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Using the HEGY Procedure When Not All Roots Are Present

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Abstract:

Empirical studies have shown little evidence to support the presence of all unit roots present in the Δ_4 filter in quarterly seasonal time series. This paper analyses the performance of the Hylleberg, Engle, Granger and Yoo (1990) (HEGY) procedure when the roots under the null are not all present. We exploit the Vector of Quarters representation and cointegration relationship between the quarters when factors $(1-L),(1+L),(1+L^2),(1-L^2)$ and $(1+L+L^2+L^3)$ are a source of nonstationarity in a process in order to obtain the distribution of tests of the HEGY procedure when the underlying processes have a root at the zero, Nyquist frequency, two complex conjugates of frequency $\pi/2$ and two combinations of the previous cases. We show both theoretically and through a Monte-Carlo analysis that the t-ratios $t_{\hat{\pi}_1}$ and $t_{\hat{\pi}_2}$ and the F-type tests used in the HEGY procedure have the same distribution as under the null of a seasonal random walk when the root(s) is/are present, although this is not the case for the t-ratio tests associated with unit roots at frequency $\pi/2$.

Key words: Seasonality, Vector of Quarters, unit root tests, HEGY tests

JEL classification: C22, C12.

Resumen:

Existe poca evidencia empírica que apoye el supuesto de que todas las raíces del filtro Δ_4 estén presentes en las series temporales trimestrales. Este trabajo analiza el funcionamiento del procedimiento propuesto por Hylleberg, Engle, Granger and Yoo (1990) (HEGY) cuando no todas las raíces unitarias bajo la hipótesis nula están presentes. Explotando la representación multivariante de las series temporales y las relaciones de cointegración existentes entre los trimestres de las series cuando los siguientes filtros $(1-L),(1+L),(1+L^2),(1-L^2)$ y $(1+L+L^2+L^3)$ son la fuente de no estacionariedad de los procesos, para poder obtener la distribución de los contrastes del procedimiento HEGY cuando los procesos analizados tienen raíces en la frecuencia cero, "Nyquist", dos conjugadas complejas en la frecuencia $\pi/2$ y dos combinaciones de los casos previos.

Mostramos analíticamente y mediante ejercicios de simulación que los contrastes tipo t $t_{\hat{\pi}_1}$ y $t_{\hat{\pi}_2}$ y los tipo F usados en el procedimiento HEGY tienen la misma distribución que bajo la hipótesis nula general consistente en que la serie analizada sigue un paseo aleatorio estacional cuando la raíz o raíces están presentes, pero este no es caso para los contrastes tipo t asociados a las raíces unitarias de la frecuencia $\pi/2$.

1.- Introduction.

This paper derives the distribution of Hylleberg, Engle, Granger and Yoo (1990) or HEGY seasonal unit root test statistics whether or not the *data generating process* (DGP) of the time series admits unit roots at the zero, Nyquist (frequency π) or complex conjugates associated with frequency $\pi/2$. Under the null hypothesis considered by HEGY, the time series follows a quarterly seasonal random walk:

$$y_{s\tau} = y_{s,\tau-1} + u_{s\tau} \quad s = 1, 2, 3, 4, \quad \tau = 1, 2, \cdots, N \qquad (1)$$

where, for observation $y_{s\tau}$ the first subscript refers to the season (s) and the second subscript to the year (τ). When s = 1, it is understood that $y_{s-1,\tau} = y_{4,\tau-1}$. Also for ease of presentation, we assume that observations are available for precisely N years, and so the total sample size is $T = 4N$. The seasonal random walk requires the use of the seasonal difference operator $\Delta_4 = \left(1 - L^4\right)$ (where L is the usual lag operator $L^k y_{s\tau} = y_{s-k,\tau}$), hence four unit roots are present in the process.

Empirical studies have shown little evidence to support the presence of all the unit roots of Δ_4 (see, among others, Ghysels and Osborn (2001), and Hylleberg *et al* (1993)). As Rodrigues and Taylor (2004a) state, "A substantial body of empirical evidence …. supports the view that seasonal patterns in macroeconomic time series evolve slowly over time displaying unit root behavior at some, but not necessarily all the seasonal frequencies"(pp 36) .

To our knowledge only Boswijk and Franses (1996), Taylor (2003), Smith and Taylor (1999), del Barrio Castro (2006) and del Barrio Castro and Osborn (2004) analyze seasonal unit root tests when the roots under the null are not all present. Boswijk and Franses (1996) obtain the distribution of the Dickey, Haza and Fuller (1984) (DHF) test statistic when the underlying process is periodically integrated (*PI*) ($y_{s\tau} = \alpha_s y_{s-1,\tau} + \varepsilon_{s\tau}$ with $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$) and therefore also as a specific case of a random walk ($\alpha_s = 1$ process (2.1)). Taylor (2003) obtains the distribution of the DHF test statistic when applied to process (2.1), while del Barrio Castro (2006) extended the result to processes (2.2) to (2.5). Del Barrio Castro and Osborn (2004) present the distribution of the tests used in the HEGY procedure when the underlying process is a periodic integrated (*PI*) process and also for a standard random walk process (2.1). Finally, Smith and Taylor (1999) present a characterization theorem which clarifies the null and alternative sub-hypotheses that are tested by the HEGY procedure for a general *S*. Using the characterization theorem as a basis, they show that HEGY coefficients at complex frequencies cannot be identified with phase and length restrictions when there are not unit roots at all other frequencies and/or when the shocks are serially correlated.

This paper presents the distribution of *t*-ratio and *F*-type tests of the HEGY procedure when the DGP of the time series is one of the following:

$$y_{s\tau} = y_{s-1,\tau} + u_{s\tau} \tag{2.1}$$

$$y_{s\tau} = -y_{s-1,\tau} + u_{s\tau} \tag{2.2}$$

$$y_{s\tau} = -y_{s-2,\tau} + u_{s\tau} \tag{2.3}$$

$$y_{s\tau} = y_{s-2,\tau} + u_{s\tau} \tag{2.4}$$

$$y_{s\tau} = -y_{s-1,\tau} - y_{s-2,\tau} - y_{s-3,\tau} + u_{s\tau} \tag{2.5}$$

The error process in (2.1) and (2.5) follow a stationary AR(p) process $\phi(L)u_{s\tau} = \varepsilon_{s\tau}$ where $\phi(z) = 1 - \sum_{i=1}^{p} \phi_i z^i$ (the roots of $\phi(z) = 0$ all lie outside the unit circle $|z| = 1$). And the innovation process $\{\varepsilon_{s\tau}\}$ is a martingale difference sequence (MDS) with constant conditional variance σ^2 (see Fuller (1996) Theorem 5.3.3 for details).

The paper is organized as follows. In the first section, we present the preliminary results that are needed in order to obtain the distribution of the HEGY tests when the DGP is one of the set (2.1) to (2.5). In the second section, the distribution of the test statistics is shown for the five cases under study. In the third, some Monte-Carlo results are given while in the fourth, we present the conclusions.

In the paper, we show that the distribution of *t*-ratio tests associated with the zero and Nyquist frequency and F-type tests associated with frequency *π/2* are pivotal when all or some of the unit roots of other frequencies are not present, regardless of any serial correlation or lack of serial correlation in the shocks. This is not the case, however, for t-ratio tests associated with frequency *π/2*, because when the shocks are not serially correlated we obtain a pivotal distribution in the case of process (2.3) but not for (2.5). Hence the results of this paper complement those reported in Smith and Taylor (1999).

2.- Preliminaries.

The seasonal random walk (1) can be alternatively represented in a vector of quarters (VQ) where the observations for each year are stacked in vectors $Y_\tau = [y_{1\tau} y_{2\tau} y_{3\tau} y_{4\tau}]'$, $U_\tau = [u_{1\tau} u_{2\tau} u_{3\tau} u_{4\tau}]'$:

$$\begin{aligned} Y_\tau &= Y_{\tau-1} + U_\tau \\ (1-B)Y_\tau &= U_\tau \end{aligned} \qquad (3)$$

Here B is the annual backward operator (that is, $B^k Y_\tau = Y_{\tau-k}$). As pointed out by Dickey *et al* (1984) and Osborn (1993), a seasonal random walk is a set of S separate random walk processes, one related to each of the seasons. We will use the Vector Moving Average (VMA) representation of the annual difference of processes (2.1) to (2.5) to obtain the asymptotics of the HEGY procedure.

