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Rough Path Analysis: an introduction

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Contents

Preface

This master thesis gives an extensive introduction to the Rough Path Analysis theory presented by Terry Lyons in the late 90's, which provides a pathwise approach to stochastic calculus. The most important result of this mathematical theory states the continuity of the Itô map with respect to a general control if we move to a new framework where the notion of path is substituted by these more involved objects: the *rough paths*, and the metric also becomes more complex.

In the first chapter of this work, we give a historical introduction to topics related to the rough path theory, and also we motivate the problem of the lack of continuity of the so called Itô map by giving a counter-example where it is not continuous in the usual setting.

In the second chapter we present the concepts and basic elements of the rough path theory and we state and prove some results that will be needed in order to show the main result.

The third chapter can be seen as a direct application of the theory we present in chapter 2 for a particular case: we will compute the rough path associated to the Brownian motion. As we will notice, this will be the only point when probability appears along all the work, since the general theory is both analytical and algebraic. Probabilistic tools are only needed when we consider rough paths associated to stochastic processes.

In chapters 4 and 5, we finally show the main result: the *Universal Limit Theorem*, that states the continuity of the Itô map for rough path differential equations. For this, and also by the necessity to give a meaning to the equation in chapter 4, a new notion of integral (along a rough path) is deined and its relevant properties proved. Once the integral has been defined, we will be able to explain what a solution of a rough path differential equation is and prove the *Universal Limit Theorem.* This is carried out in chapter 5.

Finally, in chapter 6 we present the conclusions derived from this theory.

Chapter 1

Motivation

1.1 Integration theory

Consider the following differential equation

$$\begin{cases} \mathrm{d}Y(t) = f(Y(t))\,\mathrm{d}t + g(Y(t))\,\mathrm{d}X(t), \quad t > 0, \\ Y(0) = Y_0. \end{cases} \tag{1.1}$$

where Y and X are real valued functions, as much general as we can think about them for the moment. We will discuss later when the expression $\mathrm{d}X(t)$ is well defined if X is not smooth.

The function X is called a *control*, and then we say that (1.1) is a differential equation controlled by X. Equations like (1.1) are a generalization of

$$\begin{cases} \mathrm{d}Y(t) = f(Y(t))\,\mathrm{d}t, \quad t > 0, \\ Y(0) = Y_0; \end{cases} \tag{1.2}$$

which is an ordinary differential equation. If we consider $X(t) = W_t$, where $(B_t)_{t\in\mathbb{R}^+}$ is a Brownian motion, we obtain a stochastic differential equation.

We can consider these controlled equations as an extension of both the classical ordinary differential equations and the stochastic ones. That is, this is a general framework where a lot of particular cases are included.

If Y is a solution of (1.1), we would like it to satisfy

$$Y(t) = Y_0 + \int_0^t f(Y(t))\ \mathrm{d}t + \int_0^t g(Y(t))\ \mathrm{d}X_t, \quad t > 0, \tag{1.3}$$

but, we don't even know what the second integral means (for the first, recall that it is well defined as a Riemann integral if f is continuous and well defined in general if $f \circ Y$ is Lebesgue measurable).

Many integration theories have been developed in order to generalize Lebesgue integrals to be able to define expressions like the second integral in (1.3). We present briefly some of them.

1.1.1 Riemann-Stieltjes integral

First of all, let's recall the notion of *partition* on an interval. Throughout this work, I will be a closed interval of the real line, this is $I = [a, b]$, and p a real number, $1 \leq p < \infty$. For a function X defined on I, we will set $X_t = X(t)$, for $t \in I$.

Definition 1.1.1. *For a given interval $I = [a, b]$, we define a partition π over I as the sequence $\{x_i\}_{i=0}^n$ such that*

$$x_0 = a < x_1 < x_2 < \cdots < x_n = b,$$

and we will denote as $\Pi(I)$ the set of all the partitions over I.

Now, we can define the *Riemann-Stieltjes* integral.

Definition 1.1.2. *Given two functions $f, g : [a, b] \to \mathbb{R}$, and a partition $\pi_n = \{x_i\}_{i=0}^{r_n}$ with $x_0 = a$, $x_{r_n} = b$ and $x_i < x_{i+1}$, we will say that f is Riemann-Stieltjes integrable with respect to g if the sum*

$$\sum_{j=1}^{r_n} f(z_j) \left(g(x_j) - g(x_{j-1}) \right),$$

has a limit as $|\pi_n| := \max_{i=1,\ldots,r_n} |x_i - x_{i-1}| \to 0$, where $z_j \in [x_{j-1}, x_j]$. In this case, we will denote the limit by $\int_a^b f\ \mathrm{d}g$.

We can give a characterization of a situation where this sum converges in terms of the so called *bounded variation* of the g function. First recall that for a partition of $[a, b]$, $\pi = \{x_i\}_{i=0}^n$ with $x_0 = a$, $x_n = b$ and $x_i < x_{i+1}$, we can define the variation of a function f associated to π

as

$$Vf(\pi) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \, ,$$

and the total variation of f as

$$Vf = \sup\{V(f,\pi) : \pi \text{ is a partition of } [a,b]\}. \tag{1.4}$$

We will say f is of bounded variation if $Vf < \infty$.

Now we can describe easily when this Riemann-Stieltjes sum converge, and hence, when we are able to well define $\int_a^b f \, \mathrm{d}g$.

Theorem 1.1.1. *If $f : [a,b] \to \mathbb{R}$ is continuous and $g : [a,b] \to \mathbb{R}$ is of bounded variation, then the Riemann-Stieltjes sum converges.*

The proof of this theorem and more information about the Riemann-Stieltjes integral can be found in [1], theorem 7.27.

At some moment, we will want to define similar integrals in the case we are integrating with respect to stochastic processes, for instance Brownian motion, i.e. $g(t) = W_t$. That is, we want to define $\int_a^b u_t \, \mathrm{d}W_t$, where $(u_t)_{t\geq 0}$ may be also a stochastic process. However, it is a well known fact that Brownian motion paths are almost-surely not of bounded variation. Hence, the Riemann-Stieltjes integral cannot be used

Recall that in the Lebesgue integration theory we do not need the integrand to be continuous, but for these Riemann-Stieltjes sums f has to be continuous in order to them to converge. What we should expect is to improve these result and obtain convergence for more general functions f.

1.1.2 Young integral

The Young integral is defined as the same limit as Riemann-Stieltjes integral. The change that Young ([10], 1936) introduced was a different condition on f and g that assures the convergence of the sum.

In order to characterize a new class of functions f (and controls g) for which the integral $\int f \, \mathrm{d}g$ is well defined, we need to introduce the notion of p-variation, which is a generalization of the concept of the *total variation* of a function we set in (1.4)

Definition 1.1.3. *Let I be an interval and $p \in [1, \infty)$. Given a function $X : I \to \mathbb{R}^d$, the p-variation of X on I is*

$$||X||_{p,I} := \left(\sup_{\pi \in \Pi(I)} \sum_{j=0}^{r-1} \left| X_{t_j} - X_{t_{j+1}} \right|^p \right)^{1/p} \in [0, \infty].$$

If there is risk for confusions, we will simply write $||X||_p$.

If a path X satisfies that $||X||_{p,I} < \infty$ we will write $X \in \nu^p(I)$, or simply $X \in \nu^p$.

In the sequel, we will only consider continuous functions X also called continuous *paths*.

We introduce an important notion that will be used in the proof of the theorem from Young and later, in the development of the theory of rough paths, the notion of a *control function.*

Definition 1.1.4. *Let $\Delta_{[a,b]} := \{(s,t) \in [a,b]^2 : a \leq s \leq t \leq b\}$ (the simplex defined on $[a,b]$). Then, a control function on $[a,b]$ is a continuous function $\omega : \Delta_I \to \mathbb{R}^+$, which satisfies the relation*

$$\omega(s,t) + \omega(t,u) \leq \omega(s,u), \;\; \forall a \leq s \leq t \leq u \leq b, \tag{1.5}$$

and such that $\omega(t,t) = 0$ for any $a \leq t \leq b$.

Notice that, although they share the name, this control function has no relation to the control defined previously in the introduction of the integration theory.

Now we can state the result from Young [10].

Theorem 1.1.2. *Let $f, g : [0,T] \to \mathbb{R}^d$ be such that $f \in \nu^p([0,T])$ and $g \in \nu^q([0,T])$ such that $\frac{1}{p} + \frac{1}{q} > 1$. Then, f is Riemann-Stieltjes integrable with respect to g on $[0,T]$.*

Proof. Let us define an unitary renormalization of both functions f and g, that is we will deal with $\tilde{f}$ and $\tilde{g}$ defined as

$$\tilde{f}(x) := \frac{f(x)}{||f||_{p,[0,T]}}, \text{ and } \tilde{g}(x) := \frac{g(x)}{||g||_{q,[0,T]}}.$$

Let us also define the following function $\omega : \Delta_{[0,T]} \to [0,2]$ as

$$\omega(x,y) = \frac{||f||_{p,[x,y]}}{||f||_{p,[0,T]}} + \frac{||g||_{q,[x,y]}}{||g||_{q,[0,T]}}.$$

It is clear that ω is a control for the p-variation of $\tilde{f}$ and also for the q-variation of $\tilde{g}$.

We consider a partition $\pi = \{t_i\}_{i=0}^r \in \Pi([0,T])$. For this partition, we choose t^* as follows:

$$t^* := \begin{cases} t_1 & \text{if } r = 2, \\ t_i \text{ such that } \omega(t_{i-1}, t_{i+1}) \leq \frac{2}{r-1}\omega(0,T) & \text{if } r > 2. \end{cases}$$

Let's see that such t^* indeed exists. If $r = 2$ then this is clear, and the inequality proposed for $r > 2$ holds in this case too. If $r > 2$, we argue by contradiction. If that inequality was false for any $i = 1, \ldots, r-1$, we would have $2\omega(0,T) < (r-1)\omega(t_{i-1}, t_{i+1})$. Summing over every i, we have

$$\begin{aligned}
&2\cancel{(r-1)}\omega(0,T) < \cancel{(r-1)}\sum_{i=1}^{r-1} \omega(t_{i-1}, t_{i+1}) \Rightarrow \\
&\Rightarrow 2\omega(0,T) < \sum_{i=1}^{r-1} \omega(t_{i-1}, t_{i+1}) = \\
&= \begin{cases} \sum_{k=1}^{\frac{r}{2}} \omega(t_{2(k-1)}, t_{2k}) + \sum_{k=1}^{\frac{r}{2}-1} \omega(t_{2k-1}, t_{2k+1}) & \text{for } r \text{ even}, \\ \sum_{k=1}^{\frac{r-1}{2}} \omega(t_{2(k-1)}, t_{2k}) + \sum_{k=1}^{\frac{r-1}{2}} \omega(t_{2k-1}, t_{2k+1}) & \text{for } r \text{ odd}, \end{cases} \leq \\
&\overset{S.A.}{\leq} \begin{cases} \omega(0,T) + \omega(t_1, t_{r-1}) & \text{for } r \text{ even}, \\ \omega(0, t_{r-1}) + \omega(t_1, T) & \text{for } r \text{ odd}, \end{cases} \leq 2\omega(0,T).
\end{aligned}$$

where by *S.A.* we mean we have used the sub-additivity of the control ω. Thus, we obtain $\omega(0,T) < \omega(0,T)$, which is a contradiction. Hence, such t^* must exist.

Now we consider a new partition π^* obtained by removing the point t^* from the original partition, that is, $\pi^* = \{t_j^*\}_{j=1}^{r-1}$, where $t_j^* = \begin{cases} t_j & \text{for } j < i^*, \\ t_{j+1} & \text{for } j \geq i^*, \end{cases}$ where i^* is the index of the removed point. We denote $\tilde{f}_{s,t}^1 = \tilde{f}(t) - \tilde{f}(s)$ (this will be a common notation all along this work, as we will see later).

For arbitrary functions φ, ψ, the integral relative to the partition π is

$$\int_\pi \varphi \, \mathrm{d}\psi = \sum_{i=0}^{r-1} \varphi(t_i)(\psi(t_{i+1}) - \psi(t_i)),$$

so the Riemann-Stieltjes limit can be viewed as $\lim\limits_{|\pi| \to 0} \int_\pi \varphi \, \mathrm{d}\psi$.

With all this notation, we have

$$\left|\int_\pi \tilde f \,\mathrm{d}\tilde g - \int_{\pi^*} \tilde f \,\mathrm{d}\tilde g\right| = \left|\sum_{i=0}^{r-1} \tilde f(t_i)\tilde g^1_{t_i,t_{i+1}} - \sum_{j=0}^{r-2} \tilde f(t_j^*)\tilde g^1_{t_j^*,t_{j+1}^*}\right| =$$
$$= \left|\tilde f(t_{i-1})\tilde g^1_{t_{i-1},t_i} + \tilde f(t_i)\tilde g^1_{t_i,t_{i+1}} - \tilde f(t_{i-1})\tilde g^1_{t_{i-1},t_{i+1}}\right| =$$
$$= \left|\tilde f(t_{i-1})(\tilde g(t_i) - \cancel{\tilde g(t_{i-1})}) + \tilde f(t_i)(\tilde g(t_{i+1}) - \tilde g(t_i)) - \tilde f(t_{i-1})(\tilde g(t_{i+1}) - \cancel{\tilde g(t_{i-1})})\right| =$$
$$= \left|\left(\tilde f(t_i) - \tilde f(t_{i-1})\right)(\tilde g(t_{i+1}) - \tilde g(t_i))\right| \le \omega(t_{i-1},t_i)^{1/p}\omega(t_i,t_{i+1})^{1/q} \le$$
$$\le \omega(t_{i-1},t_{i+1})^{1/p+1/q} \le \left(\frac{2}{r-1}\right)^{1/p+1/q} \omega(0,T),$$

where $t_i = t^*$, and the last inequality is due to the fact that $(s,t) \to \omega(s,t)$ is increasing on t and decreasing on s.

We can continue removing points of this partition this way, until there are just two points (the boundary points) on it. Then,

$$\int_{\pi\setminus\cup_{i=1}^{r-2}\{t_i^*\}} \tilde f \,\mathrm{d}\tilde g = \tilde f(0)(\tilde g(t) - \tilde g(0)) = \int_0^t \tilde f(0)\,\mathrm{d}\tilde g.$$

Therefore,

$$\left|\int_\pi \left(\tilde f - \tilde f(0)\right)\mathrm{d}\tilde g\right| \le 2^{1/p+1/q}\sum_{k=1}^{r-2}\frac{1}{(k)^{1/p+1/q}}\omega(0,T) \le 2^{1/p+1/q}\underbrace{\sum_{k=1}^{\infty}\frac{1}{(k)^{1/p+1/q}}}_{=\zeta(1/p+1/q)}\omega(0,T) =$$
$$= 2^{1/p+1/q}\zeta(1/p+1/q)\omega(0,T) =: c_{p,q,T},$$

where ζ is the known Riemann zeta function.

Since $|f(0)| \le ||f||_\infty$, by undoing the renormalization, we have

$$\left|\int_\pi f \,\mathrm{d}g\right| \le c_{p,q,T}\left(||f||_\infty + ||f||_{p,[0,T]}\right)||g||_{q,[0,T]} \tag{1.6}$$

It is known ([6], proposition 1.14) that given $g \in \nu^q(I)$, for any $q' > q$ the linear piecewise approximation of g coinciding at the points of a partition $\pi \in \Pi(I)$, g^π converges in the norm of the q'-variation to q. So, now we take $q' > q$ such that $1/p + 1/q' > 1$ still holds, and a limiting sequence $g(n)$ converging to g in the q'-variation norm. Then, since $g(n)$ is linear-piecewise it has bounded variation for any n. By the Stieltjes result

$$\int_0^T f \, \mathrm{d}g(n)$$

is well defined, this is, the Riemann-Stieltjes series converge as $|\pi| \to 0$. On the other hand, by (1.6), we have

$$\begin{aligned} \sup_{\pi \in \Pi([0,T])} \left| \int_\pi f \, \mathrm{d}g(n) - \int_\pi f \, \mathrm{d}g \right| \leq \\ \leq c_{p,q,T} \left(||f||_\infty + ||f||_{p,[0,T]} \right) ||g(n) - g||_{q',[0,T]} \, . \end{aligned} \tag{1.7}$$

The right hand side of the last expression tends to zero as n tends to infinity since $g(n)$ converges to g in the norm of the q'-variation. Therefore, $\int_0^T f \, \mathrm{d}g$ is well defined as

$$\lim_{|\pi| \to 0} \int_\pi f \, \mathrm{d}g(n),$$

by (1.7).

□

Remark. The weaker conditions we put on the function f, the stronger will be needed on g in order to define $\int_a^b f \, \mathrm{d}g$, as a Young integral.

We can ask ourselves if we can apply Young's conditions in order to define integrals with respect to the Brownian motion. The answer is that it depends on the function we want to integrate. It is known that Brownian motion paths have finite p-variation for $p \geq 2$. So, there are some cases where the integral is not well defined, for instance

$$\int_a^b W_s \, \mathrm{d}W_s$$

cannot be defined this way (because $\frac{1}{p} + \frac{1}{p} = \frac{2}{p} \leq 1$).

1.1.3 Itô integral

Now we move to a probabilistic framework. Let us consider a probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F} = (\mathcal{F}_t, t \geq 0)$ is the natural filtration associated to a Brownian motion $W = (W_t)_{t \in [0,T]}$ for a fixed $T > 0$.

The Itò integral provides a meaning for $\int_0^t X_t \, \mathrm{d}W_t$ for a large set of stochastic processes X. The construction of this integral is similar to the construction of the Lebesgue integral, by considering simple X processes, as follows.

Definition 1.1.5. *A stochastic process $X = (X_t)_{t\in[0,T]}$ is said to be a step process if it can be written as*

$$X_t = \sum_{j=1}^{n} x_j \mathbb{I}_{[t_{j-1},t_j)}(t), \tag{1.8}$$

for some n, where $0 = t_0 < \ldots < t_j < t_{j+1} < \ldots < t_n = T$ and where x_j are $\mathcal{F}_{t_{j-1}}$-measurable random variables such that $\mathbb{E}\left[x_j^2\right] < \infty$.

The set of all step processes is denoted by $\mathcal{E}$.

For $X \in \mathcal{E}$, we can give the expression $\int_0^T X_t \, \mathrm{d}W_t$ a meaning.

Definition 1.1.6. *Let $X = (X_t)_{t\in[0,T]} \in \mathcal{E}$ and suppose that X can be written as in* (1.8)*. Then we define*

$$\int_0^T X_t \, \mathrm{d}W_t := \sum_{j=1}^{n} x_j (W_{t_j} - W_{t_{j-1}}). \tag{1.9}$$

We would like to enlarge the set $\mathcal{E}$ of stochastic process where we can define the integral. For this, we consider the following space of processes.

