Treball final de grau

GRAU DE MATEMÀTIQUES

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LIE GROUPS AND ALGEBRAS IN PARTICLE PHYSICS

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Abstract

The present document is a first introduction to the Theory of Lie Groups and Lie Algebras and their representations. Lie Groups verify the characteristics of both a group and a smooth manifold structure. They arise from the need to study continuous symmetries, which is exactly what is needed for some branches of modern Theoretical Physics and in particular for quantum mechanics.

The main objectives of this work are the following. First of all, to introduce the notion of a matrix Lie Group and see some examples, which will lead us to the general notion of Lie Group. From there, we will define the exponential map, which is the link to the notion of Lie Algebras. Every matrix Lie Group comes attached somehow to its Lie Algebra. Next we will introduce some notions of Representation Theory. Using the detailed examples of $\mathsf{SU}(2)$ and $\mathsf{SU}(3)$, we will study how the irreducible representations of certain types of Lie Groups are constructed through their Lie Algebras. Finally, we will state a general classification for the irreducible representations of the complex semisimple Lie Algebras.

Resum

Aquest treball és una primera introducció a la teoria dels Grups i Àlgebres de Lie i a les seves representacions. Els Grups de Lie són a la vegada un grup i una varietat diferenciable. Van sorgir de la necessitat d'estudiar la simetria d'estructures continues, i per aquesta raó tenen un paper molt important en la física teòrica i en particular en la mecànica quàntica.

Els objectius principals d'aquest treball són els següents. Primer de tot, presentar la noció de Grups de Lie de matrius i veure alguns exemples que ens portaran a definir els Grups de Lie de forma general. A continuació, definirem les Àlgebres de Lie, que es relacionen amb els Grups de Lie mitjançant l'aplicació exponencial. Presentarem algunes nocions de teoria de Representacions i, a través de l'explicació detallada dels exemples $\mathsf{SU}(2)$ i $\mathsf{SU}(3)$, veurem com les representacions irreductibles d'alguns tipus de Grups de Lie es construeixen a través de les seves Àlgebres de Lie. Finalment, exposarem una classificació general de les representacions irreductibles de les Àlgebres de Lie complexes semisimples.

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Contents

1 Introduction

This work is a first introduction to Lie Groups, Lie Algebras and Representation Theory. The objectives of this work are to present an organized and complete summary of the basic notions of Lie Groups and Lie Algebras, to see how they are connected to Representation Theory and to mention the applications that this structures have in Physics.

Lie Groups arose from the need to describe transformations on solutions of partial differential equations. They have the properties of both a group and a differential manifold. Thus, when working with Lie Groups, the group operations are compatible with the smooth structure. That makes them suitable to study and describe continuous symmetry of mathematical objects and structures, which is exactly what is needed for some branches of modern Theoretical Physics, and in particular for quantum mechanics.

In this work we deal with Lie Groups by using algebraic notions instead of differential manifolds. This is the reason why we start by introducing the matrix Lie Groups, for which we define the topological notions associated to Lie Groups in a very simple way. Note that not all Lie Groups are matrix Lie Groups but in this work we will only deal with the classical matrix Lie Groups. However, a general definition for Lie Groups is also given.

The notion of a Lie Algebra comes somehow associated to the idea of Lie Groups. Since the structure of Lie Algebras deals with vector spaces, it is usually more suitable to work with them. The link between Lie Groups and Lie Algebras is the exponential map.

To describe the action of Lie Groups and Lie Algebras on vector spaces we need to study their representations, which play a very important role in Physics. The structure of a physical system is described by its symmetries. Mathematically, symmetries are transformations under which the properties of the system remain invariant. By studying the effect of a set of transformations we can obtain the conservation laws of a given system and, in particular, Lie Groups are used to describe continuous transformations.

In quantum mechanics, each system is denoted by its state, which is a vector in a Hilbert space. A Hilbert space is generalization of the concept of Euclidean space. A state ideally contains all the information about the system. Nevertheless, we usually focus on a specific property of the state of the system and we consider only the Hilbert space associated to it. For instance, we could study spin or flavour of a given particle, which sit on finite-dimensional Hilbert Spaces. That makes them much more easier to study than position or momentum, which are represented on the infinite Hilbert Space of the $\mathsf{L}^2(\mathbb{R})$ functions.

Let us consider a system composed by the spin of one particle. Its state can be a combination of several eigenstates of spin. Then the particle has no definite spin, but there is a probability associated to each of the eigenstates in which the particle could collapse when we do a measurement. Hence, in quantum mechanics we work with probabilities, which are computed through the inner product of states. Since probabilities must be conserved, we are interested in the effect of unitary Lie Groups on the states of the Hilbert Space. In particular, we are interested in studying all the representations of the Lie Groups $\mathsf{SU}(2)$ and $\mathsf{SU}(3)$ and their associated Lie Algebras. The Lie Group $\mathsf{SU}(2)$ is related to the the rotational invariance and hence describes the spin of a particle, while the Lie Group $\mathsf{SU}(3)$ refers to the flavour and colour symmetry of quarks, which compose the hadrons.