If the seasonal difference operator $\Delta_4 = (1-L^4)$ is applied to processes (2.1) to (2.5) then stationarity is achieved, but noninvertible moving averages appear as a result of overdifferencing. In (4.1) to (4.5), we show this noninvertible moving average representation when the seasonal difference (or annual difference) operator is applied to processes (2.1) to (2.5) respectively:

$$\Delta_4 y_{s\tau} = \left(1 + L + L^2 + L^3\right) u_{s\tau} \qquad (4.1)$$

$$\Delta_4 y_{s\tau} = \left(1 - L + L^2 - L^3\right) u_{s\tau} \qquad (4.2)$$

$$\Delta_4 y_{s\tau} = \left(1 - L^2\right) u_{s\tau} \qquad (4.3)$$

$$\Delta_4 y_{s\tau} = \left(1 + L^2\right) u_{s\tau} \qquad (4.4)$$

$$\Delta_4 y_{s\tau} = (1 - L) u_{s\tau} \qquad (4.5)$$

Alternatively, we can express (4.1) to (4.5) in a vector moving average representation (VMA) which will be the basis of lemma 1:

$$(1 - B) Y_\tau = \left(\Theta_0^i + \Theta_1^i B\right) U_\tau \quad i = 1,2,3,4,5 \qquad (5)$$

where vectors Y_τ, U_τ and B are as previously defined, and Θ_0^i and Θ_1^i are 4×4 matrices that contain the coefficients of the MA processes. We have two different matrices for each of processes (2.1) to (2.5) or equivalently (4.1) to (4.5), as follows:

$$\Theta_0^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \Theta_1^1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ for (4.1)} \qquad (6.1)$$

$$\Theta_0^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix} \quad \Theta_1^2 = \begin{bmatrix} 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ for (4.2)} \qquad (6.2)$$

$$\Theta_0^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad \Theta_1^3 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ for (4.3)} \qquad (6.3)$$

$$\Theta_0^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \Theta_1^4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ for (4.4)} \qquad (6.4)$$

$$\Theta_0^5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad \Theta_1^5 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ for (4.5)} \qquad (6.5)$$

Following Burridge and Taylor (2001), it is possible to write:

$$U_\tau = \sum_{J=0}^{\infty} \Psi_J^* E_\tau$$

where $E_\tau = [\varepsilon_{1\tau}\varepsilon_{2\tau}\varepsilon_{3\tau}\varepsilon_{4\tau}]'$, and we define the sequence of the 4×4 matrices as:

$$\Psi_0^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \psi_1 & 1 & 0 & 0 \\ \psi_2 & \psi_1 & 1 & 0 \\ \psi_3 & \psi_2 & \psi_1 & 1 \end{bmatrix} \quad \Psi_j^* = \begin{bmatrix} \psi_{j4} & \psi_{j4-1} & \psi_{j4-2} & \psi_{j4-3} \\ \psi_{j4+1} & \psi_{j4} & \psi_{j4-1} & \psi_{j4-2} \\ \psi_{j4+2} & \psi_{j4+1} & \psi_{j4} & \psi_{j4-1} \\ \psi_{j4+3} & \psi_{j4+2} & \psi_{j4+1} & \psi_{j4} \end{bmatrix} \; j = 1,2,\cdots$$

with $\psi(z) = 1 - \sum_{j=1}^{\infty} \psi_j z^j$ being the inverse of $\phi(z)$. Finally we define $\Psi^*(1)$ as $\Psi^*(1) = \sum_{j=0}^{\infty} \Psi_j^*$. In the next lemma, we summarize the stochastic characteristics of processes (2.1) to (2.5) in terms of the number of cointegration relationships between the quarters we have in each situation.

Lemma 1:

Consider Y_τ in (2.1) to (2.5) and assuming that the elements of E_τ are independent and identically distributed with zero mean and variance σ^2 as $T/s \to \infty$:

$$\frac{1}{\sqrt{T/s}} Y_{[rT/s]} \Rightarrow B_i(r) \quad \mathrm{i} = 1,2,3,4 \text{ and } 5$$
$$B_i(r) = \sigma C_i \Psi^*(1) W(r) \qquad (7)$$

where [rT/s] denotes the integer part of rT/s, $B_i(r)$ is a 4×1 vector Brownian motion process with variance matrix $\Omega_i = \sigma^2 C_i \Psi^*(1) \Psi^*(1)' C_i'$, $W(r)$ is a 4×1 vector Brownian motion process with variance matrix $\sigma^2 I_4$, and $C_i = \Theta_0^i + \Theta_1^i$. Finally $\Rightarrow$ denotes weak convergence. It is understood that "i" corresponds to (2.i), (4.i) and (6.i) for i=1, 2, 3, 4 and 5.

Proof can be obtained along the lines of the proof of Lemma 1 in Boswijk and Franses (1996) and del Barrio Castro (2006). Note that from Theorem 5.3.5 of Fuller (1996) it is possible to establish that $\frac{1}{\sqrt{T/s}} \sum_{j=1}^{[rT/s]} E_j \Rightarrow W(r)$. Hence the fact that the four VQ series $\{Y_\tau\}$ do not have the full set of unit roots for process (2.i) is reflected in the rank of the C_i matrices. That is, the number of roots that are present in process (2.i) is equal to the rank of the corresponding matrix C, and to the number of quarters minus the number of cointegration relationships between them. Furthermore as there is cointegration among the quarters of the time series, using the following identities:

$$C_1\Psi^*(1)=\psi(1)C_1 \quad C_2\Psi^*(1)=\psi(-1)C_2 \quad C_3\Psi^*(1)=bC_3+aC_{3*}$$
$$C_4\Psi^*(1)=\frac{1}{2}[\psi(1)C_1+\psi(-1)C_2] \quad C_5\Psi^*(1)=\frac{1}{2}[\psi(-1)C_2+(b+a)C_3-(b-a)C_{3*}]$$
$$C_1=v_1v_1' \quad C_2=v_2v_2' \quad C_3=v_3v_3' \quad C_{3*}=v_3v_{3*}'$$

with:

$$a=\frac{-i}{2}[\psi(i)-\psi(-i)] \quad b=\frac{1}{2}[\psi(i)+\psi(-i)] \quad v_1'=[1 \quad 1 \quad 1 \quad 1] \quad v_2'=[-1 \quad 1 \quad -1 \quad 1]$$
$$v_3'=\begin{bmatrix} c_3' \\ c_{3*}' \end{bmatrix}=\begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \quad v_3^*{}'=\begin{bmatrix} -c_{3*}' \\ c_3' \end{bmatrix}=\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

it is possible to rewrite expression (7) as:

$$\begin{aligned}
&B_1(r)=\sqrt{4}\sigma\psi(1)v_1w_1(r) \quad B_2(r)=\sqrt{4}\sigma\psi(-1)v_2w_2(r)\\
&B_3(r)=\sqrt{2}\sigma\left(bv_3\begin{bmatrix} w_3(r) \\ w_{3*}(r) \end{bmatrix}+av_3\begin{bmatrix} -w_{3*}(r) \\ w_3(r) \end{bmatrix}\right)\\
&B_4(r)=\sigma\psi(1)v_1w_1(r)+\sigma\psi(-1)v_2w_2(r)\\
&B_5(r)=\sigma\psi(-1)v_2w_2(r)+\\
&\qquad +(2)^{-0.5}\sigma\left((b+a)v_3\begin{bmatrix} w_3(r) \\ w_{3*}(r) \end{bmatrix}-(b-a)v_3\begin{bmatrix} -w_{3*}(r) \\ w_3(r) \end{bmatrix}\right)\\
&with:\\
&w_1(r)=(4)^{-0.5}v_1'W(r) \quad w_2(r)=(4)^{-0.5}v_2'W(r)\\
&w_3(r)=(2)^{-0.5}c_3'W(r) \quad w_{3*}(r)=(2)^{-0.5}c_{3*}'W(r)
\end{aligned} \qquad (7.a)$$

Where $w_1(r)$, $w_2(r)$, $w_3(r)$ and $w_3^*(r)$ are standard Brownian motions and mutually orthogonal transformations of the elements of $W(r)$ (see Burridge and Taylor (2001)). Hence note that $B_i(r)$ is a function of one or more of the previous orthogonal standard Brownian motions, depending on the number of underlying common trends present in the process (2.i).

The basic regression for the HEGY test, with augmentation and no deterministic terms, is:

$$\Delta_4 y_{s\tau} = \pi_1 y_{s-1,\tau}^{(1)} + \pi_2 y_{s-1,\tau}^{(2)} + \pi_3 y_{s-2,\tau}^{(3)} + \pi_4 y_{s-1,\tau}^{(3)} + \sum_{j=1}^{p} \phi_j^* \Delta_4 y_{s-j,\tau} + \varepsilon_{s\tau} \quad (8)$$

Where $y_{s\tau}^{(1)}$, $y_{s\tau}^{(2)}$, $y_{s\tau}^{(3)}$ are auxiliary variables associated with the roots of (1 - L), (1 + L) and (1 + L^2) of the seasonal difference operator Δ_4 = (1 - L^4) = (1 - L)(1 + L)(1 + L^2). More specifically,

$$\begin{aligned} y_{s\tau}^{(1)} &= (1+L)(1+L^2)y_{s\tau} \\ y_{s\tau}^{(2)} &= -(1-L)(1+L^2)y_{s\tau} \\ y_{s\tau}^{(3)} &= -(1-L)(1+L)y_{s\tau} \end{aligned} \quad (9)$$

The overall HEGY null hypothesis of seasonal integration, $y_{s\tau} \sim SI(1)$, implies the presence of unit roots at the zero frequency (captured through π_1) and at seasonal frequencies (captured through π_2, π_3 and π_4), so that $\pi_1 = \pi_2 = \pi_3 = \pi_4 = 0$ and hence $\Delta_4 y_{s\tau} = \varepsilon_{s\tau}$.