Definition 1.1.7. *$L^2_{a,T}$ is the set of stochastic processes $X = (X_t)_{t\in[0,T]}$ such that*

- *X is adapted and jointly-measurable in (t, ω) with respect to the σ-field $\mathcal{B}([0,T]) \otimes \mathcal{F}$,*
- *X is square integrable, i. e. $\int_0^T \mathbb{E}\left[X_t^2\right] \, \mathrm{d}t < \infty$.*

It can be seen that $L^2_{a,T}$ is a Banach space with norm $||X||_{L^2_{a,T}} = \left[\int_0^T \mathbb{E}\left[X_t^2\right] \, \mathrm{d}t\right]^{1/2}$, and $\mathcal{E}$ is a closed subspace of $L^2_{a,T}$. Furthermore, $\mathcal{E}$ is dense on $L^2_{a,T}$ and hence, for any $X \in L^2_{a,T}$ we can find a sequence $(X(n))_{n\in\mathbb{N}} \subset \mathcal{E}$ such that

$$L^2_{a,T} - \lim_{n\to\infty} X(n) = X.$$

The proof of this can be found in [4], lemma 4.3.3.

This fact allow us to define the stochastic Itô integral for processes in $L^2_{a,T}$ as follows.

Definition 1.1.8. *Let $X = (X_t)_{t\in[0,T]} \in L^2_{a,T}$. Then, the stochastic integral of X with respect to the Brownian motion W is given by*

$$\int_0^T X_t \, \mathrm{d}W_t := L^2_{a,T} - \lim_{n\to\infty} \int_0^T X_t(n) \, \mathrm{d}W_t, \tag{1.10}$$

where $X(n) = (X_t(n))_{t\in[0,T]} \subset \mathcal{E}$ and $X = L^2_{a,T} - \lim_{n\to\infty} X(n)$.

It can be verified that this is well defined, this is, that the limit in (1.10) does not depend on the approximating sequence.

Remark. This integral, introduced by Itô in [3], is called the *Itô integral.* Notice that the integral

$$\int_0^T W_t \, \mathrm{d}W_t$$

in the Itô sense does exist, since $W = (W_t)_{t\in[0,T]} \in L^2_{a,T}$.

However, if we consider a fractional Brownian motion with Hurst parameter $H > 1/2$, W^H, the integral $\int_0^T W_t^H \, \mathrm{d}W_t^H$ is well defined in Young's sense but not in the Itô sense (the process $(W_t^H)_{t\geq 0}$ is not even a martingale). So, the Itô and the Young integrals are not related to each other.

1.2 Continuity of the Itô map

Consider the equation (1.1) with $f = 0$

$$\begin{cases} \mathrm{d}Y(t) = g(Y(t)) \, \mathrm{d}X(t), \quad t > 0, \\ Y(0) = Y_0; \end{cases} \tag{1.11}$$

and assume that there exists a unique solution Y. We can define a map I depending on X, g and Y_0, which image is the solution of the equation, this is if (1.11) holds, then

$$I(X, g, Y_0) = Y.$$

This map is called the *Itô map.*

One question that concern us is the continuity of the Itô map with respect to X. Proving that I is continuous would be a nice result. If we have two controls whose distance is small, the corresponding solutions will be also close. This is a fact that one should expect. For instance, for numerical approximations of controlled differential equations, it is crucial assuring the continuity of the Itô map, since if not, the (approximative) result $\tilde{Y}$ we obtain by using an approximation of the control X, $\tilde{X}$ may be very different from the exact solution Y of the controlled equation.

If the control X takes values in a one-dimensional space, let's say $\mathbb{R}$, then one can see that the Itô map is continuous with the usual topology (see for instance, [6]), but if the image of X is in $\mathbb{R}^2$, then there are examples which show this lack of continuity, as below.

Example. (Lack of continuity of the Itô map for $d = 2$ and uniform convergence topology)

Consider $(X_1, X_2) = X : [0,T] \to \mathbb{R}^2$, $(Y_1, Y_2, Y_3) = Y : [0,T] \to \mathbb{R}^3$, and a function $g : \mathbb{R}^3 \to \mathbb{R}^{3\times 2}$ given by

$$g \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -b & a \end{pmatrix}.$$

With this choice, equation (1.11) can be written componentwise as

$$\begin{cases} \mathrm{d}Y_1(t) = \mathrm{d}X_1(t), \\ \mathrm{d}Y_2(t) = \mathrm{d}X_2(t), \\ \mathrm{d}Y_3(t) = -Y_2(t)\,\mathrm{d}X_1(t) + Y_1(t)\,\mathrm{d}X_2(t). \end{cases} \tag{1.12}$$

Considering $Y_0 = (1, 0, 0)$, and for any $n \in \mathbb{N}$ a control

$$X^{(n)} = (X_1^{(n)}(t), X_2^{(n)}(t)) = \frac{1}{n}(\cos(n^2 t), \sin(n^2 t)),$$

we have that the solution of (1.12) is

$$Y^{(n)}(t) = (X_1^{(n)}(t), X_2^{(n)}(t), Y_3^{(n)}(t))$$

for

$$
\begin{aligned}
Y_3^{(n)}(t) = & -\int_0^t \frac{1}{\cancel{n}} \sin(n^2 t)\cancel{n} \cdot (-\sin(n^2 t)) \ \mathrm{d}t + \int_0^t \frac{1}{\cancel{n}} \cos(n^2 t)\cancel{n} \cos(n^2 t) \ \mathrm{d}t = \\
& = \int_0^t \mathrm{d}t = t
\end{aligned}
$$

since $(\mathrm{d}X_1^{(n)}(t), \mathrm{d}X_2^{(n)}(t)) = n(-\sin(n^2 t), \cos(n^2 t))$.

Therefore, the solution, for any n is $Y^{(n)}(t) = (X_1^{(n)}(t), X_2^{(n)}(t), t)$.

For our choice of X, is clear that $X^{(n)}$ converges to (0,0) uniformly as $n \to \infty$, but it does not occur the same with the convergence of $Y^{(n)}$, whose uniform limit is $(0, 0, t)$. If the Itô map were continuous, that would not happen, and therefore, for this choices and for the *uniform convergence* topology, it is not continuous.

One of the highlights of the theory of rough paths is the *Universal Limit Theorem*, which, in a sense to be made precise later, will assure the continuity of the Itô map in a new context, considering a topology different from the usual one, and a richer structure that extend the notion of path, the *rough path*.

Chapter 2

Definitions and basic results

In this chapter, we will try to explain briefly what a rough path it is. For this, we need to define several mathematical objects and to mention some facts about them. Some short proofs will be given, but, due to their lenght, others appear at the end. In this way, the reader will be able to get the definition of rough path without getting disturbed with technicalities.

2.1 Basic definitions

Throughout this section, I will denote a closed interval of the real line this is $I = [a, b]$, and p a real number, $1 \leq p < \infty$. We recall the notation $X_t := X(t)$, $t \in I$, introduced in chapter 1, and we will use the definition of partition given in Definition 1.1.1. We will also write $X_{s,t}$ for the increments $X_t - X_s$, $a \leq s \leq t \leq b$.

We will work with a particular family of continuous functions (we will talk about *paths* rather than functions) satisfying some properties. One of them is *bounded p-variation* as defined in Definition 1.1.3, and the second one is α-Hölder continuity.

2.1.1 p-variation and α-Hölder continuity

The p-variation norm defined above can be viewed as a control function over I. In fact, we have the following proposition

Proposition 2.1.1. *Let $I = [a, b]$ and let $a \leq s \leq t \leq b$. Define $\omega_X(s, t) := ||X||^p_{p,[s,t]}$, for $X \in \nu^p(I)$. Then ω_X is a control function on $[a, b]$.*

Proof. By definition of p-variation norm and since $\Pi(I) \cup \Pi(J) \subsetneq \Pi(I \cup J)$, we clearly have that relation (1.5) is satisfied. Also from the definition, we have that $\omega_X(s,s) = 0$ for $s \in [a,b]$. So, we just have to take care about the continuity of ω_X. We show that for an increasing sequence $\{t_n\}_n$ converging to t, we have $\omega_X(s,t_n) \to \omega_X(s,t)$ which will prove the left-continuity on t.

We consider a partition $\{s = \tau_0 < \tau_1 < \ldots < \tau_r = t\}$ such that $\tau_{r-1} < t_n < t$.

Then, if we denote $X_{s,t} = X_t - X_s$, we have that $|X_{s,u}| \leq |X_{s,t}| + |X_{t,u}|$ for $a \leq s \leq t \leq u \leq b$. Hence,

$$\sum_{l=1}^{r} |X_{\tau_{l-1},\tau_l}|^p = \sum_{l=1}^{r-1} |X_{\tau_{l-1},\tau_l}|^p + |X_{\tau_{r-1},t}|^p \leq \sum_{l=1}^{r-1} |X_{\tau_{l-1},\tau_l}|^p + [|X_{\tau_{r-1},t_n}| + |X_{t_n,t}|]^p =$$
$$= \sum_{l=1}^{r-1} |X_{\tau_{l-1},\tau_l}|^p + |X_{\tau_{r-1},t_n}|^p + o\left(|X_{t_n,t}|\right) \longrightarrow \sum_{l=1}^{r-1} |X_{\tau_{l-1},\tau_l}|^p + |X_{\tau_{r-1},t}|^p,$$

as $n \to \infty$ since $X_{t,t} = X_t - X_t = 0$. Therefore, considering the supremum over all the partitions in the expresion above, we have proved the left-continuity on t. A similar procedure can be done in order to show right-continuity on t and continuity on t. □

Now, we are able to show the relation between bounded p-variation of a path and Hölder continuity. First, we recall the following definition.

Definition 2.1.1. *Let $X : I \to \mathbb{R}^n$, $\alpha \in (0,1)$. We say that X is α-Hölder continuous on I if*

$$\sup_{s \neq t} \frac{|X(s) - X(t)|}{|s-t|^\alpha} < +\infty.$$

Proposition 2.1.2. *If $X : I \to \mathbb{R}^n$ is a path with finite p-variation, then there exists a reparametrisation of I such that X is $\frac{1}{p}$-Hölder continuous on I.*

Proof. Assume $I = [0,T]$ (if not, we can linearly biject I to this form). If we consider the control function defined before $\omega_X(s,t) = ||X||^p_{p,[s,t]}$, we have that the function

$$\varphi : I \longrightarrow I,$$
$$t \longrightarrow \omega_X(0,t) \frac{T}{\omega_X(0,T)},$$

is increasing. Clearly $\varphi(0) = 0$ and $\varphi(T) = T$. Hence, φ is a bijection from I into itself, so φ^{-1} is well-defined.

Therefore, for $s \leq t$ in $[0,T]$, we have

$$\left|X_{\varphi^{-1}(s)} - X_{\varphi^{-1}(t)}\right|^p \leq \omega_X(\varphi^{-1}(s), \varphi^{-1}(t)) \leq \omega_X(0, \varphi^{-1}(t)) - \omega_X(0, \varphi^{-1}(s)) = \\ = \frac{\omega_X(0,T)}{T}(t-s) =: C^p(t-s).$$

Therefore we have $\left|X_{\varphi^{-1}(s)} - X_{\varphi^{-1}(t)}\right| \leq C\,|t-s|^{1/p}$ and hence

$$\sup_{s\neq t} \frac{\left|X_{\varphi^{-1}(s)} - X_{\varphi^{-1}(t)}\right|}{|t-s|^{1/p}} \leq C < \infty.$$

□

2.1.2 Multiplicative functionals

The aim of this section is to define the notion of *multiplicative functional*, a general object from which a *rough path* will be a particular case. For this, we need some preliminaries.

Definition 2.1.2. *The space of formal series of tensors of $\mathbb{R}^d$ is defined as the space of sequences*

$$T(\mathbb{R}^d) := \{(a_0, a_1, \ldots) \text{ such that } a_n \in (\mathbb{R}^d)^{\otimes n}\, \forall n \geq 0\},$$

endowed with two operations: a sum

$$(a_0, a_1, \ldots) + (b_0, b_1, \ldots) = (a_0 + b_0, a_1 + b_1, \ldots), \tag{2.1}$$

and a product

$$(a_0, a_1, \ldots)(b_0, b_1, \ldots) = (c_0, c_1, \ldots) \tag{2.2}$$

where $c_n = \sum\limits_{k=0}^{n} a_k b_{n-k}$.

Now, we consider a *finite* version of the space of formal series of tensors of $\mathbb{R}$, and a finite version of the signature of a path.

Definition 2.1.3. *Given $n \in \mathbb{N}$, the truncated tensor algebra of order n of $\mathbb{R}^d$ is*

$$T^n(\mathbb{R}^d) := \{(a_0, a_1, \ldots, a_n) \text{ such that } a_n \in (\mathbb{R}^d)^{\times n}\, \forall n \geq 0\},$$

endowed with the two operations (2.1) *y* (2.2) *of Definition 2.1.2.*

Definition 2.1.4. *Given* $n \in \mathbb{N}$ *and a continuous map*

$$X : \Delta_{[0,T]} \to T^n(\mathbb{R}^d),$$

let $X^i_{s,t} = X^i(s,t)$*. Then, the function defined by*

$$X_{s,t} = (X^0_{s,t}, X^1_{s,t}, \ldots, X^n_{s,t}) \in T^n(\mathbb{R}^d),$$

is called a multiplicative functional of degree n *in* $\mathbb{R}^d$ *if* $X^0 \equiv 1$ *and* X *satisfies the known Chen's relation, that is,*

$$X_{s,u}X_{u,t} = X_{s,t} \quad \text{for} \ \ 0 \leq s \leq u \leq t \leq T, \tag{2.3}$$

where the product on the left hand side is defined in (2.2)*.*

Remark. Any multiplicative functional can be seen, and in fact, this is the way it will be considered, as a continuous path over I.

First of all, notice that given $(a_0, a_1, \ldots, a_n) \in T^n(\mathbb{R}^d)$ we can find an inverse element in $T^n(\mathbb{R}^d)$ defined by

$$(a_0, a_1, \ldots, a_n)^{-1} = \frac{1}{a_0} \sum_{k=0}^{n} \left((1, 0, \ldots, 0) - (1, a_1/a_0, \ldots, a_n/a_0)\right)^n,$$

whenever $a_0 \neq 0$.

But, since a multiplicative functional X is defined as satisfying $X^0 \equiv 1 \neq 0$, we can always write X^{-1}.

Hence, if we see X as a path in $T^n(\mathbb{R}^d)$, defined by

$$X : [0,T] \longrightarrow T^n(\mathbb{R}^d),$$
$$t \longrightarrow X_{0,t}$$

it turns out that we have characterised the multiplicative functional, since, by Chen's relation we have $X_{s,t} = (X_{0,s})^{-1}X_{0,t}$.

2.1.3 Spaces of rough paths

As for paths, we can define a condition on regularity of the multiplicative functionals.

Definition 2.1.5. *Given $p \in [1, \infty)$, $n \in \mathbb{N}$ and a control function ω on $[0, T]$. We say that the multiplicative functional $X : \Delta_{[0,T]} \to T^n(\mathbb{R}^d)$ has finite p-variation on Δ_T controlled by ω if*

$$\left|\left|X^i_{s,t}\right|\right| \leq \omega(s,t)^{i/p}$$

for any $i = 1, \ldots, n$ and any $(s,t) \in \Delta_{[0,T]}$, where $||\cdot||$ denotes the Hilbert-Schmidt norm.

If it is clear from the context, sometimes we will omit the reference to the control ω.

We are now able to define what a rough path is.

Definition 2.1.6. *Let $p \in [1, \infty)$. A p-rough path in $\mathbb{R}^d$ is a multiplicative functional of degree $\lfloor p \rfloor$ (where $\lfloor p \rfloor$ is the biggest integer smaller or equal than p) in $\mathbb{R}^d$ with finite p-variation. The space of p-rough paths is denoted by $\Omega_p(\mathbb{R}^d)$.*

Once we have defined the space of rough paths, we should verify that it satisfies good properties that make it useful. The first thing we would expect for a space of functions is to be a Banach space for some metric.

However, we can see that $\Omega_p(\mathbb{R}^d)$ is not a vector space endowed with the sum of rough paths as functions. In fact, given $X, Y \in \Omega_p(\mathbb{R}^d)$, since they satisfy Chen relation (2.3) we should have, for any $0 \leq s \leq u \leq t \leq T$,

$$X_{s,t} = X_{s,u} X_{u,t},$$
$$Y_{s,t} = Y_{s,u} Y_{u,t}.$$

Hence, we will have

$$X_{s,t} + Y_{s,t} = X_{s,u} X_{u,t} + Y_{s,u} Y_{u,t} \tag{2.4}$$

also, if $X + Y \in \Omega_p(\mathbb{R}^d)$, once again by Chen relation, the equation

$$X_{s,t} + Y_{s,t} = (X_{s,u} + Y_{s,u})(X_{u,t} + Y_{u,t}) = X_{s,u} X_{u,t} + Y_{s,u} Y_{u,t} + X_{s,u} Y_{u,t} + Y_{s,u} X_{u,t} \tag{2.5}$$

should hold. This would implify

$$X_{s,u}Y_{u,t} + Y_{s,u}X_{u,t} = 0,$$

which is not true in general. Therefore, $(\Omega_p(\mathbb{R}^d), +)$ is not a vector space.

Nevertheless, the space of rough paths is complete under some metric that we define below.

Definition 2.1.7. *Given two rough paths $X, Y \in \Omega_p(\mathbb{R}^d)$, we define their distance in the p-variation metric, denoted by $d_p(X,Y)$, by the expression*

$$d_p(X,Y) = \max_{1\le i\le \lfloor p\rfloor} \sup_{\pi\in\Pi([0,T])} \left(\sum_l \left| X^i_{t_{l-1},t_l} - Y^i_{t_{l-1},t_l} \right|^{p/i} \right)^{i/p}. \tag{2.6}$$

We define $||\cdot||_p = d_p(\cdot, 0)$, the associated norm.

It can be easily checked that d_p is a distance.

