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On the proof of the Upper Bound Theorem

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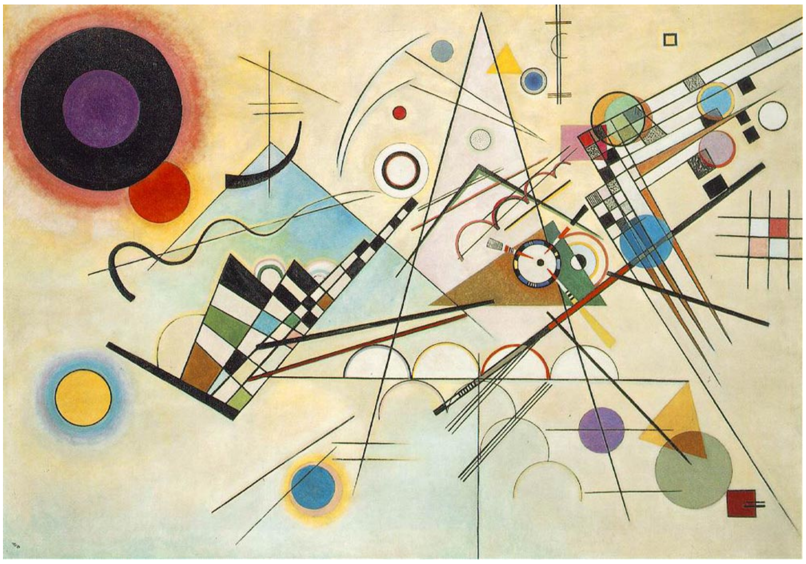
"Un matemàtic, igual que un pintor o un poeta, és algú que crea formes. Si les formes matemàtiques perduren més que les dels pintors o les dels poetes és perquè estan fetes amb idees."

Godfrey Harold Hardy



Composition VIII. Wassily Kandisnky

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Notation

$\mathbb{N}$ non-negative integers
$\mathbb{R}$ real numbers
$\mathbb{Z}$ integers
$\mathbb{Q}$ rational numbers
$\mathbb{R}^+$ non-negative real numbers
$k[X_1, \ldots, X_n]$ polynomial ring over k
$H_R(t)$ Hilbert series of R, 8
$H(R, n)$ Hilbert function, 8
$n^{(k)}$, $n^{\langle k \rangle}$ k-canonical representation of n, 9
Δ simplicial complex
$|\cdot|$ cardinal or geometric realization, according to context, 30
$\#S$ cardinal of a finite set S
$k[\Delta]$ Stanley-Reisner ring, 14
I_Δ Stanley-Reisner ideal, 14
$f(\Delta)$, $h(\Delta)$ f-vector and h-vector of Δ respectively
$(N_1 :_R N_2)$ ideal quotient $(N_1 :_R N_2) = \{x \in R | xN_2 \subset N_1\}$
Rad I radical of I
Supp M support of a module M
Supp(a) support of an element $a \in \mathbb{Z}^n$,18
Ass(M) associated primes of M
2^F powerset of a set F
$\mathcal{F}(\Delta)$ partial ordered face set, 26
$H(y, \alpha)$ hyperplane, 28
$\langle , \rangle$ usual inner product
$\widetilde{H}_i(\Delta, G)$ reduced homology group, 37
$\mathrm{st}_\Delta F$ star of a face F, 38
$\mathrm{lk}_\Delta F$ link of a face F, 38
$\mathrm{cn}(\Delta)$ cone of a complex, 39
$\Delta_1 * \Delta_2$ join of two simplices, 39
$R_\mathfrak{m}$ localization of R in $\mathfrak{m}$
$H^i_\mathfrak{m}(M)$ i-th local cohomology of a module M, 41
$\varinjlim M_i$ direct limit of modules M_i
$(R, \mathfrak{m}, k)$ local ring R with maximal ideal $\mathfrak{m}$ and residue class field $k = R/\mathfrak{m}$
$C(n, d)$ cyclic polytope, 31
∂F boundary of a simplex
$\Delta(n, d)$ boundary simplicial complex of a cyclic polytope $C(n, d)$
$\bigoplus, \coprod$ direct sum (of vector space or mudules)
kX vector space over k with basis X
$\mathrm{Ext}^n_R(M, N)$ right derivaded functor of $R^n\mathrm{Hom}_R(M, N)$

1 Introduction

Let Δ be a triangulation of a $(d-1)$-dimensional sphere with n vertices. The *Upper Bound Conjecture* (UBC for short) gives an explicit bound of the number of i-dimensional faces of Δ.

This question dates back to the beginning of the 1950's, when the study of the efficiency of some linear programming techniques led to the following problem: Determine the maximal possible number of i-faces of d-polytope with n vertices.

The first statement of the UBC was formulated in 1957 by Theodore Motzkin [7]. The original result state that the number of i-dimensional faces of a d-dimensional polytope with n vertices are bound by a certain explicit number $f_i(C(n,d))$, where $C(n,d)$ is a cyclic polytope and f_i denotes the number of i-dimensional faces of the simplex. We say that P is a polytope if it is the convex hull of a finite set of points in $\mathbb{R}^d$. Moreover, we say that $C(n,d)$ is a cyclic polytope if it is the convex hull of n distinct points on the moment curve $(t, t^2, \ldots, t^d)$, $-\infty < t < \infty$. With this notation the Upper Bound Conjecture (for convex polytopes) states that cyclic polytope maximizes the number of i-dimensional faces among all polytopes.

After some proofs of special cases by Fieldhouse, Gale and Klee, finally, in 1970 Peter McMullen [6] proved the UBC for convex polytopes. McMullen's proof is based on the line shelling, introduced by Bruggesser-Mani [1] (we will see these concepts in section 3, as well as a constructive and simple proof of the Bruggesser-Mani theorem).

In 1964 Victor Klee had extended the UBC to any triangulation Δ with n vertices of a $(d-1)$-manifold. The specific case of the Klee conjecture when the geometric realization[1] of Δ is a sphere is called *upper bound conjecture for spheres* or *simplicial spheres*.

There is no clear evidence that the UBC for simplicial spheres is stronger than the UBC for convex polytopes. We know that a polytope is homeomorphic to a sphere. However there exist triangulations of spheres which are not the boundary of a simplicial convex polytope. Moreover, Kalai [4] proved that there are many more such simplicial spheres than polytopes. Therefore, the UBC for spheres is a generalization of McMullen's theorem for polytopes.

Finally, in 1975, Richard P. Stanley [8] proved the Upper Bound Conjecture for simplicial spheres. In his proof, Stanley associated an algebraic structure, the Stanley-Reisner ring, to the simplicial complex and study the Hilbert series of this ring.

Richard Stanley with all the techniques developed in his proof of the UBC transformed combinatorics form a collection of separate methods and a discipline with lack of abstraction to a structured and mature area of mathematics. In this project we will reproduce the proof of the upper bound theorem following the Richard Stanley steps. We will do a detailed study of the combinatorial aspects of the simplicial polytope, but we will focus more in the algebraic and topological structure of the complexes.

In section 2 we will provide the algebraic background essential for our study of Stanley-Reisner ring. We will introduce the notion of grade and depth that allow us to

[1] The geometric realization of Δ is a way to associate a topological space $|\Delta|$.

define the class of Cohen-Macaulay rings. Eventually, we will present some fundamental results on Hilbert series. The reader familiarized with this notion can skip this part.

In section refsection3 we will introduce the simplicial complexes and study the properties of the Stanely-Reisner ring that allows us to define the h-vector. We will present also two special cases of simplicices, the Cohen-Macaulay and the shellable one. The h-vector contains combinatorial information about the simplex, and in the particular case of the Cohen-Macaulay we have explicit bound for each component of the vector. Finally, we will define the cyclic polytopes and characterize his h and f-vector. The results about cyclic polytopes are followed form the book citezieglerlectures.

In section 4 we will study the homology group of the simplicial complexes and the local cohomology of the Stanley-Reisner ring. Finally, we will prove the Reisner criterion which determine when a simplicial complex is Cohen-Macaulay. This result is elementary in our proof of the UBC. We will follow the proof of Reisner's theorem from [5].

Finally, in the last section we will give the proof of the upper bound theorem for simplicial spheres.

Proof of the upper bound theorem:

To read the proof of the upper bound theorem for simplicial spheres only the results from section 3 the Dehn-Sommerville relation (Theorem 3.27) and Theorem 3.11 are needed. From section 4 we use Reisner's Theorem 4.9 to prove that a simplicial sphere satisfy the Dehn-Sommervile equations. All these results are complemented by Corollary 3.23 and Proposition 3.9.

2 Basic Concepts

In this section we will give a brief introduction to all the algebraic tools that we will use in this project. Since we have seen all these results on the course of "Local Algebra" in the Master of Advanced Mathematics from "Universitat de Barcelona" we will skip almost all the proofs. Although it can be found in the book of "Cohen-Macaulay rings", see [2].

Let us start with a remainder of regular sequences.

2.1 Regular Sequence, Grade and Depth

Let M be a module over a ring R. We say that $x \in R$ is an *M-regular element* if x is not a zero-divisor on M, that is, if $xz = 0$ for $z \in M$ implies that $z = 0$.

Definition 2.1. A sequence $\boldsymbol{x} = x_1, \ldots, x_n$ of elements on R is called an *M-regular sequence* or simply an *M-sequence* if the following conditions are satisfied:

(i) x_i is an $M/(x_1, \ldots, x_{i-1})M$-regular element for $i = 1, \ldots, n$,

(ii) $M/\boldsymbol{x}M \neq 0$.

A *weak M-sequence* is only required to satisfy condition (i).

The next result shows that a regular sequence has a good behaviour respect to an exact sequence. Numerous arguments of this project will be based on this property.

Proposition 2.1. *Let R be a ring, M an R-module, and $\boldsymbol{x}$ a weak M-sequence. Then an exact sequence*

$$N_2 \xrightarrow{\varphi_2} N_1 \xrightarrow{\varphi_1} N_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

of R-modules induces an exact sequence of R-modules

$$N_2/\boldsymbol{x}N_2 \longrightarrow N_1/\boldsymbol{x}N_1 \longrightarrow N_0/\boldsymbol{x}N_0 \longrightarrow M/\boldsymbol{x}M \longrightarrow 0.$$

Another property of the regular sequences is that any permutation on the order of its elements is again a regular sequence.

Let R be a Noetherian ring and M an R-module. If $\boldsymbol{x} = x_1, \ldots, x_n$ is an M-sequence, then the sequence

$$(x_1) \subset (x_1, x_2) \subset \cdots \subset (x_1, \ldots, x_n)$$

is a strictly ascending sequence of ideals. Therefore, an M-sequence can be extended to a maximal such sequence. We say that an M-sequence $\boldsymbol{x}$ is *maximal* if $x_1, \ldots, x_{n+1}$ is not an M-sequence for any $x_{n+1} \in R$. Next, we will see a result, in a quite general case, where all the maximal M-sequences has the same length.

Theorem 2.2. *(Rees). Let R be a Noetherian ring, M a finite R-module, and I an ideal such that $IM \neq I$. Then the common length of the maximal M-sequences in I have the same length n given by*

$$n = \min\{i : Ext^i_R(R/I, m) \neq 0\}$$

This invariant of the length of such sequences, allows us to introduce the notion of grade and depth.

Definition 2.2. Let R be a Noetherian ring, M a finite R-module, and I an ideal such that $IM \neq M$. Then the common length of the maximal M-sequences in I is called the *grade of I on M*, denoted by

$$\mathrm{grade}(I, M).$$

We set $\mathrm{grade}(I, M) = \infty$ if $IM = M$.

On the special case when R is a local ring we will call, this common length, depth of the regular sequence.

Definition 2.3. Let $(R, \mathfrak{m}, k)$ be a local ring, and M a finite R-module. Then the grade of $\mathfrak{m}$ on M is called the *depth of M*, denoted

$$\text{depth } M.$$

Let us see some useful formulas to compute grade and depth.

Proposition 2.3. *Let R be a Noetherian ring, $I \subset R$ an ideal, and $0 \to N \to M \to L \to 0$ an exact sequence of R-modules. Then*

$$\begin{aligned} &grade(I, M) \leq \min\{grade(I, N), grade(I, L)\}, \\ &grade(I, N) \leq \min\{grade(I, M), grade(I, L) + 1\}, \\ &grade(I, L) \leq \min\{grade(I, N) - 1, grade(I, M)\}, \end{aligned}$$

As a direct sequence of this result we have the depth lemma.

Lemma 2.4. *(Depth lemma). Let R be a Noetherian local ring, and $0 \to N \to M \to L \to 0$ an exact sequence of R-modules. Then:*

(a) if depth $M <$ depth L, then depth $N =$ depth M;

(b) if depth $M =$ depth L, then depth $N \geq$ depth M;

(c) if depth $M >$ depth L, then depth $N =$ depth $L - 1$.

Remark 2.1. In the particular case when $(R, \mathfrak{m}, k)$ is a local ring and $0 \to N \to M \to L \to 0$ an exact sequence of R-modules such that depth $L =$ depth $N = d$ then depth $M = d$.

In the next proposition we give more formulas and properties related to the grade of an ideal.

Proposition 2.5. *Let R be a Noetherian ring, I, J ideal of R, and M a finite R-module. Then*

(a) $grade(I, M) = \inf\{depth\ M_{\mathfrak{p}} | \mathfrak{p} \in V(I)\}$,

(b) *grade*$(I, M) =$ *grad*$(\text{Rad } I, M)$,

(c) *grade*$(I \cap J, M) = \min\{\textit{grade}(I, M), \textit{grade}(J, M)\}$,

(d) *if* $\boldsymbol{x} = x_1, \ldots, x_n$ *is an* M*-sequence in* I*, then*

$$\textit{grade}(I/(\boldsymbol{x}), M/\boldsymbol{x}M) = \textit{grade}(I, M/\boldsymbol{x}M) = \textit{grade}(I, M) - n,$$

(e) *if* N *is a finite* R*-module with Supp* $N = V(I)$*, then*

$$\textit{grade}(I, M) = \inf\{i : \textit{Ext}^i_R(N, M) \neq 0\}.$$

Definition 2.4. Let R be a Noetherian ring and $M \neq 0$ a finite R-module. Then the grade of M is given by

$$\text{grade } M = \min\{i : \text{Ext}^i_R(M, R) \neq 0\}.$$

We also set

$$\text{grade } I = \text{grade } R/I = \text{grade}(I, R),$$

for an ideal $I \subset R$.

In the following result we see how the depth and the Krull dimension (for short dimension) of a module are related.

Proposition 2.6. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring and* $M \neq 0$ *a finite* R*-module. Then*

$$\textit{depth } M \leq \dim M.$$

Let R be a ring and $\mathfrak{p}$ a prime ideal of R. The *height* of $\mathfrak{p}$, denoted by $\text{ht}(\mathfrak{p})$ is the supremum of the lengths of all chains of prime ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p},$$

which ends at $\mathfrak{p}$. Also define *height* of I as

$$\text{ht}(I) = \min\{\text{ht}(\mathfrak{p}) | I \subset \mathfrak{p} \text{ and } \mathfrak{p} \in \text{Spec}(R)\}.$$

Established these definitions we have the following result.

Proposition 2.7. *Let* R *be a Noetherian ring and* $I \subset R$ *an ideal. Then*

$$\textit{grade } I \leq \textit{height } I.$$

2.2 Graded Rings and Modules

The polynomial ring admit a decomposition into homogeneous components ordered in terms of their degree. We can apply this idea to rings and modules that admit a decomposition of their elements into homogeneous components. This notion of the decomposition of a ring will be fundamental in our study of Hilbert series and the local cohomology of the Stanley-Reisner rings.

Definition 2.5. A *graded ring* is a ring R together with a decomposition $R = \bigoplus_{i \in \mathbb{Z}} R_i$ (as a $\mathbb{Z}$-module) such that $R_i R_j = R_{i+j}$ for all $i, j \in \mathbb{Z}$.

Moreover, a *graded R-module* is an R-module M together with a decomposition $M = \bigoplus_{i \in \mathbb{Z}} M_i$ (as a $\mathbb{Z}$-module) such that $R_i M_j \subset M_{i+j}$ for all $i, j \in \mathbb{Z}$.

One calls M_i the i-th *homogeneous components* of M.

An elements of M_i is sad to be *homogeneous* (of degree i). The elements of R_i are called i-forms.

Note that R_0 is a ring with $1 \in R_0$, and R_n is an R_0-module for all n.

Let us see an example.

Example 2.1. Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. A natural way to grade R is to assign degree d_i to the variables x_i, where $d_1, \ldots, d_n$ are positive integers. For $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$ we set $x^a = x_1^{a_1} \cdots x_n^{a_n}$ and $|a| = a_1 d_1 + \cdots + a_n d_n$. Let $R_i = \bigoplus_{|a|=i} kx^a$, for $i \leq 0$ and for $i < 0$ we set $R_i = 0$. The induces $\mathbb{Z}$-grading is

$$R = \bigoplus_{i=0}^{\infty} R_i$$

Now let $I \subset R$ be an ideal. We say that I is *homogeneous* of *graded*, if there are homogeneous polynomials $f_1, \ldots, f_t$ such that $I = (f_1, \ldots, f_t)$. Then the ideal I is graded by

$$I_i = I \cap R_i = f_1 R_{i - \deg f_1} + \cdots f_r R_{i - \deg f_t}.$$

Therefore, R/I inherits a structure of $\mathbb{Z}$-graded R-module whose components are given by $(R/I)_i = R_i / I_i$.

When we let $d_i = 1$ on a polynomial ring we call it the *standard grading.*

2.3 Cohen-Macaulay Rings

Definition 2.6. Let R be a Noetherian local ring. A finite R-module $M \neq 0$ is a *Cohen-Macaulay module* if deph $M = \dim M$. If R itself is a Cohen-Macaulay as an R-module, then it is called a *Cohen-Macaulay ring.*

Moreover, an ideal I of R is Cohen-Macaulay if R/I is a Cohen-Macaulay R-module.

An important equivalence between dimension and depth for the case of Cohen-Macaulay rings is given by the following proposition.