According to HEGY, the regressors in (8) are, by construction, asymptotically orthogonal under this null hypothesis. Thus, the associated asymptotic distributions of the HEGY test statistics can be obtained by considering the three factors of Δ_4, one by one. Based on the above, normalized bias $T\hat{\pi}_i$ and *t*-ratio $t_{\hat{\pi}_i}$ tests are proposed to test the nulls $H_0: \pi_i \quad i = 1, 2, 3, 4$, and an *F*-type statistic is proposed to test the joint null of H_0: $\pi_3 = \pi_4 = 0$, *F34.* Ghysels, Lee and Noh (1994) also proposed *F*-type statistics to test the nulls of H_0: $\pi_2 = \pi_3 = \pi_4 = 0$ and H_0: $\pi_1 = \pi_2 = \pi_3 = \pi_4 = 0$, and *F234* and *F1234* respectively. For more details of the definition and asymptotic distributions of the tests, see Smith and Taylor (1998) and Osborn and Rodrigues (2002).

2.- Asymptotics.

The results of the distribution of the statistics in regression (8), when the underlying processes are (2.1), (2.2), (2.3), (2.4) or (2.5), are presented in the next theorem, where we use the notation introduced by Osborn and Rodrigues (2002) and also the following two functionals $A(i, j) = \int w_i(r) dw_j(r)$ and $B(i) = \int [w_i(r)]^2 dr$, in which $w_i(r)$ for *i = 1, 2, 3* and *3** are the mutually independent standard Brownian motions defined in (7.a), see Burridge and Taylor (2001).

THEOREM 1

(a) Assume $y_{s\tau}$ *follows (2.1). Then for HEGY regression (8), the asymptotic distribution of* $t_{\hat{\pi}_1}$ *is given by*:

$$t_{\hat{\pi}_1} \Rightarrow \frac{\int W^*(r)' C_1 dW^*(r)}{\sqrt{\int W^*(r)' C_1 W^*(r) dr}} = \frac{A(1,1)}{\sqrt{B(1)}} \qquad (10)$$

Where $W^*(r) = 1/\sqrt{4}W(r)$.

(b) Assume $y_{s\tau}$ *follows (2.2). Then for HEGY regression (8), the asymptotic distribution of* $t_{\hat{\pi}_2}$ *is given by*:

$$t_{\hat{\pi}_2} \Rightarrow \frac{\int W^*(r)' C_2 dW^*(r)}{\sqrt{\int W^*(r)' C_2 W^*(r) dr}} = \frac{A(2,2)}{\sqrt{B(2)}} \qquad (11)$$

Where $W^*(r) = 1/\sqrt{4}W(r)$.

(b) Assume $y_{s\tau}$ *follows (2.3). Then for HEGY regression (8), the asymptotic distributions of* $t_{\hat{\pi}_3}$, $t_{\hat{\pi}_4}$ *and F34 are given by*:

$$\begin{aligned} t_{\hat{\pi}_3} &\Rightarrow \frac{b\int W^*(r)' C_3 dW^*(r) - a\int W^*(r)' C_{3*} dW^*(r)}{\sqrt{(a^2+b^2)\int W^*(r)' C_3 W^*(r) dr}} \\ &= \frac{b(A(3,3)+A(3*,3*)) - a(A(3*,3)-A(3,3*))}{\sqrt{(a^2+b^2)(B(3)+B(3*))}} \\ t_{\hat{\pi}_4} &\Rightarrow \frac{b\int W^*(r)' C_3^* dW^*(r) - a\int W^*(r)' C_3 dW^*(r)}{\sqrt{(a^2+b^2)\int W^*(r)' C_3 W^*(r) dr}} \\ &= \frac{b(A(3*,3)-A(3,3*)) - a(A(3,3)+A(3*,3*))}{\sqrt{(a^2+b^2)(B(3)+B(3*))}} \end{aligned} \qquad (12)$$

$$F_{34} \Rightarrow \frac{1}{2}\frac{\left(\int W^*(r)' C_3 dW^*(r)\right)^2 + \left(\int W^*(r)' C_3^* dW^*(r)\right)^2}{\int W^*(r)' C_3 W^*(r) dr}$$

$$= \frac{1}{2}\frac{(A(3,3)+A(3*,3*))^2 + (A(3*,3)-A(3,3*))^2}{(B(3)+B(3*))}$$

Where $W^*(r) = 1/\sqrt{2}W(r)$.

(d) Assume $y_{s\tau}$ *follows (2.4). Then for HEGY regression (8), the asymptotic distributions of* $t_{\hat{\pi}_1}$ *and* $t_{\hat{\pi}_2}$ *are given by*:

$$t_{\hat{\pi}_i} \Rightarrow \frac{\int W^*(r)' C_i dW^*(r)}{\sqrt{\int W^*(r)' C_i W^*(r) dr}} = \frac{A(i,i)}{\sqrt{B(i)}} \quad i = 1 \text{ and } 2 \qquad (13)$$

Where $W^*(r) = 1/\sqrt{4}W(r)$.

(e) Assume $y_{s\tau}$ *follows (2.5). Then for HEGY regression (8), the asymptotic distributions of* $t_{\hat{\pi}_2}, t_{\hat{\pi}_3}, t_{\hat{\pi}_4}$ *and F34 are given by*:

$$t_{\hat{\pi}_2} \Rightarrow \frac{\int W^*(r)' C_2 dW^*(r)}{\sqrt{\int W^*(r)' C_2 W^*(r) dr}} = \frac{A(2,2)}{\sqrt{B(2)}} \qquad (14.a)$$

Where $W^*(r) = 1/\sqrt{4}W(r)$*, and:*

$$t_{\hat{\pi}_4} \Rightarrow \frac{\frac{1}{2}(b+a)\int W^*(r)'C_3 dW^*(r)+\frac{1}{2}(b-a)\int W^*(r)'C_3^* dW^*(r)}{\sqrt{\frac{(a^2+b^2)}{2}\int W^*(r)'C_3W^*(r)dr}}$$

$$=\frac{\frac{1}{2}(b+a)(A(3,3)+A(3*,3*))+\frac{1}{2}(b-a)(A(3*,3)-A(3,3*))}{\sqrt{\frac{(a^2+b^2)}{2}(B(3)+B(3*))}}$$

$$t_{\hat{\pi}_3} \Rightarrow \frac{\frac{1}{2}(b+a)\int W^*(r)'C_3^* dW^*(r)-\frac{1}{2}(b-a)\int W^*(r)'C_3 dW^*(r)}{\sqrt{\frac{(a^2+b^2)}{2}\int W^*(r)'C_3W^*(r)dr}}$$

$$=\frac{\frac{1}{2}(b+a)(A(3*,3)-A(3,3*))-\frac{1}{2}(b-a)(A(3,3)+A(3*,3*))}{\sqrt{\frac{(a^2+b^2)}{2}(B(3)+B(3*))}}$$

$$F34=\frac{1}{2}\frac{\left(\int W^*(r)'C_3 dW^*(r)\right)^2+\left(\int W^*(r)'C_3^* dW^*(r)\right)^2}{\int W^*(r)'C_3W^*(r)dr} \qquad (14.b)$$

$$=\frac{1}{2}\frac{(A(3,3)+A(3*,3*))^2+(A(3*,3)-A(3,3*))^2}{(B(3)+B(3*))}$$

Where $W^*(r)=1/\sqrt{2}W(r)$.

