma,\Gamma]|$. By (2) in Lemma 2.11 this implies that Γ_B is cyclic, because the exponent of $[\Gamma,\Gamma]$ is not greater than the exponent of Γ_B, and $[\Gamma,\Gamma]$ is cyclic. Then (1) in Lemma 2.11 tells us that Γ is abelian.

Now suppose that $p>3$ is prime and that $2p\leq\lambda(\omega)$. By the arguments in the proof of Theorem 1.2 (see Subsection 3.5) there is a subgroup of $\mathrm{Symp}(T^2\times S^2,\omega)$ isomorphic to Γ_{2p}. The group Γ_p is isomorphic to

$$\langle X,Y,Z\mid X^p=Y^p=Z^p=[X,Z]=[Y,Z]=1,\ [X,Y]=Z\rangle,$$

so it suffices to prove that Γ_{2p} has a subgroup isomorphic to Γ_p. The map

$$d:T(\mathbb{Z},\mathbb{Z})\to T(\mathbb{Z},\mathbb{Z}),\quad d(A(x,y,z))=A(2x,2y,4z)$$

is an injective morphism of groups and $d^{-1}(T(\mathbb{Z},2p\mathbb{Z}))=T(\mathbb{Z},p\mathbb{Z})$. Hence, d gives an injection

$$\Gamma_p=T(\mathbb{Z},\mathbb{Z})/T(\mathbb{Z},p\mathbb{Z})\hookrightarrow T(\mathbb{Z},\mathbb{Z})/T(\mathbb{Z},2p\mathbb{Z})=\Gamma_{2p}$$

(in fact, computing cardinals it is clear that we can identify the image of this map with a p-Sylow subgroup of Γ_{2p}).

5. Proof of Corollary 1.5

Let (M,ω) be a symplectic manifold diffeomorphic to an S^2-fibration over a compact Riemann surface Σ. If $\chi(\Sigma)\neq 0$, then $\chi(M)\neq 0$, so by the main result in [22] the diffeomorphism group of M is Jordan. A fortiori, so is $\mathrm{Symp}(M,\omega)$. The only case not covered by [22] is precisely when $\Sigma=T^2$. In this case, M is either the trivial fibration $T^2\times S^2$ or a twisted fibration. In the first case Theorem 1.1 applies. In the second case, we can consider a degree 2 unramified covering $\mu:T^2\to T^2$ and take the pullback $\mu^*M\to T^2$ of the fibration $M\to T^2$. There is a degree 2 unramified covering $\nu:\mu^*M\to M$. Then $\mu^*M\simeq T^2\times S^2$, so $\mathrm{Symp}(\mu^*M,\nu^*\omega)$ is Jordan by Theorem 1.1, and the arguments in [21, §2.3] imply, using ν, that $\mathrm{Symp}(M,\omega)$ is also Jordan.

Suppose now that (M,ω) is a symplectic manifold with M diffeomorphic to the product of two Riemann surfaces of genuses g and h. If $\chi(M)\neq 0$, then [22] implies as before that $\mathrm{Symp}(M,\omega)$ is Jordan. Now suppose that $\chi(M)=0$. Then

$1 \in \{g, h\}$, so suppose that $g = 1$. If $h = 0$, then $M \simeq T^2 \times S^2$, so by Theorem 1.1 $\mathrm{Symp}(M, \omega)$ is Jordan. Finally, if $h \geq 1$, then one may find cohomology classes $\alpha_1, \ldots, \alpha_4 \in H^1(M; \mathbb{Z})$ such that $\alpha_1 \cup \cdots \cup \alpha_4 \neq 0$, so by [21] the diffeomorphism group of M is Jordan. Consequently, $\mathrm{Symp}(M, \omega)$ is Jordan in this case as well.

Appendix A. Automorphisms of Heisenberg groups and geometry

Here we interpret geometrically the automorphism $h \in \mathrm{Aut}(T(\mathbb{R}, \mathbb{R}))$ of §3.3 in terms of iterated integrals and the monodromy of certain fibration over the moduli space of elliptic curves. (See [25] for a group theoretical approach to $\mathrm{Aut}(T(\mathbb{Z}, \mathbb{Z}))$ and [26] for an approach based on 2-dimensional local fields.)