Theorem 2.8. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and $M \neq 0$ a Cohen-Macaulay R-module. Then* *(a)* $\dim R/\mathfrak{p} =$ *depth* M *for all* $\mathfrak{p} \in$ *Ass* M*,*

(b) *grade*$(I, M) = \dim M - \dim M/IM$ *for all ideal* $I \subset \mathfrak{m}$*,*

(c) $\boldsymbol{x} = x_1, \ldots, x_t$ *is an* M*-sequence if and only if* $\dim M/\boldsymbol{x}M = \dim M - t$*,*

Theorem 2.9. *Let* R *be a Noetherian ring, and* M *a finite* R*-module.*

(a) *Suppose* $\boldsymbol{x}$ *is an* M*-sequence. If* M *is a Cohen-Macaulay module, then* $M/\boldsymbol{x}M$ *is Cohen-Macaulay (over* R *or* $R/(\boldsymbol{x})$*). The converse holds if* R *is local.*

(b) *Suppose that* M *is Cohen-Macaulay. Then for every multiplicatively closed set* S *in* R *the local module* M_S *is also Cohen-Macaulay. In particular,* $M_\mathfrak{p}$ *is Cohen-Macaulay for every* $\mathfrak{p} \in$ *Spec* R*. If* $M_\mathfrak{p} \neq 0$*, then*

$$\textit{depth } M_\mathfrak{p} = \textit{grade}(\mathfrak{p}, M) \quad \textit{and} \quad \dim M = \dim M_\mathfrak{p} + \dim M/\mathfrak{p}M.$$

Proposition 2.10. *Let* R *be a Cohen-Macaulay ring, and* $I \neq R$ *an ideal. Then grade* $I =$ *height* I*, and if* R *is local,*

$$\textit{height } I + \dim R/I = \dim R.$$

Proposition 2.11. *Let* R *be a positively graded polynomial ring over a field* k *and* I *a graded ideal of* R*. If* R/I *is Cohen-Macaulay, then* $ht(I) = ht(\mathfrak{P})$ *for all* $\mathfrak{P} \in Ass_R(R/I)$*.*

2.4 Hilbert Series

The Hilbert functions $H(M, n)$ of a graded module M measure the dimension of its n-th homogeneous pieces. Let R be a graded ring and M an R-graded module. We will assume that R_0 is an Artinian local ring, and that R is finitely generated over R. Notice that in this case, the homogeneous components M_n of M are finite R_0-modules, and hence have finite length. We define the length of a module M_n, denoted $l(M_n)$, as the largest length of any chain contained in M_n.

Definition 2.7. Let M be a finite graded R-module. The numerical function $H(M, -) : \mathbb{Z} \to \mathbb{Z}$ with $H(M, n) = l(M_n)$ for all $n \in \mathbb{Z}$ is the *Hilbert function*, and $H_M(t) = \sum_{n \in \mathbb{Z}} H(M, n)t^n$ is the *Hilbert series* of M.

We say that a numerical function $F : \mathbb{Z} \to \mathbb{Z}$ is of *polynomial type* of degree d is there exists a polynomial $P(X) \in \mathbb{Q}[X]$ of degree d such that $F(n) = P(n)$ for all $n \gg 0$. We define the *difference operator* Δ on the set of numerical functions by setting $(\Delta F)(n) = F(n+1) - F(n)$ for all $n \in \mathbb{Z}$.

Theorem 2.12. *(Hilbert). Let* M *be a finite* R*-module of dimension* d*.* $H(M, n)$ *is of polynomial type of degree* $d - 1$*.*

Definition 2.8. Let M be a finite graded $R-$module of dimension d. The unique polynomial $P_M(X) \in \mathbb{Q}[X]$ for which $H(M, n) = P_M(n)$ for all $n \leq 0$ is called the *Hilbert polynomial* of M. We write

$$P_M(X) = \sum_{i=0}^{d-1} (-1)^{d-1-i} e_{d-1-i} \binom{X+i}{i}.$$

Then the *multiplicity* of M is defined to be

$$e(M) = \begin{cases} e_0 & \text{if } d > 0; \\ l(M) & \text{if } d = 0. \end{cases}$$

Lemma 2.13. *Let $H(t) = \sum a_n t^n$ be a formal Laurent series with integers coefficients, and $a_i = 0$ for $i \ll 0$. Further, let $d > 0$ be an integer. Then the following conditions are equivalent:*

(a) There exists a polynomial $P(X) \in \mathbb{Q}[X]$ of degree $d-1$ such that $P(n) = a_n$ for large n;

(b) $H(t) = Q(t)/(1-t)^d$ where $Q(t) \in \mathbb{Z}[t, t^{-1}]$ and $Q(1) \neq 0$.

As a result of this lemma we have the following important corollary:

Theorem 2.14. *Let $M \neq 0$ be a finite graded $R-$module of dimension d. Then there exists a unique $Q_M(t) \in \mathbb{Z}[t, t^{-1}]$ with $Q_M(1) \neq 0$ such that*

$$H_M(t) = \frac{Q_M(t)}{(1-t)^d}.$$

Moreover, if $Q_M(t) = \sum_i h_i t^i$, then min$\{i : h_i \neq 0\}$ is the least number such that $M_i \neq 0$.

Remark 2.2. Let $R = k[X_1, \ldots, X_n]$ be a polynomial ring over a field k. Then the Hilbert function $H(R, i)$ equals the number of monomials of total degree i. It can be proved by induction on n that this number is $\binom{n+i-1}{n-1}$. Therefore, the Hilbert series is

$$H_R(t) = \sum_{i \in \mathbb{Z}} \binom{n+i-1}{n-1} t^i = \frac{1}{(1-t)^n}.$$

Corollary 2.15. *Let $M \neq 0$ be a finite graded R-module of dim d. Then*

(a)

$$e_i = \frac{Q_M^{(i)}(1)}{i!} \quad \textit{for } i = 0, \ldots, d-1.$$

Furthermore, $e(M) = Q_M(1)$.

(b) If M is also Cohen-Macaulay and assume $Q_M(t) = \sum h_i t^i$. Then $h_i \geq 0$ for all i. Moreover, $e_i \geq 0$ for all i if $M_j = 0$ for all $j < 0$.

Remark 2.3. Let M be a Cohen-Macaulay finite graded R-module and let $\mathbf{x}$ be a M-sequence of elements of degree 1. Then we have the exact sequence

$$0 \to M(-1) \xrightarrow{\mathbf{x}} M \to M/\mathbf{x}M \to 0.$$

From this we obtain that $(1-t)H_M(t) = H_{M/(\mathbf{x})M}(t)$. Hence,

$$Q_M(t) = Q_{M/(\mathbf{x})M}(t).$$

In order to present the following results on the Hilbert Series we need to introduce some combinatorial properties about how we can represent an integer. First, given two integers k and n, we introduce the k-canonical representation of n.

Lemma 2.16. *For all integers $n, k > 0$ there exists a unique representation*

$$n = \binom{n_k}{k} + \binom{n_{k-1}}{k-1} + \cdots + \binom{n_j}{j}$$

where $n_k > n_{k-1} > \cdots > n_j \geq j \geq 1$.

Proof. First, to prove the existence of such a sequence, we will do induction on k. For $k = 1$ choose $n_k = n$. Assume that it is true for $k-1$.

Choose n_k maximal such that $\binom{n_k}{k} \leq n$. If $n = \binom{n_k}{k}$, then

$$n = \sum_{i=1}^{d} \binom{n_k}{i},$$

with $n_i = i - 1$ for $i = 1, \ldots, d-1$.

Now assume $n' = n - \binom{n_k}{k} > 0$. By the induction hypothesis we may assume that $n' = \sum_{i=1}^{k-1} \binom{n_i}{i}$ with $n_{k-1} > n_{k-2} > \cdots > n_1 \geq 0$. It remains to show that $n_k > n_{k-1}$. By the maximality of n_k we obtain $\binom{n_k+1}{k} > n$, therefore

$$\binom{n_k}{k-1} = \binom{n_k+1}{k} - \binom{n_k}{k} > n - \binom{n_k}{k} > n' \geq \binom{n_{k-1}}{k-1}.$$

Hence $n_k > n_{k-1}$.

Next, we prove the uniqueness. Assume that we have another k-canonical decomposition of n:

$$n = \binom{m_k}{k} + \binom{m_{k-1}}{k-1} + \cdots + \binom{m_j}{j},$$

where $m_k > \cdots > m_j \geq j \geq 1$. Let s be the largest index such that $n_s \neq m_s$.

Let s be the largest index such that $n_l \neq m_l$. Recall that our choice n_{k-i} is such that

$$n_{k-i} = \max\{a : n \geq \binom{n_k}{k} + \binom{n_{k-1}}{k-1} + \cdots \binom{n_{k-i+1}}{k-i+1} + \binom{a}{k-i}\}.$$

Then, for a given $i < s$, n_i is the maximum possible. Therefore, we must have $m_s < n_s$ and m_i are decreasing as i decreases. Hence, we can conclude that m_j are bound by $n_i - i - 1 + j$.

Since the binomial coefficients are non-negative we have

$$\begin{aligned}
n &= \binom{m_k}{k} + \cdots + \binom{m_{i+1}}{i+1} + \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \cdots + \binom{m_j}{j} \\
&\leq \binom{n_k}{k} + \cdots + \binom{n_{i+1}}{i+1} + \binom{n_i - 1}{i} + \binom{n_i - 2}{i-1} + \cdots + \binom{n_i - i - 1 + j}{j} \\
&\leq \binom{n_k}{k} + \cdots + \binom{n_{i+1}}{i+1} + \binom{n_i - 1}{i} + \binom{n_i - 2}{i-1} + \cdots + \binom{n_i - i}{1}.
\end{aligned}$$

In the last inequality we have just added some non-negative terms to the right side.

Now binding the last i binomial coefficients by $\binom{n_i}{i}$ we have

$$n < \binom{n_k}{k} + \cdots + \binom{n_{i+1}}{i+1} + \binom{n_i}{i} \leq n.$$

which is a contradiction.

$\square$

This representation for an integer was introduced by Macaulay. We call $n_k, \ldots, k_j$ the *k-Macaulay coefficients of n.*

With this lemma we can now give the two following definitions that will be useful for the Hilbert Series but also for the number of faces of a simplicial complex.

Definition 2.9. Let $n, k \in \mathbb{N} \setminus \{0\}$. And let $n_k, \ldots, n_j$ be the k-Macaulay coefficients of n. Then we define

$$n^{(k)} = \binom{n_k}{k+1} + \binom{n_{k-1}}{k} + \cdots + \binom{n_j}{j+1}$$

$$n^{\langle k \rangle} = \binom{n_k + 1}{k+1} + \binom{n_k}{k} + \cdots + \binom{n_j + 1}{j+1}$$

We set $0^{(0)} = 0^{\langle 0 \rangle} = 0$.

One of the main results of this section is the Macaulay theorem, that describe exactly those numerical functions which occur as the Hilbert functions $H(R, n)$ of a homogeneous k-algebra R over a field k. This theorem says that in a Hilbert series $H_R(t)$ the coefficients $H(R, n+1)$ are bound in terms of $H(R, n)$. Therefore, the Hilbert Series of a k-homogeneous algebra is in some way controlled.

Before we can state the Macaulay's theorem we need to introduce the idea of multicomplex.

Definition 2.10. A *multicomplex* Γ on $V = \{x_1, \ldots, x_n\}$ is a collection of monomials $x_1^{a_1} \cdots x_n^{a_n}$ such that when $m \in \Gamma$ every divisor of m is also in Γ.

Define the h-vector of Γ by $h(\Gamma) = (h_0, h_1, \ldots)$ where

$$h_i = |\{m \in \Gamma| \deg m = i\}|.$$

A sequence $(h_0, h_1, \ldots)$ which is the h-vector of some non-empty multicomplex Γ will be called M-vector[2].

A h-vector of a multicomplex may be infinite, and if $\Gamma \neq \emptyset$, then $h_0 = 1$.

Theorem 2.17. *(Macaulay). Let k be a field, and let $h : \mathbb{N} \to \mathbb{N}$ be a numerical function. The following conditions are equivalent:*

(a) $h(0), h(1), \ldots$ is an M-vector;

(b) there exists a homogeneous k-algebra R with Hilbert function $H(R, n) = h(n)$ for all $n \geq 0$;

(c) one has $h_0 = 1$ and $0 \leq h_{n+1} \leq h_n^{\langle n \rangle}$ for all $n \geq 1$.

Proof. (Sketch)

The implication $(a) \Rightarrow (b)$ follows from another Macaulay result. This result asserts that for a given $\mathbb{N}$-graded k-algebra R generated by $x_1, \ldots, x_n$, then R has a k basis which is a multicomplex on $\{x_1, \ldots, x_n\}$.

For the equivalence (a) and (c) one has to construct a multicomplex Γ_h and verify that the following conditions are equivalent:

(i) $h = (h(0), h(1), \ldots)$ is an M-vector;

(ii) Γ_h is a multicomplex;

(iii) $0 \leq h(n+1) \leq h_n^{\langle n \rangle}$, for $n \geq 1$.

Given h we define $\Gamma_h = \cap_{i \geq 0}\{$ first h_i monomials of degree i in the reverse lexicographic order$\}$. Here the difficult implication is $(i) \Rightarrow (ii)$.

Concerning the implication $(b) \Rightarrow (c)$ one can use Green's theorem. This result affirms that for a given homogeneous k-algebra R, with k an infinite field. Then

$$H(R/gR, n) \leq H(R, n)_{\langle n \rangle},$$

where $n \leq 1$, and g is a general linear form. The notation $n_{\langle k \rangle}$ denotes the sum $\binom{n_k - 1}{k} + \cdots + \binom{n_j - 1}{j}$.

Then, we construct an exact sequence

$$0 \to gR_n \to R_{n+1} \to R/gR \to 0,$$

[2] In honor of F. S. Macaulay.

that yields the inequality $H(R, n+1) \leq H(R, n) + H(R/gR, n+1)$. It remains to be proved that $H(R, n)^{\langle n \rangle} \geq H(R, n+1)$. This fact follows from properties of the canonical representation of an integer. (For further details, see Theorem 4.2.10. [2].)

□

Macaulay's theorem is valid for any homogeneous k-algebra. It is not surprising that for stronger restraints on R we can obtain further constraints of the Hilbert series of R.

The following result characterizes the h-vector of Cohen-Macaulay homogeneous algebras.

Proposition 2.18. *Let k be a field, and $h_0, \ldots, h_t$ a finite sequence of integers. The following conditions are equivalent:*

(a) There exists an integer d, and a Cohen-Macaulay reduced homogeneous k-algebra R of dimension d such that

$$H_R(t) = \frac{\sum_{i=0}^{t} h_i t^i}{(1-t)^d};$$

(b) $h_0 = 1$, and $0 \leq h_{i+1} \leq h_i^{\langle i \rangle}$ for all $i = 1, \ldots, t-1$.

Proof. Let R be as in (a). Since R is Cohen-Macaulay, there exists an R-sequence $\mathbf{x} = x_1, \ldots, x_d$ formed by elements of degree 1. By the theorem 2.14, we can write

$$H_R(t) = \frac{Q_R(t)}{(1-t)^d},$$

where $Q_R(t) = \sum_{i=0}^{s} h_i t^t$.

Let $\overline{R} = R/\mathbf{x}R$; then $H_{\overline{R}}(t) = (1-t)^d H_R(t) = Q_R(t)$. By comparing coefficients it follows that $h_n = H(\overline{R}, n)$ for all $n \geq 0$. Therefore, applying theorem 2.17 we have the result.

For the reverse implication, we again apply Theorem 2.17. Then, there exists a homogeneous k-algebra $R = k[x_1, \ldots, x_n]/I$, where I is generated by monomials such that $H_R(t) = \sum_{i=0}^{s} h_i t^i$. Since R is of dimension 0, R is Cohen-Macaulay k-algebra.

To see that R is reduced, we can take an reduced homogeneous k-algebra S, whose defining ideal is generated by square-free monomials, and an S-sequence $\mathbf{y}$ of elements of degree 1 such that $R \cong S/\mathbf{y}S$[3]. Hence, R is also a reduced k-homological algebra.

□

[3]The proof of the existence of such a reduced k-algebra S can be found in [bibliographic]

3 Simplicial Complexes and the Face Ring

In this section the simplicial complexes are introduced, the combinatorial objects to which an algebraic object, the Stanley-Reisner rings, are assigned.

Definition 3.1. A simplicial complex Δ on a finite set $V = \{v_1, \ldots, v_n\}$ is a collection of subsets of V such that

(i) If $v \in V$, then $\{v\} \in \Delta$.

(ii) If $F \in \Delta$ and $G \subset F$, then $G \in \Delta$.

The elements of Δ are called faces. Let Δ be a simplicial complex and F a face of Δ. Define the *dimension* of F and Δ by

$$\dim(F) = |F| - 1 \quad \text{and} \quad \dim(\Delta) = \sup\{\dim(F)|\ F \in \Delta\}$$

respectively.

Note that for any non empty simplicial complex, the empty set $\emptyset$ is a face of dimension -1. Faces of dimension 0 and 1 are called vertices and edges, respectively. The maximal faces under inclusion are called the *facets* of the simplicial complex.

Definition 3.2. The *f-vector* of a $d-$simplicial complex Δ is

$$f(\Delta) = (f_0, f_1, \ldots, f_d),$$

where $f_i = |\{F \in \Delta : \dim F = i\}|$.

So the f_i counts the number of $i-$dimensional faces of the simplex. Observe that $f_{-1} = 1$, since $\emptyset \in \Delta$ and $f_0 = |V|$, the number of vertices.

Example 3.1. Consider the set of vertices $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and let Δ be the simplicial complex in Figure 1. The f-vector of this octahedron is $f(\Delta) = (6, 12, 8)$. The faces of this simplex are the vertices, the edges and the facets $\{v_1, v_3, v_4\}$, $\{v_1, v_4, v_5\}$, $\{v_1, v_5, v_6\}$, $\{v_2, v_3, v_4\}$, $\{v_2, v_4, v_5\}$, $\{v_2, v_5, v_6\}$ and $\{v_2, v_3, v_5\}$.

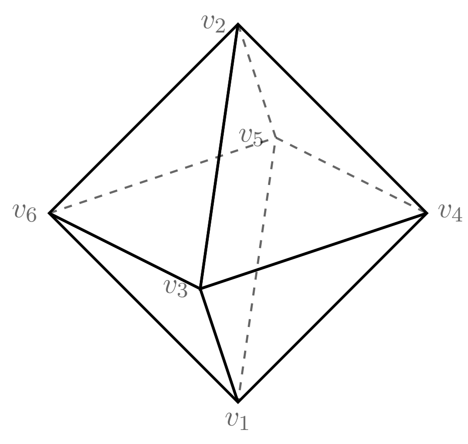

Figure 1: Octahedron

The main tool which we will work on is the Stanley-Reisner ring.

Definition 3.3. Let Δ be a simplicial complex on the vertex set $V = \{v_1, \dots, v_n\}$, and k be a ring. The Stanley-Reisner ring of the complex Δ is the homogeneous $k-$algebra

$$k[\Delta] = k[X_1, \dots, X_n]/I_\Delta$$

where $I_\Delta = (\{X_{i_1} \cdots X_{i_r} \mid i_1 < \cdots < i_r, \{v_{i_1}, \dots, v_{i_r}\} \notin \Delta\})$[4].

Observe that by definition I_Δ is generated by square-free monomials. Let see an example.

Example 3.2. Let Δ be the simplicial complex on the Figure 2 on the set of vertices $V = \{v_1, v_2, v_3, v_4\}$.

Is a complex of dimension 2. Its f-vector is $f(\Delta) = (4, 5, 1)$ and the Stanley-Reisner ideal is $I_\Delta = (x_1x_2, x_1x_2x_4)$. Notice that the face $\{v_2, v_3, v_4\}$ is a maximal face of maximal dimension.

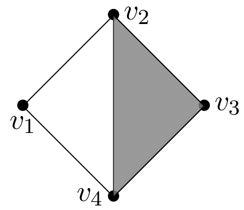

Figure 2:

Now, the Krull dimension of the Stanley-Reisner rings can be determined easily from the following result.