Theorem 2.1.1. $(\Omega_p(\mathbb{R}^d), d_p)$ *is a complete space.*

Proof. Let $(X(n))_n$ be a Cauchy sequence in $(\Omega_p(\mathbb{R}^d), d_p)$. Then, there exists a subsequence $(X(n_k))_k \subset (X(n))_n$ such that we have, for any k,

$$\sum_{i=1}^{\lfloor p\rfloor} \sup_{\pi\in\Pi[0,T]} \sum_l \left| X(n_k)^i_{t_{l-1},t_l} - X(n_{k-1})^i_{t_{l-1},t_l} \right|^{p/i} \le \frac{1}{2^{2k}}.$$

Let define

$$\omega(s,t) = \sup_{k\ge 0} \sum_{i=1}^{\lfloor p\rfloor} \sup_{\pi\in\Pi([0,T])} \sum_j \left| X(n_k)^i_{t_{j-1},t_j} \right|^{p/i} + \\ + \sum_{k=1}^{\infty} \sum_{i=1}^{\lfloor p\rfloor} \sup_{\pi\in\Pi([0,T])} \sum_j \left| X(n_k)^i_{t_{j-1},t_j} - X(n_{k-1})^i_{t_{j-1},t_j} \right|^{p/i}.$$

By proposition (2.1.1) and by Dini theorem[1] applied to the sequence

$$\left(\sum_{k=1}^{j} \sum_{i=1}^{\lfloor p\rfloor} \sup_{\pi\in\Pi([0,T])} \sum_j \left| X(n_k)^i_{t_{j-1},t_j} - X(n_{k-1})^i_{t_{j-1},t_j} \right|^{p/i} \right)_{j\in\mathbb{N}}$$

as $j \to \infty$, we have that ω is a control function.

[1] If we have a monotone sequence of continuous functions converging to a continuous function on a compact space, we can assure that the convergence is uniform

Clearly, for any $i = 1, \ldots, \lfloor p \rfloor$, and for any pair $(s,t) \in \Delta_{[0,T]}$, we have (by the definition of rough path)

$$\left| X(n_k)^i_{s,t} \right| \le \omega(s,t)^{i/p}, \tag{2.7}$$

and it can also be proved, for $k \ge 1$,

$$\left| X(n_k)^i_{s,t} - X(n_{k-1})^i_{s,t} \right| \le \frac{1}{2^{ik/p}} \omega(s,t)^{i/p}.$$

Hence, the following uniform limit exists and we can define

$$X^i_{s,t} := \lim_{k\to\infty} X(n_k)^i_{s,t},$$

which is a multiplicative functional (since Chen's relation passes to the limit). The bound (2.7) also holds in the limit, i. e. $\left| X^i_{s,t} \right| \le \omega(s,t)^{i/p}$, which implies that X is a p-rough path, since this is true for any $(s,t) \in \Delta_{[0,T]}$ and any $i = 1, \ldots, \lfloor p \rfloor$.

By means of triangular inequality, and by adding and substracting terms, it can be seen that

$$\lim_{n\to\infty} \sup_{\pi \in \Pi([0,T])} \sum_k \left| X(n)^i_{s,t} - X^i_{s,t} \right|^{p/i} = 0,$$

that is, d_p–$\lim X(n) = X$. □

We summarize the "control results" in the next proposition.

Proposition 2.1.3. *Let $(X(n))_n \in \Omega_p(\mathbb{R}^d)$ converging to X in the distance d_p. Then, there is a subsequence $X(n_k)$ and a control ω such that*

1. $\left| X(n_k)^i_{s,t} \right| \le \omega(s,t)^{i/p}$,
2. $\left| X^i_{s,t} \right| \le \omega(s,t)^{i/p}$,
3. $\left| X(n_k)^i_{s,t} - X^i_{s,t} \right| \le \frac{1}{2^k} \omega(s,t)^{i/p}$,

for any $i = 1, \ldots, \lfloor p \rfloor$, any $k \in \mathbb{N}$ and any pair $(s,t) \in \Delta_{[0,T]}$.

Since the distance d_p defined above is quite difficult to use, in order to study convergence of rough paths, and motivated by the last proposition, we give a new definition of convergence.

Definition 2.1.8. *Let* $(X(n))_n \subset \Omega_p(\mathbb{R}^d)$*. Then we will say it converges to* $X \in \Omega_p(\mathbb{R}^d)$ *if there exist a control* ω *and a function* $a(n)$ *with* $\lim_{n\to\infty} a(n) = 0$*,* $a(n)$ *depending on* $(X(n))_n$*,* X *and* ω*, such that for any* $i = 1, \ldots, \lfloor p \rfloor$*,* $(s,t) \in \Delta_{[0,T]}$ *and* $n \geq 1$*,* $n \in \mathbb{N}$*, we have*

- $\left| X(n)^i_{s,t} \right| \leq \omega(s,t)^{i/p}$,
- $\left| X^i_{s,t} \right| \leq \omega(s,t)^{i/p}$,
- $\left| X(n)^i_{s,t} - X^i_{s,t} \right| \leq a(n)\omega(s,t)^{i/p}$.

Canonical example: iterated integrals

We give now the first and most important example of a rough path. In some sense, as we will see, it is the canonical one, and it is used in the construction of any other rough path.

Given a *smooth* path X, which in this context means, a path X with finite 1-variation; $X : [0,T] \to \mathbb{R}^d$, for any $n \in \mathbb{N}$, we can define the following element $\boldsymbol{X} \in T^n(\mathbb{R}^d)$

$$\boldsymbol{X}_{s,t} = (1, X^1_{s,t}, \ldots, X^n_{s,t})$$

where, for $k \in \{1, \ldots, n\}$,

$$X^k_{s,t} := \int_{s<t_1<\ldots<t_k<t} \mathrm{d}X_{t_1} \cdot \ldots \cdot \mathrm{d}X_{t_k}. \tag{2.8}$$

Notice that, since X has bounded variation, this integrals are well-defined in the Riemann-Stieltjes sense.

Next, we give an expression for $X^k_{s,t}$ as a limit of sums. The proof is done recursively. For $k = 1$, we clearly have

$$X^1_{s,t} = X_t - X_s.$$

For $k > 1$, consider a partition $\{t^*_l\}_{l=1}^m \in \Pi([0,T])$. Fix $l \in \{1, \ldots, m\}$ and assume $t_i <$

$t^*_{l-1} < t_{i+1}$. In this case,

$$\underbrace{\int\limits_{s<t_1<\ldots<t_i<t_{i+1}<t^*_{l-1}} \mathrm{d}X_{t_1}\cdot\ldots\cdot\mathrm{d}X_{t_i}}_{=:f(t_{i+1})}\underbrace{\left(\int\limits_{t^*_{l-1}<t_{i+1}<\ldots<t_k<t} \mathrm{d}X_{t_{i+1}}\cdot\ldots\cdot\mathrm{d}X_{t_k}\right)}_{=:g(t_{i+1})} = \\ =: \int f\ \mathrm{d}g \overset{\mathrm{RS}}{=} \lim_{|\pi|\to 0}\sum_{l=1}^{m} f(t_{l-1})\left(g(t_l)-g(t_{l-1})\right). \tag{2.9}$$

where at RS we mean we use the definition of a Riemann-Stieltjes integral. But notice that

$$f(x) = X^i_{s,x} \quad \text{and} \quad g(x)-g(y) = X^{k-i}_{y,x}.$$

Therefore, we can write as

$$\lim_{|\pi|\to 0}\sum_{l=1}^{m} X^i_{s,t_{l-1}} X^{k-i}_{t_{l-1},t_l}. \tag{2.10}$$

But we have to consider the different possibilities varying i between 1 and $k-1$, so we finally have

$$X^k_{s,t} = \lim_{|\pi|\to 0}\sum_{l=1}^{m}\sum_{i=1}^{k-1} X^i_{s,t_{l-1}} X^{k-i}_{t_{l-1},t_l}. \tag{2.11}$$

Now that we have defined these iterated integrals, we prove that they are 1-rough paths

- $\boldsymbol{X}_{s,t}$ **is a multiplicative functional**. Since the Riemann-Stieltjes integral is additive, this is $\int_s^u f\ \mathrm{d}g + \int_u^t f\ \mathrm{d}g = \int_s^t f\ \mathrm{d}g$ and since we have deduced the relation (2.11), it is clear that $\boldsymbol{X}_{s,t}$ has to satisfy Chen's relation.

- $\boldsymbol{X}_{s,t}$ **has finite 1-variation**. Following the procedure of the proof of Proposition 2.1.2, we can establish a reparametrisation such that $||X||_{1,[s,t]} = t-s$. So, we can assume this fact without loss of generality.

  Then X is Lipschitz continuous and therefore almost everywhere differentiable with $\left|\frac{\mathrm{d}X}{\mathrm{d}t}\right| = 1$ almost everywhere. Therefore

$$\left|X^k_{s,t}\right| = \left|\int\limits_{s<t_1<\ldots<t_k<t} \mathrm{d}X_{t_1}\cdot\ldots\cdot\mathrm{d}X_{t_k}\right| = \left|\int\limits_{s<t_1<\ldots<t_k<t} \frac{\mathrm{d}X_{t_1}}{\mathrm{d}t_1}\cdots\frac{\mathrm{d}X_{t_k}}{\mathrm{d}t_k}\ \mathrm{d}t_1\ldots\mathrm{d}t_k\right| \leq \\ \leq \int\limits_{s<t_1<\ldots<t_k<t} \left|\frac{\mathrm{d}X_{t_1}}{\mathrm{d}t_1}\right|\cdots\left|\frac{\mathrm{d}X_{t_k}}{\mathrm{d}t_k}\right| \mathrm{d}t_1\ldots\mathrm{d}t_k = \int\limits_{s<t_1<\ldots<t_k<t} \mathrm{d}t_1\ldots\mathrm{d}t_k = \frac{(t-s)^k}{k!},$$

where the last integral can be easily computed by induction on k. Therefore, $\boldsymbol{X}_{s,t}$ has finite 1-variation (in the sense of definition 2.1.5), with $\omega(s,t) = |t-s|$.

Remark. Notice that in the case of a smooth path, all the information of the iterated integrals is in X^1. With this, we can compute any of the rest.

We put a name to the set of rough paths introduced before.

Definition 2.1.9. *Let $X \in \Omega_p(\mathbb{R}^d)$. We say that X is a smooth rough path if $t \to X_t \equiv X^t_{0,t}$ is a continuous path with finite 1-variation and for $i = 1, \dots, \lfloor p \rfloor$, $X^i_{s,t}$ is given by*

$$X^i_{s,t} = \int_{s<t_1<\ldots<t_i<t} \mathrm{d}X_{t_1} \cdot \ldots \cdot \mathrm{d}X_{t_i}.$$

Definition 2.1.10. *A rough path $X \in \Omega_p(\mathbb{R}^d)$ is a geometric rough path if there exists a sequence $(X(n))_n \subset \Omega_p(\mathbb{R}^d)$ of smooth rough paths such that $\lim_{n\to\infty} d_p(X(n), X) = 0$. We denote the space of geometric rough paths with roughness p by $G\Omega_p(\mathbb{R}^d)$.*

2.1.4 Roughness of a path and degree of the associated rough path

We have seen a way to construct rough paths given a function of bounded variation by considering the iterated integrals. In general, one may wonder if we have to compute infinitely many of those integrals in order to uniquely determine the original path knowing the associated geometric rough path, or if, instead of this, there is some way to determine the minimum level, to be reached in order to have this definition. The answer comes from the following result, which is a consequence of the Chen's relation.

Lemma 2.1.1. *Let $m \in \mathbb{N}$ and let X, Y be two multiplicative functionals such that $X^i_{s,t} = Y^i_{s,t}$ for any $i = 0, \dots, m$ and any $(s,t) \in \Delta_{[0,T]}$. Then,*

$$\psi^{m+1}_{s,t} = X^{m+1}_{s,t} - Y^{m+1}_{s,t}$$

is additive, i.e.,

$$\psi^{m+1}_{s,u} = \psi^{m+1}_{s,t} + \psi^{m+1}_{t,u},$$

for any $s \le t \le u$.

Conversely, if Z is a multiplicative functional and $\Psi : [0,T] \to (\mathbb{R}^n)^{\times(m+1)}$ is an additive function, then the map

$$(s,t) \longrightarrow Z_{s,t} + \Psi_{s,t}$$

is a multiplicative functional.

Now consider two p-rough paths (controlled by ω) X and Y and apply the previous lemma for $m = n-1$. Then we know the function $\psi^n_{s,t} = X^n_{s,t} - Y^n_{s,t}$ is additive. Since they are p-rough paths controlled by ω, we know that

$$\left|\left|X^n_{s,t}\right|\right|, \left|\left|Y^n_{s,t}\right|\right| \leq \omega(s,t)^{n/p}.$$

With all of this, we can see (with a similar reasoning than we will do later in the proof of theorem 2.1.2) that we can write $\psi^n_{s,t} = \rho(s) - \rho(t)$, for some function ρ which is n/p-Hölder continuous.

Then, if $n > \lfloor p \rfloor$ (i. e. $n/p > 1$) we have that ρ is constant and hence $\psi^n_{s,t} = 0$ which implies that $X^n_{s,t} = Y^n_{s,t}$. As we can reason this way for any $n > \lfloor p \rfloor$ this means that a p-rough path is determined by its first $\lfloor p \rfloor$ components. As a path with finite p-variation (we have seen it for $p = 1$, the case of Lipschitz paths and the rough path constructed by iterated integrals) determines a p-rough path, this means that we have to study just up to the $\lfloor p \rfloor$-th degree. For instance, in chapter 3, we study the Brownian rough path and as we known sample paths of Brownian motion are of finite p-variation for any $p > 2$ we will need to consider up to the second level for this case. For a fractional Brownian motion with Hurst parameter H, with $1/3 < H < 1/4$, whose sample paths are α-Hölder continuous for $1/3 < \alpha < 1/4$ we should take into account also the third level when constructing a rough path associated to it.

2.1.5 Almost rough paths

We next address the question about the construction of rough paths using approximations. The motivation for this will become clear in chapter 4, where the integration (along a rough path) will be introduced. We introduce a new notion: *almost* rough paths.

Definition 2.1.11. *Let $p \geq 1$. A function $X : \Delta_{[0,T]} \to T^n(\mathbb{R}^d)$ is said to be an almost rough path of roughness p, or p-almost rough path if it is of finite p-variation, $X^0 \equiv 1$ and such that* *there exist a constant $\theta > 1$ and a control ω satisfying*

$$\left|(X_{s,t}X_{t,u})^i - X^i_{s,u}\right| \leq \omega(s,u)^\theta, \tag{2.12}$$

for any $(s,t),(t,u) \in \Delta_{[0,T]}$ and any $i = 1,\ldots,\lfloor p \rfloor$. The expression (2.12) *is known as the almost Chen relation.*

The importance of this object is made clear in the following theorem. Basically, it states that any almost rough path can be extended to a rough path.

Theorem 2.1.2. *If $X : \Delta_{[0,T]} \to T^{\lfloor p \rfloor}(\mathbb{R})$ is an θ-almost rough path of roughness p, then there is a unique p-rough path $\hat{X}$, such that there exists a control ω and a constant $\theta > 1$ satisfying*

$$\sup_{i=1,\ldots,\lfloor p \rfloor} \sup_{s \neq t} \frac{\left|\hat{X}^i_{s,t} - X^i_{s,t}\right|}{\omega(s,t)^\theta} < \infty, \tag{2.13}$$

for any $i = 1,\ldots,\lfloor p \rfloor$ and any pair $(s,t) \in \Delta_{[0,T]}$.

Proof (Uniqueness). Let $\hat{X}$ and $\tilde{X}$ be two p-rough paths satisfying the statement of the theorem. We have to see that $\hat{X} = \tilde{X}$, this is, that

$$\hat{X}^i = \tilde{X}^i \tag{2.14}$$

for any $i = 1,\ldots,\lfloor p \rfloor$.

We will use an induction argument. For $i = 0$, this is clear, since $\hat{X}^0 = \tilde{X}^0 = 1$. Now we assume that there exists $j \in \{1,\ldots,\lfloor p \rfloor\}$ such that (2.14) holds for $i = 1,\ldots,j-1$ and we see that this implies that it also holds for $i = j$.

Since $\hat{X}$ is a multiplicative functional, we should have

$$\hat{X}^j_{s,t} = (\hat{X}_{s,u}\hat{X}_{u,t})^j = \sum_{k=0}^{j} \hat{X}^k_{s,u}\hat{X}^{j-k}_{u,t}, \quad \forall 0 \leq s \leq u \leq t \leq T.$$

and the same for $\tilde{X}^j_{s,t}$. Considering the difference between them at level j we have

$$\begin{aligned}\hat{X}^j_{s,t} - \tilde{X}^j_{s,t} &= \sum_{k=0}^{j} \left(\hat{X}^k_{s,u}\hat{X}^{j-k}_{u,t} - \tilde{X}^k_{s,u}\tilde{X}^{j-k}_{u,t} \right) \overset{(2.14) \text{ up to } j-1}{=} \\ &= \hat{X}^0_{s,u}\hat{X}^j_{u,t} + \hat{X}^j_{s,u}\hat{X}^0_{u,t} - \tilde{X}^0_{s,u}\tilde{X}^j_{u,t} - \tilde{X}^j_{s,u}\tilde{X}^0_{u,t} \overset{(\hat{X}^0 \equiv \tilde{X}^0 \equiv 1)}{=} \\ &= \hat{X}^j_{u,t} - \tilde{X}^j_{u,t} + \hat{X}^j_{s,u} - \tilde{X}^j_{s,u} \quad \forall 0 \le s \le u \le t \le T.\end{aligned}$$

That is, if we set $\delta(s,t) = \hat{X}^j_{s,t} - \tilde{X}^j_{s,t}$, we see that δ is additive ($\delta(s,t) = \delta(s,u) + \delta(u,t) \quad \forall 0 \le s \le u \le t \le T$), and if we define $\rho(t) := \delta(0,t)$ we have that

$$\hat{X}^j_{s,t} - \tilde{X}^j_{s,t} = \rho(t) - \rho(s) \text{ for } 0 \le s \le t \le T.$$

By the triangular inequality, we have

$$|\rho(t) - \rho(s)| \le \left| \hat{X}^j_{s,t} - X^j_{s,t} \right| + \left| \tilde{X}^j_{s,t} - X^j_{s,t} \right| \le 2K\omega(s,t)^\theta,$$

and this implies ρ is a constant. Indeed,

$$\begin{aligned}|\rho(t) - \rho(s)| &\le \sum_{\substack{\pi \in \Pi([s,t]) \\ \pi \text{dyadic}}} |\rho(t_k) - \rho(t_{k-1})| \le 2K \sum_{\substack{\pi \in \Pi([s,t]) \\ \pi \text{dyadic}}} \omega(t_k, t_{k-1})\omega(t_{k-1}, t_k)^\theta \\ &\le 2K \sup_k \omega(t_{k-1}, t_k)^{\theta - 1}\omega(s,t) \to 0\end{aligned}$$

as $k \to \infty$. We can conclude that $\hat{X}^j = \tilde{X}^j$ and use induction on j to assure $\hat{X} = \tilde{X}$.

□

A property that one may expect from the map that sends an almost rough path to the rough path associated to it in the sense of the previous theorem is to be continuous. That will be needed in chapter 4 to assure that two close-in-distance almost rough paths establish two close-in-distance rough paths and with this, we will be able to define properly the notion of the integral along a rough path, as we will see later. In fact, we have the following theorem that quantifies the continuity of this correspondence.