It is clear from expressions (10) to (14.b) that the distributions of $t_{\hat{\pi}_1}, t_{\hat{\pi}_2}$ and *F34* in the absence of all or some of the unit roots associated with the other frequencies are pivotal, but those of $t_{\hat{\pi}_3}$ and $t_{\hat{\pi}_4}$ depend on nuisance parameters. It is evident that, in the absence of serial correlation, $\phi(L)=1$, *a* = 0 and *b* = 1, hence (12) and (14.b) reduce to:

$$t_{\hat{\pi}_3} \Rightarrow \frac{\int W^*(r)' C_3 dW^*(r)}{\sqrt{\int W^*(r)' C_3 W^*(r) dr}} = \frac{(A(3,3)+A(3*,3*))}{\sqrt{(B(3)+B(3*))}}$$

$$t_{\hat{\pi}_4} \Rightarrow \frac{\int W^*(r)' C_3^* dW^*(r)}{\sqrt{\int W^*(r)' C_3 W^*(r) dr}} = \frac{(A(3*,3)-A(3,3*))}{\sqrt{(B(3)+B(3*))}} \qquad (12.1)$$

$$t_{\hat{\pi}_3} \Rightarrow \frac{\frac{1}{2}\left(\int W^*(r)' C_3 dW^*(r) + \int W^*(r)' C_3^* dW^*(r)\right)}{\sqrt{\frac{1}{2}\int W^*(r)' C_3 W^*(r) dr}}$$

$$= \frac{\frac{1}{2}[(A(3,3)+A(3*,3*))+(A(3*,3)-A(3,3*))]}{\sqrt{\frac{1}{2}(B(3)+B(3*))}}$$

$$t_{\hat{\pi}_4} \Rightarrow \frac{\frac{1}{2}\left(\int W^*(r)' C_3^* dW^*(r) - \int W^*(r)' C_3 dW^*(r)\right)}{\sqrt{\frac{1}{2}\int W^*(r)' C_3 W^*(r) dr}}$$

$$= \frac{\frac{1}{2}[(A(3*,3)-A(3,3*))-(A(3,3)+A(3*,3*))]}{\sqrt{\frac{1}{2}(B(3)+B(3*))}}. \qquad (14.b.1)$$

Then, without serial correlation, the distribution of $t_{\hat{\pi}_3}$ and $t_{\hat{\pi}_4}$ remain pivotal in the absence of all unit roots associated with other frequencies. However, if the unit root associated with frequency π is present, the distribution of both t-ratio tests differ from those obtained under the overall null hypothesis.

Although results in Theorem 1, (12.1) and (14.b.1) do not consider the inclusion of deterministic terms (such as seasonal dummies and linear trend) in the HEGY regression, our results could be easily extended to this situation. More specifically, with the inclusion of seasonal dummies or seasonal dummies and a linear trend our results will carry over when expressed using de-meaned and de-trended Brownian motions. Further, as shown by Smith and Taylor (1998), the inclusion of seasonal dummies in the HEGY regression makes the test statistics invariant to starting values and the seasonal trends further invariance to seasonal drifts, and these results apply also in our case.

In the simplest case of serial correlation, i.e. $\phi(L)=(1-\phi L)$ followed by the innovation u_t of processes (2.1) to (2.5), it is easy to see that the coefficient associated with the stationary regressors in (8) converges to:

$$\begin{aligned}
&\hat{\Pi}_1^S = [\hat{\pi}_2,\hat{\pi}_3,\hat{\pi}_4,\phi^*]' \rightarrow \\
&\quad \rightarrow [-1/2(1+\phi), -1/2(1-\phi), -1/2(1+\phi), 0]' \quad \text{for} \quad (2.1) \\
&\hat{\Pi}_2^S = [\hat{\pi}_1,\hat{\pi}_3,\hat{\pi}_4,\phi^*]' \rightarrow \\
&\quad \rightarrow [-1/2(1-\phi), -1/2(1+\phi), 1/2(1-\phi), 0]' \quad \text{for} \quad (2.2) \\
&\hat{\Pi}_3^S = [\hat{\pi}_1,\hat{\pi}_2,\phi^*]' \rightarrow [-1/2(1-\phi), -1/2(1+\phi), 0]' \quad \text{for} \quad (2.3) \\
&\hat{\Pi}_4^S = [\hat{\pi}_3,\hat{\pi}_4,\phi^*]' \rightarrow [-1, -\phi, 0]' \quad \text{for} \quad (2.4) \\
&\hat{\Pi}_5^S = [\hat{\pi}_1,\phi^*]' \rightarrow [-1+\phi, 0]' \quad \text{for} \quad (2.5)
\end{aligned} \qquad (15)$$

The previous results are obtained using (2.1) to (2.5), (4.1) to (4.5), and (9), based on the orthogonality of the nonstationary and stationary regressors of (8). Note that in the absence of serial correlation, (15) implies that the scaled estimator $T\hat{\pi}_j$ for j = 2, 3, 4 diverges to - ∞ as $T \rightarrow \infty$ except in the case of $T\hat{\pi}_4$ for process (2.4), hence we could expect good performance, in power terms, for the t-ratio tests except for the one associated with π_4 for process (2.4). For positive values of ϕ, we do not expect good performance in terms of power for the t-ratio tests associated with π_3 for process (2.1), π_1 for processes (2.2), (2.3) and (2.5) and π_4 for process (2.2). For negative values of ϕ, we do not expect good performance in terms of power for the t-ratio associated with π_2 for processes (2.1) and (2.3), π_4 for process (2.1) and π_3 for process (2.2). Finally, from (15), using (4.1) to (4.5), it is easy to confirm that the innovations of regression (8) will follow a white noise process when the regression is augmented by one lag of $\Delta_4 y_{s\tau}$.

3.- Monte-Carlo results.

The Monte-Carlo results are shown in Tables 1.a to 1.c. These tables reflect the proportion of times that the null hypothesis is rejected for *t*-ratio tests $t_{\hat{\pi}_1}, t_{\hat{\pi}_2}, t_{\hat{\pi}_3}$ and $t_{\hat{\pi}_4}$ and for *F*-type tests *F34, F234* and *F1234* for processes (2.1) to (2.5) and also for process (1) with $\phi(L) = (1-\phi L)$, using the following combination of parameters: $\phi = \{0.9, 0.5, 0, -0.5, -0.9\}$. The results are based on 50,000 replications and are computed for sample sizes of 100 and 200 observations (25 and 50 years), with a nominal size of 5%. For each sample size, we report the results based on regression (8) with *p*=1, and on the following regression which also includes seasonal dummies:

$$\Delta_4 y_{s\tau} = \mu_s + \pi_1 y^{(1)}_{s-1,\tau} + \pi_2 y^{(2)}_{s-1,\tau} + \pi_3 y^{(3)}_{s-2,\tau} + \pi_4 y^{(3)}_{s-1,\tau} + \phi^* \Delta_4 y_{s-1,\tau} + \varepsilon_{s\tau} \quad (16)$$

Note first, that for process (1) we obtain similar results to those reported in Burridge and Taylor (2001). Note too that the effect of the nuisance parameter on the empirical size of $t_{\hat{\pi}_3}$ and $t_{\hat{\pi}_4}$ is more clearly observable in the case of the HEGY regression that includes the seasonal dummies (16). This situation was also observed for the rest of the processes.

The results of Tables 1.a to 1.c clearly reflect the forecasts of the asymptotic analysis, even though we are using small sample sizes, making the results relevant for practical situations. With regard to the empirical size, it is clear that we observe size distortions for only $t_{\hat{\pi}_3}$ and $t_{\hat{\pi}_4}$ Note also that the same kind of problems observed for process (1) are also observed for process (2.3), as expected, because in (12) we have the same distributions for $t_{\hat{\pi}_3}$ and $t_{\hat{\pi}_4}$ as those obtained by Burridge and Taylor (2001) for process (1) in the presence of serial correlation. Furthermore, the effect of the nuisance parameters disappears when there is an absence of serial correlation, as predicted in (12.1). Finally for process (2.5), the non-pivotal distributions $t_{\hat{\pi}_3}$ and $t_{\hat{\pi}_4}$ (14.b and (14.b.1)) are only clearly reflected in the empirical size for the following values of $\phi = \{0, -0.5, -0.9\}$.

If we take a look at the empirical power, it is also clear that all the forecasts made in the previous section, based on (15), are reflected in the empirical power reported in Tables 1.a to 1.c. Note that the final effect of the t-ratio test on the empirical power is dependent on the absolute value of parameter ϕ (since distortions tend to increase as the absolute value of ϕ increases) and also on the sample size (since problems of empirical power tend to decrease as the sample size increases).

4.- Concluding remarks.

In this paper, we obtain the distribution of the *t*-ratio and *F*-type tests of the HEGY procedure when the unit roots of the overall null hypothesis are not all present. We show that in these situations the distribution of $t_{\hat{\pi}_1}$, $t_{\hat{\pi}_2}$ and *F34* remain pivotal but those of $t_{\hat{\pi}_3}$ and $t_{\hat{\pi}_4}$ are affected by nuisance parameters. Hence we provide additional analytical evidence and Monte-Carlo results to support the recommendation of Burridge and Taylor (2001) and Rodrigues and Taylor (2004b) to use the joint *F34* test instead of t-ratio tests $t_{\hat{\pi}_3}$ and $t_{\hat{\pi}_4}$. Finally, the results of this paper complement those reported by Smith and Taylor (1999).