Proposition 3.1. *Let Δ be a simplicial complex, and k a field. Then*

$$I_\Delta = \bigcap_F \mathfrak{P}_F$$

where the intersection is taken over all facets F of Δ, and $\mathfrak{P}_F$ denotes the prime ideal generated by all X_i such that $v_i \notin F$.

Before the proof of this proposition we need some fundamental lemmas about the primary decomposition of the monomial ideals.

Lemma 3.2. *Let k be a field and $I \subset k[X_1, \dots, X_n]$ an ideal generated by square-free monomials. Then $k[X_1, \dots, X_n]/I$ is reduced.*

Proof. Let $R = k[X_1, \dots, X_n]$. We know that a quotient ring R/I is reduced if and only if I is radical. We also have the following fact. Let k be a field and let I be a monomial ideal in $k[X_1, \dots, X_n]$ then Rad $= I$ if and only if I is generated by square-free monomials.

Since I_Δ is generated by square-free monomials then I_Δ is radical, therefore R/I_Δ is reduced.

[4]We will call the ideal I_Δ the Stanley-Reisner ideal.

First, we shall prove that I_Δ is radical, Rad $(I_\Delta) = I_\Delta$.

The inclusion $I_\Delta \subseteq \text{Rad}(I)$ is by definition of radical, for the other inclusions observe that $\forall f \in \text{Rad}(I_\Delta)$ then $\exists m \in \mathbb{Z}$ such that $g^m \in I_\Delta$, since I_Δ is generated by square-free monomials then $g \in I_\Delta$. So I_Δ is radical.

Now we will prove that for a radical ideal I_Δ, the quotient ring R/I_Δ is reduced.

Let I_Δ be a radical ideal, we have to show that if $x+I_\Delta \in R/I_\Delta$ such that $(x+I_\Delta)^n = 0_{R/I_\Delta}$ for some positive integer n then $x + I = 0_{R/I_\Delta}$. Let $(x + I_\Delta)^n = 0_{R/I_\Delta}$ then $x^n + I_\Delta = 0_{R/I_\Delta}$ and therefore $x^n \in I_\Delta$. That is $x + I_\Delta = 0$. Hence R/I_Δ is reduced. $\square$

Lemma 3.3. *Let I be a radical ideal in a ring R, then*

$$I = \bigcap_{\mathfrak{p} \supset I} \mathfrak{p},$$

where $\mathfrak{p}$ is a prime ideal containing I.

Proof. Let $\mathfrak{p}$ be a prime ideal containing I. If $r \in R$ such that $r^n \in I$, then $r^n \in \mathfrak{p}$, so $r \in \mathfrak{p}$ since $\mathfrak{p}$ is a prime ideal. Thus $\text{Rad}(I) \in \bigcap_{\mathfrak{p} \supset I} \mathfrak{p}$.

Conversely, if $r \notin \text{Rad}(I)$, then $r^n \notin I$, for any k. So $S = \{1, r, r^2, \ldots\}$ is a multiplicative closed set disjoint from I. We know that there exists a prime ideal $\mathfrak{p}_S \subset R \setminus S$ and containing I. Since $r \notin \mathfrak{p}_S$, we get that $r \notin \bigcap_{\mathfrak{p} \supset I} \mathfrak{p}$. $\square$

Definition 3.4. A *face ideal* is an ideal $\mathfrak{p}$ of R generated by a subset of the set of variables, that is, $\mathfrak{p} = (x_{i_1}, \ldots, x_{i_k})$ for some variables x_{i_j}.

Lemma 3.4. *Let $R = k[X_1, \ldots, X_n]$ be a polynomial ring over a field k.*

(i) If $I \subset R$ is generated by monomials, then every associated prime of I is generated by a subset of variables $(X_1, \ldots, X_n)$.

(ii) If $I \subset R$ is an ideal generated by square-free monomials and $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ are the associated primes of I, then

$$I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s.$$

Proof. (i): Let $R = k[x_1, \ldots, x_n]$. We will prove it by induction on the set of variables tat are generators of I. Recall that by Dickson's lemma, a monomial ideal I is minimally generated by a unique finite set of monomials.

Let $\mathfrak{m} = (x_1, \ldots, x_n)$ and $\mathfrak{p}$ be an associated prime of I. If $\text{Rad}(I) = \mathfrak{m}$, then $\mathfrak{p} = \mathfrak{m}$. Therefore, we can assume $\text{Rad}(I) \neq \mathfrak{m}$. Choose a variable x_1 not in $\text{Rad}(I)$ and consider the ascending chain of ideals

$$I_0 = I \qquad \text{and} \qquad I_{i+1} = (I_i : x_1) = \{a \in R | a \cdot x_1 \in I_1\} \quad (i \geq 0).$$

Since R is Noetherian, this ascending chain is stationary. Hence, there exists k such that $I_k = (I_k : x_1)$.

New we separate in two cases. First, if $\mathfrak{p}$ is an associated prime of (I_i, x_1) for some i. Then we can write $(I_i, x_1) = (I'_i, x_1)$ where I'_i is an ideal minimally generated by a finite set of monomials in the variables $x_2, \ldots, x_n$. Therefore, we can write

$$\mathfrak{p} = x_1 R + \mathfrak{p}' R,$$

where $\mathfrak{p}'$ is a prime ideal of $k[x_2, \ldots, x_n]$. By the induction hypothesis $\mathfrak{p}'$ is a face ideal, then $\mathfrak{p}$ is a face ideal also.

Now assume that $\mathfrak{p}$ is not an associated prime ideal of $(I_i : x_1)$ for any i. Then, for each i consider the short exact sequence

$$0 \longrightarrow R/(I_i : x_1) \xrightarrow{x_1} R/I_i \longrightarrow R/(I_i, x_1) \longrightarrow 0.$$

Making an recursive use of the property that the associated prime ideals of R/I_i is a subset of the associated primes ideals of the extremes of the exact sequence, that is,

$$\operatorname{Ass}(R/I_i) \subset \operatorname{Ass}(R/(I_i : x_1)) \cup \operatorname{Ass}(R/(I_i, x_1)),$$

and applying the first case, one obtain that $\mathfrak{p}$ is an associated prime of I_i for all i.

From the stationariness of the chain $(I_i : x_1)$ it follows that x_1 is a regular element on R/I_k. Hence, I_k is an ideal minimally generated by monomials in variables $x_2, \ldots, x_n$. Repeating the same argument as in the first case, and applying the induction hypothesis we have that $\mathfrak{p}$ is face ideal.

(ii): Since $I \subset \mathfrak{p}$ for any $\mathfrak{p}$ that is an associated prime of I we only have to show that $\cap_{i=1}^{s} \mathfrak{p}_i \subset I$.

Let $f = x_{i_1}^{a_1} \cdots x_{i_r}^{a_r} \in \cap_{i=1}^{s} \mathfrak{p}_i$ be a monomial, where $i_1 < \cdots < i_r$ and $a_i > 0$ for all i. Then by Lemma 3.3 $f \in \operatorname{Rad}(I)$ therefore, there exits $k \geq 1$ such that $f^k \in I$. Since I is generated by square-free monomials we obtain that $x_{i_1} \cdots x_{i_r} \in I$. Hence, $f \in I$. Now since the intersection of monomial ideals is a monomial ideal again, we obtain that $\cap_{i=1}^{s} \mathfrak{p}_i$ is a monomial ideal. So every element in the intersection if of the form $\lambda_1 f_1 + \cdots + \lambda_m f_m$, where f_i is a monomial like before and $\lambda_i \in R$. Therefore, we can extend this argument for all this elements, so $\cap_{i=1}^{s} \mathfrak{p}_i \subseteq I$. As we desire.

□

Now we are ready to prove the that the Stanley-Reisner ideal decompose into prime ideals.

Proof. (Proposition 3.1). Let Δ be a simplicial complex, and let $F \in \Delta$. Let $\mathfrak{P}_F = (\{X_i | \, v_i \notin F\})$. By Lemma 3.2 $k[\Delta]$ is reduced since I_Δ is generated by square-free monomials.

Now $k[\Delta] = k[X_1, \ldots, X_n]/I_\Delta$ is reduced if and only if I_Δ is a radical ideal. Since I_Δ is a radical ideal, then I_Δ is the intersection of all the minimal prime ideals of $k[X_1, \ldots, X_n]$ that contain I_Δ.

By Lemma 3.4 (ii) all these ideals are generated by subset of $\{X_1, \ldots, X_n\}$. Let $\mathfrak{p}_F = (X_{i_1}, \ldots, X_{i_s})$. New $I_\Delta \subset \mathfrak{P}_F$ if and only if $\{v_1, \ldots, v_n\} \backslash \{v_{i_1}, \ldots, v_{i_s}\}$ is a face of Δ, this is just by definition of I_Δ and $\mathfrak{p}_F$.

Moreover, $\mathfrak{P}_F$ is a minimal prime ideal of I_Δ if and only if $\{v_1, \ldots, v_n\} \backslash \{v_{i_1}, \ldots, v_{i_s}\}$ is a facet. Therefore by Lemma 3.4 (ii), since $\mathfrak{P}_F$ are the associated primes of I_Δ,

$$I_\Delta = \bigcap_F \mathfrak{P}_F.$$

□

Theorem 3.5. *Let Δ be a $(d-1)-$dimensional simplicial complex and let k be a field with $|k| = \infty$. Then*

$$\dim(k[\Delta]) = d.$$

Proof. Let F be a face of Δ with d vertices. By the previous Proposition 3.1 the ideal $\mathfrak{P}_F$ is generated by variables X_i such that $v_i \notin F$. This ideal $\mathfrak{P}_F$ has height equal to the height of I_Δ,

$$\mathrm{ht}(\mathfrak{P}_F) = \{\mathrm{ht}(\mathfrak{P}_F) | I_\Delta \subset \mathfrak{P}_F \text{ and } \mathfrak{P}_F \text{ is prime ideal } \}.$$

Hence $\mathrm{ht}(I_\Delta) = n - d$ and by the formula $\dim(R/I) = \dim(R) - \mathrm{ht}(I)$ we get that

$$\dim(k[\Delta]) = d.$$

□

Definition 3.5. Let Δ be a simplicial complex of $\dim \Delta = d$. We say that Δ is *pure* if all of its facets are of the same dimension d.

We say that Δ is *Cohen-Macaulay complex* if the ring $k[\Delta]$ is Cohen-Macaulay over some field k.

In view of Theorem 3.5 and Proposition 3.1, we immediately obtain the following Corollary.

Corollary 3.6. *If $k[\Delta]$ is Cohen-Macaulay then Δ is pure simplicial complex.*

Proof. By the Proposition 3.1 $I_\Delta = \cap_F \mathfrak{P}_F$ for F facet of Δ, so $\mathfrak{P}_F$ are the associated primes of I_Δ. From the Proposition 2.11, we know that for any Cohen-Macaulay ring S, $\dim S/P = \dim P$ for all minimal prime ideals P of S. Since $k[\Delta]$ is Cohen-Macaulay, $\dim k[X_1, \ldots, X_n]/\mathfrak{P}_F = \dim \mathfrak{P}_F = \dim F + 1 = \dim \Delta + 1$. for all F facet of Δ. Hence all the facets have the same dimension.

□

Hilbert Series of $k[\Delta]$

For a Stanley-Reisner ring $k[\Delta]$ there are explicit formulas for its Hilbert Series in terms of f-vector, the combinatorial data of the simplicial complex.

To relate the Hilbert function of the Stanley-Reisner ring with the f-vector of a simplicial complex Δ we need to introduce a fine $\mathbb{Z}^n$-grading on $k[\Delta]$.

Let $(G,+)$ be an abelian group. A G-grading ring R is a decomposition of the ring $R=\bigoplus_{a\in G} R_a$ (as a $\mathbb{Z}$-module) such that $R_aR_b = R_{a+b}$ for all $a,b\in G$. Similarly, one defines a G-graded R-module. If R is an $G-$graded ring, an R-module M is G-graded if there is a decomposition $M=\bigoplus_{a\in G} M_a$, such that $R_aM_a \subset M_{a+b}$ for all $a,b\in G$.

In our case, for the polynomial ring $R=k[X_1,\dots,X_n]$ the fine grading is given in a natural way by the degree of monomials. Now, let $a=(a_1,\dots,a_n)\in\mathbb{Z}^n$, $a_i\leq 0$ for $i=1,\dots,n$, we set $X^a = X_1^{a_1}\cdots X_n^{a_n}$. Let $R_a=\{cX^a | c\in k\}$ be the a-th homogeneous component of R. Set $R_a=0$ if $a_i<0$ for some i. The $\mathbb{Z}^n$-graded ideals in R are just the ideals generated by monomials, and the $\mathbb{Z}^n$-graded prime ideals are just the many finite ideals which are generated by subsets of $\{X_1,\dots,X_n\}$.

Let $I\subset R$ be an ideal generated by monomials. Therefore I is $\mathbb{Z}^n$-graded. So we get an induced $\mathbb{Z}^n$-grading on the quotient ring R/I. The inherited $\mathbb{Z}^n$-grading is given by $(R/I)_a = R_a/I_a$ for all $a\in\mathbb{Z}^n$. Therefore, the Stanley-Reisner rings are $\mathbb{Z}^n$-graded. Given simplicial complex Δ, we denote by x_i the residue classes of the indeterminates X_i in $k[\Delta]$. With this notation we can express the fine grading of the Stanley-Reisner ring as follows:

$$k[\Delta] = R/I_\Delta = \bigoplus_{a:\mathrm{Supp}(a)\in\Delta} kx^a. \text{ [5]} \tag{1}$$

The $\mathbb{Z}^n$-graded polynomial ring $R=k[X_1,\dots,X_n]$ has the Hilbert series

$$H_R(\mathbf{t}) = \sum_{a\in\mathbb{N}^n}\mathbf{t}^a = \prod_{i=1}^{n}(1-t_i)^{-1}.$$

Our next goal will be to compute the Hilbert Series for the face ring of a simplicial complex as a homogeneous $\mathbb{Z}$-graded algebra. Grouping the $a\in\mathbb{Z}^n$ in $i=|a|=a_1+\cdots+a_n$ for $i\in\mathbb{Z}$, we get a $\mathbb{Z}$-grading for $k[\Delta]$:

$$k[\Delta] = \bigoplus_{i\in\mathbb{Z}} k[\Delta]_i = \bigoplus_{i\in\mathbb{Z}}\left(\bigoplus_{a\in\mathbb{Z}^n,\ |a|=i} k[\Delta]_a\right).$$

Since we are working with monomials of the form x^a, the tool that relates these monomials with the faces of a simplicial complex is the support. Let $a\in\mathbb{Z}^n$, we define the support of a in Δ by

$$\mathrm{Supp}(a) = \{v_i | a_i > 0\}.$$

And for a non-zero monomial $x\in R$, we set $\mathrm{Supp}(x^a)=\mathrm{Supp}(a)$.

With this notation established we are ready to compute the Hilbert Series of the face ring of a simplicial complex.

Theorem 3.7. *Let Δ be a simplicial complex with f-vector $(f_0,\dots,f_{d-1})$. Then*

$$H_{k[\Delta]}(t) = \sum_{i=-1}^{d-1}\frac{f_it^{i+1}}{(1-t)^{i+1}}.$$

[5]In this last expresion, k is a field.

Proof. Assume that Δ is of dimension d. Let $V = \{v_1, \ldots, v_n\}$ be the set of vertices of Δ. From the equation (1) we get that

$$H(k[\Delta], \mathbf{t}) = \sum_{a\colon \operatorname{Supp}(a) \in \Delta} \mathbf{t}^a$$

By partitioning the faces of Δ we get

$$H(k[\Delta], \mathbf{t}) = \sum_{F \in \Delta} \left(\sum_{a\colon \operatorname{Supp}(a) = F} \mathbf{t}^a \right).$$

For each $a \in \mathbb{N}^n$ such that $\operatorname{Supp}(a) = F$, $\mathbf{t}^a$ must contain $\prod_{i : v_i \in F} t_i$ as well as any number of additional powers of t_i for each $v_i \in F$. Therefore we have

$$\begin{aligned} H(k[\Delta], \mathbf{t}) &= \sum_{F \in \Delta} \left(\Big(\prod_{i : v_i \in F} t_i \Big) \Big(\prod_{i : v_i \in F} (1 + t_i + t_i^2 + \cdots \Big) \right) \\ &= \sum_{F \in \Delta} \left(\Big(\prod_{i : v_i \in F} t_i \Big) \Big(\prod_{i : v_i \in F} \frac{1}{1 - t_i} \Big) \right). \end{aligned}$$

By replacing all the t_i by t, we obtain the following formula

$$H(k[\Delta], t) = \sum_{F \in \Delta} t^{|F|} \cdot \frac{1}{(1-t)^{|F|}}. \tag{2}$$

By partitioning the faces of Δ by dimension, we can write this sum in terms of the face number of Δ to get the desired formula:

$$H(k[\Delta], t) = \sum_{i=0}^{d} f_{i-1} \frac{t^i}{(1-t)^i}. \tag{3}$$

□

From the Hilbert Series we can read off its Hilbert function.

Proposition 3.8. *The Hilbert Function is given by*

$$H(k[\Delta], n) = \sum_{i=0}^{d} \binom{n-1}{i} f_i, \quad \textit{for } n \leq 1 \ \textit{ and } H(k[\Delta], 0) = 1.$$

Proof. Recall that $H(k[\Delta], n) = \dim_k k[\Delta]_n$ since the dimension and the length coincide over vector spaces of finite dimension. Therefore, for the case $n = 0$ it is clear that $H(k[\Delta], 0) = 1$.

Let $n > 0$. The Hilbert Series is given by the power series

$$1 + f_0 \left(\frac{t}{1-t} \right) + \cdots + f_{d-1} \left(\frac{t}{1-t} \right)^d. \tag{4}$$

Then $\dim k[\Delta]_n$ is given by the coefficient of t^n in the expansion of the power series in (4).

Using the identity $(1-t)^{-(n+1)} = \sum_{i=0}^{\infty} \binom{n+i}{i} t^i$, we get

$$\begin{aligned}\dim k[\Delta]_n &= f_0 + f_1 \binom{1+n-2}{n-2} + f_2 \binom{2+n-3}{n-3} + \cdots + f_{d-1} \binom{d-1+n-d}{n-d} \\ &= \sum_{i=0}^{d-1} f_i \binom{n-i}{i}\end{aligned}$$

□

Remark 3.1. Observe that $H(k[\Delta], n])$ is a polynomial function for $n > 0$ and by unicity coincides with the Hilbert polynomial for all $n > 0$. Evaluating the polynomial

$$\sum_{i=0}^{d-1} f_i \binom{n-i}{i}$$

at $n = 0$ gives $\chi(\Delta) = f_0 - f_1 + \cdots + (-1)^{d-1} f_{d-1}$, the *Euler characteristic* of Δ.

Thus the Hilbert polynomial and the Hilbert function of $k[\Delta]$ coincide if and only if $\chi(\Delta) = 1$ for all $n \leq 0$.

Other information that we can obtain from the formula of the Hilbert Series is that the order of its poles is d, and we recover that $\dim_k k[\Delta] = d$. We also see that the multiplicity of the Stanley-Reisner ring, $e(k[\Delta])$ is f_{d-1}, the number of facets of Δ.