Theorem 2.1.3. *Let $p \ge 1$ and $\theta > 1$. Let $X, Y : \Delta_{[0,T]} \to T^{\lfloor p \rfloor}(\mathbb{R})$ be θ-almost rough paths* *controlled by* ω. *Let* $\hat{X}, \hat{Y}$ *be the associated rough paths in the sense of the theorem 2.1.2. If there exists a constant* $\varepsilon > 0$ *such that for any* $i = 1, \ldots, \lfloor p \rfloor$, *and any* $(s,t) \in \Delta_{[0,T]}$ *the relation*

$$\left|\left|X^i_{s,t} - Y^i_{s,t}\right|\right|_p \leq \varepsilon \omega^{i/p}$$

holds, then there exists a number B *depending on* ε *(but not exclusively) such that* $\lim_{\varepsilon \to 0} B = 0$ *and such that for any* $i = 1, \ldots, \lfloor p \rfloor$, *and any* $(s,t) \in \Delta_{[0,T]}$ *we have*

$$\left|\left|\hat{X}^i_{s,t} - \hat{Y}^i_{s,t}\right|\right|_p \leq B \omega^{i/p}.$$

Proof. The proof (in a more general setting) can be seen in [7], p. 45-47. □

Chapter 3

Rough path associated to the Brownian Motion

In this chapter we present the construction of a rough path associated to a particular continuous path, namely the Brownian motion. In order to prove the results, some technical lemmas are needed. We will focus on the main result.

Recall that the (one dimensiona) Brownian motion is a stochastic process $(W_t)_{t\in\mathbb{R}^+}$ satisfying

(a) $W_0 = 0$ almost surely, and

(b) for any $0 \leq s < t$, the random variable $W_t - W_s$ is independent of the σ-field generated by $(W_r)_{0\leq r\leq s}$ and $W_t - W_s$ is a normal variable with zero mean and variance $t - s$.

Although it does not seem obvious from the definition, using Kolmogorov's continuity criterion we can assure the existence of a version of the process whose sample paths $t \to W_t(\omega)$ are not just continuous but also α-Hölder continuous for any $\alpha \in (0, \frac{1}{2})$. However, it is also known that Brownian motion sample paths are not α-Hölder continuous for any $\alpha > 1/2$, which in particular implies that they are not differentiable. All this will be important in our discussion.

We would like to look at the Brownian motion as a geometric p-rough path X (see Definition 2.1.10) in the sense that at its first level one has

$$X^1_{s,t} = W_t - W_s.$$

According to Definition 2.1.10, we will need an approximating sequence of smooth rough paths. For this, we will first give approximations of the paths of the Brownian motion that have more regularity than the Brownian itself.

In this chapter we will sometimes consider that $I = [0,1]$ instead of $I = [a,b]$. By a reparametrization one can easily check that this done without loss of generality.

3.1 The framework: dyadic polygonal approximations

We introduce the dyadic partitions of the interval $I = [s,t]$.

Definition 3.1.1. *Let $m \in \mathbb{N}\backslash\{0\}$. The m-th dyadic partition of $[s,t]$ is the sequence $\{t_i^m\}$, where*

$$t_i^m = s + (t-s)\frac{i}{2^m}, \quad i = 0, \ldots, 2^m.$$

We define the m-th dyadic polygonal approximation as the linear piecewise interpolation of the paths of the Brownian motion.

Definition 3.1.2. *Given $m \in \mathbb{N}\backslash\{0\}$, we define the m-th dyadic polygonal approximation of the Brownian motion is the continuous function given by*

$$W(m)_t = W_{t_{l-1}^m} + 2^m(t - t_{l-1}^m)\Delta_l^m W, \tag{3.1}$$

for $t_{l-1}^m \leq t \leq t_l^m$, where $(\{t_j^m\})_{0\leq j\leq 2^m}$ is the m-th dyadic partition of $[s,t]$, and $\Delta_l^m W = W_{t_l^m} - W_{t_{l-1}^m}$.

The first level of the multiplicative functional associated to $W(m)$ is defined as

$$X(m)_{s,t}^1 = W(m)_t - W(m)_s. \tag{3.2}$$

Since $t \to W(m)_t$ is a piecewise linear and continuous map, it has finite 1-variation and therefore, we can construct the multiple iterated path integrals

$$X(m)_{s,t}^j = \int_{s<t_1<\ldots<t_j<t} \mathrm{d}W(m)_{t_1} \cdot \ldots \cdot \mathrm{d}W(m)_{t_j}, \tag{3.3}$$

which, as we said before, since $W(m)$ is a Lipschitz path, are well defined as a recurrence by

$$X(m)^j_{s,t} = \lim_{|\pi|\to 0} \sum_l \sum_{i=1}^{j-1} X(m)^i_{s,t-1} X(m)^{j-1}_{t_{l-1},t_l} \tag{3.4}$$

for any $j \geq 2$.

Now we focus our attention on the first degree of the multiplicative functional, this is, $X(m)^1$.

3.2 Study of $X(m)^1$

First of all, we are going to prove that, $X(m)^1$ has finite p-variation for any $p > 2$.

Notice that throughout this chapter, two different partitions appear.

- A dyadic partition $\mathcal{D}^m = \{\frac{i}{2^m}, i \in \{0, \ldots, 2^m\}\}$.
- A finite partition of $[s,t]$, $\pi = \{t_l, l = 0, \ldots, n\}$.

Proposition 3.2.1. *Let $(W_t)_{t\in[0,1]}$ be a d dimensional Brownian motion and let $p > 2$. Then the dyadic polygonal approximations $X(m)^1_{s,t}$, defined in (3.1) and (3.2), have finite p-variation uniformly in m. Moreover, for any $\gamma > p/i - 1$ there exists a constant C depending on p, and the dimension d, such that*

$$\mathbb{E}\left[\sup_m \sup_{\pi\in\Pi([s,t])} \sum_{\substack{l \\ t_l\in\pi}} \left|X(m)^1_{t_{l-1},t_l}\right|^p\right] \leq C\,|t-s|^{p/2}, \tag{3.5}$$

for $0 \leq s \leq t \leq 1$. In particular, almost surely we have

$$\sup_m \sup_{\pi\in\Pi([s,t])} \sum_l \left|X(m)^1_{t_{l-1},t_l}\right|^p < \infty. \tag{3.6}$$

In order to prove this proposition above we need the following lemmas.

Lemma 3.2.1. *Let $X = (1, X^1, X^2, \ldots, X^N)$ be a multiplicative functional. Then for any $i = 1, \ldots, N$ and any p such that $p/i > 1$, there exists a constant $C = C(i,p,\gamma)$ such that*

$$\sup_{\pi\in\Pi([0,1])} \sum_l \left|X^i_{t_{l-1},t_l}\right|^{p/i} \leq C \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} \sum_{j=1}^{i} \left|X^j_{t^n_{k-1},t^n_k}\right|^{p/j}.$$

Proof. It can be found in [7], p. 62. □

Remark. The previous lemma will be used in order to show the convergence of first level of the rough paths associated to the approximation. So, it will be applied to a difference of the form $X(m)^1 - X^1$. Notice the following: the difference of multiplicative functionals, at its first level, it is still a multiplicative functional, that is, it still satisfies Chen's relation. Consider a simple multiplicative functional $(1, X^1_{a,b})$. Then, Chen's relation $(1, X^1_{s,u})(1, X_{u,t}) = (1, X_{s,t})$ implies that

$$X^1_{s,t} = X^1_{s,u} + X^1_{u,t},$$

where the right hand side is defined in (2.1).

Consider, X and Y two multiplicative functionals. Then

$$\begin{aligned}(X-Y)^1_{s,t} &= (X-Y)_t - (X-Y)_s = (X_t - Y_t) - (X_s - Y_s) = \\ &= (X_t - X_s) - (Y_t - Y_s) = X^1_{s,t} - Y^1_{s,t}.\end{aligned}$$

Therefore, lemma 3.2.1 applies to a difference of multiplicative functionals if $i = 1$.

Lemma 3.2.2. *Let $(W_t)_{t\in[0,1]}$ be a continuous path in $\mathbb{R}^n$ and let $p \geq 1$. Consider the dyadic polygonal approximations $W(m)$ given in* (3.1) *and the corresponding multiplicative functionals $X(m)$ defined in* (3.3) *and* (3.4)*. Then, for any fixed $n \geq 1$, the function*

$$m \to \sum_{k=1}^{2^n} \left| X(m)^1_{t^n_{k-1},t^n_k} \right|^p \tag{3.7}$$

is monotone increasing. Hence,

$$\sup_m \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} \left| X(m)^1_{t^n_{k-1},t^n_k} \right|^p = \lim_{m\to\infty} \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} \left| X(m)^1_{t^n_{k-1},t^n_k} \right|^p .$$

Proof. We will distinguish, as it will be usually done in this section, between the cases $n \leq m$ and $n > m$.

- If $n \leq m$ then $X(m)(t^n_k) = W_{t^n_k}$ and we simply have $X(m)^1_{t^n_{k-1},t^n_k} = \Delta^n_k W$ for any $k =$

$0, \dots, 2^n$. Hence,

$$\sum_{k=1}^{2^n} \left| X(m)^1_{t^n_{k-1}, t^n_k} \right|^p = \sum_{k=1}^{2^n} |\Delta^n_k W|^p \,,$$

which does not depend on m. Hence, in this case, (3.7) is a constant function.

- If $n > m$, for fixed k, we can choose the index l such that $t^m_{l-1} \leq t^n_{k-1} < t^n_k < t^m_l$. By (3.1), we have

$$X(m)_{t^n_k} = W_{t^m_{l-1}} + 2^m (t^n_k - t^m_{l-1}) \Delta^m_l W,$$
$$X(m)_{t^n_{k-1}} = W_{t^m_{l-1}} + 2^m (t^n_{k-1} - t^m_{l-1}) \Delta^m_l W.$$

Considering the difference between the two expressions above, and using (3.2), we obtain

$$X(m)^1_{t^n_{k-1}, t^n_k} = X(m)_{t^n_k} - X(m)_{t^n_{k-1}} = 2^m \underbrace{(t^n_k - t^n_{k-1})}_{= \frac{1}{2^n}} \Delta^m_l W = 2^{m-n} \Delta^m_l W. \tag{3.8}$$

Hence,

$$\begin{aligned} \sum_{k=1}^{2^n} \left| X(m)^1_{t^n_{k-1}, t^n_k} \right|^p &\overset{\text{eq.3.8}}{=} \sum_{l=1}^{2^m} \sum_{\substack{k \in \mathbb{N} \\ t^m_{l-1} \leq t^n_k < t^m_l}} \left| 2^{m-n} \Delta^m_l W \right|^p = \\ &= 2^{(m-n)p} \sum_{l=1}^{2^m} \sum_{\substack{k \in \mathbb{N} \\ t^m_{l-1} \leq t^n_k < t^m_l}} |\Delta^m_l W|^p \overset{(*)}{=} \\ &= 2^{(m-n)p} 2^{n-m} \sum_{l=1}^{2^m} |\Delta^m_l W|^p = \\ &= 2^{n(1-p)} 2^{m(p-1)} \sum_{l=1}^{2^m} |\Delta^m_l W|^p =: 2^{n(1-p)} g(m), \end{aligned}$$

where in (*) we have to used that the cardinal of $\{k \in \mathbb{N} : t^m_{l-1} \leq t^n_k \leq t^m_l\}$ is 2^{n-m}. Now, we focus on $g(m)$, and we prove that it is increasing in m. For this, we notice that

$$\begin{aligned} \Delta^m_l W &= W_{t^m_l} - W_{t^m_{l-1}} = W_{t^{m+1}_{2l}} - W_{t^{m+1}_{2l-2}} = \\ &= \left(W_{t^{m+1}_{2l}} - W_{t^{m+1}_{2l-1}} \right) + \left(W_{t^{m+1}_{2l-1}} - W_{t^{m+1}_{2l-2}} \right) = \Delta^{m+1}_{2l} + \Delta^{m+1}_{2l-1}. \end{aligned} \tag{3.9}$$

Thus,

$$
\begin{aligned}
g(m) = 2^{m(p-1)} \sum_{l=1}^{2^m} |\Delta_l^m W|^p &\overset{\text{eq. (3.9)}}{=} 2^{m(p-1)} \sum_{l=1}^{2^m} \left|\Delta_{2l}^{m+1} W + \Delta_{2l-1}^{m+1} W\right|^p \overset{(**)}{\leq} \\
&\leq 2^{(m+1)(p-1)} \sum_{l=1}^{2^m} \left[\left|\Delta_{2l}^{m+1} W\right|^p + \left|\Delta_{2l-1}^{m+1} W\right|^p\right] = \\
&= 2^{(m+1)(p-1)} \sum_{k=1}^{2^{m+1}} \left|\Delta_{k}^{m+1} W\right|^p = g(m+1),
\end{aligned}
$$

where at (**) we use the inequality $|a+b| \leq 2^{p-1}(|a|^p + |b|^p)$. Hence the proof is finished.

□

We are now able to prove the proposition 3.2.1.

Proof of the proposition 3.2.1. Since $X(m)^1$ is a multiplicative functional of degree 1 and $p > 2$, we can apply lemmas 3.2.1 and 3.2.2. This yields,

$$
\begin{aligned}
&\mathbb{E}\left[\sup_m \sup_{\pi \in \Pi([s,t])} \sum_l \left|X(m)^1_{t_{l-1},t_l}\right|^p\right] \overset{\text{lemma 3.2.1}}{\leq} C\mathbb{E}\left[\sup_m \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} \left|X(m)^1_{t^n_{k-1},t^n_k}\right|^p\right] \leq \\
&\overset{\text{lemma 3.2.2}}{\leq} C\mathbb{E}\left[\lim_{m\to\infty} \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} \left|X(m)^1_{t^n_{k-1},t^n_k}\right|^p\right] \overset{MCT}{=} C\mathbb{E}\left[\sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} \left|X^1_{t^n_{k-1},t^n_k}\right|^p\right],
\end{aligned}
\tag{3.10}
$$

where at MCT we have used the Monotone Convergence Theorem.

At this point we recall that for a Brownian motion $(W_t)_{t\in\mathbb{R}^+}$, since the law of the increments is Gaussian, we have that $\mathbb{E}\left[|W_t - W_s|^p\right] = C_1 |t-s|^{p/2}$ for some constant $C_1 = C_1(p,d)$, and hence

$$
\mathbb{E}\left[\left|X^1_{t^n_{k-1},t^n_k}\right|^p\right] \leq C_1 \left(\frac{t-s}{2^n}\right)^{p/2} = C_1 (t-s)^{p/2} \left(\frac{1}{2^n}\right)^{p/2}. \tag{3.11}
$$

Hence, we can use this on (3.10) and obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{m}\sup_{\pi\in\Pi([s,t])}\sum_{l}\left|X(m)^1_{t_{l-1},t_l}\right|^p\right] &\leq C\mathbb{E}\left[\sum_{n=1}^{\infty}n^{\gamma}\sum_{k=1}^{2^n}\left|X^1_{t^n_{k-1},t^n_k}\right|^p\right] \overset{MCT}{=} \\
&= C\sum_{n=1}^{\infty}n^{\gamma}\sum_{k=1}^{2^n}\mathbb{E}\left[\left|X^1_{t^n_{k-1},t^n_k}\right|^p\right] \overset{(3.11)}{\leq} \\
&\leq C\sum_{n=1}^{\infty}n^{\gamma}\sum_{k=1}^{2^n}C_1(t-s)^{p/2}\left(\frac{1}{2^n}\right)^{p/2} = \\
&= C_2(t-s)^{p/2}\left(\sum_{n=1}^{\infty}n^{\gamma}2^n\left(\frac{1}{2^n}\right)^{p/2}\right) = \\
&= C_2(t-s)^{p/2}\left(\sum_{n=1}^{\infty}n^{\gamma}\left(\frac{1}{2^n}\right)^{p/2-1}\right)
\end{aligned} \tag{3.12}$$

where $C_2 = C \cdot C_1$. Since $p > 2$, we have $p/2 - 1 > 0$ and hence the last series converges to a finite value, and where we have applied at MCT the Monotone Convergence Theorem in order to commute the expectation and the series, since

$$\sum_{n=1}^{s}n^{\gamma}\sum_{k=1}^{2^n}\left|X^1_{t^n_{k-1},t^n_k}\right|^p$$

form a monotone increasing sequence of functions as s increases. The proof of the proposition is complete □

Our next goal is to see that $X(m)^1$ converges to X^1 in the norm of the p-variation.