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Table 1.a

Proportion of times that the null is rejected (nominal size 5%).

regression	4N	ϕ	DGP	(1-L)	(1+L)	(1+L^2)	$t_{\hat{\pi}_1}$	$t_{\hat{\pi}_2}$	$t_{\hat{\pi}_3}$	$t_{\hat{\pi}_4}$	*F34*	*F234*	*F1234*
(8)	100	0.9	(1)	X	X	X	0.048	0.048	0.045	0.055	0.050	0.053	0.053
(16)			(1)	X	X	X	0.057	0.045	0.026	0.173	0.047	0.047	0.056
(8)	100	0.5	(1)	X	X	X	0.050	0.047	0.048	0.053	0.050	0.052	0.052
(16)			(1)	X	X	X	0.048	0.047	0.040	0.102	0.049	0.050	0.055
(8)	100	0	(1)	X	X	X	0.045	0.047	0.051	0.049	0.050	0.051	0.050
(16)			(1)	X	X	X	0.046	0.047	0.049	0.051	0.048	0.049	0.053
(8)	100	-0.5	(1)	X	X	X	0.046	0.047	0.047	0.052	0.049	0.050	0.049
(16)			(1)	X	X	X	0.045	0.045	0.041	0.102	0.049	0.051	0.053
(8)	100	-0.9	(1)	X	X	X	0.046	0.049	0.046	0.055	0.049	0.054	0.052
(16)			(1)	X	X	X	0.045	0.055	0.030	0.172	0.050	0.056	0.058
(8)	200	0.9	(1)	X	X	X	0.051	0.051	0.046	0.056	0.049	0.049	0.051
(16)			(1)	X	X	X	0.055	0.049	0.029	0.181	0.048	0.047	0.052
(8)	200	0.5	(1)	X	X	X	0.051	0.051	0.048	0.053	0.050	0.049	0.048
(16)			(1)	X	X	X	0.052	0.049	0.040	0.110	0.050	0.048	0.050
(8)	200	0	(1)	X	X	X	0.048	0.048	0.052	0.051	0.050	0.051	0.051
(16)			(1)	X	X	X	0.050	0.051	0.050	0.057	0.050	0.050	0.050
(8)	200	-0.5	(1)	X	X	X	0.050	0.049	0.049	0.055	0.050	0.050	0.049
(16)			(1)	X	X	X	0.049	0.050	0.039	0.112	0.048	0.049	0.048
(8)	200	-0.9	(1)	X	X	X	0.050	0.049	0.045	0.057	0.049	0.052	0.050
(16)			(1)	X	X	X	0.050	0.059	0.029	0.186	0.050	0.052	0.053
(8)	100	0.9	(2.1)	X			0.052	1.000	0.072	1.000	1.000	1.000	1.000
(16)			(2.1)	X			0.058	1.000	0.004	1.000	1.000	1.000	1.000
(8)	100	0.5	(2.1)	X			0.047	1.000	0.585	1.000	1.000	1.000	1.000
(16)			(2.1)	X			0.045	1.000	0.144	1.000	1.000	1.000	1.000
(8)	100	0	(2.1)	X			0.048	1.000	0.998	0.997	1.000	1.000	1.000
(16)			(2.1)	X			0.045	1.000	0.948	0.992	1.000	1.000	1.000
(8)	100	-0.5	(2.1)	X			0.046	1.000	1.000	0.533	1.000	1.000	1.000
(16)			(2.1)	X			0.046	0.979	1.000	0.390	1.000	1.000	1.000
(8)	100	-0.9	(2.1)	X			0.048	0.632	1.000	0.064	1.000	1.000	1.000
(16)			(2.1)	X			0.048	0.234	1.000	0.032	1.000	1.000	1.000
(8)	200	0.9	(2.1)	X			0.051	1.000	0.090	1.000	1.000	1.000	1.000
(16)			(2.1)	X			0.055	1.000	0.004	1.000	1.000	1.000	1.000
(8)	200	0.5	(2.1)	X			0.048	1.000	0.867	1.000	1.000	1.000	1.000
(16)			(2.1)	X			0.050	1.000	0.405	1.000	1.000	1.000	1.000
(8)	200	0	(2.1)	X			0.049	1.000	1.000	1.000	1.000	1.000	1.000
(16)			(2.1)	X			0.049	1.000	1.000	1.000	1.000	1.000	1.000
(8)	200	-0.5	(2.1)	X			0.049	1.000	1.000	0.839	1.000	1.000	1.000
(16)			(2.1)	X			0.050	1.000	1.000	0.740	1.000	1.000	1.000
(8)	200	-0.9	(2.1)	X			0.051	0.984	1.000	0.083	1.000	1.000	1.000
(16)			(2.1)	X			0.052	0.729	1.000	0.042	1.000	1.000	1.000

Based on 50,000 replications. Original DGPs $(1)\ y_{s\tau} = y_{s,\tau-1} + u_{s\tau}$, $(2.1)\ y_{s\tau} = y_{s-1,\tau} + u_{s\tau}$, and with $y_{s0} = 0$ and $(1-\phi L)u_{s\tau} = \varepsilon_{s\tau}$. t_{π_1}, $t_{\hat{\pi}_2}$, $t_{\hat{\pi}_3}$ and $t_{\hat{\pi}_4}$ tests for $H_0: \pi_1 = 0$, $H_0: \pi_2 = 0$, $H_0: \pi_3 = 0$ and $H_0: \pi_4 = 0$. *F34, F234* and *F1234* tests for $H_0: \pi_3 = \pi_4 = 0$, $H_0: \pi_2 = \pi_3 = \pi_4 = 0$ and $H_0: \pi_1 = \pi_2 = \pi_3 = \pi_4 = 0$.

Table 1.b
Proportion of times that the null is rejected (nominal size 5%).

regression	4N	ϕ	DGP	(1-L)	(1+L)	$(1+L^2)$	$t_{\hat{\pi}_1}$	$t_{\hat{\pi}_2}$	$t_{\hat{\pi}_3}$	$t_{\hat{\pi}_4}$	*F34*	*F234*	*F1234*
(8)	100	0.9	(2.2)		X		0.636	0.049	1.000	0.066	1.000	1.000	1.000
(16)			(2.2)		X		0.235	0.048	1.000	0.032	1.000	1.000	1.000
(8)	100	0.5	(2.2)		X		1.000	0.048	1.000	0.529	1.000	1.000	1.000
(16)			(2.2)		X		0.978	0.047	1.000	0.387	1.000	1.000	1.000
(8)	100	0	(2.2)		X		1.000	0.048	0.998	0.997	1.000	1.000	1.000
(16)			(2.2)		X		1.000	0.046	0.947	0.992	1.000	1.000	1.000
(8)	100	-0.5	(2.2)		X		1.000	0.047	0.587	1.000	1.000	1.000	1.000
(16)			(2.2)		X		1.000	0.045	0.147	1.000	1.000	1.000	1.000
(8)	100	-0.9	(2.2)		X		1.000	0.050	0.072	1.000	1.000	1.000	1.000
(16)			(2.2)		X		1.000	0.058	0.004	1.000	1.000	1.000	1.000
(8)	200	0.9	(2.2)		X		0.984	0.051	1.000	0.083	1.000	1.000	1.000
(16)			(2.2)		X		0.729	0.051	1.000	0.042	1.000	1.000	1.000
(8)	200	0.5	(2.2)		X		1.000	0.049	1.000	0.838	1.000	1.000	1.000
(16)			(2.2)		X		1.000	0.051	1.000	0.738	1.000	1.000	1.000
(8)	200	0	(2.2)		X		1.000	0.050	1.000	1.000	1.000	1.000	1.000
(16)			(2.2)		X		1.000	0.049	1.000	1.000	1.000	1.000	1.000
(8)	200	-0.5	(2.2)		X		1.000	0.051	0.865	1.000	1.000	1.000	1.000
(16)			(2.2)		X		1.000	0.049	0.409	1.000	1.000	1.000	1.000
(8)	200	-0.9	(2.2)		X		1.000	0.050	0.092	1.000	1.000	1.000	1.000
(16)			(2.2)		X		1.000	0.057	0.004	1.000	1.000	1.000	1.000
(8)	100	0.9	(2.3)			X	0.667	1.000	0.045	0.056	0.049	1.000	1.000
(16)			(2.3)			X	0.228	1.000	0.032	0.168	0.051	1.000	1.000
(8)	100	0.5	(2.3)			X	1.000	1.000	0.048	0.053	0.050	1.000	1.000
(16)			(2.3)			X	0.994	1.000	0.043	0.104	0.049	1.000	1.000
(8)	100	0	(2.3)			X	1.000	1.000	0.049	0.051	0.050	1.000	1.000
(16)			(2.3)			X	1.000	1.000	0.052	0.052	0.051	0.994	1.000
(8)	100	-0.5	(2.3)			X	1.000	1.000	0.048	0.053	0.050	0.998	1.000
(16)			(2.3)			X	1.000	0.993	0.044	0.102	0.051	0.876	1.000
(8)	100	-0.9	(2.3)			X	1.000	0.666	0.045	0.056	0.051	0.343	1.000
(16)			(2.3)			X	1.000	0.229	0.030	0.169	0.050	0.150	1.000
(8)	200	0.9	(2.3)			X	0.990	1.000	0.046	0.058	0.053	1.000	1.000
(16)			(2.3)			X	0.751	1.000	0.030	0.184	0.051	1.000	1.000
(8)	200	0.5	(2.3)			X	1.000	1.000	0.050	0.052	0.051	1.000	1.000
(16)			(2.3)			X	1.000	1.000	0.043	0.108	0.051	1.000	1.000
(8)	200	0	(2.3)			X	1.000	1.000	0.050	0.051	0.049	1.000	1.000
(16)			(2.3)			X	1.000	1.000	0.052	0.054	0.051	1.000	1.000
(8)	200	-0.5	(2.3)			X	1.000	1.000	0.049	0.054	0.051	1.000	1.000
(16)			(2.3)			X	1.000	1.000	0.042	0.110	0.051	1.000	1.000
(8)	200	-0.9	(2.3)			X	1.000	0.990	0.046	0.057	0.051	0.833	1.000
(16)			(2.3)			X	1.000	0.755	0.030	0.181	0.051	0.414	1.000