The h-vector of a Simplicial Complex

Recall that the Hilbert series of a homogeneous k-algebra R of dimension d can be expressed as a quotient of a polynomial $Q_R(t)$ by $(1-t)^d$ (see 2.14). If we write the polynomial $Q_R(t)$ as $h_0 + h_1 t + \cdots h_d t^d$, we know that these coefficients are integers. For a simplicial complex Δ, write

$$H_{h[\Delta]}(t) = \frac{h_0 + h_1 t + \cdots}{(1-t)^d}$$

and these coefficients are called the $h-$vector of the simplicial complex.

Definition 3.6. Let Δ be a simplicial complex of dimension $d-1$. The h-vector of Δ are the coefficients of $Q_{k[\Delta]}(t)$. We write $h(\Delta) = (h_0, h_1, \cdots, h_d)$.

We can get more information about the simplicial complex by studying the h-vector. It is more easy to manipulate, since it has a direct relation with the algebraic properties of the simplex.

Comparing the coefficients of the expression in (3) with the h-vector we get an explicit formula that relates the f-vector with the h-vector.

Proposition 3.9. *The f-vector and the $h-$vector of a $(d-1)$-dimensional simplicial complex Δ are related by*

$$\sum_{i=0}^{d} h_i t^i = \sum_{i=0}^{d} f_{i-1} t^i (1-t)^{d-i}. \tag{5}$$

In particular $h_j = 0$ for $j > d$, and for $j = 0, \ldots, d$,

$$h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{j-i} f_{i-1} \quad \textit{and} \quad f_{j-1} = \sum_{i=0}^{j} \binom{d-i}{j-i} h_i. \tag{6}$$

Proof. By multiplying the expression in (3) by $(1+t)^d$ we get the first identity of polynomials.

Now, by using the relation $(1+x)^n = \sum_{k=0} \binom{n}{k} x^k$ we can expand the right side in the above identity. By, comparing the coefficients on both sides we get the formula for the h_j in terms of f_i.

In order to prove the formula for the f_i's, replacing t by $s/(1+s)$ in the polynomial identity, we get an analogue expression

$$\sum_{i=0}^{d} h_i s^i (1+s)^{d-i} = \sum_{i=0}^{d} f_{i-1} s^i.$$

If we compare the coefficients, we get the desired result.

□

From the above equation we have some interesting special cases:

Corollary 3.10. *Let Δ be a simplicial complex of dimension d, then*

$$h_0 = 1, \qquad h_1 = f_0 - d, \qquad h_d = (-1)^{d-1}(\chi(\Delta) - 1) \quad \textit{and} \quad \sum_{i=0}^{d} h_i = f_{d-1}.$$

To compute the h-vector of a simplicial complex we can follow the procedure of R. Stanley. In the example of the octahedron 1, one has $f(\Delta) = (6, 12, 8)$. Write down the entries of the f-vector diagonally, and put a 1 to the left of f_0.

$$\begin{array}{cccc} 1 & 6 & & \\ & & 12 & \\ & & & 8 \end{array}$$

Complete the array constructing a table, by placing below a consecutive pair of entries the difference between them, and by placing 1 on the left-hand side:

$$
\begin{array}{rcccccccc}
 & & & & 1 & & 6 & & \\
 & & & 1 & & 5 & & 12 & \\
 & & 1 & & 4 & & 7 & & 8 \\
\hline
h(\Delta) = & (1 & & 3 & & 3 & & 1) &
\end{array}
$$

After completing the table the next row of differences will be the $h-$vector.

Since the f-vector and the $h-$vector determine each other, we are interested in studying special case of bound of these two vectors. For the case of Cohen-Macaulay complex we have a bound for h-vector. Now, we present a result that plays an important role in the proof of upper bound conjecture.

Theorem 3.11. *Let Δ be a $(d-1)$-dimensional Cohen-Macaulay complex with n vertices and h-vector $(h_0, \cdots, h_d)$. Then*

$$0 \le h_i \le \binom{n-d+i-1}{i}, \qquad 0 \le i \le d.$$

Proof. First we prove that $h_i \le 0$. We may assume that k is infinite. Since $R = k[\Delta]$ is Cohen-Macaulay, there exists $\mathbf{x}$ an R-sequence formed by elements of degree 1. Set $\overline{R} = R/(\mathbf{x})R$, this quotient module is of Krull dimension 0. Now, from the Remark 2.3, $h_i = H(\overline{R}, i)$ for all i. Then, applying Corollary 2.15 (b) we get the result, $h_i \ge 0$ for all i.

For the second inequality observe that $\overline{R}$ is generated over k by $n-d$ elements of degree 1. New the Hilbert function of $\overline{R}$ is bound by the Hilbert function of a polynomial ring in $n-d$ variables. And

$$H(k[y_1, \dots, y_m], i) = \binom{m+i-1}{m-1}$$

taking $m = n-d$

$$H(\bar{R}, i) \le H(k[y_1, \dots, y_{n-d}], i) = \binom{n-d+i-1}{n-d-1} = \binom{n-d+i-1}{i}.$$

$\square$

3.1 Shellable simplicial complex

The previous theorem shows us the importance of the Cohen-Macaulay class of complexes. In order to detect when a complex is Cohen-Macaulay we will introduce some special types of simplicial complexes.

In this section we will introduce the class of shellable simplicial complexes, and we will see that these complexes are Cohen-Macaulay.

For a simplicial complex Δ, write Δ_j for the subcomplex of Δ generated by $F_1, \dots, F_t$:

$$\Delta_j = 2^{F_1} \cup 2^{F_2} \cup \cdots \cup 2^{F_j},$$

where $2^F = \{G \in \Delta | G \subset F\}$[6] and we set $\Delta_0 = \emptyset$.

We also define the *face poset* of a simplicial complex Δ to be the set of all faces of Δ ordered by inclusion. We will denote by $\mathcal{F}(\Delta)$.

Definition 3.7. A pure simplicial complex Δ of dimension d is *shellable* if there exists a total order[7] of the facets of Δ, $F_1, \ldots, F_t$, such that for all $2 \leq i \leq t$

$$2^{F_i} \bigcap \Delta_{i-1}$$

is pure simplicial complex of dimension $d-1$.

If Δ is shellable, $F_1, \ldots, F_s$ is called a *shelling order*.

Before giving an example, we shall present two more equivalent conditions for shellability.

Proposition 3.12. *Let Δ be a pure simplicial complex and let $F_1, \ldots, F_t$ be the facets of Δ. Then the following conditions are equivalent:*

(i) *Δ is shellable.*

(ii) *There exists a total order of the facets of $F_1, \cdots, F_t$ such that there is a unique minimal element in*

$$2^{F_i} \setminus \Delta_{i-1}$$

for every i, $1 \leq i \leq t$. This minimal element is called the restriction *of F_i and is denoted $r(F_i)$. We set $r(F_1) = \emptyset$.*

(iii) *There exists a total order of the facets of $F_1, \cdots, F_t$ such that for all $1 \leq j < i \leq t$, there is $v \in F_i \setminus F_j$ and $k < i$ with $F_i \cap F_j \subset F_i \cap F_k = F_i \setminus \{v\}$.*

Proof. Let $V = \{v_1, \ldots, v_n\}$ be the set of vertices of Δ. Suppose that Δ is a pure shellable simplicial complex. First, we will prove $(i) \Rightarrow (ii)$.

Without loss of generality we may assume that $F_i = \{v_1, \ldots, v_t\}$. Since $2^{F_i} \cap \Delta_{i-1}$ is generated by non-empty set of maximal proper faces, we can assume that it is generated by faces of the form $\{v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_t\}$ for all $1 \leq j \leq r \leq t$. Then the unique minimal element in the set $\Gamma_i = \{F \in \Delta_i | F \notin \Delta_{i-1}\}$ is $\{v_1, \ldots, v_r\}$.

For the proof of $(ii) \Rightarrow (iii)$ we let G be the unique minimal element in Γ. Since $G \not\subset F_j$ for any $j \leq i$, then there exists $v \in G \setminus F_j$. Hence $v \in F_i \setminus F_j$. If $F_i \setminus F_k \neq \{v\}$ for all $k < i$, then $F_i - \{v\} \not\subset F_k$ for all $k < i$, therefore $F_i \setminus \{v\} \in \Gamma_i$. Since every element of Γ_i contains G, we must have $G \subset F_i \setminus \{v\}$, a contradiction.

Now we can prove the last implication, $(iii) \Rightarrow (i)$. Let $F \in 2^{F_i} \cap \Delta_{i-1}$. Then $F \subset F_j$ for some $j < i$. Let $v \in F_i \setminus F_j$ such that $F_i \setminus F_k = \{v\}$ for some $k < i$, as in (iii). Then $F_i \setminus \{v\}$ is a maximal proper face o 2^{F_i} that belongs in $2^{F_i} \cap \Delta_{i-1}$ and contains F. This proves that all conditions are equivalent.

□

[6]The notation 2^F comes from the fact that the *powerset*, $\mathcal{P}$, of a set with n elements has cardinal 2^n.

[7]A total order satisfies the antisymmetry, transitivity and totality properties (all elements of the set are comparable).

To illustrate the importance of all these different definitions of shellability we must analyse some simplicial complexes.

Example 3.3. In Figure 3 is illustrated an example of a non-shellable simplicial complex, while the simplicial complex in Figure 4 is shellable.

Let Δ_1 and Δ_2 the simplicial complex on the Figure 3 and Figure 4, respectively. We can apply directly the definition to conclude that Δ_1 is not shellable. Notice that to pass from the facet on the left to the facet on the right we always have to cross $\{v_1\}$ a face of dimension 0. Hence, Δ_1 is not shellable.

On the other hand, all of the facets on the complex Δ_2 share a face of co-dimension 1. Therefore, we can find a shelling order.

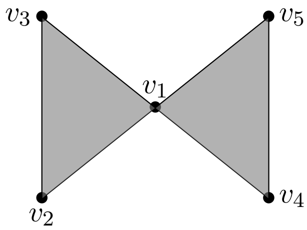

Figure 3: Non shellable

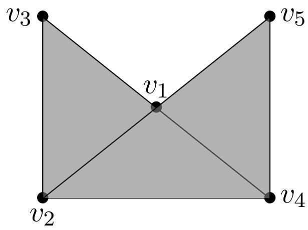

Figure 4: Shellable

Example 3.4. For the simplicial complex Δ in Figure 5 we have the following shelling:

The facets of this simplicial complex are F_1, F_2, F_3, F_4 and this is a shelling order. For this order the restriction of each facet is: $r(F_1) = \emptyset$, $r(F_2)$ is the vertex $\{4\}$, $r(F_3)$ is the edge $\{2, 4\}$, and $r(F_4)$ is the vertex $\{5\}$.
Note that F_1 and F_4 can not be consecutive in any sequence of shelling orders.

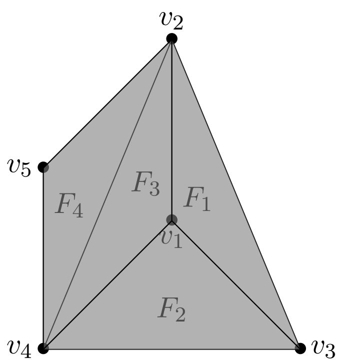

Figure 5: Shellable and Restrictions

Theorem 3.13. *A shellable simplicial complex is Cohen-Macaulay over every field.*

Proof. Let Δ be a simplicial complex d-dimensional, with vertex set $V = \{v_1, \dots, v_n\}$. Consider $R = k[x_1, \dots, x_n]$ a polynomial ring over a field k.

Assume that $F_1, \dots, F_t$ is a total order of the facets of Δ, such that $2^{F_i} \cap \Delta_{i-1}$ is pure of dimension $d-1$ for all $i \leq 2$, this is just the definition of shellable simplicial complex.

Set

$$\sigma = \{v_i \in F_s |\ F_s \setminus \{v_i\} \in 2^{F_s} \cap \Delta_{s-1}\},$$

then we have $\sigma \in 2^{F_s}$. Therefore, we can assume $\sigma = \{v_1, \dots, v_r\}$. Let $f = x_1 \cdots x_r$ then

$$0 \longrightarrow R/(I_\Delta : f) \xrightarrow{\varphi_f} R/I_\Delta \xrightarrow{\varphi} R/(I_\Delta, f) \longrightarrow 0 \tag{7}$$

is a short exact sequence, where φ is the projection map and φ_f is the multiplication by f map. The ideal $(I_\Delta : f)$ is defined as $\{a \in R | a \cdot f \in I_\Delta\}$ and if I_Δ is prime then $(I_\Delta : f) = R$ or $(I_\Delta : f) = (f)$. That (7) is an exact sequence is because φ_f is injective and φ is exhaustive, by definition.

First, we need to show the equality $R/(I_\Delta : f) = k[2^{F_t}]$ where $k[2^{F_t}]$ is the Stanley-Reisner ring associated to the simplicial complex 2^{F_t}. Since F_t is a facet of Δ it has dimension d and $k[2^{F_t}]$ is a polynomial ring in $d+1$ variable. So $k[2^{F_t}]$ is Cohen-Macaulay.

We know that we can write $I_\Delta = \cap_{i=1}^t \mathfrak{P}_i$, where $\mathfrak{P}_i$ is generated by $V \setminus F_i$ for all i and they are all different.

Actually we arrived that $I_\Delta = \cap_{F \in \Delta} \mathfrak{P}_F$ but if $G \in \Delta$ is not a facet then, $\mathfrak{P}_G \supset \mathfrak{P}_F$ where F is a facet that contains G. So

$$\bigcap_{G \in \Delta} \mathfrak{P}_G = \bigcap_{\substack{F \in \Delta \\ F \text{ facet}}} \mathfrak{P}_F = \bigcap_{i=1}^t \mathfrak{P}_{F_i} = \bigcap_{i=1}^t \mathfrak{P}_i.$$

Now

$$\begin{aligned} (I_\Delta : f) &= \{a \in R | a \cdot f \in I_\Delta\} = \{a \in R | a \cdot f \in \bigcap_{i=1}^t \mathfrak{P}_i\} \\ &= \{a \in R | a \cdot f \in \mathfrak{P}_i \text{ for all } i = 1, \dots, t\} \\ &= \bigcap_{i=1}^t (\mathfrak{P}_i : f) = \bigcap_{F_i \supset \sigma} \mathfrak{P}_i. \end{aligned} \tag{8}$$

The last equality holds because if $\sigma \not\subset F$ then $\sigma \subset V \setminus F$, so $f \in \mathfrak{P}_F$ and

$$(\mathfrak{P}_F : f) = \{a \in R | a \cdot f \in \mathfrak{P}_i\} = R.$$

Hence, we can take only the prime ideals such that $\sigma \subset F_i$.

Next, we should prove that $\sigma \in 2^{F_t}$ and $\sigma \notin F_j$ for all $1 \le j < t$. If $\sigma \in F_j$ then $\sigma \in 2^{F_t} \cap \Delta_{t-1}$, but since this simplicial complex is pure of dimension $d-1$, all of its facets are of dimension $d-1$.

So there is $v \in \sigma$ such that $\sigma \in F_t \setminus \{v\}$, but this can not happen, therefore $\sigma \in F_j$ if $j = s$.

Hence, we just proved that

$$(I_\Delta : f) = (\mathfrak{P}_t : f) = \mathfrak{P}_t \quad \text{and} \quad R/(I_\Delta : f) = R/\mathfrak{P}_t = k[2^{F_t}].$$

Following on from this, we will show that

$$R/(I_\Delta, f) = k[2^{F_1} \cup \cdots \cup 2^{F_{t-1}}].$$

The previous argument also shows us that $f \in \mathfrak{P}_i$ for $i = 1, \dots, t-1$; because if $f \notin \mathfrak{P}_i$ for some $i < t$, then $\sigma \subset F_i$ so we can repeat the argument and again we arrive to a contradiction.

If $\mathfrak{P}$ is a minimal prime of (I_Δ, f) different form $\mathfrak{P}_1, \dots, \mathfrak{P}_{t-1}$, then $\mathfrak{P}_t \subset \mathfrak{P}$ and $X_i \in \mathfrak{P}$ for some $v_i \in \sigma$, thus by construction of σ one has $F_t \setminus \{v_i\} \subset F_k$ for some $k < t$. Therefore $V \setminus F_k \subset \{v_i\} \cup (V \setminus F_t)$, from this we can deduce that $\mathfrak{P}_k \subset (\mathfrak{P}_t, x_i) \subset \mathfrak{P}$, and by the minimality of $\mathfrak{P}$ one has $\mathfrak{P}_k = \mathfrak{P}_t$ and this is a contradiction. Hence we can conclude that $\mathfrak{P}_1, \dots, \mathfrak{P}_{t-1}$ are the minimal prime of (I_Δ, f) and $(I_\Delta, f) = \bigcap_{i=1}^{s-1} \mathfrak{P}_i$, as required.

This gives us an exact sequence induced by (7):

$$0 \longrightarrow k[2^{F_t}] \longrightarrow k[\Delta] \longrightarrow k[2^{F_1} \cup \dots \cup 2^{F_{t-1}}] \longrightarrow 0. \tag{9}$$

We finish the proof doing induction on t.

For $t = 1$, we have the following exact sequence:

$$0 \longrightarrow k[2^{F_1}] \longrightarrow k[\Delta] \longrightarrow 0.$$

Since $k[2^{F_t}]$ is Cohen-Macaulay, it is implied that $k[\Delta]$ is also Cohen-Macaulay.

For $t = 2$, we have the exact sequence:

$$0 \longrightarrow k[2^{F_2}] \longrightarrow k[\Delta] \longrightarrow k[2^{F_1}] \longrightarrow 0.$$

Since $k[2^{F_i}]$ are Cohen-Macaulay, for $i = 1, 2$; by the depth lemma, depth $k[\Delta]$ =depth $k[2^{F_2}]$. Thus, $k[\Delta]$ is Cohen-Macaulay.[8]

Finally, we come to the induction step. Suppose it is true for t, as before we have an exact sequence as in (9), with $s = s+1$. By induction hypothesis depth $k[2^{F_1} \cup \dots \cup 2^{F_t}] = d+1$. By applying again the depth lemma we get that depth $k[\Delta] = d+1$. Hence $k[\Delta]$ is Cohen-Macaulay.

□

From the second equivalent definition of shellability in proposition 3.12 (ii), the partial ordered face set, $\mathcal{F}(\Delta)$ of a shellable simplicial complex can be broken into a disjointed union of intervals

$$\mathcal{F}(\Delta) = \bigsqcup_{1 \leq i \leq t} [r(F_i), F_i)].$$

This gives us a relationship between the h-vector of a simplicial complex and the size of the restriction faces, $r(F)$, of a shelling order.

Theorem 3.14. *Let Δ be a pure $(d-1)$-dimensional shellable simplicial complex with $\mathcal{F}(\Delta) = \bigsqcup_{1 \leq i \leq t} [r(F_i), F_i)]$. Then*

$$h_i(\Delta) = |\{j : |r(F_j)| = i\}|.$$

[8]Here we also apply the Theorem 3.5, to get that dim $k[\Delta] = d$.

Proof. We sort the faces of the simplex Δ by the interval of the partition in which they appear. So, we look at the faces $G \in \Delta$ such that $r(F_j) \subset G$ and $G \subset F_j$ for all $j = 1, \dots, t$.