Proposition 3.2.2. *Under the same assumptions as in Proposition 3.2.1, for $0 \leq s \leq t \leq 1$ and fixed $m \geq 1$, we have*

$$\mathbb{E}\left[\sup_{\pi\in\Pi([s,t])}\sum_{l}\left|X(m)^1_{t_{l-1},t_l} - X^1_{t_{l-1},t_l}\right|^p\right] \leq C\left(\frac{1}{2^m}\right)^{p/4-1/2}|t-s|^{p/2}, \tag{3.13}$$

for some $C = C(p, h, d)$. Since $p > 2$, this implies

$$\sum_{m=1}^{\infty}\sup_{\pi\in\Pi([0,1])}\left(\sum_{l}\left|X(m)^1_{t_{l-1},t_l} - X^1_{t_{l-1},t_l}\right|^p\right)^{1/p} < \infty, \quad a.\ s. \tag{3.14}$$

Proof. First, notice the following. If $n \leq m$ then $W(m)_{t^n_k} = W_{t^n_k}$ for any $k = 0, \ldots, 2^n$ and thus, $X(m)^1_{t^n_{k-1},t^n_k} - X^1_{t^n_{k-1},t^n_k} = 0$. On the other hand, for $n > m$ we can apply the fact that

$|a \pm b|^p \leq 2^{p-1}(|a|^p + |b|^p)$ to bound the following difference

$$\left|X(m)^1_{t^n_{k-1},t^n_k} - X^1_{t^n_{k-1},t^n_k}\right|^p \leq 2^{p-1}\left(\left|X(m)^1_{t^n_{k-1},t^n_k}\right|^p + \left|X^1_{t^n_{k-1},t^n_k}\right|^p\right). \tag{3.15}$$

Thus, we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{\pi\in\Pi([s,t])}\sum_l \left|X(m)^1_{t_{l-1},t_l} - X^1_{t_{l-1},t_l}\right|^p\right] \overset{\text{lemma3.2.1 applied to } X(m)^1 - X^1}{\leq} \\
&\leq C\mathbb{E}\left[\sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n}\left|X(m)^1_{t^n_{k-1},t^n_k} - X^1_{t^n_{k-1},t^n_k}\right|^p\right] = \\
&= C\mathbb{E}\left[\sum_{n=1}^{m} n^\gamma \sum_{k=1}^{2^n}\left|X(m)^1_{t^n_{k-1},t^n_k} - X^1_{t^n_{k-1},t^n_k}\right|^p\right] + \\
&+ C\mathbb{E}\left[\sum_{n=m+1}^{\infty} n^\gamma \sum_{k=1}^{2^n}\left|X(m)^1_{t^n_{k-1},t^n_k} - X^1_{t^n_{k-1},t^n_k}\right|^p\right] = \\
&= C\mathbb{E}\left[\sum_{n=m+1}^{\infty} n^\gamma \sum_{k=1}^{2^n}\left|X(m)^1_{t^n_{k-1},t^n_k} - X^1_{t^n_{k-1},t^n_k}\right|^p\right] \overset{(3.15)}{\leq} \\
&\leq C\mathbb{E}\left[\sum_{n=m+1}^{\infty} n^\gamma \sum_{k=1}^{2^n} 2^{p-1}\left(\left|X(m)^1_{t^n_{k-1},t^n_k}\right|^p + \left|X^1_{t^n_{k-1},t^n_k}\right|^p\right)\right] = \\
&= 2^{p-1}C\left(\mathbb{E}\left[\sum_{n=m+1}^{\infty} n^\gamma \sum_{k=1}^{2^n}\left|X(m)^1_{t^n_{k-1},t^n_k}\right|^p\right] + \mathbb{E}\left[\sum_{n=m+1}^{\infty} n^\gamma \sum_{k=1}^{2^n}\left|X^1_{t^n_{k-1},t^n_k}\right|^p\right]\right) \overset{eq.(3.11)}{\leq} \\
&\leq 2^p C\,|t-s|^{p/2}\sum_{n=m+1}^{\infty} n^\gamma\left(\frac{1}{2^n}\right)^{p/2-1} \overset{(***)}{\leq} \bar{C}\,|t-s|^{p/2}\left(\frac{1}{2^m}\right)^{p/4-1/2}\sum_{n=m+1}^{\infty} n^\gamma\left(\frac{1}{2^n}\right)^{p/4-1/2},
\end{aligned} \tag{3.16}$$

where at (***) we have used that

$$\left(\frac{1}{2^n}\right)^{p/2-1} = \left(\frac{1}{2^n}\right)^{p/4-1/2}\left(\frac{1}{2^n}\right)^{p/4-1/2} \leq 2^{-m(p/4-1/2)}2^{-}n(p/4-1/2), \text{ if } m \geq n.$$

The last series converges to a finite constant D. Setting $\tilde{C} = \bar{C}D$, we have proved that

$$\mathbb{E}\left[\sup_{\pi\in\Pi([s,t])}\sum_l \left|X(m)^1_{t_{l-1},t_l} - X^1_{t_{l-1},t_l}\right|^p\right] \leq \tilde{C}\left(\frac{1}{2^m}\right)^{p/4-1/2}|t-s|^{p/2}, \tag{3.17}$$

proving (3.13).

Finally, we have

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{m=1}^{\infty}\left(\sup_{\pi\in\Pi([s,t])}\sum_{l}\left|X(m)^1_{t_{l-1},t_l}-X^1_{t_{l-1},t_l}\right|^p\right)^{1/p}\right] \overset{\text{MCT}}{=} \\
&=\sum_{m=1}^{\infty}\mathbb{E}\left[\left(\sup_{\pi\in\Pi([s,t])}\sum_{l}\left|X(m)^1_{t_{l-1},t_l}-X^1_{t_{l-1},t_l}\right|^p\right)^{1/p}\right] \overset{\text{Hölder's in. for the expectation}}{\leq} \\
&\leq\sum_{m=1}^{\infty}\left[\mathbb{E}\left[\left(\left(\sup_{\pi\in\Pi([s,t])}\sum_{l}\left|X(m)^1_{t_{l-1},t_l}-X^1_{t_{l-1},t_l}\right|^p\right)^{1/p}\right)^p\right]\right]^{1/p} = \\
&=\sum_{m=1}^{\infty}\left[\mathbb{E}\left[\sup_{\pi\in\Pi([s,t])}\sum_{l}\left|X(m)^1_{t_{l-1},t_l}-X^1_{t_{l-1},t_l}\right|^p\right]\right]^{1/p} \overset{(3.17)}{\leq} \\
&\leq\tilde{C}\,|t-s|^{1/2}\sum_{m=1}^{\infty}\left(\frac{1}{2^m}\right)^{\frac{1}{4}-\frac{1}{2p}},
\end{aligned}
$$

where at MCT we have used the Monotone Convergence Theorem applied to the sequence

$$
\left(\sum_{m=1}^{r}\left(\sup_{\pi\in\Pi([s,t])}\sum_{l}\left|X(m)^1_{t_{l-1},t_l}-X^1_{t_{l-1},t_l}\right|^p\right)^{1/p}\right)_r,
$$

which is increasing in r.

Since $\frac{1}{4}-\frac{1}{2p}>0$, the last series converges. Hence, we have

$$
\sum_{m=1}^{\infty}\left(\sup_{\pi\in\Pi([s,t])}\sum_{l}\left|X(m)^1_{t_{l-1},t_l}-X^1_{t_{l-1},t_l}\right|^p\right)^{1/p}<\infty,\quad\text{almost surely.}
$$

In particular,

$$
\lim_{m\to\infty}\left(\sup_{\pi\in\Pi([s,t])}\sum_{l}\left|X(m)^1_{t_{l-1},t_l}-X^1_{t_{l-1},t_l}\right|^p\right)^{1/p}=0,\quad\text{almost surely,}
$$

which shows that $X(m)^1$ converges to X^1 in the norm of the p-variation with probability 1. □

3.3 Study of $X(m)^2$

As we have noticed through the study of $X(m)^1$, it is useful to consider separately the cases $n>m$ and $n\leq m$. First, we will assume that $n>m$. The case $n\leq m$ demands a lot of technicalities. A complete discussion on the other case can be found in [7], chapter 4; and we will say something on it at the end of the section.

Since $W(m)$ defines a differentiable function, we can compute $X(m)^2_{s,t}$ as an iterated integral.

$$\begin{aligned}
X(m)^2_{s,t} &= \iint_{s<t_1<t_2<t} \mathrm{d}W(m)_{t_1}\,\mathrm{d}W(m)_{t_2} \\
&= \int_s^t \left(\int_s^{t_2} \mathrm{d}W(m)_{t_1}\right) \mathrm{d}W(m)_{t_2} \\
&= \int_s^t \left(W(m)_{t_2} - W(m)_s\right) \mathrm{d}W(m)_{t_2} \\
&= \left|\frac{1}{2}\left[W(m)_{t_2}\right]^2 - W(m)_s W(m)_{t_2}\right|_{t_2=s}^{t_2=t} \\
&= \frac{1}{2}\left[W(m)_t\right]^2 - W(m)_s W(m)_t - \frac{1}{2}\left[W(m)_s\right]^2 + \left[W(m)_s\right]^2 = \\
&= \frac{1}{2}\left(W(m)_t - W(m)_s\right)^2 .
\end{aligned}$$

Take $s = t^n_{k-1}$ and $t = t^n_k$ and l such that $t^m_{l-1} \leq t^n_{k-1} < t^n_k \leq t^m_l$ (notice that such an l exists since we assume $n > m$). Substituting in the last equation $W(m)$ by its expression given in (3.1), we obtain,

$$\begin{aligned}
X(m)^2_{t^n_{k-1},t^n_k} &= \frac{1}{2}\left(\cancel{W^n_{t_{l-1}}} + 2^m(t^n_k - t^m_{l-1})\Delta^m_l W - \cancel{W^n_{t_{l-1}}} - 2^m(t^n_{k-1} - t^n_{l-1})\Delta^m_l W\right)^2 \\
&= \frac{1}{2}\left(2^m\left(t^n_{k-1} - \cancel{t^m_{l-1}} + t^n_k + \cancel{t^m_{l-1}}\right)\Delta^m_l W\right)^2 \underbrace{=}_{t^n_{k-1}-t^n_k=\frac{1}{2^n}} \\
&= \frac{1}{2}2^{2m}2^{-2n}(\Delta^m_l W)^2 = 2^{2(m-n)-1}(\Delta^m_l W)^2.
\end{aligned}$$

From this, we infer

$$\begin{aligned}
X(m+1)^2_{t^n_{k-1},t^n_k} - X(m)^2_{t^n_{k-1},t^n_k} &= 2^{2(m+1-n)-1}(\Delta^{m+1}_l W)^2 - 2^{2(m-n)-1}(\Delta^m_l W)^2 \\
&= 2^{2(m-n)+1}\left[(\Delta^{m+1}_l W)^2 - \frac{1}{4}(\Delta^m_l W)^2\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{k=1}^{2^n}\mathbb{E}\left[\left|X(m+1)^2_{t^n_{k-1},t^n_k} - X(m)^2_{t^n_{k-1},t^n_k}\right|^{p/2}\right] = \\
&= 2^{(2(m-n)+1)p/2}\sum_{k=1}^{2^n}\mathbb{E}\left[\left|(\Delta_l^{m+1}W)^2 - \frac{1}{4}(\Delta_l^m W)^2\right|^{p/2}\right] \underbrace{\leq}_{\text{triang. ineq.}} \\
&\leq \hat{C}_p 2^{(2(m-n)+1)p/2}\sum_{k=1}^{2^n}\mathbb{E}\left[\left|\Delta_l^{m+1}W\right|^p\right] + \frac{1}{2^p}\mathbb{E}\left[|\Delta_l^m W|^p\right] \underbrace{\leq}_{\Delta_l^m W \text{ Gaussian}} \\
&\leq c_{p,d}2^{(2(m-n)+1)p/2}\sum_{k=1}^{2^n}\left(\frac{1}{2^{m+1}}\right)^{p/2} + \frac{1}{2^p}\left(\frac{1}{2^m}\right)^{p/2} = \\
&= C_{p,d}2^{(m-n)p}\sum_{k=1}^{2^n}2^{-mp/2} = C_{p,d}2^{(m-n)p}2^n 2^{-mp/2}. = \\
&= C_{p,d}2^{mp/2}2^{n(1-p)} = C_{p,d}\left(\frac{2^m}{2^n}\right)^{p/2}\left(\frac{1}{2^n}\right)^{p/2-1} \underbrace{\leq}_{n>m\Rightarrow \frac{1}{2^n}<\frac{1}{2^m}} \\
&\leq C_{p,d}\left(\frac{2^m}{2^n}\right)^{p/2}\left(\frac{1}{2^m}\right)^{p/2-1}
\end{aligned}$$

where $c_{p,d}$ and $C_{p,d}$ are constants depending on p and d.

Now, we give some details concerning the case $n \leq m$. We have a lemma that gives an expression for $X(m+1)^2_{t^n_{k-1},t_k} - X(m)^2_{t^n_{k-1},t^n_k}$.

Lemma 3.3.1. *For $m > n$, we have*

$$X(m+1)^2_{t_{k-1},t_k} - X(m)^2_{t_{k-1},t_k} = \frac{1}{2}\sum_{l=2^{m-n}(k-1)+1}^{2^{m-n}k}\left(\Delta_{2l-1}^{m+1}W\Delta_{2l}^{m+1}W - \Delta_{2l}^{m+1}W\Delta_{2l-1}^{m+1}W\right).$$

Proof. See [7], pages 69-70. □

Recall that in our case $W = (w^1, w^2, \dots, w^d)$ where each w^i is a one-dimensional Brownian motion and such that w^i is independent of w^j if $i \neq j$. With this, and using Lemma 3.3.1, we can estimate the expectation $\mathbb{E}\left[\left|\left(X(m+1)^2_{t_{k-1},t_k} - X(m)^2_{t_{k-1},t_k}\right)\right|^2\right]$. Indeed,

$$
\begin{aligned}
&2\mathbb{E}\left[\left|\left(X(m+1)^2_{t_{k-1},t_k} - X(m)^2_{t_{k-1},t_k}\right)\right|^2\right] \overset{\text{lemma 3.3.1}}{=} \\
&= \mathbb{E}\left[\left|\left(\sum_{l=2^{m-n}(k-1)+1}^{2^{m-n}k} \left(\Delta^{m+1}_{2l-1}W\Delta^{m+1}_{2l}W - \Delta^{m+1}_{2l}W\Delta^{m+1}_{2l-1}W\right)\right)\right|^2\right] = \\
&= \sum_{i\neq j}\mathbb{E}\left[\left(\sum_{l=2^{m-n}(k-1)+1}^{2^{m-n}k}\left[\Delta^{m+1}_{2l-1}w^i\Delta^{m+1}_{2l}w^j - \Delta^{m+1}_{2l}w^i\Delta^{m+1}_{2l-1}w^j\right]\right)^2\right] = \\
&= 2\sum_{i\neq j}\sum_{l,r=2^{m-n}(k-1)+1}^{2^{m-n}k}\mathbb{E}\left[\Delta^{m+1}_{2r-1}w^i\Delta^{m+1}_{2l-1}w^i\right]\mathbb{E}\left[\Delta^{m+1}_{2r}w^j\Delta^{m+1}_{2l}w^j\right] - \\
&- 2\sum_{i\neq j}\sum_{l,r=2^{m-n}(k-1)+1}^{2^{m-n}k}\mathbb{E}\left[\Delta^{m+1}_{2r-1}w^i\Delta^{m+1}_{2l}w^i\right]\mathbb{E}\left[\Delta^{m+1}_{2r}w^j\Delta^{m+1}_{2l-1}w^j\right].
\end{aligned}
\tag{3.18}
$$

Since the increments of a Brownian motion are independent, we have

$$
\mathbb{E}\left[(w^i_t - w^i_s)^2\right] = t - s.
$$

Hence, in the last expression in (3.18) the second sum is equal to 0 and for the first sum the terms different from 0 appear only when $l = r$. Therefore,

$$
\begin{aligned}
&2\mathbb{E}\left[\left(X(m+1)^2_{t_{k-1},t_k} - X(m)^2_{t_{k-1},t_k}\right)^2\right] = \\
&= 2\sum_{i\neq j}\sum_{s=2^{m-n}(k-1)+1}^{2^{m-n}k}\left(\underbrace{\mathbb{E}\left[\left(\Delta^{m+1}_s w\right)^2\right]}_{=\frac{1}{2^{m+1}}}\right)^2 = 4\sum_{i\neq j}\sum_{s=2^{m-n}(k-1)+1}^{2^{m-n}k}\frac{1}{2^{2m+2}} = \\
&= 2\sum_{i\neq j}\left[2^{m-n}k - 2^{m-n}(k-1) - 1\right]\frac{1}{2^{2m+2}} = \\
&= 2(d^2-d)\left[\frac{1}{2^{m+n+2}} - \frac{1}{2^{m+2}}\right].
\end{aligned}
\tag{3.19}
$$

Notice that the second level of the difference of two multiplicative functionals is not anymore multiplicative, so we cannot apply the Proposition 3.2.1. However, we have the following result.

Proposition 3.3.1. *For $p > 2$ and for $\gamma > p/2 - 1$, there exists a constant C depending on p*

and γ, $C > 0$ *such that*

$$\sup_{\pi\in\Pi([0,T])}\sum_{l}\left|X^2_{t_{l-1},t_l}-Y^2_{t_{l-1},t_l}\right|^{p/2}\leq$$

$$\leq C\left(\sum_{n=1}^{\infty}n^{\gamma}\sum_{k=1}^{2^n}\left|X^1_{t^n_{k-1},t^n_k}-X^1_{t^n_{k-1},t^n_k}\right|^p\right)^{1/2}\left(\sum_{n=1}^{\infty}n^{\gamma}\sum_{k=1}^{2^n}\left|X^1_{t^n_{k-1},t^n_k}\right|^p+\left|X^1_{t^n_{k-1},t^n_k}\right|^p\right)^{1/2}+$$

$$+\,C\sum_{n=1}^{\infty}n^{\gamma}\sum_{k=1}^{2^n}\left|X^2_{t^n_{k-1},t^n_k}-X^2_{t^n_{k-1},t^n_k}\right|^p.$$

Applying the proposition above carefully to $X = X(m+1)$ and $Y = X(m)$ and using on the right hand side the estimates we have induced in (3.19) (and the previous estimates used in the study of $X(m)^1$) and applying Hölder inequality several times, we can see that the the sequence $(X(m)^2)_m$ is convergent in the p-variation metric. Further details on this computation can be seen in [7], chapter 3.

3.4 Characterization of the rough path

We have proved that $(X(m)^1)_{m\in\mathbb{N}}$ converges almost surely to X^1 where $X^1_{s,t} = W_t - W_s$ in the topology of the p-variation norm. We also have seen that the second levels of the rough path of the dyadic approximation of the Brownian motion $(X(m)^2)_{m\in\mathbb{N}}$ is a sequence that converges as $m \to \infty$ to some element X^2,

$$X^2 : \Delta_{[0,T]} \to T^2(\mathbb{R}^n).$$

We can characterize this element (see [7], Theorem 4.4.3) in terms of an well known object: the *Stratonovich* integral. In fact, we have that

$$X^2_{s,t} = \int_{s<t_1<t_2<t} \circ\, \mathrm{d}W_{t_1} \circ \mathrm{d}W_{t_2},$$

where $\circ$ indicates that the integral has to be taken in the *Stratonovich sense*.

Chapter 4

Integration (of one-forms) along rough paths

Recall that we want to give a meaning to controlled differential equations in a general framework. So far, we have constructed an analytical and algebraic object, the rough path (of roughness $\lfloor p \rfloor$), that can be associated to any path with finite p-variation, for any $p > 1$. From now on, we consider controlled differential equations of the form

$$\mathrm{d}\boldsymbol{Y} = f(\boldsymbol{X})\ \mathrm{d}\boldsymbol{X}, \tag{4.1}$$

where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are rough paths. We interpret (4.1) in the integral for as

$$\boldsymbol{Y} = \int f(\boldsymbol{X})\ \mathrm{d}\boldsymbol{X},$$

but the integral with respect to a rough path has not been defined yet. This chapter is devoted to give a proper definition of this integral, and to describe its basic properties. We will see that $f(X)$ should be a more general object than a function.

4.1 Integral of a one-form along a (geometric) p-rough path

We sketch here the steps we are going to follow in order to do the construction. First, we will assume that $\boldsymbol{X}$ is a smooth path and try to define the integral in a natural way. After some computations, we will realize that the definition does not need the rough path to be smooth.

4.1.1 Integral of a one-form along a smooth path

The class Lip(γ)

As we have said, now we deal with more general objects than functions. First, recall the notion of one-form.

Definition 4.1.1. *A $\mathbb{R}^m$-valued one-form on $\mathbb{R}^n$, θ, is a function on $\mathbb{R}^n$ whose value at each point is a linear homomorphism from $\mathbb{R}^n$ to $\mathbb{R}^m$. That is,*

$$\theta : \mathbb{R}^n \longrightarrow \mathit{Hom}(\mathbb{R}^n, \mathbb{R}^m).$$

We need the notion of the differential of a one-form. The motivation of the following definition is out of the scope of this work, but more information can be found in [9]. For our purposes, it is sufficient to consider the following definition.