It is easy to prove that any automorphism of $\Gamma_{\mathbb{R}} = T(\mathbb{R}, \mathbb{R})$ lifting the automorphism of $\Gamma_{\mathbb{R}}/[\Gamma_{\mathbb{R}}, \Gamma_{\mathbb{R}}] \simeq \mathbb{R}^2$ given by $(x, y) \mapsto (-y, x + y)$ has to coincide with h up to adding linear combinations $\alpha x + \beta y$ to the third term, and all such lifts have order 6. Varying α and β corresponds to the action on $\mathrm{Aut}(\Gamma_{\mathbb{R}})$ of the inner automorphisms of $\Gamma_{\mathbb{R}}$, which act trivially on $\Gamma_{\mathbb{R}}/[\Gamma_{\mathbb{R}}, \Gamma_{\mathbb{R}}]$. In particular, the automorphism h' of $T(\mathbb{R}, \mathbb{R})$ defined as

$$h'(A(x, y, z)) = A\left(-y, x + y, z - xy - \frac{y^2 - y}{2}\right)$$

represents the same class in $\mathrm{Out}(\Gamma_{\mathbb{R}})$ as h. Clearly h' preserves[1] $\Gamma = T(\mathbb{Z}, \mathbb{Z})$, and since $\Gamma_{\mathbb{R}} = \Gamma \otimes \mathbb{R}$ as nilpotent groups, one can recover h' from its restriction to Γ. The latter belongs to $\mathrm{Aut}^+(\Gamma)$, the group of automorphisms of Γ acting trivially on $Z(\Gamma) \simeq \mathbb{Z}$.

Let $\mathrm{Int}^+(\Gamma) \subset \mathrm{Aut}^+(\Gamma)$ be the inner automorphisms. The outer automorphism group $\mathrm{Out}^+(\Gamma) = \mathrm{Aut}^+(\Gamma)/\mathrm{Int}^+(\Gamma)$ maps to $\mathrm{SL}(2, \mathbb{Z})$ via its action on $\Gamma/[\Gamma, \Gamma]$, and one can prove easily that this morphism $\eta : \mathrm{Out}^+(\Gamma) \to \mathrm{SL}(2, \mathbb{Z})$ is injective. To prove that η is also surjective, consider a principal S^1-bundle $p : M \to T^2$ of degree 1. For any $x_0 \in M$ we have $\pi_1(M, x_0) \simeq \Gamma$. Let $F \in \mathrm{SL}(2, \mathbb{Z})$ and let $\phi : T^2 \to T^2$ be a diffeomorphism whose mapping class coincides with F. Since $\det F = 1$, ϕ acts trivially on $H^2(T^2)$ and hence admits a lift $\psi : M \to M$ which is a principal bundle automorphism. Then ψ defines an element $\psi_* \in \mathrm{Out}^+(\Gamma)$ which only depends on F and which is mapped to F by η. Therefore η is surjective.

The preceding construction allows us to interpret the quadratic terms in h and h' in terms of Chen's iterated integrals (see e.g. [14]). Let u, v denote the standard coordinates in $\mathbb{R}^2$ and let du, dv be the induced 1-forms in $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Denote also for simplicity by du, dv the pullbacks to M. Let $\alpha \in \Omega^1(M, \mathbf{i}\mathbb{R})$ be a connection form whose curvature $d\alpha$ is equal to $-2\pi\mathbf{i} du \wedge dv$. For any smooth loop γ in M define

$$x(\gamma) = \int_\gamma du, \quad y(\gamma) = \int_\gamma dv, \quad z(\gamma) = \frac{-\mathbf{i}}{2\pi} \int_\gamma \alpha + \int_\gamma du\, dv.$$

$z(\gamma)$ is a homotopy functional by [14, Proposition 3.1], i.e. it only depends on the homotopy class of γ. One proves easily that

$$\pi_1(M, x_0) \ni \gamma \mapsto A(x(\gamma), y(\gamma), z(\gamma)) \in \Gamma$$

[1] We remark that in Subsection 3.3 we use h instead of h' because h preserves $T(\mathbb{Z}, n\mathbb{Z})$ for even n, whereas h' does not preserve $T(\mathbb{Z}, n\mathbb{Z})$ if n is even.

is an isomorphism of groups. Choose a lift $f \in \mathrm{Aut}^+(\Gamma)$ of ψ_*. We have $f(A(x,y,z)) = A(x',y',z+g(x,y,z))$, where $(x',y') = F(x,y)$. Our aim is to compute $g(x,y,z)$ up to linear terms. Suppose that $x = x(\gamma)$, $y = y(\gamma)$ and $z = z(\gamma)$. We have

$$g(x,y,z) = \frac{-\mathbf{i}}{2\pi}\int_\gamma (\psi^*\alpha - \alpha) + \int_\gamma (\psi^* du\, \psi^* dv - du\, dv).$$