First, by changing variables in the equation (5), and then using the separation in the intervals explained above, we get

$$\sum_{i=0}^{d} h_i t^{d-i} = \sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{j=1}^{t} \left(\sum_{G: r(F_j) \subseteq G \subseteq F_j} (t-1)^{d-|G|} \right).$$

Now we can also separate the faces G by their dimension. Moving k in $|r(F_j)| + k$ we choose all the faces that are in the previous summation .

Since F_j is a simplex, a face of maximal dimension, the number of simplexes G of size $|r(F_j)| + k$ such that $r(F_j) \subseteq G \subseteq F_j$ is equal to the number of ways of choosing k of the vertices of $F_j \setminus r(F_j)$. Since $|F_j \setminus r(F_j)| = d - |r(F_j)|$, this number is equal to $\binom{d-|r(F_j)|}{k}$. Therefore we have,

$$\sum_{i=0}^{d} h_i x^{d-i} = \sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{j=1}^{t} \left(\sum_{k=0}^{d-|r(F_j)|} \binom{d-|r(F_j)|}{k} (x-1)^{d-|r(F_j)|-k} \right).$$

Then using the binomial theorem[9], the sum in brackets is equal to $(x-1+1)^{d-|r(F_j)|}$, so we have

$$\sum_{i=0}^{d} h_i x^{d-i} = \sum_{j=1}^{t} x^{d-|r(F_j)|}.$$

We can rearrange this sum such that the expression $\{j : |r(F_j)| = i\}$ appear as a coefficient, grouping the exponents by the size of $r(F_j)$, then yields

$$\sum_{i=0}^{d} h_i x^{d-i} = \sum_{i=1}^{d} x^{d-i} \cdot |\{j : |r(F_j)| = i\}|.$$

By comparing coefficients in the both side we have the desired equality.

$\square$

This theorem shows that the number of restriction faces of a certain size is independent of the choice of the shelling order, in other words, all the shelling orders are equivalent.

One of the main results in this section is the Stanley Theorem, that gives us a characterization of the h-vector.

Theorem 3.15. *(Stanley) Let $s = (h_0, \dots, h_d)$ be a sequence of integers. Then, the following conditions are equivalent:*

---

[9] $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$

(a) $h_0 = 1$ *and* $0 \leq h_{i+1} \leq h_i^{\langle i \rangle}$ *for all* $i, 1 \leq i \leq d-1$*;*

(b) s *is the* h*-vector of a shellable simplicial complex;*

(c) s *is the* h*-vector of a Cohen-Macaulay complex.*

Proof. The implication $(a) \Rightarrow (b)$ is a purely combinatorial result. Given h satisfying the condition $0 \leq h_{i+1} \leq h_i^{\langle i \rangle}$ and $h_0 = 1$, take $n = h_1 + d$[10] and $V = \{1, 2, \ldots, n\}$. Let $\mathcal{F}$ be the family of subsets of V with cardinal d. Observe that $\mathcal{F}$ has $\binom{n}{d}$ these kind of elements.

We order the elements of $\mathcal{F}$ in such a way that for, $F, G \in \mathcal{C}$, $F < G$ if $\max\{F \triangle G\} \in G$, i.e. the largest element in their symmetric difference is in G.

Let $\mathcal{F}_i$ be the subfamily formed by those $F \in \mathcal{F}$ such that $d+1-i$ is the smallest element of V not in F. Observe that $\mathcal{F}_{i+1}$ has at least $h_i^{\langle i \rangle}$ elements.

Now for each i, $0 \leq i \leq d$, choose the first (in the given order) h_i members of $\mathcal{F}_i$, and call the resulting collection $\mathcal{C}$. Therefore, $\mathcal{C} = \{F_1, \ldots, F_t\}$ consists of the facets of the desired shellable simplicial complex Δ. The given order on $\mathcal{F}$ induces the shelling order on Δ, and $2^{F_i} \setminus (2^{F_1} \cup \cdots \cup 2^{F_{i-1}})$ has a unique minimal element for all i, $2 \leq i \leq t$, by the condition established on $\mathcal{F}_i$. So $\Delta := \langle F_1, \ldots, F_t \rangle$ is the desired shellable simplicial complex.

The implication $(b) \Rightarrow (c)$ is the Theorem 3.13.

The equivalence $(a) \Leftrightarrow (c)$ follows from the Proposition 2.18, taking $R = k[\Delta]$, the reduced homogeneous k-algebra. Since Δ is a Cohen-Macaulay complex, $k[\Delta]$ is also a Cohen-Macaulay ring.

□

3.2 Polytopes

A cyclic polytope, denoted $C(n, d)$, is the convex hull of n distinct points on the moment curve in $\mathbb{R}^d$. The upper bound theorem states that the boundary of a cyclic polytope, denoted $\Delta(n, d)$, maximizes the number of i-dimensional faces among all simplicial spheres.

In this section we will briefly introduce polytopes and state some basic results. The methods employed in this section are non-algebraic and, for the most part of the statements will be given without complete proof. Nevertheless, all theorems will be well referenced.

We consider $\mathbb{R}^d$ as a d-dimensional vector space with the standard product $\langle x, y \rangle$, for $x, y \in \mathbb{R}^d$. Recall that a subset $X \subset \mathbb{R}^d$ is *convex* if for any two distinct points $x_0, x_1 \in X$, the set of *convex combination*[11] of these points $x = (1-\lambda)x_0 + \lambda x_1$, $\lambda \in \mathbb{R}$, $0 \leq \lambda \leq 1$, belongs to X. The intersection of any family of convex sets in $\mathbb{R}^d$ is again convex. Therefore, for any subset $X \subset \mathbb{R}^d$, the intersection of all the convex set containing X is the *convex hull* of X, denoted $\operatorname{conv}(X)$.

The convex hull of a subset of $\mathbb{R}^d$ can also be described in the following way:

[10] Recall that $h_1 = f_0 - d$.

[11] In general, a convex combination of n points is a linear combination, $\sum \lambda_i x_i$, such that $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$.

Theorem 3.16. *Let $X \subset \mathbb{R}^d$. The convex hull is the set of all convex combinations of points from X.*

Definition 3.8. A *polytope* is the convex hull of a finite set of points in $\mathbb{R}^d$.

An example of polytope is the octahedron in Figure 1.
Given $y \in \mathbb{R}^d \setminus \{0\}$ and $\alpha \in \mathbb{R}$, we can define the hyperplane

$$H(y, \alpha) = \{x \in \mathbb{R}^d : \langle x, y \rangle = \alpha\},$$

and the two closed half-spaces

$$H^+(y, \alpha) = \{x \in \mathbb{R}^d : \langle x, y \rangle \geq \alpha\} \text{ and } H^-(y, \alpha) = \{x \in \mathbb{R}^d : \langle x, y \rangle \leq \alpha\}.$$

We can also define the *supporting hyperplane*, $H \subset \mathbb{R}^d$, of a subset $X \subset \mathbb{R}^d$ to be the hyperplane that contains X in one of the two closed half-spaces bounded by H and $X \cap H \neq \emptyset$.

Definition 3.9. A polyhedron is a set P in $\mathbb{R}^d$ which is the intersection of a finite number of closed half-spaces of $\mathbb{R}^d$.

The *dimension*, dim P, of a polyhedron P is the dimension of its affine hull[12]. A polytope is just a bounded polyhedron.

Definition 3.10. A *proper face* of a polyhedron P is a set $F \subset P$ such that there is a supporting hyperplane $H(x, \alpha)$ such that

(a) $F = P \cap H(y, \alpha) \neq \emptyset$,

(b) $P \not\subset H(y, \alpha)$, and $P \subset H^+(y, \alpha)$ or $P \subset H^-(y, \alpha)$.

We get the following result that helps us to understand the structure of faces of a polyhedron

Theorem 3.17. *Let P be a polyhedron.*

(a) P has only a finite number of faces.

(b) Let F be a face of P and F' a face of F. Then F' is a face of P.

(c) Any proper face of P is a face of some facet of P.

(d) The set of faces of P, ordered by inclusion, is a lattice.

Definition 3.11. A d-simplex is the convex hull of $d+1$ affinely independent points. A polytope is called simplicial if all its proper faces are simplices.

The next result explains what the faces of a simplex are.

[12] An affine hull of a set is all the affine combination of the points on the set.

Proposition 3.18. *Every j-face of a d-simplex P is a j-simplex, and every $j+1$ vertices of P are the vertices of a j-face of P.*

Let P be a simplicial polytope on the vertex set V. We define the *vertex scheme*, $\Delta(P)$, of P to be the collection of subsets of V consisting of the empty set and the vertices of the proper faces of P.

Proposition 3.19. *Let P be a simplicial polytope with vertex set V. Then $\Delta(P)$ is a simplicial complex.*

Not every simplicial complex is the vertex scheme of some simplicial polytope P. However, we can associate a geometric object to any simplicial complex that is the inverse construction of the scheme vertex. The idea is to choose the base elements of the vector space $\mathbb{R}^n$ as vertices of the simplex.

Definition 3.12. Let Δ be a simplicial complex with vertices $V = \{v_1, \dots, v_n\}$. Let e_i be the $i-$th unit coordinate vector in $\mathbb{R}^n$. Given a face $F \in \Delta$ set

$$|F| = \text{conv } \{e_i | v_i \in F\}.$$

Define the geometric realization, denoted $|\Delta|$, of the simplicial complex Δ by

$$|\Delta| = \bigcup_{F \in \Delta} |F|.$$

We can generalize this definition to any injective map $\rho : V \to \mathbb{R}^n$, such that elements of $p(F)$ are independents for all $F \in \Delta$. By choosing $\rho(v_i) = x_i$ we have our definition.

The geometric realization inherits the topology of the Euclidean space $\mathbb{R}^n$. Observe that any two geometric realizations are homeomorphic with the induced topology.

After this brief introduction to polytopes, we now move on to a special type of polytopes, that play a major role in the poof of the upper bound conjecture.

Cyclic polytopes

One of the main goals in this section is to prove the *Dehn-Sommerville Equations* which give a symmetry of the $h-$vector of a cyclic polytope. This result has a direct implication in the proof of the upper bound conjecture.

Consider the algebraic curve of degree $d \geq 2$, $\mathcal{M}_d \subset \mathbb{R}^d$, defined parametrically by

$$x : \mathbb{R} \longrightarrow \mathbb{R}^d, \qquad t \longmapsto x(t) = (t, t^2, \dots, t^d).$$

$\mathcal{M}_d$ is called the *moment curve.*

Every hyperplane intersects the moment curve in a finite set of almost d points. These properties allow us to define cyclic polytopes. Moreover, let $t_1, \dots, t_n$ be n distinct real numbers and consider $n \geq d+1$, then the n-family $(x(t_1), \dots, x(t_n))$ of points from $\mathbb{R}^d$ is affinely independent.

Definition 3.13. A *cyclic polytope*, denoted $C(n,d)$, is the convex hull of any n distinct points on $\mathcal{M}_d$.

Note that a cyclic polytope with $n = d+1$, $C(d+1,d)$ is a d-simplex, since the points $x(t_i)$ are affinely independent. Let us see the proof of this fact.

Proposition 3.20. *Any $d+1$ distinct points on $\mathcal{M}_d$ are affinely independent. In particular, $C(n,d)$ is a simplicial d-polytope.*

Proof. Let $t_1 < \cdots < t_n$ be the distinct parameters of these points. To show that the vectors, $x(t_1) - x(t_0), \ldots, x(t_d) - x(t_0)$ are linearly independent we will see that the determinant of the corresponding matrix of the row vectors is different from zero.

This matrix is the Vandermonde one, with determinant

$$\delta = \begin{vmatrix} 1 & t_0 & t_0^2 & \ldots & t_0^n \\ 1 & t_1 & t_1^2 & \ldots & t_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & t_n & t_n^2 & \ldots & t_n^n \end{vmatrix} = \prod_{0 \le j < i \le n} (t_i - t_j).$$

Since t_i are pairwise distinct, this determinant is non-zero.

Therefore,

$$\operatorname{aff}\{x(t_1), \ldots, x(t_{d+1})\} = \mathbb{R}^d,$$

implying that dim conv$\{x(t_1), \ldots, x(t_n)\} = d$.

$\square$

The notation $C(n,d)$ of a cyclic d-polytope is motivated by the fact that the points $x(t_i)$ are vertices, and the class of the polytope does not depend on the specific choice of the parameters t_i. The following results encode the combinatorial structure of the cyclic polytopes.

Proposition 3.21. *Let $n > d \geq 2$ and let $t_1 < t_2 < \cdots < t_n$. A $d-$subset $S = (t_{i_1} < \cdots < t_{i_d})$ forms a facet of $C(n,d)$ if and only if for any $i < j$ that $t_i, t_j \notin S$ then*

$$2 | \sharp\{k \in \mathbb{N} : t_k \in S, t_i < t_k < t_j\}.$$

Proof. Consider the linear function $F_S : \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$F_S(x) = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x & x(t_{i_1}) & \cdots & x(t_{i_d}) \end{pmatrix}.$$

This function vanishes at the points $x(t_{i_s})$, for $t_{i_j} \in S$. Moreover, $F_S(x(t))$ is polynomial in t of degree d.

Now, let the point $x(t)$ move on the moment curve $\{x(t) : t \in \mathbb{R}\}$, since $F_S(x(t))$ vanishes at $t = t_{i_s}$ has d different zeroes, and has a change of sign at each of them.

Observe that S is a facet if and only if $F_S(x(t_i))$ has the same sign for all the points $x(t_i)$ with $t_i \notin S$. That is if $F_S(x(t))$ has an even number of sign changes between $t = t_i$ and t_j, for $i < j$ and $t_i, t_j \notin S$.

$\square$

If a polytope P has the maximal number of i-faces when every set of $i+1$ vertices is the vertex set of a proper face of P[13].

Corollary 3.22. *Any* $\left\lfloor \frac{d}{2} \right\rfloor$*-set of vertices is the vertex set of a proper face of a cyclic polytope* $C(n,d)$.

Proof. Let $t_1 < \cdots < t_n$ be the parameters for the vertices on the moment curve. Let $T = \{i_1, \ldots, i_k\} \in \mathbb{N}^k$ with $k \leq \lfloor \frac{d}{2} \rfloor$. Choose $\epsilon > 0$ and D such that $t_i < t_i + \epsilon < t_{i+1}$ for all $i < n$, and $M > t_n + \epsilon$.

As in the argument of the previous theorem, we shall consider the linear function $F_T(x)$ defined by

$$\det\ (x, x(t_{i_1}), x(t_i + \epsilon), \ldots, x(t_{i_k}), x(t_{i_k} + \epsilon), x(D+1), \ldots, x(D+d-2k)).$$

Since this function is a polynomial in t of degree d, and has as distinct zeroes,

$$t_{i_1}, t_{i_1} + \epsilon, \ldots, t_{i_k}, t_{i_k} + \epsilon, M+1, \ldots, M+d-2k.$$

If $F_T(x(t))$ has a zero at t_l then it also has a zero at $t = t_l + \epsilon$. Therefore, there is an even number of zeros between t_i and t_j for $0 \leq i, j \leq n$ that are not in T. Thus $F_T(x)$ has the same sign on all the points $x(t_i)$ such that $0 \leq i \leq n$, and $i \notin T$. By repeating the previous argument, the corresponding vertexes of T is a face.

□

An important consequence of this result is the following corollary:

Corollary 3.23. *Let* $C(n,d)$ *be a cyclic polytope. Then*

$$f_i = \binom{n}{i+1} \quad \text{for} \quad 0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor.$$

Moreover,

$$h_k = \binom{n+k-d-1}{k} \quad \text{for} \quad 0 \leq k \leq \left\lfloor \frac{d}{2} \right\rfloor.$$

Proof. The previous result is a direct implication of the first equality. Since any $(i+1)$-subset of vertices is a i-dimensional face.

For the second, let $0 \leq k \leq \left\lfloor \frac{d}{2} \right\rfloor$. One has

$$\begin{aligned} h_k &= \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1} = \sum_{i=0}^{k} (-1)^{2(k-i)} \binom{k-d-2}{k-i} \binom{n}{i} \\ &= \binom{n-d+k-1}{k}. \end{aligned} \tag{10}$$

[13] In this case it is said that P is $(i+1)-$neighbourly.

Where for the second equality we use $\binom{r}{k} = (-1)^k \binom{-r+k-1}{k}$, and for the one last we use the Vandermonde Convolution:

$$\sum_{k=0}^{b} = \binom{a}{k}\binom{c}{b-k} = \binom{a+c}{b}, \quad \text{for } a, c \in \mathbb{Z}.$$

□

Recall that the boundary of a cyclic polytope $C(n,d)$ is a simplicial complex and we will denote by $\Delta(n,d)$. To simplify the notation we will write $f_i(C(n,d))$ for the f-vector of the boundary a cyclic polytope $\Delta(n,d)$.

Theorem 3.24. *(Bruggesser-Mani) Every polytope is shellable.*

Before the start of the proof we shall outline a constructive way to do the shelling order of the facets of a polytope P. This is called *line shelling* and it was introduced by Bruggesser-Mani [bibliography].

Recall that we can see P as a bounded polyhedron. Therefore, P is the intersection of closed half-spaces. Assume that $F_1, \ldots, F_t$ are the facets of P. For each facet F_i, take a_i as his normal vector, then we can assume that

$$P = \{x \in \mathbb{R}^d : \langle a_i, x \rangle \leq 1\}, \ \ 0 \leq i \leq t.$$

Next, take a vector c such that $\langle a_i, x \rangle \neq 0$, and order the facets in such a way that $\langle a_1, c \rangle > \langle a_2, c \rangle > \cdots > \langle a_t, c \rangle$. Our claim is that $F_1, \ldots, F_t$ is a shelling order of P.

We can give a geometric interpretation of this line shelling. First, choose an oriented line L intersecting the polytope P that is not parallel to any facet of the polytope. With this we mean that the line L does not pass through the intersections of the facets and also that it does not pass through the intersections of the supporting hyperplane of the facets.[14] We give this line the outward orientation with respect to the polytope, is the same orientation as the one induced by the vector c. This is equivalent to the previous condition, $\langle a_i, c \rangle \neq 0$. We also choose a point x on the line L in the interior of the polytope. We will move x in the direction of c. Note that a line with these properties always exists.

For a point x we define the visible part of the polytope as the union of the first intersections of the line segments that pass through x and the vertices of the polytope, as well as the faces formed by these vertices.

The order of this shelling is the order as we see them from x moving along the line. See Figure 6.

[14] We call such a line a *generic oriented line.*

The first facet in this shelling order is the face where the line intersects the polytope with the given orientation c.
Then, we add the facets one by one in the order as x passes through the intersections with the supporting hyperplane of the corresponding facet.
After we have passed through all the intersections of the line with the supporting hyperplanes in the direction c, we then start listing the remaining facets in the reverse order as we move in the direction $-c$.
The last facet in this shelling order is the face in which the line intersects the polytope with an inward orientation.