Definition 4.1.2. *The differential of a $\mathbb{R}^m$-valued one-form θ on $\mathbb{R}^n$ is a mapping* $\mathrm{d}\theta$

$$\mathrm{d}\theta : \mathbb{R}^n \longrightarrow \mathit{Hom}(\mathbb{R}^n, \mathit{Hom}(\mathbb{R}^n, \mathbb{R}^m) \cong \mathit{Hom}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^m),$$

which satisfies (for smooth paths and conventional integrals)

$$\int_s^t \mathrm{d}\theta(x_u)(\mathrm{d}x_u, v) = \theta(x_t)(v) - \theta(x_s)(v).$$

Remark. By iteration, we can define differentials of higher order of a one-form as the function

$$\mathrm{d}^k\theta : \mathbb{R}^n \longrightarrow \mathrm{Hom}(\underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{\text{k times}}, \mathbb{R}^m)$$

such that (for smooth paths and usual integrals) satisfies

$$\int_s^t \mathrm{d}^k\theta(x_u)(\mathrm{d}x_u, v_2, \ldots, v_k) = \mathrm{d}^{k-1}\theta(x_t)(v_2, \ldots, v_k) - \mathrm{d}^{k-1}\theta(x_s)(v_2, \ldots, v_k).$$

Notation. We will denote $\theta^k := \mathrm{d}\theta^{k-1} = \mathrm{d}^k\theta$.

We will not consider general one-forms here. Our aim is to define the integral of a one-form of some specific type along a geometric rough path. For this, we introduce the class Lip(γ).

Definition 4.1.3. *Let $\gamma > 1$. θ is a Lip($\gamma - 1$) one-form with norm less or equal than M if for any $j \in \mathbb{N}$, $1 \leq j < \gamma$, one has*

$$\theta^j(x_t)(v_1, v_2, \ldots, v_j) = \sum_{0 \leq i < \gamma - j} \theta^{j+i}(x_0)(x^i_{0,t}, v_1, v_2, \ldots, v_j) + R^j(x_0, x_t)(v_1, v_2, \ldots, v_j), \quad (4.2)$$

where $x^i_{0,t}$ is the iterated i-th integral of the (smooth) path $x : [0,T] \to \mathbb{R}^n$, and $\theta^i(x)$, $R^i(x,y)$ satisfies the following bounds

$$\begin{cases} ||\theta^i(x)|| \leq M, \\ ||R^i(x,y)|| \leq M\, ||x-y||^{\gamma - i}, \end{cases} \quad (4.3)$$

$i = 0, 1, \ldots, \gamma - j$.

The Lip-norm of the form θ, that we denote by $||\theta||_{Lip}$, is the minimum of the constants M satisfying (4.3).

The expression (4.2) is known as the Taylor series expression for θ^j because of its similarity to the Taylor expansion and the term R the remainder of the Taylor expression. We can think of a Lip($\gamma - 1$) form θ as a one-form *differentiable* up to $\lfloor \gamma \rfloor$-th level and whose "Taylor series" converges.

Construction of the integral

Consider a smooth path (i. e., with finite 1-variation) X. As we have seen in chapter 1, it is possible to construct a geometric rough path $\boldsymbol{X}_{s,t} = (1, X^1_{s,t}, \ldots, X^{\lfloor p \rfloor}_{s,t})$ based on X, by considering

$$X^j_{s,t} = \int_{s<u_1<\ldots<u_j<t} \mathrm{d}X_{u_1} \ldots \mathrm{d}X_{u_j}. \quad (4.4)$$

Consider a one-form $\theta \in$ Lip($\gamma - 1$). We define the integral of θ along X, by $\boldsymbol{Y} = (1, Y^1, \ldots, Y^{\lfloor p \rfloor})$, where

$$Y^i_{s,t} := \int_{s<u_1<\ldots<u_i<t} \left(\sum_{l_1=1}^{\lfloor p \rfloor} \theta^{l_1}(X_s)(\mathrm{d}X^{l_1}_{s,u_1}) \right) \cdot \ldots \cdot \left(\sum_{l_i=1}^{\lfloor p \rfloor} \theta^{l_i}(X_s)(\mathrm{d}X^{l_i}_{s,u_i}) \right). \quad (4.5)$$

Since $\theta^{l_j}(X_s)$ are constant functions for any j (we are varying u_k in (4.5)), we have

$$Y^i_{s,t} = \sum_{l_1=1}^{\lfloor p \rfloor} \cdots \sum_{l_i=1}^{\lfloor p \rfloor} \theta^{l_1}(X_s) \cdot \ldots \theta^{l_i}(X_s) \underbrace{\int_{s<u_1<\ldots<u_i<t} \mathrm{d}X^{l_1}_{s,u_1} \ldots \mathrm{d}X^{l_i}_{s,u_i}}_{A}.$$

We will use a combinatorial argument to simplify the underbraced expression A. Indeed, using (4.4) we obtain

$$A = \int_D \underbrace{\mathrm{d}X_{u_1,1} \cdot \ldots \cdot \mathrm{d}X_{u_1,l_1}}_{\mathrm{d}X^{l_1}_{s,u_1}} \cdot \ldots \cdot \underbrace{\mathrm{d}X_{u_i,1} \cdot \ldots \cdot \mathrm{d}X_{u_i,l_i}}_{\mathrm{d}X^{l_i}_{s,u_i}}, \tag{4.6}$$

where D is the domain (it is a product of simplexes) defined by the following relations

$$\begin{cases} s < u_1 & < \ldots < u_i < t, \\ s < u_{1,1} & < \ldots < u_{1,l_1} = u_1, \\ \ldots & \\ s < u_{i,1} & < \ldots < u_{i,l_i} = u_i. \end{cases} \tag{4.7}$$

Then for any $L = (l_1, \ldots, l_i)$ and any $U = \{u_{k,m_k}\}_{k=1,\ldots,i;\ m=1,\ldots,l_k}$ satisfying the relations for D in (4.7), we consider the permutation $\pi_U \in \Sigma_{|L|}$ (where $|L| = l_1 + \ldots + l_i$) such that when applied to U itself, it produces an increasingly ordered sequence of numbers. Finally, let's denote Π_L the range of π_U as an element of $\Sigma_{|L|}$.

These definitions allow us to express A as a linear combination of integrals whose indexes are well ordered, and thanks to that, being able to express A depending on the geometric rough paths associated to the iterated integrals directly. That is,

$$
\begin{aligned}
A &\overset{(4.6)}{=} \int_D \mathrm{d}X_{u_1,1} \cdot \ldots \cdot \mathrm{d}X_{u_1,l_1} \cdot \ldots \cdot \mathrm{d}X_{u_i,1} \cdot \ldots \cdot \mathrm{d}X_{u_i,l_i} \\
&= \sum_{\pi \in \Pi_L} \left(\int\limits_{s<v_1<\ldots<v_{|L|}<t} \mathrm{d}X_{v_{\pi(1)}} \cdot \ldots \cdot \mathrm{d}X_{v_{\pi(|L|)}} \right) \\
&= \sum_{\pi \in \Pi_L} \pi \left(\int\limits_{s<v_1<\ldots<v_{|L|}<t} \mathrm{d}X_{v_1} \cdot \ldots \cdot \mathrm{d}X_{v_{|L|}} \right) = \\
&= \sum_{\pi \in \Pi_L} \pi(X^{|L|}_{s,t}).
\end{aligned} \tag{4.8}
$$

Therefore, substituting (4.8) into (4.6), we finally have

$$
Y^i_{s,t} = \sum_{l_1,\ldots,l_i=1}^{\lfloor p \rfloor} \theta^{l_i}(X_s) \cdot \ldots \cdot \theta^{l_i}(X_s) \sum_{\pi \in \Pi_L} \pi(X^{|L|}_{s,t}), \tag{4.9}
$$

which does not need the higher levels of the rough path X come from iterated integrals This suggests the following definition for any geometric p-rough path $\boldsymbol{X}$.

Definition 4.1.4. *Let $\boldsymbol{X}$ be a geometric p-rough path and $\theta \in Lip(\gamma - 1)$, with $\gamma > p$. Then, $\boldsymbol{Y} = (1, Y^i, \ldots, Y^{\lfloor p \rfloor})$, where*

$$
Y^i_{s,t} = \sum_{l_1,\ldots,l_i=1}^{\lfloor p \rfloor} \theta^{l_i}(X_s) \cdot \ldots \cdot \theta^{l_i}(X_s) \sum_{\pi \in \Pi_L} \pi(X^{|L|}_{s,t}).
$$

Now we have constructed an element $\boldsymbol{Y} \in T^{\lfloor p \rfloor}(\mathbb{R}^n)$. A very involved computation (see [5], theorem 3.2.1.) shows that in fact $\boldsymbol{Y}$ is an almost p-rough path (this is, it can be shown that $\boldsymbol{Y}$ is almost multiplicative, i. e., it satisfies relation (2.12), and $\boldsymbol{Y}$ has finite p-variation.

Recall that the theorem 2.1.2 states the existence and uniqueness of a p-rough path associated to any given almost p-rough path. This allow us to define the integral of a one-form as a true rough path. More precisely, we have the following definition.

Definition 4.1.5. *Given a p-rough path $\boldsymbol{X}$ and a one-form $\theta \in Lip(\gamma - 1)$ with $\gamma > p$, the integral of θ along $\boldsymbol{X}$, $\hat{\boldsymbol{Y}}$; denoted by $\hat{Y}$ is*

$$
\hat{\boldsymbol{Y}}_{s,t} = \int\limits_{s<u<t} \theta(X_u) \delta \boldsymbol{X}_u
$$

(or by $\delta\hat{\boldsymbol{Y}} = \theta(X)\delta\boldsymbol{X}$); is the p-rough path associated to the almost p-rough path constructed by (4.9).

4.2 Continuity of the integral

Remember that we are trying to establish a continuity result for the Itô map. It seems quite clear that as a previous step, we need the integral along a rough path to be also continuous in the following sense: if we have two geometric rough paths of the same roughness p, $\boldsymbol{X}$ and $\boldsymbol{X'}$, then, given a one-form θ, the distance (in the p-variation norm) of the integrals associated to them, let's put $\delta\boldsymbol{Y} = \theta(X)\,\delta\boldsymbol{X}$ and $\delta\boldsymbol{Y'} = \theta(X')\,\delta\boldsymbol{X'}$ is controlled by the distance (in the p-variation norm) of the rough paths themselves.

This is actually true. More precisely, we have the following result.

Theorem 4.2.1. *Let $\boldsymbol{X}$ and $\boldsymbol{X'}$ be geometric p-rough paths controlled by ω with $||\omega||_\infty < K$ and θ is a Lip($\gamma - 1$), $\gamma > p$ one-form with norm less or equal than M. Assumere there exists ε such that*

1. *for any $i \leq \lfloor p \rfloor$,*
$$\left|\left|(\boldsymbol{X}_{s,t} - \boldsymbol{X'}_{s,t})\right|\right|_p < \varepsilon \frac{\omega(s,t)^{i/p}}{\beta(i/p)!}; \tag{4.10}$$
where
$$\beta = p^2 \left(1 + \sum_{r=3}^{\infty} \left(\frac{2}{r-2}\right)^{\frac{\lfloor p \rfloor + 1}{p}}\right)$$
and$(i/p)!$ *is the generalized factorial defined by the Gamma function. And also,*

2. $||X_u - X'_u|| \leq \varepsilon$.

Then there is a function $\delta(\varepsilon, K, M, p)$ satisfying

$$\lim_{\varepsilon \to 0} \delta(\varepsilon, L, M, p) = 0,$$

and such that for any $i \leq \lfloor p \rfloor$,

$$\left|\left|\left(\int_{s<u<t} \theta(X_u)\delta\boldsymbol{X} - \int_{s<u<t} \theta(X'_u)\delta\boldsymbol{X'}\right)^i\right|\right|_p < \delta(\varepsilon, K, M, p)\frac{\omega(s,t)^i}{(i/p)!}.$$

Then the mapping $\boldsymbol{X} \to \int \theta(X)\,\delta\boldsymbol{X}$ *is continuous in the p-variation topology.*

Proof. Fix $\varepsilon > 0$ satisfying properties 1. and 2.

We have seen in Theorem 2.1.3 that the association of a rough path with an almost rough path is continuous. Therefore, it suffices to prove the continuity for the almost-rough path intermediate in the definition of the integral.

Recall that for almost rough paths, we have

$$Y^i_{s,t} = \sum_{l_1,\dots,l_i=1}^{\lfloor p\rfloor} \theta^{l_i}(X_s)\cdot\ldots\cdot\theta^{l_i}(X_s) \sum_{\pi\in\Pi_L} \pi(X^{|L|}_{s,t}).$$

Notice that the product $\theta^{l_1}(\cdot)\times\cdots\times\theta^{l_i}(\cdot)$ is uniformly continuous for some modulus of continuity σ depending on ε, M, p. Then, we have

$$\left|\left|\mathbf{Y}^i_{s,t} - \mathbf{Y}^i_{s,t}\right|\right|_p \leq \underbrace{\left[\sum_{l_1,\dots,l_i=1}^{\lfloor p\rfloor} \left\{\varepsilon M^{|L|}\,|\Pi_L| + \sigma(\varepsilon, M, p)\right\} \frac{\omega(s,t)^{\frac{|L|-i}{p}}}{\beta\left(\frac{|L|}{p}\right)!}\right]}_{:=\delta(\varepsilon,M,K,p)\cdot(i/p)!} \omega(s,t)^{i/p} =$$
$$=: \delta(\varepsilon, M, K, p)\frac{\omega(s,t)^i}{(i/p)!}.$$

From our definition of δ is clear that it is continuous at $\varepsilon = 0$ and takes the value 0, which is what we wanted. □

4.3 Integral of a one-form along a (general) p-rough path, $2 \leq p < 3$

Now that we have constructed the integral for a geometric rough path we will reduce to the case of a p-rough path with $2 \leq p < 3$, which is the most interesting case for us, since it includes the Brownian rough path we have constructed in chapter 3, and we will see that in this case, the integral is well defined even in the case when the rough path $\boldsymbol{X}$ is not geometric. We will skip the intermediate computations, that can be done with an algebraic manipulator.

Let us consider a p-rough path with $2 < p < 3$ and $\theta \in \mathrm{Lip}(\alpha)$ with $p - 1 < \alpha < 2$. We can consider as the previous almost rough path associated to the integral $\boldsymbol{Y} \in T^2(\mathbb{R}^n)$, defined by

$$\boldsymbol{Y}_{s,t} = \{\boldsymbol{Y}^0_{s,t}, \boldsymbol{Y}^1_{s,t}, \boldsymbol{Y}^2_{s,t}\} = \left\{1,\ \theta(X_s)(\boldsymbol{X}^1_{s,t} + \frac{1}{2}(\mathrm{d}\theta)(X_s)(\boldsymbol{X}^2_{s,t}),\ \left(\theta(X_s) \times \theta(X_s)\right)(\boldsymbol{X}^2_{s,t})\right\}. \tag{4.11}$$

By the Taylor expansion of θ (since it is of the Lip(α) class for $\alpha < 2$, see (4.2)), we have

$$\begin{cases} \theta(X_t) = \theta(X_s) + \frac{1}{2}(\mathrm{d}\theta)(X_s)(\boldsymbol{X}^1_{s,t}) + r_1(t,s), \\ \mathrm{d}\theta(X_t) = \mathrm{d}\theta(X_s) + r_2(t,s); \end{cases} \tag{4.12}$$

with

$$\begin{cases} ||r_1(t,s)||_p < M\omega(t,s)^{\gamma/p}, \\ ||r_2(t,s)||_p < M\omega(t,s)^{(\gamma-1)/p}. \end{cases} \tag{4.13}$$

From the first equations in (4.12) and (4.13), it is easy to see that $\boldsymbol{Y}$, defined in (4.11) has finite p-variation controlled by $2M\omega$. Now, our aim is to check that it is actually almost-multiplicative. By algebraic computations, we can obtain that

$$\boldsymbol{Y}_{s,u} - \boldsymbol{Y}_{s,t}\boldsymbol{Y}_{t,u} = (0, A, B),$$

with

$$\begin{aligned} A = {} & \theta(X_s)(\boldsymbol{X}^1_{s,u}) + \frac{1}{2}(\mathrm{d}\theta)(X_s)(\boldsymbol{X}^2_{s,u}) - \left(\theta(X_s)(\boldsymbol{X}^1_{s,u}) + \frac{1}{2}(\mathrm{d}\theta)(X_s)(\boldsymbol{X}^2_{s,t})\right) - \\ & - \left(\theta(X_t)(\boldsymbol{X}^1_{t,u}) + \frac{1}{2}(\mathrm{d}\theta)(X_t)(\boldsymbol{X}^2_{t,u})\right); \\ B = {} & (\theta(X_s) \times \theta(X_s))(\boldsymbol{X}^2_{s,u}) - \\ & - \left(\left(\theta(X_s)(\boldsymbol{X}^1_{s,t}) + \frac{1}{2}(\mathrm{d}(\theta)(X_s)(\boldsymbol{X}^2_{s,t})\right) \times \left(\theta(X_s)(\boldsymbol{X}^1_{s,u}) + \frac{1}{2}(\mathrm{d}(\theta)(X_s)(\boldsymbol{X}^2_{s,u})\right)\right) - \\ & - \left(\theta(X_s)(\boldsymbol{X}^1_{s,t}) + \frac{1}{2}(\mathrm{d}\theta)(X_s)(\boldsymbol{X}^2_{s,t}) + \theta(X_t)(\boldsymbol{X}^1_{t,u}) + \frac{1}{2}(\mathrm{d}\theta)(X_t)(\boldsymbol{X}^2_{t,u})\right). \end{aligned} \tag{4.14}$$

We bound explicitly the term A. From (4.12),it is not difficult to see that

$$A = r_1(s,t)\boldsymbol{X}^1_{t,u} + r_2(s,t)\boldsymbol{X}^2_{t,u}.$$

Now, from (4.13) we have

$$
\begin{aligned}
||A||_p &= \left|\left|r_1(s,t)\boldsymbol{X}^1_{t,u} + r_2(s,t)\boldsymbol{X}^2_{t,u}\right|\right|_p \leq \\
&\leq ||r_1(s,t)||_p \left|\left|\boldsymbol{X}^1_{t,u}\right|\right|_p + ||r_2(s,t)||_p \left|\left|\boldsymbol{X}^2_{t,u}\right|\right|_p \overset{\text{eqs. (4.13) and control on}\boldsymbol{X}}{\leq} \\
&\leq M\left[\omega(s,t)^{\gamma/p}\omega(t,u)^{1/p} + \omega(s,t)^{(\gamma-1)/p}\omega(t,u)^{2/p}\right] \leq \\
&\leq 2M\omega(s,u)^{(\gamma+1)/p}.
\end{aligned}
\tag{4.15}
$$

The term B can be computed using (4.12), and rewritten as a sum of fifteen terms. Each of them is bounded by

$$M^2\omega(s,u)^{3/p}.$$

Therefore

$$||B||_p \leq 15M^2\omega(s,u)^{3/p}.$$

Finally, notice that since $p-1<\alpha$, then $\frac{\alpha+1}{p}>1$, and since $p<3$ we will also have $3/p>1$. This means that, using the triangular inequality, we can bound the second level of $\boldsymbol{Y}_{s,u}-\boldsymbol{Y}_{s,t}\boldsymbol{Y}_{t,u}$ by a control to a power greater than one.