Based on 50,000 replications. Original DGPs $(2.2)\; y_{s\tau} = -y_{s-1,\tau} + u_{s\tau}$, $(2.3)\; y_{s\tau} = -y_{s-2,\tau} + u_{s\tau}$ with $y_{s0} = 0$ and $(1-\phi L)u_{s\tau} = \varepsilon_{s\tau}$. t_{π_1}, $t_{\hat{\pi}_2}$, $t_{\hat{\pi}_3}$ and $t_{\hat{\pi}_4}$ tests for $H_0: \pi_1 = 0$, $H_0: \pi_2 = 0$, $H_0: \pi_3 = 0$ and $H_0: \pi_4 = 0$. *F34, F234* and *F1234* tests for $H_0: \pi_3 = \pi_4 = 0$, $H_0: \pi_2 = \pi_3 = \pi_4 = 0$ and $H_0: \pi_1 = \pi_2 = \pi_3 = \pi_4 = 0$.

Table 1.c

Proportion of times that the null is rejected (nominal size 5%).

Regression	4N	ϕ	DGP	(1-L)	(1+L)	(1+L^2)	$t_{\hat{\pi}_1}$	$t_{\hat{\pi}_2}$	$t_{\hat{\pi}_3}$	$t_{\hat{\pi}_4}$	*F34*	*F234*	*F1234*
(8)	100	0.9	(2.4)	X	X		0.050	0.049	1.000	1.000	1.000	1.000	1.000
(16)			(2.4)	X	X		0.059	0.050	0.999	0.999	1.000	1.000	1.000
(8)	100	0.5	(2.4)	X	X		0.050	0.047	1.000	0.908	1.000	1.000	1.000
(16)			(2.4)	X	X		0.048	0.050	1.000	0.839	1.000	1.000	1.000
(8)	100	0	(2.4)	X	X		0.049	0.048	1.000	0.051	1.000	1.000	1.000
(16)			(2.4)	X	X		0.049	0.049	1.000	0.025	1.000	1.000	1.000
(8)	100	-0.5	(2.4)	X	X		0.047	0.048	1.000	0.908	1.000	1.000	1.000
(16)			(2.4)	X	X		0.050	0.049	1.000	0.840	1.000	1.000	1.000
(8)	100	-0.9	(2.4)	X	X		0.049	0.048	1.000	1.000	1.000	1.000	1.000
(16)			(2.4)	X	X		0.049	0.057	0.999	0.999	1.000	1.000	1.000
(8)	200	0.9	(2.4)	X	X		0.048	0.048	1.000	1.000	1.000	1.000	1.000
(16)			(2.4)	X	X		0.058	0.051	1.000	1.000	1.000	1.000	1.000
(8)	200	0.5	(2.4)	X	X		0.050	0.050	1.000	0.998	1.000	1.000	1.000
(16)			(2.4)	X	X		0.051	0.052	1.000	0.993	1.000	1.000	1.000
(8)	200	0	(2.4)	X	X		0.050	0.049	1.000	0.049	1.000	1.000	1.000
(16)			(2.4)	X	X		0.051	0.052	1.000	0.024	1.000	1.000	1.000
(8)	200	-0.5	(2.4)	X	X		0.050	0.050	1.000	0.997	1.000	1.000	1.000
(16)			(2.4)	X	X		0.053	0.051	1.000	0.993	1.000	1.000	1.000
(8)	200	-0.9	(2.4)	X	X		0.049	0.049	1.000	1.000	1.000	1.000	1.000
(16)			(2.4)	X	X		0.050	0.055	1.000	1.000	1.000	1.000	1.000
(8)	100	0.9	(2.5)		X	X	0.694	0.048	0.049	0.049	0.048	0.049	0.310
(16)			(2.5)		X	X	0.258	0.045	0.050	0.049	0.050	0.049	0.148
(8)	100	0.5	(2.5)		X	X	1.000	0.050	0.049	0.051	0.048	0.049	1.000
(16)			(2.5)		X	X	1.000	0.045	0.046	0.075	0.050	0.051	0.928
(8)	100	0	(2.5)		X	X	1.000	0.048	0.045	0.056	0.049	0.051	1.000
(16)			(2.5)		X	X	1.000	0.045	0.027	0.180	0.049	0.051	1.000
(8)	100	-0.5	(2.5)		X	X	1.000	0.049	0.034	0.060	0.050	0.052	1.000
(16)			(2.5)		X	X	1.000	0.047	0.008	0.306	0.047	0.051	1.000
(8)	100	-0.9	(2.5)		X	X	1.000	0.050	0.029	0.058	0.048	0.053	1.000
(16)			(2.5)		X	X	1.000	0.056	0.003	0.342	0.047	0.056	1.000
(8)	200	0.9	(2.5)		X	X	0.995	0.050	0.049	0.052	0.049	0.050	0.781
(16)			(2.5)		X	X	0.803	0.050	0.051	0.054	0.050	0.048	0.373
(8)	200	0.5	(2.5)		X	X	1.000	0.049	0.051	0.052	0.052	0.050	1.000
(16)			(2.5)		X	X	1.000	0.050	0.048	0.080	0.051	0.050	1.000
(8)	200	0	(2.5)		X	X	1.000	0.050	0.045	0.058	0.051	0.052	1.000
(16)			(2.5)		X	X	1.000	0.050	0.027	0.197	0.048	0.049	1.000
(8)	200	-0.5	(2.5)		X	X	1.000	0.050	0.036	0.061	0.050	0.050	1.000
(16)			(2.5)		X	X	1.000	0.049	0.008	0.328	0.049	0.049	1.000
(8)	200	-0.9	(2.5)		X	X	1.000	0.051	0.030	0.060	0.051	0.054	1.000
(16)			(2.5)		X	X	1.000	0.057	0.003	0.361	0.049	0.052	1.000

Based on 50,000 replications. Original DGPs $(2.4)\ y_{s\tau} = y_{s-2,\tau} + u_{s\tau}$, $(2.5)\ y_{s\tau} = -y_{s-1,\tau} - y_{s-2,\tau} - y_{s-3,\tau} + u_{s\tau}$ with $y_{s0} = 0$ and $(1-\phi L)u_{s\tau} = \varepsilon_{s\tau}$. t_{π_1}, $t_{\hat{\pi}_2}$, $t_{\hat{\pi}_3}$ and $t_{\hat{\pi}_4}$ tests for $H_0 : \pi_1 = 0$, $H_0 : \pi_2 = 0$, $H_0 : \pi_3 = 0$ and $H_0 : \pi_4 = 0$. *F34, F234* and *F1234* tests for $H_0 : \pi_3 = \pi_4 = 0$, $H_0 : \pi_2 = \pi_3 = \pi_4 = 0$ and $H_0 : \pi_1 = \pi_2 = \pi_3 = \pi_4 = 0$.