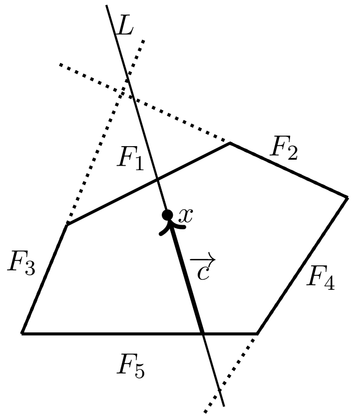

Figure 6:

We claim that this total order is in fact a shelling order of the polytope P. We prove this by using the second equivalent definition of shelling.

Proof. (Bruggesser-Mani)

We have to show that $2^{F_i} \cap \Delta_{i-1}$ is pure for all $i \geq 2$.

We will separate in two cases. First, when F_i was added to the shelling order before we passed through the infinity point.

Let G be a face in $2^{F_i} \cap \Delta_{i-1}$, that is not a facet of ∂F_i[15]. Now G must be visible form x. We will show that G is always contained in a visible facet of ∂F_i. Therefore $2^{F_i} \cap \Delta_{i-1}$ is pure, since the facets of a simplex have the same dimension.

First, observe that the sub-faces of a visible face are also visible. Now we know that G is contained in a facet of ∂F_i, if this facet is visible, then it is in $2^{F_i} \cap \Delta_{i-1}$, and we are done. If it is not, then the facet is not visible, and G is not visible either. Therefore, is not in the intersection, a contradiction with the hypothesis.

If F_i is added to the shelling after we have passed the infinity point in our line the situation is reversed. In this case $2^{F_i} \cap \Delta_{i-1}$ is the union of all the faces of ∂F_i that are contained in any face of ∂F_i not visible from x.

Again, let G be a face in $2^{F_i} \cap \Delta_{i-1}$ that is not a facet of ΔF_i. From previous observation, G must be contained in a face G' of ∂F_i not visible from x. Since G' is not visible form x, all the facets of ∂F_i that contain G' are also not visible from x. Hence, G is contained in a facet of ∂F_i not visible from x, and therefore it is in the intersection. □

Corollary 3.25. *Let $F_1, \ldots, F_t$ be a line shelling of the polytope P. Then $F_t, F_{t-1}, \ldots, F_1$ is a line shelling of P, too.*

Proof. Assume that the line shelling $F_1, \ldots, F_t$ is induced by the c and the line L. Then $F_t, \ldots, F_1$ is a line shelling induced by $-c$ and L. □

[15] ∂F_i denotes the boundary of the face F.

Lemma 3.26. *Given a shelling order $F_1, \dots, F_t$ of a simplicial complex Δ the restriction F_i is given by*

$$r(F_i) = \{v \in F_i | \ \exists\, j < i \text{ such that } (F_i \setminus v) \subseteq F_j\}.$$

Proof. Let $\widetilde{r(F_i)}$ be the restriction of F_i in the definition (ii) of 3.12. Let $v \in \widetilde{r(F_i)}$, then $v \in 2^{F_i} \setminus \Delta_{i-1}$. Hence, there exists $j < i$ such that $v \in F_i \setminus F_j$,then by the second equivalent definition of shellability in (iii) 3.12 $(F_j \setminus v) \subseteq F_k$ for some $k < i$. Therefore $v \in r(F_i)$.

For the reverse implication, let $v \in r(F_i)$, then $v \in 2^{F_i}$ and $v \notin \Delta_{i-1}$ since there exists $j < i$ such that $(F_i \setminus v) \subseteq F_j$. Therefore, $v \in \widetilde{r(F_i)}$. The minimality and unicity of $r(F_i)$ are clear by construction.

□

Theorem 3.27. *(Dehn-Sommerville). Let $\Delta(P)$ be the boundary complex of a simplicial d-polytope P. Let $(h_0, \dots, h_d)$ be the h-vector of $\Delta(P)$. Then*

$$h_i = h_{d-i}, \quad \text{for } 0 \leq i \leq d.$$

Proof. Let $G_1, \dots, G_t$ be any labelling of the facets of $\Delta(P)$. Let L be a generic line of the polytope P with orientation c. We will denote by $r_c(G_i)$ the restriction to the line shelling induced by c, and by $r_{-c}(G_i)$ the restriction induced by $-c$. Considering all this two line shelling over the line L.

Since a shelling of P induces a shelling of $\Delta(P)$ then there is a unique $G_j \neq G_i$ such that $(G_i \setminus v) \subseteq G_j$. Therefore, this face G_j will appear before G_i in the line shelling generated by c or in the line shelling generated by $-c$ and after G_i in the other line shelling.

By the previous lemma, we know that v is either in $r_c(G_i)$ or in $r_{-c}(G_i)$. Therefore $r_c(G_i) = G_i \setminus r_{-c}(G_i)$, and the cardinal is $|r_c(G_i)| = |G_i| - |r_{-c}(G_i)| = d - |r_{-c}(G_i)|$.

Now, we will apply Theorem 3.14 combined with the equality $r_c(G_i) = G_i \setminus r_{-c}(G_i)$ to compute h_i:

$$h_i(\Delta) = |\{j : |r_c(G_j)| = i\}| = |\{j : |r_{-c}(G_j)| = d - i\}| = h_{d-1}(\Delta).$$

□

Let us resume some properties of cyclic polytopes:

- $C(n, d)$ is simplicial, so the boundary $\partial C(n, d)$ defines an abstract simplicial complex $\Delta(n, d)$ such that $|\Delta(n, d)| \cong S^{d-1}$.
- $f_i(\Delta(n, d)) = \binom{n}{i+1}$ for $0 \leq i < \lfloor \frac{d}{2} \rfloor$.
- From Dehn-Sommerville equations we get that $f_0, f_1, \dots, f_{\lfloor \frac{d}{2} \rfloor - 1}$ determine $f_{\lfloor \frac{d}{2} \rfloor}, f_{\lfloor \frac{d}{2} \rfloor + 1}, \dots, f_{\lfloor \frac{d}{2} \rfloor - 1}$.

4 Local cohomology of Stanley-Reisner Rings

In this section we will briefly introduce the *reduced simplicial homology* of a simplicial complex Δ. The reduced simplicial homology is a slight modification made to simplicial homology designed to make a point have all its homology group zero. This simplification is made using an *augmented chain complex* $\mathscr{C}(\Delta)$.

Next, we shall introduce the *local cohomology* of the Stanley-Reisner ring. We will see the two most important alternative definitions of local cohomology, although we will not prove the equivalence. The fine grading inherited from the Stanley-Reisner ring allows us to decompose the local cohomology groups. This decomposition has an important role in one of the basic results of this section, Hochster's Theorem. This theorem yields to the Reisner Criterion, which provides a characterization of Cohen-Macaulay simplicial complexes in terms of the topology properties of Δ, using the reduced homology.

4.1 Reduced Simplicial Homology

We start by defining an orientation in a face of a simplicial complex. This orientation will induce an orientation in the chain complex of our simplex.

Assume that Δ is a $d-1$-dimensional simplicial complex on the set of vertices $V = \{v_1, \ldots, v_n\}$. An *orientation* on Δ is a total order on V. A simplicial complex together with an orientation on the set of vertices is an *oriented simplicial complex*.

Assume that Δ is an oriented simplicial complex. Let $F \in \Delta$ be an q-face, the orientation on Δ induce an orientation on F, and we write $F = [v_{i_1}, \ldots, v_{i_q}]$ if $F = \{v_{i_1}, \ldots, v_{i_q}\}$ and $v_{i_1} < v_{i_2} < \cdots < v_{i_q}$. If $F = \emptyset$ we write $F = [\,]$.

Now, for any field k we define the free R-module $\mathscr{C}_q(\Delta)$ with basis consisting of the oriented q-simplices in Δ. The dimension of this module, $\dim_k(\mathscr{C}_q(\Delta))$, is equal to the number of q-simplices of Δ. Thus $\mathscr{C}_q = 0$ for $q < 0$ and for $q \geq 0$ $\mathscr{C}_q(\Delta)$ is a free R-module with rank equal to the number of q-simplices of Δ.

Observe that we for each $i = 0, \ldots, d$ we can write

$$\mathscr{C}_i(\Delta) = \bigoplus_{\substack{F \in \Delta \\ \dim F = i}} \mathbb{Z}F.$$

Having introduced these notations, we can define the standard *oriented chain complex* of Δ,

$$\mathscr{C}(\Delta):\ 0 \longrightarrow \mathscr{C}_{d-1}(\Delta) \xrightarrow{\partial_{d-1}} \mathscr{C}_{d-2}(\Delta) \xrightarrow{\partial_{d-2}} \cdots \longrightarrow \mathscr{C}_0(\Delta) \longrightarrow 0.$$

For $q \geq 1$ we define the homomorphism

$$\partial_q : \mathscr{C}(\Delta) \longrightarrow \mathscr{C}_{q-1}(\Delta)$$

induced by

$$\partial_q([v_0, v_1, \ldots, v_q]) = \sum_{i=0}^{q} (-1)^i [v_0, v_1, \ldots, v_{i-1}, \widehat{v}_i, v_{i+1}, \ldots, v_q]^{16}.$$

[16]The $\widehat{v}_i$ denotes that v_i is missing.

It is simple to check that $\partial_q \circ \partial_{q+1}$:

$$\partial_q \circ \partial_{q+1}[v_0, \dots, v_{q+1}] = \partial_q \left(\sum_{i=0}^{q+1} (-1)^i [v_0, \dots, \widehat{v}_i, \dots, v_{q+1}] \right)$$
$$= \sum_{i=0}^{q+1} (-1)^i \left(\sum_{\substack{j=0 \\ j \neq i}} (-1)^j [v_0, \dots, \widehat{v}_j, \dots, \widehat{v}_i, \dots, v_{q+1}] \right) = 0.$$

Now we can extend ∂_q to a homomorphism $\mathscr{C}_q \to \mathscr{C}_{q-1}$, and since $\partial_q \circ \partial_{q+1} = 0$ it follows that $\mathscr{C}(\Delta) = \{\mathscr{C}_q(\Delta), \partial_q\}$ is the orientated chain complex of Δ.

First, let us see that this definition does not depend on the orientation given to Δ.

Lemma 4.1. *Define $\widetilde{\mathscr{C}}'(\Delta)$ in the same way as $\widetilde{\mathscr{C}}(\Delta)$, but with respect to a different orientation of Δ. Then, there exists an isomorphism of complexes $\widetilde{\mathscr{C}}(\Delta) \cong \widetilde{\mathscr{C}}(\Delta)$.*

Proof. Let $<$ and $\prec$ be two different total orders on the vertex set of V, and the augmented oriented chain complex induced by this total orders are $\widetilde{\mathscr{C}}(\Delta)$ and $\widetilde{\mathscr{C}}'(\Delta)$, respectively. Given a face of Δ, $F = \{v_0, \dots, v_i\}$, with $v_0 < \cdots < v_i$, there exists a permutation τ of the vertices of F such that

$$v_{\tau(0)} \prec v_{\tau(1)} \prec \cdots \prec v_{\tau(i)}.$$

Now, let $\psi : \widetilde{\mathscr{C}}(\Delta) \longrightarrow \widetilde{\mathscr{C}}'(\Delta)$ defined by $\psi(F) = \operatorname{sgn}(\tau)F$. By construction this map is bijective. Hence, the two oriented chain complex are isomorphic.

□

Recall that the n-th homology group is defined by

$$H_n(\Delta) = \ker(\partial_n)/\operatorname{im}(\partial_{n+1}).$$

If $\Delta \neq 0$, the simplicial complex Δ contains $\emptyset$ as a face (of dimension -1). Let $\mathscr{C}_{-1}(\Delta)$ be the free R-module with basis $\{\emptyset\}$, and define an augmentation

$$\begin{aligned} \epsilon : \mathscr{C}_0(\Delta) &\longrightarrow \mathscr{C}_{-1}(\Delta) \cong R, \qquad \text{for all } v \in V \\ v &\longmapsto \ \emptyset. \end{aligned}$$

The *augmented oriented chain complex* of Δ over R is the complex

$$\widetilde{\mathscr{C}}(\Delta):\ 0 \longrightarrow \mathscr{C}_{d-1}(\Delta) \xrightarrow{\partial_{d-1}} \mathscr{C}_{d-2}(\Delta) \xrightarrow{\partial_{d-2}} \cdots \longrightarrow \mathscr{C}_0(\Delta) \xrightarrow{\epsilon} \mathscr{C}_{-1}(\Delta) = R \longrightarrow 0,$$

and it will be denoted by dented by $(\widetilde{\mathscr{C}}(\Delta), \epsilon)$.

Let G be an abelian group. We set

$$\widetilde{H}_i(\Delta, G) = H_i(\widetilde{C}(\Delta) \otimes G), \quad \text{for } i = -1, \dots, d-1.$$

We understand $H_i(\widetilde{C}(\Delta)\otimes G)$ as the i-th homology group of associated chain complex

$$0 \longrightarrow \mathscr{C}_{d-1}\otimes_{\mathbb{Z}} G \xrightarrow{\partial_{d-1}\otimes\mathbb{1}} \mathscr{C}_{d-2}\otimes_{\mathbb{Z}} G \xrightarrow{\partial_{d-2}\otimes\mathbb{1}} \cdots \longrightarrow \mathscr{C}_0\otimes_{\mathbb{Z}} G \xrightarrow{\epsilon\otimes\mathbb{1}} \mathscr{C}_{-1}\otimes_{\mathbb{Z}} G \longrightarrow 0,$$

looking at $\mathscr{C}_i$ and R as abelian groups and $\mathscr{C}_i\otimes_{\mathbb{Z}} G$ is the tensor product of groups.

We call $\widetilde{H}_i(\Delta, G)$ the i-th *reduced simplicial homology* of Δ.

Since any linear orientation on the vertex set V of a simplicial complex Δ yields to an isomorphism of the chain complexes induced by these orientations, it follows that the reduced simplicial homology is invariant under orientations.

The i-th *reduced simplicial cohomology* of Δ with values in G is defined to be

$$\widetilde{H}^i(\Delta; G) = H^i(\mathrm{Hom}_{\mathbb{Z}}(\widetilde{\mathscr{C}}(\Delta), G)) \quad \text{for } i=-1,\dots,d-1.$$

We set $\widetilde{H}_i(\Delta)=\widetilde{H}_i(\Delta;\mathbb{Z})$ and $\widetilde{H}^i(\Delta)=\widetilde{H}^i(\Delta;\mathbb{Z})$ for all $i=-1,\dots,d-1$.

Observe that we could also compute the homology $\widetilde{H}_i(\Delta)$ of the chain $\widetilde{\mathscr{C}}(\Delta)$ and then compute $\widetilde{H}_i(\Delta;G)$ using universal coefficients theorem.

If $G=k$ is also a field, then the reduced simplicial homology and cohomology groups are k-vector spaces, and there are canonical isomorphisms:

$$\widetilde{H}^i(\Delta;k)\cong \mathrm{Hom}_k(\widetilde{H}_i(\Delta;k),k), \qquad \widetilde{H}_i(\Delta;k)\cong \mathrm{Hom}_k(\widetilde{H}^i(\Delta;k),k)$$

for all $i=-1,\dots,d-1$.

Since $\mathscr{C}_i\otimes k$ is a vector space of dimension f_i for all i, it follows that

$$\sum_{i=-1}^{d-1}(-1)^i \dim_k \widetilde{H}_i(\Delta;k) = \sum_{i=-1}^{d-1}(-1)^i f_i = \widetilde{\chi}(\Delta) = \chi(\Delta)-1.$$

We call $\widetilde{\chi}(\Delta)$ the *reduced Euler characteristic* of Δ.

A fundamental result in algebraic topology is that the reduced singular homology $\widetilde{H}_i(X;k)$ of a topological space X with triangulation Δ is isomorphic to the reduced simplicial homology of Δ, $\widetilde{H}_i(\Delta;k)$.

Theorem 4.2. *Let X be a topological space with triangulation Δ. Then*

$$\widetilde{H}_i(X;k)\cong \widetilde{H}_i(\Delta;k) \quad \textit{for all } i.$$

For more details about singular homology see [3].

Definition 4.1. A simplicial complex Δ or a topological space X is *acyclic* if its reduced homology with coefficients in $\mathbb{Z}$ vanishes in all degrees q, i.e. $\widetilde{H}_\bullet(\Delta)=0$.

The following notation will be very useful in the analysis of the local cohomology of a Stanley-Reisner ring.

Definition 4.2. Let Δ be a simplicial complex. and $F\in\Delta$. The *star* of F is the set

$$\mathrm{st}_\Delta F = \{G\in\Delta | G\cup F\in\Delta\},$$

and the *link* of F is the set

$$\mathrm{lk}_\Delta F = \{G\in\Delta | G\cup F\in\Delta, G\cap F=\emptyset\}.$$

We will write st and lk instead of st_Δ and lk_Δ to simplify the notation if is possible to avoid misunderstandings.

Let us illustrate this in an example.

Example 4.1. Let Δ be the simplicial complex associated to the hexagon in Figure 7 with vertices $\{v, a_1, a_2, \dots, v_6\}$ the 1-faces the edges of the triangulation of the hexagon and with facets (in this case all the 2-faces) are all the faces of the form $\{v, a_i, a_{i+1}\}$ for the $i = 1, \dots, 5$ and $\{v, a_6, a_1\}$.

The star of the vertex v, $\mathrm{st}_\Delta(v)$ is the full simplicial complex, while the link of v is the subsimplicial complex constituting the boundary of the hexagon:

$$\begin{aligned}\mathrm{lk}_\Delta(v) = \{&\{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}, \{a_5\}, \{a_6\}, \{a_1, a_2\},\\ &\{a_2, a_3\}, \{a_3, a_4\}, \{a_4, a_5\}, \{a_5, a_6\}, \{a_6, a_1\}\}.\end{aligned}$$

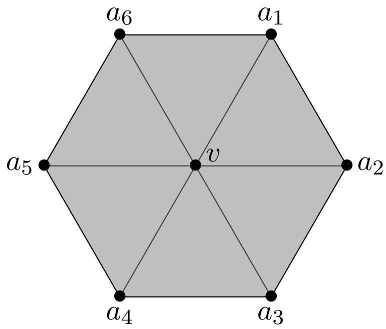

Figure 7:

Definition 4.3. Let Δ_1 and Δ_2 be two simplicial complexes on disjoint vertex sets V and W respectively. The *join*

$$\Delta_1 * \Delta_2$$

is the simplicial complex on the vertex set $V \cup W$ with faces $F \cup G$ where $F \in \Delta_1$ and $G \in \Delta_2$.

The *cone* is the join of a point $\Pi = \{v_0\}$ with Δ,

$$\mathrm{cn}\ (\Delta) = \Pi * \Delta.$$

Observe that the cone construction can be iterated. For $j > 1$ we set $\mathrm{cn}^j(\Delta) = \mathrm{cn}(\mathrm{cn}^j(\Delta))$.

Lemma 4.3. *Let F be a face of the simplicial complex Δ, and $G \in lk\ F$. Then*

(a) $F \in lk\ F$ and $F \in lk_{st\ G}F = \langle G \rangle * lk_{lk\ G}F$;

(b) $lk_{st\ G}F$ is acyclic, *if $G \neq 0$.*

Proof. (a): The first part is by definition. For the second part, let $H \in \mathrm{lk}_{\mathrm{st}\ G}F$. Now consider $H \setminus G$ and observe that $H \setminus G \in \mathrm{lk}_{\mathrm{lk}\ G}F$. Therefore we have one inclusion. The reverse inclusion is clear, since $\mathrm{lk}_{\mathrm{st}\ G}F$ contains $\langle G \rangle$ and $\mathrm{lk}_{\mathrm{lk}\ G}F$.