Summarizing, we have seen that there exists some constant $K>0$ and some $\beta>1$ such that

$$||\boldsymbol{Y}_{s,u} - \boldsymbol{Y}_{s,t}\boldsymbol{Y}_{t,u}||_p \leq K\omega(s,t)^{\beta};$$

that is, Y is an almost p-rough path.

Remark. We want to insist on the following fact. Throughout the above computations, the algebraic manipulations do not require $\boldsymbol{X}$ to be geometric, since the terms up to the second level are “easier” to handle. For an hypothetic third level, due to its complexity , it would probably be very difficult to go beyond the case of geometric rough paths.

Chapter 5

Universal Limit Theorem

Once we have constructed an integration theory along rough paths, we can now discuss controlled differential equations as (1.1) in a more general sense. In the first part of this chapter, we will explain how to put (1.1) in the rough paths differential equations setting. Then, we will be able to discuss the most fundamental theorem of the theory, the *Universal Limit Theorem* that assures existence, uniqueness and *continuity* with respect to the control (i. e., continuity of the Itô map) of the solution for these equations under some regularity conditions. This was the main motivation for the theory.

5.1 Rough path differential equations

First of all, in order to put the equation

$$\mathrm{d}Y_t = f(Y_t)\ \mathrm{d}X_t$$

in the rough path setting, it seems important to know what means the image of a rough path by a function, that is, we need to define $f(\boldsymbol{Y})$ for $\boldsymbol{Y}$ a p-rough path. Notice that, given a function $f : \mathbb{R}^n \to \mathbb{R}^m$, the differential $\mathrm{d}f$ is a one-form with one less degree of regularity. More precisely, if f has $\lfloor\gamma\rfloor$ derivatives $\{f^{(0)}, f^{(1)}, \ldots, f^{(\lfloor\gamma\rfloor)}\}$ and $f^{(k)}$ is $(\gamma - k)$-Hölder continuous for any $p > 1$ (we denote this class of function as $\mathrm{Lip}(\gamma, \mathbb{R}^m)$), then $\mathrm{d}f$ is a $\mathrm{Lip}(\gamma - 1)$ one-form in the sense of the definition 4.1.3.

Definition 5.1.1. *Let $p > 1$. Let $\boldsymbol{Z}$ be a p-rough path if $2 < p < 3$, or a geometric rough path* *if $p \geq 3$. Let $f : \mathbb{R}^n \to \mathbb{R}^m \in$Lip($\gamma$) function for some $\gamma > p$. Then we can define the image of $\boldsymbol{Z}$ by f as*

$$f(\boldsymbol{Z}) := \int \mathrm{d}f(\boldsymbol{Z})\ \mathrm{d}\boldsymbol{Z}.$$

Remark. Notice that thanks to the results in chapter 4, $f(\boldsymbol{Z})$ is well defined as an element of $\Omega_p(\mathbb{R}^m)$, since $\mathrm{d}f \in \mathrm{Lip}(\gamma - 1)$ and $\gamma > p$.

5.1.1 Examples

Image by a linear function.

Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be linear. Then, given $k \in \mathbb{N}$, A induces a map $\tilde{A}^k$ satisfying

$$\tilde{A}^k : (\mathbb{R}^n)^{\otimes k} \longrightarrow (\mathbb{R}^m)^{\otimes k},$$
$$x_1 \times \ldots \times x_k \longrightarrow A(x_1) \times \ldots \times A(x_k).$$

Then, for any $l \in \mathbb{N}$, by combinating the family $\{\tilde{A}^k\}_{k=0}^l$ we can define a map between spaces of multiplicative functionals

$$T(A) : T^{(l)}(\mathbb{R}^n) \longrightarrow T^{(l)}(\mathbb{R}^m)$$

such that $T(A) = (Id, \tilde{A}^1, \ldots, \tilde{A}^l)$ and hence we have simply, for any $\boldsymbol{X} \in \Omega_p(\mathbb{R}^n)$ f $p < 3$ or $\boldsymbol{X} \in \mathrm{G}\Omega_p(\mathbb{R}^n)$ if $p \geq 3$,

$$A(\boldsymbol{X}) : \Delta_{[0,T]} \longrightarrow T^{\lfloor p \rfloor}(\mathbb{R}^m),$$
$$(s,t) \longrightarrow A(\boldsymbol{X})_{s,t} := T(A)(\boldsymbol{X}_{s,t}).$$

The projection map.

It is a particular case of a linear function, but it is worth taking a look at the projection map, since it will be used to define what a solution for a rough path differential equation is.

Let $\boldsymbol{Z} \in \Omega_p(\mathbb{R}^n \oplus \mathbb{R}^m)$ (or $\boldsymbol{Z} \in \mathrm{G}\Omega_p(\mathbb{R}^n \oplus \mathbb{R}^m)$ if $p \geq 3$), then we can consider the projections

$$\pi_n : \mathbb{R}^n \oplus \mathbb{R}^m \longrightarrow \mathbb{R}^n,$$

$$\pi_m : \mathbb{R}^n \oplus \mathbb{R}^m \longrightarrow \mathbb{R}^m.$$

By definition $\boldsymbol{X} = \pi_n(\boldsymbol{Z})$ and $\boldsymbol{Y} = \pi_m(\boldsymbol{Z})$ are p-rough paths (over $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively).

Remark. Notice that having $\boldsymbol{Z} = (\boldsymbol{X}, \boldsymbol{Y}) \in \Omega(\mathbb{R}^n \oplus \mathbb{R}^m)$, we can determine $\boldsymbol{X}$ and $\boldsymbol{Y}$ as rough paths by projecting $\boldsymbol{Z}$. But, given two rough paths $\boldsymbol{X}$, $\boldsymbol{Y}$, we cannot determine $\boldsymbol{Z} = (\boldsymbol{X}, \boldsymbol{Y})$ as a rough path in general. For instance, if $\boldsymbol{X}$ and $\boldsymbol{Y}$ are smooth rough paths,the iterated integrals of $(\boldsymbol{X}, \boldsymbol{Y})$ need cross iterated integrals of both $\boldsymbol{X}$ and $\boldsymbol{Y}$ which do not depend on $\boldsymbol{X}$ and $\boldsymbol{Y}$ separately, which is the information we would know if we knew them as a rough paths.

We can now state the definition of a rough path differential equation and also define the notion of solution.

Definition 5.1.2. *Let f be a Lip($\gamma - 1$) one-form. Let $\boldsymbol{X} \in G\Omega_p(\mathbb{R}^n)$, and $Y_0 \in \mathbb{R}^n$. Set $f_{Y_0}(\cdot) = f(\cdot + Y_0)$. Define h as the map*

$$\begin{aligned} h : \mathbb{R}^n \oplus \mathbb{R}^m &\longrightarrow \mathit{End}(\mathbb{R}^n \oplus \mathbb{R}^m) \\ (x, y) &\longrightarrow \begin{pmatrix} Id_{\mathbb{R}^n} & 0 \\ f_{Y_0}(y) & 0 \end{pmatrix} \end{aligned} \tag{5.1}$$

Then, we say that $\boldsymbol{Z}(\boldsymbol{X}, \boldsymbol{Y})$ is a (strong) solution of the differential equation

$$\begin{cases} \mathrm{d}\boldsymbol{Y}_t = f(\boldsymbol{Y}_t)\ \mathrm{d}\boldsymbol{X}_t, \\ \boldsymbol{Y}(0) = \boldsymbol{Y}_0, \end{cases} \tag{5.2}$$

if and only if,

(1) $\boldsymbol{Z} = \int h(\boldsymbol{Z})\ \mathrm{d}\boldsymbol{Z}$, and

(2) $\pi_{\mathbb{R}^n}(\boldsymbol{Z}) = \boldsymbol{X}$.

Remark. Notice that this reformulation gives us indeed an usual solution, since the equation (5.2) is equivalent to

$$\begin{cases} \mathrm{d}Y_t = f_{Y_0}(Y_t)\ \mathrm{d}X_t, \quad Y_0 = 0, \\ \mathrm{d}X_t = \mathrm{d}X_t \end{cases} \tag{5.3}$$

and if we define $\boldsymbol{Z} = (\boldsymbol{X}, \boldsymbol{Y})$ and h are given in (5.1) and (5.2) respectively, then (5.3) reads

$$\begin{cases} \mathrm{d}\boldsymbol{Z}_t = h(\boldsymbol{Z}_t)\ \mathrm{d}\boldsymbol{Z}_t, \\ \boldsymbol{Z}_0 = 0. \end{cases}$$

In addition $\pi_{\mathbb{R}^n}(\boldsymbol{Z}) = \boldsymbol{X}$. Then, being a solution in the strong sense implies being a usual solution for the controlled equation.

5.2 Statement and proof of the Theorem

On this section we state the *Universal Limit Theorem.* We need some technical lemmas to prove the theorem. The proof of some lemmas can be found at the end of the chapter and the rest can be found in [6], chapter 5.

5.2.1 Statement

Theorem 5.2.1 (Universal Limit Theorem). *Let $p \geq 1$ and $\gamma > p$. Let $f : \mathbb{R}^n \to$ Hom$(\mathbb{R}^n, \mathbb{R}^m)$ be a Lip(γ) one-form. Then, for any given control $\boldsymbol{X} \in G\Omega_p(\mathbb{R}^n)$ and any initial condition $Y_0 \in \mathbb{R}^n$ we have the following.*

(1) The equation

$$\begin{cases} \mathrm{d}\boldsymbol{Y}_t = f(\boldsymbol{Y}_t)\ \mathrm{d}\boldsymbol{X}_t, \\ \boldsymbol{Y}(0) = \boldsymbol{Y}_0; \end{cases}$$

admits an unique (strong) solution $\boldsymbol{Z} = (\boldsymbol{X}, \boldsymbol{Y}) \in G\Omega_p(\mathbb{R}^n \oplus \mathbb{R}^m)$, in the sense of Definition 5.1.2.

(2) The map $(\boldsymbol{X}, \boldsymbol{Y}_0) \to \boldsymbol{Z}$ is continuous in the p-variation topology and

$$I_f : G\Omega_p(\mathbb{R}^n) \times \mathbb{R}^m \longrightarrow G\Omega_p(\mathbb{R}^m)$$

is the unique extension of the Itô map which is continuous in the p-variation topology.

(3) The rough path $\boldsymbol{Y}$ is the limit of a sequence $\{\boldsymbol{Y}(n)\}_{n\in\mathbb{N}}$ of the form $\boldsymbol{Y}(n) = \pi_{\mathbb{R}^m}(\boldsymbol{Z}(n))$ and

where $\{\boldsymbol{Z}(n)\}_{n\in\mathbb{N}}$ *is built iteratively by a Picard-type iteration*

$$\boldsymbol{Z}(n+1) = \int h(\boldsymbol{Z}(n))\,\mathrm{d}\boldsymbol{Z}(n), \quad \boldsymbol{Z}(0) = (\boldsymbol{X}, 0).^{1} \tag{5.4}$$

(4) If $\boldsymbol{X}$ *is controlled by* ω, *then* $\forall\rho > 1$ *there exists* $T_\rho \in (0, T]$ *such that*

$$\left|\left|\boldsymbol{Y}(n)^i_{s,t} - \boldsymbol{Y}(n+1)^i_{s,t}\right|\right|_p \leq 2^i \frac{\omega(s,t)^{i/p}}{\beta(i/p)!}\rho^{-n} \tag{5.5}$$

for any $(s,t) \in \Delta_{[0,T_\rho]}$, *for any* $i = 0, 1, \ldots, \lfloor p \rfloor$ *and where* $T_\rho = T_\rho(||f||_{Lip(\gamma)}, p, \gamma, \omega)$. *This is, we can control the rate of convergence.*

Remark. Notice that we require one more degree of smoothness for f than to just define the integral to apply the Theorem.

5.2.2 Lemmas needed in the proof of Theorem 5.2.1

The first lemma is referred to a division property of one-forms, and is independent from the rough path theory.

Lemma 5.2.1. *Fix* $\gamma > 1$, *and let* $f \in Lip(\gamma)$, $f : \mathbb{R}^n \to Hom(\mathbb{R}^n, \mathbb{R}^m)$. *Then, there exists* $g : \mathbb{R}^n \times \mathbb{R}^n \to Hom(\mathbb{R}^n, \mathbb{R}^m)$, $g \in Lip(\gamma - 1)$ *such that*

$$f(x) - f(y) = g(x,y)(x-y),$$

and

$$||g||_{Lip(\gamma-1)} \leq C\,||f||_{Lip(\gamma)}$$

for some $C = C(\gamma, n)$.

Proof. See [6], page 18. □

The following two lemmas refer to rough paths. The first one is a general property and will not be proved. The second one is an important step concerning the proof of the Universal Limit Theorem. We claim it to be true and the proof is postponed to the end of the chapter.

[1] Here, by 0, we mean the trivial rough path (1,0,. . . ,0).

Lemma 5.2.2. *Let* $\boldsymbol{Z} = (\boldsymbol{Z}_1, \boldsymbol{Z}_2) \in G\Omega_p(\mathbb{R}^k \oplus \mathbb{R}^k)$*. Fix* $\varepsilon > 0$ *and define*

$$\begin{aligned} D : \mathbb{R}^k \oplus \mathbb{R}^k &\longrightarrow \mathbb{R}^k \oplus \mathbb{R}^k \\ (x, y) &\longrightarrow \left(x, \frac{y-x}{\varepsilon}\right). \end{aligned}$$

If the rough path $D(\boldsymbol{Z}_1, \boldsymbol{Z}_2)$ *is controlled by some control* ω *on* $[0, T]$*, then we have the following estimate for the distance between* $\boldsymbol{Z}_1$ *and* $\boldsymbol{Z}_2$*:*

$$\left\|(\boldsymbol{Z}_1)^i_{s,t} - (\boldsymbol{Z}_2)^i_{s,t}\right\| \leq ((1+\varepsilon)^i - 1)\frac{\omega(s,t)^{i/p}}{\beta(i/p)!},$$

for any $(s, t) \in \Delta_{[0,T]}$*,* $i = 0, 1, \ldots, \lfloor p \rfloor$

Proof. See [6], page 86. □

In order to state the last lemma notice the following: by continuity of the integral, there exists a constant $\bar{M}$ depending on h_0, h_1 and h_2 such that, if $\boldsymbol{Z}$ is controlled by ω then $\int h_i(\boldsymbol{Z})\, \mathrm{d}\boldsymbol{Z}$ is controlled by $\bar{M}\omega$ for any $i = 0, 1, 2$. Now we choose $M = \max(1, \bar{M})$ and set $\varepsilon = M^{-\lfloor p \rfloor/p}$. With that choices, if ω_0 is a control for the p-variation of $\boldsymbol{X}$ and T_ρ is such that $\omega(0, T_\rho) = \varepsilon^p$, by letting $\omega = \varepsilon^{-p}\omega_0$, we have that $\varepsilon^{-1}\boldsymbol{X}$ is controlled by ω, and $\omega(0, T_\rho) \leq 1$.

Now, with all these assumptions and choices of constants, we can state the following.

Lemma 5.2.3. *Given* $\rho > 1$*, we have that the p-variation of the rough paths*

$$\begin{aligned} &(\varepsilon^{-1}X, Y(n)), \\ &(\varepsilon^{-1}X, Y(n), Y(n+1), \rho^n(Y(n+1) - Y(n))) \end{aligned}$$

is controlled by ω *on* $[0, T_\rho]$*, where* T_ρ *is defined above.*

Basically, this lemma states that the Picard iterations used in the proof of Theorem 5.2.1 are controlled by the same control as the signal, which clearly will be a critical point in order to check the continuity of the solution of the differential equation.

5.2.3 Proof of the Theorem 5.2.1

Proof. First of all, we see that the condition (2) in Definition 5.1.2 is satisfied. In fact, we will see that for any $k \in \mathbb{N}$, $\pi_{\mathbb{R}^n}(\boldsymbol{Z}(k)) = \boldsymbol{X}$, where $(\boldsymbol{Z}(k))_{k \geq 0}$ are the Picard iteration defined in

(5.4) Now we claim that

$$\pi_{\mathbb{R}^n}(\boldsymbol{Z}) = \pi_{\mathbb{R}^n}\left(\int h(\boldsymbol{Z})\ \mathrm{d}\boldsymbol{Z}\right). \tag{5.6}$$

By definition of the map h, this is true if $\boldsymbol{Z}$ is a 1-rough path, but since 1-rough paths are dense on the space of geometric p-rough paths by definition, and the integration is continuous in the p-variation metrics (see theorem 4.2.1) and also the projection is continuous, we have that by a density argument, this also holds for any $\boldsymbol{Z} \in \mathrm{G}\Omega_p(\mathbb{R}^n \oplus \mathbb{R}^m)$. Now we apply induction on k. If $k = 0$ we know it by definition of $\boldsymbol{Z}(0) = (\boldsymbol{X}, 0)$. Now, if we assume this is true for a given $k \in \mathbb{N}$ we have

$$\pi_{\mathbb{R}^n}(\boldsymbol{Z}(k+1)) \overset{\text{def. of } \boldsymbol{Z}(k+1)}{=} \pi_{\mathbb{R}^n}\left(\int h(\boldsymbol{Z}(k))\ \mathrm{d}\boldsymbol{Z}(k)\right) \overset{\text{IH}}{=} \pi_{\mathbb{R}^n}(\boldsymbol{Z}(k)) = \boldsymbol{X},$$

where by IH we mean that we are applying the induction hypothesis.