Appendix

Due to the $I(1)$ property of one or two of the auxiliary variables $y_{s\tau}^{(1)}$, $y_{s\tau}^{(2)}$ and $y_{s\tau}^{(3)}$, when the underlying process is one of the set (2.1) to (2.5) and the remaining variables in the regression are stationary, the coefficients associated with these variables converge at different rates when (8) is estimated. To reflect this, it is useful to define the $(4+p) \times (4+p)$ scaling matrices M_1, M_2, M_3, M_4 and M_5, as:

$$M_1 = M_2 = diag\left[T, T^{1/2}, \cdots, T^{1/2}\right]; \quad M_3 = M_4 = diag\left[T, T, T^{1/2}, \cdots, T^{1/2}\right];$$

$$M_5 = diag\left[T, T, T, T^{1/2}, \cdots, T^{1/2}\right].$$

It is straightforward to see that the scaled OLS estimators for HEGY regression (8) can be summarized into the five different cases, as follows:

$$M_i \hat{\Pi}_i = \begin{bmatrix} T\hat{\Pi}_i^{NS} \\ T^{1/2}\hat{\Pi}_i^{S} \end{bmatrix}$$

$$= \begin{bmatrix} T^{-2}\sum_{s=1}^{4}\sum_{\tau=1}^{N}(Y_{s-1,\tau}^{(NSi)})(Y_{s-1,\tau}^{(NSi)})' & T^{-3/2}\sum_{s=1}^{4}\sum_{\tau=1}^{N}Y_{s-1,\tau}^{(NSi)}(Y_{s-1,\tau}^{(Si)})' \\ T^{-3/2}\sum_{s=1}^{4}\sum_{\tau=1}^{N}(Y_{s-1,\tau}^{(Si)})'Y_{s-1,\tau}^{(NSi)} & T^{-1}\sum_{s=1}^{4}\sum_{\tau=1}^{N}Y_{s-1,\tau}^{(Si)}(Y_{s-1,\tau}^{(Si)})' \end{bmatrix}^{-1} \times$$

$$\times \begin{bmatrix} T^{-1}\sum_{s=1}^{4}\sum_{\tau=1}^{N}Y_{s-1,\tau}^{(NSi)}\Delta_4 y_{s\tau} \\ T^{-1/2}\sum_{s=1}^{4}\sum_{\tau=1}^{N}Y_{s-1,\tau}^{(Si)}\Delta_4 y_{s\tau} \end{bmatrix} \quad i = 1, 2, 3, 4 \ and \ 5$$

$where:$

$$\hat{\Pi}_1^{NS} = [\hat{\pi}_1]; \hat{\Pi}_1^{S} = [\hat{\pi}_2, \hat{\pi}_3, \hat{\pi}_4, \phi_1^*, \cdots, \phi_p^*]'; Y_{s-1,\tau}^{(NS1)} = [y_{s-1,\tau}^{(1)}];$$

$$Y_{s-1,\tau}^{(S1)} = [y_{s-1,\tau}^{(2)}, y_{s-2,\tau}^{(3)}, y_{s-1,\tau}^{(3)}, \Delta_4 y_{s-1,\tau}, \cdots, \Delta_4 y_{s-p,\tau}]'$$

$$\hat{\Pi}_2^{NS} = [\hat{\pi}_2]; \hat{\Pi}_2^{S} = [\hat{\pi}_1, \hat{\pi}_3, \hat{\pi}_4, \phi_1^*, \cdots, \phi_p^*]'; Y_{s-1,\tau}^{(NS2)} = [y_{s-1,\tau}^{(2)}];$$

$$Y_{s-1,\tau}^{(S2)} = [y_{s-1,\tau}^{(1)} y_{s-2,\tau}^{(3)} y_{s-1,\tau}^{(3)}, \Delta_4 y_{s-1,\tau}, \cdots, \Delta_4 y_{s-p,\tau}]'$$

$$\hat{\Pi}_3^{NS} = [\hat{\pi}_3 \hat{\pi}_4]'; \hat{\Pi}_3^{S} = [\hat{\pi}_1, \hat{\pi}_2, \phi_1^*, \cdots, \phi_p^*]'; Y_{s-1,\tau}^{(NS3)} = [y_{s-2,\tau}^{(3)} y_{s-1,\tau}^{(3)}]';$$

$$Y_{s-1,\tau}^{(S3)} = [y_{s-1,\tau}^{(1)} y_{s-1,\tau}^{(2)}, \Delta_4 y_{s-1,\tau}, \cdots, \Delta_4 y_{s-p,\tau}]'$$

$$\hat{\Pi}_4^{NS} = [\hat{\pi}_1\hat{\pi}_2]'; \hat{\Pi}_4^{S} = [\hat{\pi}_3, \hat{\pi}_4, \phi_1^*, \cdots, \phi_p^*]'; Y_{s-1,\tau}^{(NS4)} = [y_{s-1,\tau}^{(1)} y_{s-1,\tau}^{(2)}]';$$
$$Y_{s-1,\tau}^{(S4)} = [y_{s-2,\tau}^{(3)} y_{s-1,\tau}^{(3)}, \Delta_4 y_{s-1,\tau}, \cdots, \Delta_4 y_{s-p,\tau}]'$$
$$\hat{\Pi}_5^{NS} = [\hat{\pi}_2\hat{\pi}_3\hat{\pi}_4]'; \hat{\Pi}_5^{S} = [\hat{\pi}_1, \phi_1^*, \cdots, \phi_p^*]'; Y_{s-1,\tau}^{(NS5)} = [y_{s-1,\tau}^{(2)} y_{s-2,\tau}^{(3)} y_{s-1,\tau}^{(3)}]';$$
$$Y_{s-1,\tau}^{(S5)} = [y_{s-1,\tau}^{(1)}, \Delta_4 y_{s-1,\tau}, \cdots, \Delta_4 y_{s-p,\tau}]'$$

As in the previous cases, i = 1, 2, 3, 4 and 5, corresponding to processes (2.1), (2.2), (2.3), (2.4) and (2.5) respectively. Note that due to the nonstationary elements of $Y_{s-1,\tau}^{(NSi)}$ and the stationary elements of $Y_{s-1,\tau}^{(Si)}$ it follows that $T^{-3/2}\sum_{s=1}^{4}\sum_{\tau=1}^{N} Y_{s-1,\tau}^{(NSi)}(Y_{s-1,\tau}^{(Si)})' \xrightarrow{p} 0$. Note also that when i = 1 and 2, $T^{-2}\sum_{s=1}^{4}\sum_{\tau=1}^{N}(Y_{s-1,\tau}^{(NSi)})(Y_{s-1,\tau}^{(NSi)})'$ is a 1×1, and when i = 3 and 4, it is a 2×2 diagonal matrix. Finally, when i =5, it is a 3×3 diagonal matrix due to the orthogonality of the nonstationary HEGY auxiliary variables.

To prove parts (10), (11) and (12), first note that the distribution of $t_{\hat{\pi}_1}, t_{\hat{\pi}_2}, t_{\hat{\pi}_3}$ and $t_{\hat{\pi}_4}$ could be obtained using:

$$t_{\hat{\pi}_i} = \frac{(4N)^{-1} Y_{-1}^{(i)}{}' Q_i \Delta_4 Y}{\hat{\sigma}\sqrt{(4N)^{-2} Y_{-1}^{(i)}{}' Q_i Y_{-1}^{(i)}}} \quad i = 1,2, \quad t_{\hat{\pi}_3} = \frac{(4N)^{-1} Y_{-2}^{(3)}{}' Q_3 \Delta_4 Y}{\hat{\sigma}\sqrt{(4N)^{-2} Y_{-2}^{(3)}{}' Q_3 Y_{-2}^{(3)}}}$$

$$t_{\hat{\pi}_4} = \frac{(4N)^{-1} Y_{-1}^{(3)}{}' Q_3 \Delta_4 Y}{\hat{\sigma}\sqrt{(4N)^{-2} Y_{-1}^{(3)}{}' Q_3 Y_{-1}^{(3)}}}$$

Where $Y_{-1}^{(1)}, Y_{-1}^{(2)}, Y_{-1}^{(3)}$ and $\Delta_4 Y$ are 4N×1 vectors with generic elements $y_{s-1,\tau}^{(1)}$, $y_{s-1,\tau}^{(2)}$, $y_{s-1,\tau}^{(3)}$ and $\Delta_4 y_{s\tau}$ respectively and Q_i is a 4N×4N matrix $Q_i = I - X_i (X_i' X_i)^{-1} X_i'$ with the columns of X_i having the elements of the stationary HEGY regressors in each case and the first p lags of $\Delta_4 y_{s\tau}$.