(b): This proof has two parts. One is to show that the join of a q-simplex with a simplicial complex Λ is a q iterated cone, $\mathrm{cn}^q(\mathrm{cn}^{q-1}(\Lambda))$. The second part is to prove that the reduced cohomology of a cone is zero in all dimensions, $\widetilde{H}_i(\mathrm{cn}\ (\Lambda)) = 0$ for all i. We omit the proof of this fact, since is completely combinatorial, see Proposition 5.2.5 [9].

□

4.2 Local Cohomology

As explained in the introduction of this section, we present two equivalent definitions of local cohomology.

Let us begin with the construction of local cohomology via injective resolutions. First, we recall some definitions.

Let R be a commutative ring and $\mathcal{M}(R)$ the category of left R-modules. Let M be any module in $\mathcal{M}(R)$. An injective resolution of M is an exact sequence of R-modules

$$E_M^\bullet : \quad 0 \to M \to E^0(M) \xrightarrow{\varphi_0} E^1(M) \xrightarrow{\varphi_1} E^2(M) \xrightarrow{\varphi_2} \cdots ,$$

where each $E^i(M)$ is an injective R-module.

We now come to the basic definition of this first construction of local cohomology.

Definition 4.4. For an ideal I in a commutative ring R and M an R-module we define the *submoduele supported on* I by

$$\Gamma_I M = (0 :_M I^\infty) = \{y \in M | I^r y = 0 \ \text{ for some } r \in \mathbb{N}\}.$$

An element of $\Gamma_I M$ is said to have support on I.

Now $\Gamma_I(\_) : \mathcal{M}(R) \longrightarrow \mathcal{M}(R)$ is a left exact additive functor between the categories of R-modules. We know that for a left exact additive function we can consider its derived functor

$$H_I^i(\_) = R^i\Gamma_I(\_).$$

Definition 4.5. Let $M \in \mathcal{M}(R)$ be an R-module. The *local cohomology* of M, denoted by $H_I^i(M)$, are the right derived functors of $\Gamma_I(E_M^\bullet)$, i.e.

$$H_I^i(M) := H^i(\Gamma_I(E_M^\bullet) \quad \text{for all } i \in \mathbb{Z}.$$

To simplify this notation we will write $\Gamma_I(M)$ or $\Gamma_I(E^\bullet)$ instead of $\Gamma_I(E_M^\bullet)$.

Another way to see this construction is that, for any injective resolution $E^\bullet$ of M, Γ_I induces an injective resolution

$$\Gamma_I(E^\bullet) : \quad 0 \to \Gamma_I(M) \to \Gamma_I(E^0) \to \Gamma(E^1) \to \cdots .$$

Then, the i-th local cohomology module of M with support on I is the module $H_I^i(M)$ obtained from any injective resolution $E^\bullet$ of M by taking the i-th cohomology of its subcomplex $\Gamma_I(E^\bullet)$ supported on I.

Remark 4.1. Let M be an R-module; then $H_I^i(M) \cong \Gamma_I(M)$, and $H_I^i(M) = 0$ for $i < 0$. So, the interesting case is for $i > 0$.

Before introducing the alternative definition of local cohomology, let us make another observation.

Let (R, m, k) be a Noetherian local ring and M an R-module. Let $\mathcal{F} = (I_k)_{k \geq 0}$ be a family of ideals of R such that $I_j \subset I_k$ for all $j > k$. Then $\mathcal{F}$ defines a topology on R.

Moreover, $\mathcal{F}$ gives the m-adic topology on R is and only if for each I_k there is a $j \in \mathbb{N}$ such that $m^j \subset I_k$, and for each m^i there is a $l \in \mathbb{N}$ such that $I_l \subset m^i$.

With this notation we arrive at the following

$$\Gamma_m(M) = \{x \in M | I_k x = 0 \text{ for all } k \geq 0\}.$$

Let $\boldsymbol{x} = x_1, \dots, x_n$ be a sequence of elements in R generating an m-primary ideal.

The family $(\boldsymbol{x}^k) = (x_1^k, \dots, x_n^k)$ gives the m-adic topology on R, and so

$$\Gamma_m = \{y \in M | (\boldsymbol{x}^k)y = 0 \text{ for some } k \geq 0\}.$$

Since $\operatorname{Hom}_R(R/I, M) = \{x \in M | Ix = 0\} = (0 :_M I)$ for any ideal I in R, we obtain natural isomorphisms

$$\Gamma_m(M) \cong \varinjlim \operatorname{Hom}_R(R/m^k, M) \cong \varinjlim \operatorname{Hom}_R(R/(\boldsymbol{x}^k), M).$$

Now $\Gamma_m(\_)$ is a left exact additive functor, so again we can consider its derived functor to obtain the cohomology functor $H^i_m(\_)$.

As expected, if $E^\bullet$ is an injective resolution of an R-module M, then

$$H^i_m(M) \cong H^i(\Gamma_m(E^\bullet)) \text{ for all } i \geq 0.$$

Therefore, for a Noetherian ring R we have a more explicit construction of the local cohomology.

Local Cohomology and the Čech Complex

After this brief introduction to local cohomology via injective resolution, we now are ready to present the alternative definition of the cohomology using the *Čech complex*. The equivalence between this two definitions is proved using the *Koszul complex*, a far more complicated construction, that we will omit.

Let Δ be a simplicial complex on the set of vertices $V = \{v_1, \dots, v_n\}$, and $R = k[X_1, \dots, X_n]/I_\Delta$ the Stanley-Reisner ring of Δ over a field k. Let $m = (x_1, \dots, x_n)$ be the maximal ideal generated by the residual classes x_i of the indeterminates X_i.

We are interested in computing the local cohomology of R_m[17], the localization of R at m.

Let $C^\bullet$ be the complex defined by

$$C^\bullet : \quad 0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots \longrightarrow C^n \longrightarrow 0,$$
$$C^t = \bigoplus_{1 \leq i_1 < \cdots < i_t \leq n} R_{x_{i_1} x_{i_2} \cdots x_{i_t}},$$

where the differentiation $d^t : C^t \longrightarrow C^{t+1}$ is given on the components

$$R_{x_{i_1} x_{i_2} \cdots x_{i_t}} \longrightarrow R_{x_{j_1} x_{j_2} \cdots x_{j_{t+1}}}$$

[17] We write R_x for the localization of R at the multiplicatively closed set closed $\{x^n\}_{n \geq 0}$, i.e. $R_x = R[x^{-1}]$.

to be the homomorphism $(-1)^{s-1} \cdot \mathrm{nat} : R_{x_{i_1}x_{i_2}\cdots x_{i_t}} \longrightarrow (R_{x_{i_1}x_{i_2}\cdots x_{i_t}})x_{j_s}$ if $\{i_1, \ldots, i_t\} = \{j_1, \ldots, \widehat{j}_s, \ldots, j_{t+1}\}$ and 0 otherwise.

The complex $C^\bullet$ is called the *Čech Complex*. The importance of this complex results from the following theorem.

Theorem 4.4. *Let M be an R-module. Then,*

$$H^i_m(M) \cong \varinjlim H^i(\boldsymbol{x}^l, M) \cong H^i(M \otimes_R C^\bullet) \quad \textit{for all } i \geq 0.$$

Proof. See [2], Theorem 3.5.6. □

Recall that for any R-module M we have $R \otimes_R M = M$, therefore, since each component of $C^\bullet$ is an R module, we can apply the previous theorem, and it follows that

$$H^i_m(R) \cong H^i(R \otimes_R C^\bullet_m) \cong H^i(C^\bullet_m) \cong H^i(C^\bullet)_m,$$

where the last equivalence follows from the fact that localization commutes with the homology group.

Now, since Supp $H^i(C^\bullet) = \{P \subset R, \text{ prime ideal } | H^i(C^\bullet)_P \neq 0\}$, implies that Supp $H^i(C^\bullet) \subset \{m\}$. Hence, we finally obtain the equivalence

$$H^i_m(R) \cong H^i(C^\bullet).$$

Remember that R is a $\mathbb{Z}^n$-graded ring. Let $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, then $R_a = \{c\boldsymbol{x}^a | c \in k\} \cong k$ if $a \in \mathbb{N}^n$ and $\{v_i \in V | a_i > 0\} \in \Delta$, otherwise we set $R_a = 0$.

Our interest is in defining a fine grading on $C^\bullet$. Since the components of C^i are of the form R_x for some homogeneous element $x \in R$, then a grading on R_x induces a grading on C^i.

One defines a $\mathbb{Z}^n$-grading on R_x by setting

$$(R_x)_a = \left\{ \frac{y}{x^m} | \ y \text{ homogeneous, deg } y - m \cdot \deg x = a \right\}.$$

Extending this $\mathbb{Z}^n$-grading on the components of C^i, the complex $C^\bullet$ becomes a $\mathbb{Z}^n$-grading. This fine grading is passed on to the local cohomology modules $H_m(R)$.

The following Lemma will also give us a $\mathbb{Z}$-grading of the complex $C^\bullet$ and the local cohomology modules.

Lemma 4.5. *Let Δ be a simplicial complex, and k a field. Let*

$$(C^i)_j = \bigotimes_{\substack{a \in \mathbb{Z}^n \\ |a| = j}} (C^i)_a.$$

Then $C^\bullet$ is a complex of $\mathbb{Z}$-graded modules. Moreover, endowing $H^i(C^\bullet)$ with the induced $\mathbb{Z}$-graded structure, then $H^i_m(k[\Delta]) \cong H^i(C^\bullet)$ and

$$H^i_m(k[\Delta])_j \cong \bigotimes_{a \in \mathbb{Z}^n, |a| = j} H^i(C^\bullet)_a \quad \textit{for all } i \textit{ and } j.$$

Proof. Let $R = k[\Delta]$ the Stanley-Reisner ring of Δ. To see that $(C^t)_j$ is a graded module is just a checking of the definition. We can write,

$$(C^t)_j = \bigoplus_{\substack{a\in\mathbb{Z}^n \\ |a|=j}} \left(\bigoplus_{1\leq i_1<\cdots<i_t\leq n} \left(R_{x_{i_1}\cdots x_{i_t}}\right)_a \right). \tag{11}$$

We can also do the same for the base ring R,

$$R_k = \bigoplus_{\substack{b\in\mathbb{Z}^n \\ |b|=k}} R_b.$$

Now, by writing the above expression in brackets in the expression (11), and using the fact that the degree of a product of monomials is the sum of the degrees of each monomial, we obtain that $R_k(C^t)_j = (C^t)_{j+k}$.

It remains to prove that the composition of two consecutive boundary maps vanishes. Since we have boundary maps, d^t, from

$$d^t : \bigotimes_{\substack{a\in\mathbb{Z}^n \\ |a|=j}} (C^t)_a \longrightarrow \bigotimes_{\substack{b\in\mathbb{Z}^n \\ |b|=j}} (C^{t+1})_b,$$

as a restriction of the boundary maps of the Čech complex. Hence, by extending these maps linearly, we obtain the vanishing condition.

For the second part of the the isomorphism $H^i_m(k[\Delta]) \cong H^i(C^\bullet)$ follows directly from Theorem 4.4.

Now, the $\mathbb{Z}^n$-graded chain complex

$$C^\bullet : \quad 0 \to \bigoplus_{a\in\mathbb{Z}^n} (C^0)_a \to \bigoplus_{a\in\mathbb{Z}^n} (C^1)_a \to \cdots \to \bigoplus_{a\in\mathbb{Z}^n} (C^n)_a \to 0,$$

and the $\mathbb{Z}$-graded complex

$$C'^\bullet : \quad 0 \to \bigoplus_{j\in\mathbb{Z}} (C^0)_j \to \bigoplus_{j\in\mathbb{Z}} (C^1)_j \to \cdots \to \bigoplus_{j\in\mathbb{Z}} (C^n)_j \to 0$$

are isomorphic. Hence, is obtained the desire isomorphism

$$H^i_m(k[\Delta])_j \cong \bigotimes_{a\in\mathbb{Z}^n, |a|=j} H^i(C^\bullet)_a \quad \text{for all } i \text{ and } j.$$

□

Now, we would like to analyse when $(R_x)_a \neq 0$ for $a \in \mathbb{Z}^n$. For this we introduce some further notations. Given $x = x_{i_1}, \ldots, x_{i_r}$ with $i_1 < \cdots < i_r$, we set $F = \{v_{i_1}, \ldots, v_{i_r}\}$ and define

$$G_a = \{v_i : a_i < 0\} \quad \text{and} \quad H_a = \{v_i : a_i > 0\}.$$

In the next two lemmas x represents a monomial as before, $x = x_{i_1}, \ldots, x_{i_r}$.

Lemma 4.6. *Let Δ be a simplicial complex, and $(R_x)_a$ as before.*

(a) The Krull dimension, $dim_k(R_x)_a \leq 1$ for all $a \in \mathbb{Z}^n$.

(b) $(R_x)_a \cong k$ if and only if $F \supset G_a$ and $F \cup H_a \in \Delta$.

Proof. Let us start with the proof of (a). We will show that if $(R_x)_a \neq 0$ then $(R_x)_a \cong k$.

Assume that $(R_x)_a \neq 0$. Let y_1/x^{n_1} and y_2/x^{n_2} be non-zero elements in $(R_x)_a$. Then $x^{n_2}y_1$ and $x^{n_1}y_2$ are homogeneous of the same degree. Since $(R_x)_a$ is a vector space, these two elements are linearly dependent over k. Then, there exists $c \in k$ such that $c(x^{n_2}y_1) = x^{n_1}y_2$. Therefore $c(y_1/x_1^{n_1}) = y_2/x^{n_2}$, so $(R_x)_a$ has dimension at most 1 over k as a vector space. Therefore $(R_x)_a \cong k$, as we desire.

For the proof of (b), first we will study the case when $(R_x)_a \neq 0$. Hence, part (a) implies that $(R_x)_a$ automatically is isomorph to k.

Now, observe that $(R_x)_a \neq 0$ if and only if $\exists v \in R$, homogeneous, and $l \in \mathbb{Z}$ such that

(i) $x^m v \neq 0 \in (R_x)$ for all $m \in \mathbb{N} \setminus \{0\}$, and

(ii) $\deg v/x^l = a$.

Under condition (ii), condition (i) is equivalent to (i') $v/x^l \neq 0 \in (R_x)_a$

From (i) we have that $x^m v \neq 0$ if and only if Supp $x^m v \in \Delta$. Now since, $m > 0$, Supp x^m = Supp $m = \{v_i | m > 0\} = F$, then the support of $x^m v$ is $F \cup$ Supp v. Moreover, Supp $a = G_a \cup H_a$ and the condition (ii) implies that Supp $(\deg v - \deg x^l) = G_a \cup H_a$, then $G_a \subset F$ and $H_a \subset$ Supp v. In particular, $F \cup H_a \in \Delta$. Hence, if $(R_x)_a \neq 0$ then $F \supset G_a$ and $F \cup H_a \in \Delta$.

Conversely, suppose $F \supset G_a$ and $F \cup H_a \in \Delta$. Set $v = \prod_{a_i > 0} x_i^{a_i}$ and $w = \prod_{a_i < 0} x_i^{-a_i}$. Since $G \subset F$, there exists an integer $l \in \mathbb{Z}$ and a monomial u with non-negative exponents in x_i, such that $x^l = uw$.

Since $F \cup H_a \in \Delta$, we get $\dfrac{vu}{x^l} \neq 0$, and it follows that

$$\deg \frac{vu}{x^l} = a,$$

so $(R_x)_a \neq 0$.

□

From the previous lemma we can deduce a basis for $(C^i)_a$. This follows from the fact that $\dim_k(R_x)_a \leq 1$ for all $a \in \mathbb{Z}^n$, and the equality occurs when $F \supset G$ and $F \cup H_a \in \Delta$, then let b_F be the k-base of $(R_x)_a$ when it has dimension 1. Hence, the basis of $(C^i)_a$ is formed by the set

$$\{b_F : \ F \supset G_a, \ F \cup H_a \in \Delta, \ |F| = i\}.$$

In order to prove Hochster's theorem we need one more lemma.

Lemma 4.7. *For all $a \in \mathbb{Z}^n$ there exists an isomorphism of complexes:*

$$\varphi^i : (C^i)_a \longrightarrow Hom_{\mathbb{Z}}(\widetilde{\mathscr{C}}(lk_{st\ H_a} G_a)_{i-|G_a|-1}, k).$$

Here we set $lk_{st\ H_a} G_a = \emptyset$ if $G_a \notin st\ H_a$, and let $\widetilde{\mathscr{C}}(\emptyset)$ be the zero complex.

Proof. Let $j = |G_a|$. Consider the sets

$$\mathscr{B} = \{F \in \Delta |\ F \supset G_a, F \cup H_a \in \Delta, |F| = i\}$$
$$\mathscr{B}' = \{F' \in \Delta |\ F' \in \mathrm{lk}_{\mathrm{st}\ H_a} G_a, |F'| = i - j\}.$$

Observe that $\mathscr{B}$ is a basis of the complex $(C^i)_a$, and $\mathscr{B}'$ is a basis of $\widetilde{\mathscr{C}}(\mathrm{lk}_{\mathrm{st}\ H_a} G_a)_{i-|G_a|-1}$.

We have a bijection between $\mathscr{B}$ and $\mathscr{B}'$ given by

$$\begin{aligned} \alpha^i : \Delta &\longrightarrow \Delta \\ F &\longmapsto F' = F \setminus G_a. \end{aligned}$$

The injectivity is clear because if $\alpha^i(F_1) = \alpha^i(F_2)$ then $F_1 \setminus G_a = F_2 \setminus G_a$, and this implies $F_a = F_2$.

For the exhaustivity take $F' \in \mathscr{B}'$, and consider $F = F' \cup G_a$. The condition $F' \in \mathrm{lk}_{\mathrm{st}\ H_a} G_a$ ensure us that $F \cup H_a \in \Delta$. Therefore, α^i is bijective for all i.

Now, α^i can be extended linearly to an isomorphism φ^i between vector spaces

$$\begin{aligned} \varphi^i : (C^i)_a &\longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\widetilde{\mathscr{C}}(\mathrm{lk}_{\mathrm{st}\ H_a} G_a)_{i-|G_a|-1}, k) \\ b_F &\longmapsto \psi_{F \setminus G_a}, \end{aligned}$$

where $\psi_{F'}$ is defined by

$$\psi_{F'}(F'') = \begin{cases} 1 & \text{if } F' = F'', \\ 0 & \text{otherwise.} \end{cases}$$

To ensure that α^i is a complex homomorphism we can adjust the orientation of Δ in such a way that elements in G_a are last in the linear order of the vertices set of Δ. Furthermore, we give the subcomplex $\mathrm{lk}_{\mathrm{st}\ H_a} G_a$ the induced orientation.

□

Now, we are ready to prove the main result of this section.