We can define two new one-forms h_1, h_2. associated to the controlled differential equation. Let h_1 be defined as

$$\begin{aligned} h_1 : \mathbb{R}^n \oplus \mathbb{R}^m \oplus \mathbb{R}^m &\longrightarrow \mathrm{End}(\mathbb{R}^n \oplus \mathbb{R}^m \oplus \mathbb{R}^m) \\ \begin{pmatrix} x \\ y_1 \\ y_2 \end{pmatrix} &\longrightarrow \begin{pmatrix} Id|_{\mathbb{R}^n} & 0 & 0 \\ 0 & 0 & Id|_{\mathbb{R}^m} \\ f_{Y_0}(y_2) & 0 & 0 \end{pmatrix}. \end{aligned}$$

On the other hand, by the Lemma 5.2.1, given $f_{\mathbf{Y}_0} \in \mathrm{Lip}(\gamma)$ defined by $f_{\mathbf{Y}_0}(\cdot) = f(\cdot + \mathbf{Y}_0)$, there exists $g \in \mathrm{Lip}(\gamma - 1)$ such that $f_{\mathbf{Y}_0}(x) - f_{\mathbf{Y}_0}(y) = g(x, y)(x - y)$. Then, given $\rho > 1$, we define

$$\begin{aligned} h_2 : \mathbb{R}^n \oplus \mathbb{R}^m \oplus \mathbb{R}^m \oplus \mathbb{R}^m &\longrightarrow \mathrm{End}(\mathbb{R}^n \oplus \mathbb{R}^m \oplus \mathbb{R}^m \oplus \mathbb{R}^m) \\ \begin{pmatrix} x \\ y_1 \\ y_2 \\ d \end{pmatrix} &\longrightarrow \begin{pmatrix} Id|_{\mathbb{R}^n} & 0 & 0 & 0 \\ 0 & 0 & Id|_{\mathbb{R}^m} & 0 \\ f_{Y_0}(y_2) & 0 & 0 & 0 \\ \rho g(y_1, y_2)(d) & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

In the same way that h allow us to define by recurrence $\boldsymbol{Z}(n)$, we can define two rough paths associated to the one-forms h_1 and h_2 as

$$\boldsymbol{Z}_i(k+1) = \int h_i(\boldsymbol{Z}_i(k))\,\mathrm{d}\boldsymbol{Z}_i(k), \quad k = 1, 2;$$

with

$$\begin{cases} \boldsymbol{Z}_1(0) & = (\boldsymbol{X}, 0, \boldsymbol{Y}(1)), \\ \boldsymbol{Z}_2(0) & = (\boldsymbol{X}, 0, \boldsymbol{Y}(1), \boldsymbol{Y}(1)). \end{cases}$$

With this, if we consider projections of $\boldsymbol{Z}_i$ for some $i = 1, 2$; as has been done with $\boldsymbol{Z}(k)$, we can see that

$$\begin{cases} \boldsymbol{Z}_1(n) & = (\boldsymbol{X}, \boldsymbol{Y}(n), \boldsymbol{Y}(n+1)), \\ \boldsymbol{Z}_2(n) & = (\boldsymbol{X}, \boldsymbol{Y}(n), \boldsymbol{Y}(n+1), \rho^n(\boldsymbol{Y}(n+1) - \boldsymbol{Y}(n))). \end{cases}$$

Now we focus our attention on the first requirement in the definition of strong solution. Let us firstly check the property (4) of the theorem and also the **existence** of solution.

Let $0 < \varepsilon < 1$. Then, since $\varepsilon^{-1} > 1$, the p-variation of

$$\boldsymbol{Z}_2(n) = (\boldsymbol{X}, \boldsymbol{Y}(n), \boldsymbol{Y}(n+1), \rho^n(\boldsymbol{Y}(n+1) - \boldsymbol{Y}(n))), \tag{5.7}$$

is smaller than the p-variation of $(\varepsilon^{-1}\boldsymbol{X}, \boldsymbol{Y}(n), \boldsymbol{Y}(n+1), \rho^n(\boldsymbol{Y}(n+1) - \boldsymbol{Y}(n)))$, which is controlled by ω on $[0, T_\rho]$, by Lemma 5.2.3.

If we apply the linear map

$$T : \mathbb{R}^n \oplus (\mathbb{R}^m \oplus \mathbb{R}^m \oplus \mathbb{R}^m) \longrightarrow (\mathbb{R}^n \oplus \mathbb{R}^m) \oplus (\mathbb{R}^n \oplus \mathbb{R}^m),$$
$$(a, b, c, d) \longrightarrow ((a, b), (0, d)),$$

to $\boldsymbol{Z}_2(n)$ we have (since the norm of T is 1) that

$$(\,(\boldsymbol{X}, \boldsymbol{Y}(n))\,,\; \rho^n[(\boldsymbol{X}, \boldsymbol{Y}(n+1)) - (\boldsymbol{X}, \boldsymbol{Y}(n))]\,)$$

is also controlled by ω.

Now, we can apply Lemma 5.2.2 with $\mathbb{R}^k = \mathbb{R}^{n+m} \cong \mathbb{R}^n \oplus \mathbb{R}^m$, $\boldsymbol{Z}_1 = (\boldsymbol{X}, \boldsymbol{Y}(n))$, $\boldsymbol{Z}_2 =$

$(\boldsymbol{X}, \boldsymbol{Y}(n+1)))$ and $\varepsilon = \rho^{-n}$ to obtain

$$\begin{aligned} \left\|(\boldsymbol{X}, \boldsymbol{Y}(n))^i_{s,t} - (\boldsymbol{X}, \boldsymbol{Y}(n+1))^i_{s,t}\right\|_p &\leq ((1+\rho^{-n})^i - 1)\frac{\omega(s,t)^{i/p}}{\beta(i/p)!} \\ &\leq 2^i \rho^{-n} \frac{\omega(s,t)^{i/p}}{\beta(i/p)!}, \end{aligned} \tag{5.8}$$

which is the estimate for the speed of convergence (5.5) for $[0, T_\rho]$.

With this estimate, we have that $(\,(\boldsymbol{X}, \boldsymbol{Y}(n))\,)_{n\in\mathbb{N}}$ converges in p-variation to a rough path $\boldsymbol{Z} = (\boldsymbol{X}, \boldsymbol{Y})$ that should satisfy $\boldsymbol{Z} = \int h(\boldsymbol{Z})\, \mathrm{d}\boldsymbol{Z}$ by definition of $\boldsymbol{Z}(n)$.

So far, we have proved existence of a local solution on $[0, T_\rho] \subset [0, T]$. In order to define a solution on the whole $[0, T]$ we can set a new problem starting at T_ρ by a translation of time, and paste this new local solution as many times as needed.

Next, we check the **uniqueness** of solution. For this, consider $\overline{\boldsymbol{Z}} = (\boldsymbol{X}, \overline{\boldsymbol{Y}})$, a solution of the equation different from the one obtained by the Picard iteration, which we denote by $(\boldsymbol{X}, \boldsymbol{Y})$. Set

$$\begin{aligned} h_3 : \mathbb{R}^n \oplus \mathbb{R}^m \oplus \mathbb{R}^m \oplus \mathbb{R}^m &\longrightarrow \mathrm{End}(\mathbb{R}^n \oplus \mathbb{R}^m \oplus \mathbb{R}^m \oplus \mathbb{R}^m) \\ \begin{pmatrix} x \\ y \\ \overline{y} \\ \overline{d} \end{pmatrix} &\longrightarrow \begin{pmatrix} Id|_{\mathbb{R}^n} & 0 & 0 & 0 \\ f_{Y_0}(y) & 0 & 0 & 0 \\ 0 & 0 & Id|_{\mathbb{R}^m} & 0 \\ \rho g(y, \overline{y})(\overline{d}) & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

and as usual $\boldsymbol{Z}_3(n+1) = \int h_3(\boldsymbol{Z}_3(n))\, \mathrm{d}\boldsymbol{Z}_3(n)$ with $\boldsymbol{Z}_3(0) = (\boldsymbol{X}, 0, \overline{\boldsymbol{Y}}, \overline{\boldsymbol{Y}})$. Then, similarly as in Lemma 5.2.3, we can see that $\boldsymbol{Z}_3(n) = (\boldsymbol{X}, \boldsymbol{Y}(n), \overline{\boldsymbol{Y}}, \rho^n(\overline{\boldsymbol{Y}} - \boldsymbol{Y}(n)))$ is controlled uniformly on n by ω on a time interval $[0, T_\rho]$.

By Lemma 5.2.2 we have that $\boldsymbol{Y} := \lim_{n\to\infty} \boldsymbol{Y}(n) = \overline{\boldsymbol{Y}}$ and hence, the solution is unique.

Finally, we prove the **continuity** of the Itô map I_f. Given $n \geq 0$, we define $F_n(\boldsymbol{X}, \boldsymbol{Y}_0) = (\boldsymbol{X}, \boldsymbol{Y}(n))$. So far, we have seen that $F_n(\boldsymbol{X}, \boldsymbol{Y}_0)$ converges to $I_f(\boldsymbol{X}, \boldsymbol{Y}_0)$.

By the continuity of the integral we proved in Theorem 4.2.1, $F_n(\boldsymbol{X}, \boldsymbol{Y}_0)$ is continuous on $\mathrm{G}\Omega_p(\mathbb{R}^n) \times \mathbb{R}^m$ and therefore, it suffices to check that F_n converges uniformly towards the Itô map I_f.

Observe that, by the definition of T_ρ on page 56, it depends on f and the Lip-norms of

h, h_1 and h_2, and these Lip-norms depend in turn on the Lip-norms of $f_{\mathbf{Y}_0}$ and g. But clearly a translation does not change that norm, i. e. $||f_{\mathbf{Y}_0}||_{\text{Lip}} = ||f||_{\text{Lip}}$, and by the Lemma 5.2.1 we have that $||g||_{\text{Lip}}$ is comparable to $||f||_{\text{Lip}}$. Summing up, T_ρ depends on p, γ, $||f||_{\text{Lip}}$ and ω.

Given a control ω, we denote by $G\Omega_p^\omega(\mathbb{R}^n)$ the set of geometric rough paths on $\mathbb{R}^n$ controlled by that particular ω. The bound (5.8) assures that F_n converges uniformly towards the Itô map I_f on the set $G\Omega_p^\omega(\mathbb{R}^n) \times \mathbb{R}^m$, and on a time interval that might depend on ω. However, the length of this time interval can be bounded from below. Therefore, it can be extended to the whole $[0, T]$.

Finally, we have to extend the uniform convergence on $G\Omega_p^\omega(\mathbb{R}^n \times \mathbb{R}^m)$ to $G\Omega_p(\mathbb{R}^n) \times \mathbb{R}^m$. For this, consider a sequence of geometric rough paths $(\boldsymbol{X}(n))_{n\in\mathbb{N}}$ converging to $\boldsymbol{X}$ and a sequence $(\boldsymbol{Y}_{0,n})_{n\in\mathbb{N}}$ converging to $\boldsymbol{Y}_0$ as n goes to infinity. Then, there exists a control ω such that $\boldsymbol{X}$ and $\boldsymbol{X}(n)$ are controlled by ω for any n. Hence

$$I_f(\boldsymbol{X}(n), \boldsymbol{Y}_{0,n}) \to I_f(\boldsymbol{X}, \boldsymbol{Y}_0)$$

on $G\Omega_p(\mathbb{R}^n) \times \mathbb{R}^m$ and we have that I_f is continuous and hence the Theorem is proved.

□

5.3 Proof of lemma B

For the proof of Lemma 5.2.3 we need an auxiliary lemma, known as the *Scaling Lemma.*

Lemma 5.3.1 (Scaling Lemma)**.** *Let $\boldsymbol{Z} \in G\Omega_p(\mathbb{R}^n \oplus \mathbb{R}^m)$ be a rough path, $\boldsymbol{Z} = (\boldsymbol{X}, \boldsymbol{Y})$. Let ω be a control and $M \geq 1$ a real constant. Suppose $\boldsymbol{X}$ and $\boldsymbol{Z}$ are controlled by ω and $M\omega$ respectively.*

Then $(X, \varepsilon Y)$ is controlled also by ω whenever $0 \leq \varepsilon \leq M^{-\lfloor p \rfloor/p}$.

Now we are able to prove Lemma 5.2.3.

Proof. : We use induction on n in both parts. The way we have defined ε and T_ρ before the statement of the lemma guarantees that $(\varepsilon^{-1}\boldsymbol{X}, 0)$ is controlled by ω on $[0, T_\rho]$. Assume that $(\varepsilon^{-1}X, Y(n))$ is controlled by ω and we will prove $(\varepsilon^{-1}X, Y(n+1))$ is then controlled by ω too. Consider

$$(\boldsymbol{U}_0, \boldsymbol{U}_1) = \int h(\varepsilon^{-1}\boldsymbol{X}, \boldsymbol{Y}(n))\ \mathrm{d}(\varepsilon^{-1}\boldsymbol{X}, \boldsymbol{Y}(n)).$$

Then, by definition of the form h, we have

$$\mathrm{d}\boldsymbol{U}_0 = \varepsilon^{-1}\ \mathrm{d}\boldsymbol{X} \Rightarrow \boldsymbol{U}_0 = \varepsilon^{-1}\boldsymbol{X},$$

and

$$\mathrm{d}\boldsymbol{U}_1 = f_{\mathbf{Y}_0}(\boldsymbol{Y}(n))\,\mathrm{d}(\varepsilon^{-1}\boldsymbol{X}) = \varepsilon^{-1}f_{\mathbf{Y}_0}(\boldsymbol{Y}(n))\ \mathrm{d}\boldsymbol{X} \Rightarrow \boldsymbol{U}_1 = \varepsilon^{-1}\boldsymbol{Y}(n+1),$$

by definition of $\boldsymbol{Y}(n+1)$ with respect to $\boldsymbol{Y}(n)$. Now, by definition of M, we have that $(\varepsilon^{-1}\boldsymbol{X}, \varepsilon^{-1}\boldsymbol{Y}(n+1))$ is controlled by $M\omega$ and since $\varepsilon^{-1}\boldsymbol{X}$ is controlled by ω, we can apply the Scaling Lemma and conclude that

$$(\varepsilon^{-1}\boldsymbol{X}, \cancel{\varepsilon\varepsilon^{-1}}\boldsymbol{Y}(n+1)) = (\varepsilon^{-1}\boldsymbol{X}, \boldsymbol{Y}(n+1))$$

is controlled by ω on $[0, T_\rho]$.

For the second rough path we repeat the argument of induction on n. For $n = 0$, we have that the rough path $(\varepsilon^{-1}\boldsymbol{X}, 0, \boldsymbol{Y}(1), \boldsymbol{Y}(1))$ is controlled by ω since $(\varepsilon^{-1}\boldsymbol{X}, \boldsymbol{Y}(n))$ it is for any n. Now assume the statement is true for a fixed n, this is, that ω is a control for $(\varepsilon^{-1}\boldsymbol{X}, \boldsymbol{Y}(n), \boldsymbol{Y}(n+1), \rho^n(\boldsymbol{Y}(n+1) - \boldsymbol{Y}(n)))$ and let

$$(\boldsymbol{U}_0, \boldsymbol{U}_1, \boldsymbol{U}_2, \boldsymbol{U}_3) =$$
$$= \int h_2(\varepsilon^{-1}X, Y(n), Y(n+1), \rho^n(Y(n+1) - Y(n)))\ \mathrm{d}(\varepsilon^{-1}X, Y(n), Y(n+1), \rho^n(Y(n+1) - Y(n))).$$

Similarly as before, we have

$$\mathrm{d}\boldsymbol{U}_0 = \mathrm{d}(\varepsilon^{-1}\boldsymbol{X}) = \varepsilon^{-1}\,\mathrm{d}\boldsymbol{X} \Rightarrow \boldsymbol{U}_0 = \varepsilon^{-1}\boldsymbol{X},$$
$$\mathrm{d}\boldsymbol{U}_1 = \mathrm{d}\boldsymbol{Y}(n+1) \Rightarrow \boldsymbol{U}_1 = \boldsymbol{Y}(n+1),$$
$$\mathrm{d}\boldsymbol{U}_2 = f_{\mathbf{Y}_0}(\boldsymbol{Y}(n+1))\,\mathrm{d}(\varepsilon^{-1}\boldsymbol{X}) = \varepsilon^{-1}f_{\mathbf{Y}_0}(\boldsymbol{Y}(n+1))\,\mathrm{d}\boldsymbol{X} \Rightarrow \boldsymbol{U}_2 = \varepsilon^{-1}\boldsymbol{Y}(n+2),$$
$$\mathrm{d}\boldsymbol{U}_3 = \varepsilon^{-1}\rho^{n+1}(f_{\mathbf{Y}_0}(\boldsymbol{Y}(n+1)) - f_{\mathbf{Y}_0}(\boldsymbol{Y}(n)))\ \mathrm{d}\boldsymbol{X} \Rightarrow \boldsymbol{U}_3 = \varepsilon^{-1}\rho^{n+1}(\boldsymbol{Y}(n+2) - \boldsymbol{Y}(n+1)).$$

by definition of $\boldsymbol{Y}(n+1)$ with respect to $\boldsymbol{Y}(n)$ once again.

Now, by the first part we have that $(\boldsymbol{U}_0, \boldsymbol{U}_1, \boldsymbol{U}_2, \boldsymbol{U}_3)$ is controlled by $M\omega$ and we also knew that ω is a control for $(\varepsilon^{-1}\boldsymbol{X}, \boldsymbol{Y}(n+1))$. By the Scaling Lemma,

$$\begin{aligned} &((\varepsilon^{-1}\boldsymbol{X}, \boldsymbol{Y}(n+1)), \cancel{\varepsilon}(\cancel{\varepsilon^{-1}}\boldsymbol{Y}(n+2), \cancel{\varepsilon^{-1}}\rho^{n+1}(\boldsymbol{Y}(n+2) - \boldsymbol{Y}(n+1)))) = \\ =&(\varepsilon^{-1}\boldsymbol{X}, \boldsymbol{Y}(n+1), \boldsymbol{Y}(n+2), \rho^{n}(\boldsymbol{Y}(n+2) - \boldsymbol{Y}(n+1))) \end{aligned}$$

is controlled by ω on $[0, T_\rho]$ and the Lemma is proved.

□

Summary/Conclusions

We summarize here the most important points of the work.

- The expression

$$\int f \, \mathrm{d}g \tag{5.9}$$

  is well defined in the Riemann-Stieltjes sense for f of bounded variation and g continuous. Also, (5.9) is well defined in the Young sense for f of finite p-variation and g of finite q-variation whenever $1/p + 1/q > 1$.

- It is not enough to know how a path $X : [0,T] \to \mathbb{R}^d$ behaves to assure that the solution Y of the equation

$$\mathrm{d}Y_t = f(Y_t) \, \mathrm{d}X_t$$

  depend continuously on it, at least if $d \geq 2$.

- For X a path of finite p-variation, the amount of information needed in order to restore the continuity of the Itô map is related to $\lfloor p \rfloor$.

- This information we have to consider is related with the iterated integrals (see equation (2.8)) if the path X is smooth. The information is stored in the *rough path*.

- The Rough Path theory is both analytical and algebraic (see Chen's relation, equation (2.3)). Probability only appears when we compute a rough path associated to a stochastic process, like it is done in Chapter 3 for the Brownian Motion.

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