Take $\phi : T^2 \to T^2$ to be the map induced by the linear transformation $F : \mathbb{R}^2 \to \mathbb{R}^2$. Then $\phi^*(du \wedge dv) = du \wedge dv$ because $\det F = 1$, so $\psi^* d\alpha = d\alpha$. Hence $\psi^*\alpha - \alpha$ is a closed 1-form, so its contribution to $g(x,y,z)$ is a linear term on x, y (closed 1-forms only *see* $\Gamma/[\Gamma,\Gamma] = H_1(M) = H_1(T)$). It follows that the integral involving $\psi^* du\, \psi^* dv - du\, dv$ is a homotopy functional which coincides with g up to linear terms. By naturality we have $\int_\gamma (\psi^* du\, \psi^* dv - du\, dv) = \int_{p\circ\gamma} (\psi^* du\, \psi^* dv - du\, dv)$, and since this is a homotopy functional it only depends on the homotopy class of γ. In particular we may assume that γ comes from a linear map $\mathbb{R} \ni t \mapsto (\lambda t, \mu t) \in \mathbb{R}^2$. Then $\lambda = \int_\gamma du = x$, $\mu = \int_\gamma dv = y$, and if $F = \begin{pmatrix} \alpha & \beta \\ \delta & \epsilon \end{pmatrix}$, a straightforward computation gives

$$\int_{p\circ\gamma} (\psi^* du\, \psi^* dv - du\, dv) = \frac{\alpha\delta x^2 + (\alpha\epsilon + \beta\delta - 1)xy + \beta\epsilon y^2}{2}.$$

Taking $\alpha = 0$, $\beta = -1$, $\delta = 1$ and $\epsilon = 1$ we obtain the quadratic terms in h and h'.

We close this appendix constructing a morphism of groups $\xi : \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{Aut}^+(\Gamma)$ which is a section of $\mathrm{Aut}^+(\Gamma) \to \mathrm{Out}^+(\Gamma) \simeq \mathrm{SL}(2,\mathbb{Z})$. This gives a conceptual explanation of the existence of elements of $\mathrm{Aut}(\Gamma)$ of order 6 (such as h'). Let $\mathcal{M} = \mathcal{M}_{1,1}$ be the moduli orbifold/stack of elliptic curves over $\mathbb{C}$, let $p : \mathcal{C} \to \mathcal{M}$ be the universal curve (all bundles here are to be understood in the orbifold/stack sense), let $\sigma : \mathcal{M} \to \mathcal{C}$ be the section corresponding to the marked point, let $\mathcal{D} = \sigma(\mathcal{M})$, let $\mathcal{L} = \mathcal{O}(\mathcal{D}) \to \mathcal{C}$, and let $\lambda \in H^0(\mathcal{L})$ be a section transverse to the zero section and satisfying $\lambda^{-1}(0) = \mathcal{D}$. Let $\mathcal{T} = \sigma^* T^{\mathrm{ver}}\mathcal{C} \to \mathcal{M}$, where $T^{\mathrm{ver}}\mathcal{C}$ is the vertical tangent bundle of $\mathcal{C}$. Let $\Lambda = \mathcal{L} \otimes p^*\mathcal{T}^*$ and let $\Lambda^* \subset \Lambda$ be the complementary of the zero section. Since λ vanishes transversely along $\mathcal{D}$, its derivative defines a nonvanishing section of $\mathrm{Hom}(\mathcal{T}, \sigma^*\mathcal{L})$, which can be interpreted as a section $b : \mathcal{M} \to \Lambda^*$ lifting σ. Let $r : \Lambda^* \to \mathcal{M}$ be the projection and let $e_0 \in \mathcal{M}$ be any point. We have $\pi_1(r^{-1}(e_0), b(e_0)) \simeq \Gamma$, and, thanks to the existence of the section b, the monodromy defines a map $\mathrm{SL}(2,\mathbb{Z}) \simeq \pi_1^{\mathrm{orb}}(\mathcal{M}) \to \mathrm{Aut}^+(r^{-1}(e_0), b(e_0)) \simeq \mathrm{Aut}^+(\Gamma)$, which is the desired morphism ξ.

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