Theorem 4.8. *(Hochster). Let Δ be a simplicial complex, and k a field. Then the Hilbert series of the local cohomology modules of $k[\Delta]$ with respect to the fine grading is given by*

$$H_{H^i_m(k[\Delta])}(\boldsymbol{t}) = \sum_{F \in \Delta} dim_k \widetilde{H}_{i-|F|-1}(lk\ F; k) \prod_{j: v_j \in F} \frac{t_j^{-1}}{1 - t_j^{-1}}.$$

Proof. Let Δ be a simplicial complex, and k a field. Now the isomorphism φ^i in the previous Lemma induces an isomorphism

$$H^i_m(k[\Delta])_a \cong \widetilde{H}^{i-|G_a|-1}(\mathrm{lk}_{\mathrm{st}\ H_a} G_a; k). \tag{12}$$

Therefore,

$$\dim_k H^i_m(k[\Delta])_a = \dim_k \widetilde{H}^{i-|G_a|-1}(\mathrm{lk}_{\mathrm{st}\ H_a} G_a; k),$$

since $\dim_k \widetilde{H}_{i-|G_a|-1}(\mathrm{lk}_{\mathrm{st}\ H_a} G_a; k) = \dim_k \widetilde{H}^{i-|G_a|-1}(\mathrm{lk}_{\mathrm{st}\ H_a} G_a; k)$.

If $H_a \neq 0$, then by Lemma 4.3 $\mathrm{lk}_{\mathrm{st}\ H_a} G_a$ is acyclic, and if $H_a = \emptyset$, then $\mathrm{st}\ H_a = \Delta$, and so $\mathrm{lk}_{\mathrm{st}\ H_a} G_a = \mathrm{lk}_\Delta G_a$.

Let $\mathbb{Z}^n_- = \{a \in \mathbb{Z}^n | a_i \leq 0 \text{ for } i = 1, \ldots, n\}$, then $H_a = \emptyset$ if and only if $a \in \mathbb{Z}^n_-$ and it follows that

$$\begin{aligned}
H_{H^i_m(k[\Delta])}(\boldsymbol{t}) &= \sum_{a \in \mathbb{Z}^n} \dim_k H^i(k[\Delta])_a \boldsymbol{t}^a = \sum_{a \leq 0} \dim_k H^i(k[\Delta])_a \boldsymbol{t}^a \\
&= \sum_{F \in \Delta} \left(\sum_{\substack{a \leq 0 \\ \mathrm{Supp}(a) = F}} \dim_k \widetilde{H}_{i-1-|G_a|}(\mathrm{lk}_\Delta G_a; k) t^a \right) \\
&= \sum_{F \in \Delta} \left(\dim_k \widetilde{H}_{i-1-|G_a|}(\mathrm{lk}_\Delta G_a; k) \cdot \left(\sum_{\substack{a \leq 0 \\ \mathrm{Supp}(a) = F}} \boldsymbol{t}^a \right) \right).
\end{aligned}$$

Now, since the sum on the right is a geometric one, we can proceed in the same way as in the computation of the Hilbert series of the face ring (see Theorem 3.7). Hence, the Hochster's formula for the Hilbert series is

$$H_{H^i_m(k[\Delta])}(\boldsymbol{t}) = \sum_{F \in \Delta} \dim_k \widetilde{H}_{i-|F|-1}(\mathrm{lk}\ F; k) \prod_{j: v_j \in F} \frac{t_j^{-1}}{1 - t_j^{-1}}.$$

□

As we explained in the introduction of this section, Hochster's theorem yields to the Reisner criterion.

Theorem 4.9. *(Reisner). Let Δ be a simplicial complex, and let k be a field. The following conditions are equivalent:*

(a) Δ is Cohen-Macaulay over k;

(b) $\widetilde{H}_i(lk_\Delta F; k) = 0$ for all $F \in \Delta$ and all $i <$ dim $lk_\Delta F$.

Before starting the Reisner Theorem proof we need one more result.

Theorem 4.10. *(Grothendieck). Let R be a finitely generated k-algebra. Then $H^i(R)$ is equal to zero when $i <$ depth R and when $i >$ dim (R), while $H^i(R)$ is not zero for $i =$ depth R and $i =$ dim R.*

Proof. See Theorem 3.5.6 [2]. □

Proof. (Reisner's Theorem)

Let us start with the implication $(b) \Rightarrow (a)$.

Let Δ be a $(d-1)$-dimensional simplicial complex. Then for $F \in \Delta$ and $i < \dim \mathrm{lk}_\Delta F$ we have $\widetilde{H}_i(\mathrm{lk}_\Delta F; k) = 0$ by hypothesis.

First, we will show that such a simplicial complex is pure as all its facets are of the same dimension.

Assume that Δ is not pure. Let K_1 be the sub-complex of Δ consisting of all the maximal dimension faces of Δ and all sub-faces of these faces. Let K_2 be the sub-complex of Δ consisting of all the maximal faces of Δ that do not have maximal dimension, as well as all the sub-faces of these faces.

Since Δ is not pure, both $K_1 \setminus K_2$ and $K_2 \setminus K_1$ are non-empty. Moreover, $\{$facets of $K_1\} \cap \{$ facets of $K_2\} = \emptyset$ and $\{$facets of $K_1\} \cup \{$facets of $K_2\} = \{$facets of $\Delta\}$, actually this is a disjoint union since the intersection is empty.

Let G be a maximal face of $K_1 \cap K_2$ (if $K_1 \cap K_2 = \emptyset$, then $G = \emptyset$). Now consider $\mathrm{lk}_\Delta(G)$. The link is formed by faces that are in facets or the hole facet. Therefore, since the facets of K_1 and K_2 are disjoint, we can write $\mathrm{lk}_\Delta(G)$ as a disjoint union $\mathrm{lk}_{K_1}(G) \sqcup \mathrm{lk}_{K_2}(G)$.

Now, the dimension of $\mathrm{lk}_\Delta(G)$ is the dimension of the maximal face of the link in the maximal dimensional face of Δ, so

$$\mathrm{lk}_\Delta(G) = \dim \mathrm{lk}_{K_1} = \dim K_1 - \dim G - 1 = d - 1 - |G| + 1 - 1 = d - |G| - 1.$$

On the other hand, any face in G is contained in some maximal face of non-maximal dimension, therefore $|G| \leq d-2$. Hence, we obtain that $\dim \mathrm{lk}_\Delta(G) \geq 1$, and since the link can be decomposed in a disjoint union (3.5) implies that lk G is disconnected. But this is a contradiction with our assumption regarding Δ, $\widetilde{H}_0(\mathrm{lk}_\Delta(G); k) = 0$, and since

$$H_0(\mathrm{lk}_\Delta(G); k) \cong \widetilde{H}_0(\mathrm{lk}_\Delta(G); k) \otimes \mathbb{Z} = \mathbb{Z}$$

implies that $\mathrm{lk}_\Delta(G)$ is connected.

Hence, Δ must be pure, and we know that $\dim \mathrm{lk}_\Delta = d - |F| - 1$, so for $i < d$ we have $\dim\ {}_k\widetilde{H}_{i-|F|-1}(\mathrm{lk}_\Delta(F); k) = 0$. By applying Hochster's theorem, we obtain that $H^i(k[\Delta], \boldsymbol{t}) = 0$ for $i < d$. Then, by Grothendieck's theorem we know that the depth of $k[\Delta]$ is at least d.

Now, the Krull dimension of $k[\Delta]$ is d and since depth $k[\Delta] \leq$ depth $k[\Delta] = d$ implies that Δ is Cohen-Macaulay.

For the reverse implication we use an analogue argument. Assume that Δ is Cohen-Macaulay, then Δ is pure, all of its facets have maximal dimension. Moreover, we know that for a pure simplicial complex, $\dim\ \mathrm{lk}_\Delta F = d - |F| - 1$ for all $F \in \Delta$. Hence, for $i < d$ we have the strict inequality $i - |F| - 1 < \dim\ (\mathrm{lk}_\Delta(F)$.

By applying Grothendieck's theorem we obtain that $H^i(k[\Delta]) = 0$ for all $i < d$, and Hochster's theorem implies that $\dim_k \widetilde{H}_{i-1-|F|}(\mathrm{lk}_\Delta(F); k) = 0$ for all $i < d$. Hence $\widetilde{H}_j(\mathrm{lk}_\Delta(F); k) = 0$ for $j < \dim_k(\mathrm{lk}_\Delta F)$. □

5 The Upper Bound Theorem

Now we are on the final stretch on the proof of upper bound conjecture. We will separate it into two steps.

One is to prove that any simplicial sphere is an *Euler* complex. We will show this fact using a characterization of the Cohen-Macaulay complex in terms of the *relative singular homology* and a use of Reisner's criterion.

Secondly, we will show that the Euler complexes satisfy the Dehn-Sommerville equations.

Recall that the n-th singular homology group of a pair (X, Y), where X is a topological space and Y a subspace of X, is the n-th homology group of the chain complex $C_\bullet(X, y) = C_\bullet(X)/C_\bullet(Y)$. We will denote by $H_\bullet(X, Y; k)$ the relative singular homology group.

Let us introduce first the concept of Euler complex.

Definition 5.1. The simplicial complex Δ is an *Euler complex* if Δ is pure, and the reduced Euler characteristic $\widetilde{\chi}(\text{lk } F) = (-1)^{\dim \text{lk } F}$ for all $F \in \Delta$.

Now, for any Euler complex Δ and any face $F \in \Delta$ with $\dim \text{lk } F = l$, it follows from the Reisner criterion that if Δ is a Cohen-Macaulay complex then $\widetilde{H}_l(\text{lk } F; k) \cong k$ and 0 otherwise. But the opposite also holds. From this we observe the following.

Remark 5.1. Let Δ be an Euler complex, and k a field. Then Δ is Cohen-Macaulay over k if and only if for all $F \in \Delta$

$$\widetilde{H}_i(\text{lk } F; k) \cong \begin{cases} k & \text{for } i = \dim \text{lk } F, \\ 0 & \text{otherwise.} \end{cases}$$

In the next lemma we will connect the relative homology group with the reduced homology group of a a simplex. We shall omit the proof of this lemma since is a complete topological result.

Lemma 5.1. *Let Δ be a simplicial complex on the vertex set V, and k be a field. Suppose that X is the geometric realization of Δ, and that $F \in \Delta$ is a face of dimension j, and $p \in |F|$. If lk $F \neq \emptyset$, then*

$$H_i(X, X \setminus \{p\};; k) \cong \widetilde{H}_{i-j-1}(lk\ F; k) \quad \textit{for all } i,$$

and if lk $F = \emptyset$, then

$$H_i(X, X \setminus \{p\}; k) \cong \begin{cases} k & \textit{for } i = j, \\ 0 & \textit{otherwise.} \end{cases}$$

This lemma together with Reisner's criterion allow us to prove an equivalence between the Cohen-Macaulay property of Δ and the vanishing property of the relative homology group of $|\Delta|$.

Theorem 5.2. *(Munkres, Stanley). Let Δ be a $(d-1)$-dimensional simplicial complex. Let $X = |\Delta|$ be a geometric realization of Δ, and k a field. The following conditions are equivalent:*

(a) Δ is Cohen-Macaulay over k;

(b) for all $p \in X$ and all $i < \dim X$ one has

$$\widetilde{H}_i(X; k) = H_i(X, X \setminus \{p\}; k) = 0.$$

Moreover, if the equivalent conditions are satisfied, then Δ is an Euler complex if and only if

$$\widetilde{H}_{d-1}(X; k) \cong H_{d-1}(X, X \setminus \{p\}; k) \cong k \quad \text{for all } p \in X.$$

Proof. First, observe that $\mathrm{lk}\ F = \emptyset$ if and only if F is a facet, since a facet is not contained in any other face of Δ.

Now, by the assumption (b) combined with Lemma 5.1 we obtain that $H_i(X, X \setminus \{p\}; k) = 0$ if $i < \dim X$, and this is different from zero for $i = \dim F$. This can happen if and only if all the facets have maximal dimension, $d-1$. Then Δ is pure.

Now, assume that $\mathrm{lk}\ F \neq \emptyset$. Recall that for a pure simplicial complex Δ, we have $\dim \mathrm{lk}\ F = d-2-\dim F = d-2-j$.Hence, if $i < \dim \mathrm{lk}\ F$, then $i+j+1 < d-1$ and so

$$\widetilde{H}_i(\mathrm{lk}\ F; k) \cong H_{i+j+1}(X, X \setminus \{p\}; k) = 0.$$

Here, we apply assumption (b) and we change the role of i with $i-j-$ in the Lemma 5.1.

Therefore, by applying again assumption (b) it holds that $\widetilde{H}_i(\mathrm{lk}\ F; k) \cong \widetilde{H}_i(X; k) = 0$ for $i < \dim \mathrm{lk}\ F$.

Then, by Reisner's criterion it follows that Δ is Cohen-Macaulay over k.

For the reciprocal implication, $(a) \Rightarrow (b)$, assume that $\mathrm{lk}\ F = \emptyset$. Since Δ is Cohen-Macaulay by Reisner's criterion $\widetilde{H}_i(\mathrm{lk}\ F; k) = 0$ for $i < \dim \mathrm{lk}\ F$. By repeating the previous argument, and by applying Lemma 5.1 we conclude that Δ is pure. Then, all the implications in the previous argument can be reversed. Hence, the equivalent conditions are proved.

For the second part of the Theorem, observe that we only have to check $\widetilde{H}_{d-1}(X; k) = k$. By the Remark 5.1 we get the result.

□

Now, computing the relative singular homology of the sphere (a good reference work about how to compute this homology group is *Algebraic Topology*, see [3]) we obtain

$$H_i(S^{d-1}, S^{d-1} \setminus \{p\}; k) \cong \begin{cases} k & \text{for } i = d-1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, S^{d-1} is an Euler complex.

Finally we will prove the symmetry of the h-vector of a sphere. In a more general way:

Theorem 5.3. *(Dehn-Sommerville,Klee). Let Δ be an Euler complex of dimension $d-1$ with h-vector $(h_0, \ldots, h_d)$. Then $h_i = h_{d-i}$ for $i = 0, \ldots, d$.*

In order to prove this result, we shall see first a technical Lemma related to the Hilbert series of $k[\Delta]$.

Lemma 5.4. *Let Δ be a simplicial complex on the set of vertices $V = \{v_1, \ldots, v_n\}$. Then*

$$H_{k[\Delta]}(t_1^{-1}, \ldots, t_n^{-1}) = \sum_{F\in\Delta} (-1)^{\dim F} \widetilde{\chi}(lk\ F) \prod_{i:v_i\in F} \frac{t_i}{1-t_i}.$$

Proof. Observe that we can write the equation (2) from the Theorem 3.7 in the following way:

$$H_{k[\Delta]}(\boldsymbol{t}) = \sum_{F\in\Delta} \prod_{i:v_i\in F} \frac{t_i}{i-t_i}.$$

Now, if we change the variables by replacing $t_i \mapsto t_i^{-1}$ and also grouping the term in the previous product by the sub-faces of F we get

$$\begin{aligned} H_{k[\Delta]}(t_1^{-1}, \ldots, t_n^{-1}) = \sum_{F\in\Delta} \prod_{i:v_i\in F} \frac{1}{t_i - 1} &= \sum_{F\in\Delta} \prod_{i:v_i\in F} (-1)^{\dim F+1} \left(1 + \frac{t_i}{1-t_i}\right) \\ &= \sum_{F\in\Delta} (-1)^{\dim F+1} \sum_{G\subset F} \prod_{i:v_i\in G} \frac{t_i}{1-t_i} \\ &= \sum_{G\in\Delta} \left(\sum_{\substack{F\in\Delta \\ G\subset F}} (-1)^{\dim F+1} \right) \prod_{i:v_i\in G} \frac{t_i}{1-t_i}, \end{aligned}$$

where in the last equality we just change the order of the sums.

Finally we can express the sum in the brackets, on the above equation, in terms of the reduced Euler characteristic,

$$\sum_{F\in\Delta, G\subset F} (-1)^{\dim F+1} = \sum_{F\in \mathrm{lk}\ G} (-1)^{\dim F - \dim G} = (-1)^{\dim G} \widetilde{\chi}(\mathrm{lk}\ G).$$

Therefore, the Hilbert series has the desired expression.

□

Proof. (Dehn-Sommervile-Klee Theorem). Let Δ be an Euler complex, then $\widetilde{\chi}(\mathrm{lk}\ F) = (-1)^{\dim \mathrm{lk}\ F}$. Since Δ is pure, we also have $\widetilde{\chi}(\mathrm{lk}\ F) = (-1)^{d-\dim F}$.

By the previous Lemma 5.4, the Hilbert series for an Euler complex holds that

$$H_{k[\Delta]}(t_1, \ldots, t_n) = (-1)^d H_{k[\Delta]}(t_1^{-1}, \ldots, t_n^{-1}).$$

Replacing t_i by t, we get the identity $H_{k[\Delta]}(t) = (-1)^d H_{k[\Delta]}(t^{-1})$. Now, from Theorem 2.14 we can express these Hilbert series as a quotient

$$H_{k[\Delta]}(t) = \frac{Q_{k[\Delta]}(t)}{(1-t)^d}.$$

Therefore, $Q_{k[\Delta]}(t) = t^d Q_{k[\Delta]}(t^{-1})$, and by comparing coefficients on both sides we obtain the Dehn-Somerville equations.

$\square$

Finally we are ready to prove the upper bound theorem.

Theorem 5.5. *(The upper bound theorem for simplicial spheres). Let Δ be a simplicial complex with n vertices and $|\Delta| \cong S^{d-1}$. Then*

$$f_i(\Delta) \leq f_i(C(n,d)) \quad \textit{for all } i = 1, \ldots, d-1.$$

Proof. Let Δ be a simplicial complex on the set of vertices $V = \{v_1, \ldots, v_n\}$, such that $|\Delta| \cong S^{d-1}$. Let $C(n, d-1)$ be a cyclic polytope of dimension $d-1$.

If i is such that $\lfloor \frac{d}{2} \rfloor < i \leq d$, then $0 \leq d-i \leq \lfloor \frac{d}{2} \rfloor$, therefore

$$\begin{aligned} h_i(\Delta) &\overset{(a)}{=} h_{d-i}(\Delta) \overset{(b)}{\leq} \binom{n-d+d-i+i-1}{d-i} \\ &= \binom{n-1}{d-i} \overset{(c)}{=} h_{d-i}(C(n,d)) \overset{(d)}{=} h_i(C(n,d)). \end{aligned}$$

(a) Follows from Dehn-Sommerville-Klee equations, Theorem 5.

(b) Follows from Theorem 3.11.

(c) Follows from Corollary 3.23.

(d) Follows from Dehn-Sommerville equations for simplicial polytopes.

From this we get that $h_i(\Delta) \leq h_i(C(n,d))$ for $0 \leq i \leq d+1$. Therefore, we can apply Proposition 3.9 for the expression of f_{k-1} in terms of h_i to obtain the desired bound.

$$f_{k-1}(\Delta) = \sum_{i=1}^{d} \binom{d-i}{k-i} h_i(\Delta) \leq \sum_{i=0}^{d} \binom{d-i}{k-i} h_i(C(n,d)) = f_{k-1}(C(n,d))$$

for $1 \leq k \leq d$.

So, the Upper Bound Theorem is proved.

$\square$

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