First, note that the HEGY stationary variables contained in X_i will take into account the noninvertible MA process induced by the use of Δ_4 (i.e (4.1), (4.2) and (4.3)), and the p lags of $\Delta_4 y_{s\tau}$ will take into account the autoregressive serial correlation $\phi(L)u_{s\tau} = \varepsilon_{s\tau}$. Hence as in Phillips and Oularis' Theorem 4.2 (1990), it follows that:

$$(4N)^{-2} Y_{-1}^{(i)}{}' Q_i Y_{-1}^{(i)} = (4N)^{-2} Y_{-1}^{(i)}{}' Y_{-1}^{(i)} + o_p(1) \quad i = 1,2,3$$
$$(4N)^{-2} Y_{-2}^{(3)}{}' Q_3 Y_{-2}^{(3)} = (4N)^{-2} Y_{-2}^{(3)}{}' Y_{-2}^{(3)} + o_p(1)$$
$$(4N)^{-1} Y_{-1}^{(i)}{}' Q_i \Delta_4 Y = (4N)^{-1} Y_{-1}^{(i)}{}' Q_i E = (4N)^{-1} Y_{-1}^{(i)}{}' E + o_p(1) \quad i = 1,2,3$$
$$(4N)^{-1} Y_{-2}^{(3)}{}' Q_3 \Delta_4 Y = (4N)^{-1} Y_{-2}^{(3)}{}' Q_3 E = (4N)^{-1} Y_{-2}^{(3)}{}' E + o_p(1)$$

Where E is a $4N \times 1$ vector with generic element $\varepsilon_{s\tau}$. Note also that it is possible to write:

$$(4N)^{-2} Y_{-1}^{(i)}{}' Y_{-1}^{(i)} = (4N)^{-2} \sum_{\tau=1}^{N} \sum_{s=1}^{4} \left[y_{s-1,\tau}^{(i)} \right]^2 = (4N)^{-2} \sum 4 Y_\tau{}' C_i Y_\tau + d_i \quad i = 1,2$$

$$(4N)^{-2} Y_{-1}^{(3)}{}' Y_{-1}^{(3)} = (4N)^{-2} \sum_{\tau=1}^{N} \sum_{s=1}^{4} \left[y_{s-1,\tau}^{(3)} \right]^2 = (4N)^{-2} \sum 2 Y_\tau{}' C_3 Y_\tau + d_3^*$$

$$(4N)^{-2} Y_{-2}^{(3)}{}' Y_{-2}^{(3)} = (4N)^{-2} \sum_{\tau=1}^{N} \sum_{s=1}^{4} \left[y_{s-2,\tau}^{(3)} \right]^2 = (4N)^{-2} \sum 2 Y_\tau{}' C_3 Y_\tau + d_3$$

$$(4N)^{-1} Y_{-1}^{(i)}{}' E = (4N)^{-2} \sum_{\tau=1}^{N} \sum_{s=1}^{4} y_{s-1,\tau}^{(i)} \varepsilon_{s\tau} = (4N)^{-1} \sum Y_\tau{}' C_i E_\tau \quad i = 1,2$$

$$(4N)^{-1} Y_{-2}^{(3)}{}' E = (4N)^{-2} \sum_{\tau=1}^{N} \sum_{s=1}^{4} y_{s-2,\tau}^{(3)} \varepsilon_{s\tau} = (4N)^{-1} \sum Y_\tau{}' C_3 E_\tau$$

$$(4N)^{-1} Y_{-1}^{(3)}{}' E = (4N)^{-2} \sum_{\tau=1}^{N} \sum_{s=1}^{4} y_{s-1,\tau}^{(3)} \varepsilon_{s\tau} = (4N)^{-1} \sum Y_\tau{}' C_3^* E_\tau$$

Using Lemma 1 and the following identities:

$$C_1 \Psi^*(1) = \psi(1) C_1 \quad C_2 \Psi^*(1) = \psi(-1) C_2 \quad C_3 \Psi^*(1) = b C_3 + a C_{3*}$$

$$C_1{}' C_1 C_1 = 16 C_1 \quad C_1{}' C_1 = 4 C_1 \quad C_2{}' C_2 C_2 = 16 C_2 \quad C_2{}' C_2 = 4 C_2$$

$$C_3{}' C_3 C_3 = 4 C_3 \quad C_3{}' C_3 C_3^* = 4 C_3^* \quad C_3^*{}' C_3 C_3 = -4 C_3^* \quad C_3^*{}' C_3 C_3^* = 4 C_3$$

$$C_3{}' C_3 = 2 C_3 \quad C_3^*{}' C_3 = -2 C_3 \quad C_3{}' C_3^* = 2 C_3 \quad C_3^*{}' C_3^* = 2 C_3.$$

It is possible to write:

$$(4N)^{-2} \sum 4 Y_\tau{}' C_1 Y_\tau \Rightarrow \frac{\sigma^2}{4} 16 \psi(1)^2 \int W(r)' C_1 W(r) dr = \sigma^2 16 \psi(1)^2 \int W^*(r)' C_1 W^*(r) dr$$

$$(4N)^{-1} \sum Y_\tau{}' C_1 E_\tau \Rightarrow \frac{\sigma^2}{4} 4 \psi(1) \int W(r)' C_1 dW(r) = \sigma^2 4 \psi(1) \int W^*(r)' C_1 dW^*(r)$$

$$(4N)^{-2} \sum 4 Y_\tau{}' C_2 Y_\tau \Rightarrow \frac{\sigma^2}{4} 16 \psi(-1)^2 \int W(r)' C_2 W(r) dr = \sigma^2 16 \psi(-1)^2 \int W^*(r)' C_2 W^*(r) dr$$

$$(4N)^{-1} \sum Y_\tau{}' C_2 E_\tau \Rightarrow \frac{\sigma^2}{4} 4 \psi(-1) \int W(r)' C_2 dW(r) = \sigma^2 4 \psi(-1) \int W^*(r)' C_2 dW^*(r)$$

where $W^*(r) = 1/\sqrt{4} W^*(r)$. And also:

$$(4N)^{-2}\sum 2Y_\tau ' C_3 Y_\tau \Rightarrow \frac{\sigma^2}{8}\int W(r)'\left(bC_3 + aC_3^*\right)' C_3\left(bC_3 + aC_3^*\right)W(r)dr =$$

$$= \frac{4\sigma^2}{8}\left(a^2 + b^2\right)\int W(r)'C_3 W(r)dr = \sigma^2\left(a^2 + b^2\right)\int W^*(r)'C_3 W^*(r)dr$$

$$(4N)^{-1}\sum Y_\tau ' C_3 E_\tau \Rightarrow \frac{\sigma^2}{4}\int W(r)'\left(bC_3 + aC_3^*\right)' C_3 dW(r) =$$

$$= \frac{2\sigma^2}{4}\left(b\int W(r)'C_3 dW(r) - a\int W(r)'C_3^* dW(r)\right) =$$

$$= \sigma^2\left(b\int W^*(r)'C_3 dW^*(r) - a\int W^*(r)'C_3^* dW^*(r)\right)$$

$$(4N)^{-1}\sum Y_\tau ' C_3^* E_\tau \Rightarrow \frac{\sigma^2}{4}\int W(r)'\left(bC_3 + aC_3^*\right)' C_3^* dW(r) =$$

$$= \frac{2\sigma^2}{4}\left(b\int W(r)'C_3^* dW(r) + a\int W(r)'C_3 dW(r)\right) =$$

$$= \sigma^2\left(b\int W^*(r)'C_3^* dW^*(r) + a\int W^*(r)'C_3 dW^*(r)\right)$$

where $W^*(r) = 1/\sqrt{4}W^*(r)$. Finally note that as $\hat{\sigma} \to \sigma$ and $d_i \to 0$, expressions in (10), (11) and (12) are easily obtained using $C_1 = v_1 v_1'$, $C_2 = v_2 v_2'$, $C_3 = v_3 v_3'$, $C_{3*} = v_3 v_{3*}'$, (7.a) and $F34 = \frac{1}{2}\left(t_{\hat{\pi}_3} + t_{\hat{\pi}_3}\right) + o_p(1)$.

Finally, the results of (13), (14.a) and (14.b) can be obtained following the methods with which the previous results were found, using the following identities:

$$\left(C_4\Psi^*(1)\right)' C_1\left(C_4\Psi^*(1)\right) = 4\psi(1)^2 C_1 \quad \left(C_4\Psi^*(1)\right)' C_2\left(C_4\Psi^*(1)\right) = 4\psi(-1)^2 C_2$$

$$\left(C_4\Psi^*(1)\right)' C = 2\psi(1)C_1 \quad \left(C_4\Psi^*(1)\right)' C_2 = 2\psi(-1)C_2$$

$$\left(C_5\Psi^*(1)\right)' C_2\left(C_5\Psi^*(1)\right) = 4\psi(-1)^2 C \quad \left(C_5\Psi^*(1)\right)' C_2 = 2\psi(-1)^2 C_2$$

$$\left(C_5\Psi^*(1)\right)' C_3\left(C_5\Psi^*(1)\right) = 2\left(b^2 + a^2\right)C_3 \quad \left(C_5\Psi^*(1)\right)' C_3 = (b+a)C_3 + (b-a)C_3^*$$

$$\left(C_5\Psi^*(1)\right)' C_3^* = (b+a)C_3 - (b-a)C_3^*.$$