We will find all the representations of $\mathsf{SU}(3)$ and $\mathsf{SU}(2)$ through the construction of a classification for the irreducible representations of their Lie Algebras. Nevertheless, the representations of a Lie Algebra are not always related one to one to the representations of its Lie Group. This is the case of $\mathsf{SO}(3)$, whose Universal Cover is $\mathsf{SU}(2)$, and as a consequence the spin of fermions (which are particles with semi-integer spin) are not directly related to rotations in a three dimensional space.

As a final general result, we will classify all the irreducible representations of complex semisimple Lie Algebras in terms of their highest weight.

Structure of the memory

This memory is structured in two parts. The first one is about Lie Groups and Lie Algebras. In Sections 2 and 3 we will introduce **matrix Lie Groups**, present the most important examples and introduce the notion of **general Lie Group**. In Section 4 we will review the definition of the **exponential** and the **logarithm** of a matrix, which will be used to introduce the notion of **Lie Algebra of a matrix Lie Group**. Again, we will present some examples, associated to the matrix Lie Groups from Section 2, and then we are going to get a further insight into the notions of **exponential map** and **general Lie Algebras**. The second part is related to representations of Lie Groups and Lie Algebras. In Section 5 we first introduce the notions of **representations** and **complexification of Lie Algebras**. Then we present three examples of the basic representations of a matrix Lie Group or its Lie Algebra and we focus on constructing all the irreducible representations of $\mathsf{SU}(2)$ and $\mathsf{su}(2)$. In Section 5.5, we focus on how to generate new representations through **direct sums** and **tensor products**. In particular, we are interested in the idea of a **completely reducible representation** and the application of tensor products in particle Physics. In Section 5.6 we prove **Schur's Lemma**, which is an important result regarding representations of Lie Groups and Lie Algebras and it will be essential to prove the main Theorems that follow. The next step is to see the relation between Lie Groups and their Lie Algebras representations. First we deal with connected and simple connected Lie Groups in Section 6, for which there is a representation of the Lie Group associated to each representation of the Lie Algebra. Then, we generalize this notion in Section 6.1 by defining the **Universal Covering** of a Lie Group and in particular we work on the example of $\mathsf{SO}(3)$. By constructing the irreducible representations of the Lie Algebra of $\mathsf{SU}(3)$ we will introduce the notions of **roots** and **weights**, which will lead us to the classification of the irreducible representations by their **highest weight**. Finally, in Section 8 we are going to generalize this result for all **complex semisimple Lie Algebras**.

My first approach to theory of Lie Groups and Lie Algebras was through the book [4], from which I grasped an idea of how Lie Groups are used in quantum mechanics. Nevertheless, the structure of the project is mainly based on the book [7], since it has been a much more rigorous and complete guide to understand the basics of Lie Groups. Other sources like [3] or [5] have been used from time to time to complement some ideas or results.

Notation

Throughout this work we refer to the following notation. $M_{n\times n}(\mathbb{C})$ stands for all the $n \times n$ matrices with complex coefficients. We will write $I \in M_{n\times n}$ to denote the identity matrix. $\mathsf{GL}(n;\mathbb{C})$ stands for the complex general linear group, which is the group of all $n \times n$ invertible matrices A with complex coefficients together with the operation of matrix multiplication. We will write A^{Tr} to denote the transpose of A and A^* to denote the conjugate transpose of A. Finally, the norm of a vector $u = (u_1, ..., u_n)$ will be $\|u\| = \sqrt{(u_1)^2 + ... + (u_n)^2}$.

2 Matrix Lie Groups

2.1 Definition and main properties

The general linear group is the group of all $n \times n$ invertible matrices. We can consider the elements of the general linear group to have real entries $\mathsf{GL}(n;\mathbb{R})$ or complex entries $\mathsf{GL}(n;\mathbb{C})$. It holds that $\mathsf{GL}(n;\mathbb{R}) \subset \mathsf{GL}(n;\mathbb{C})$. From now on we will consider the general case of complex entries.

Definition 2.1.1. A **matrix Lie Group** G is any subgroup of $\mathsf{GL}(n;\mathbb{C})$ such that for any sequence of matrices $\{A_m\}_{m\in\mathbb{N}}$ in G that converges to some matrix A, i.e. $(A_m)_{ij}$ converges to $A_{ij}\ \forall\ 1 \leqslant i,j \leqslant n$, either $A \in G$ or A is not invertible.
This condition is equivalent to say that G is a closed subgroup of $\mathsf{GL}(n;\mathbb{C})$.

We will now define the most important topological properties of matrix Lie Groups.

Definition 2.1.2. A matrix Lie Group G is said to be **compact** if the following conditions are satisfied:

1. Any convergent sequence $\{A_m\}_{m\in\mathbb{N}}$ of elements of G converges to a matrix $A \in G$.
2. There exists a constant C such that for all $A \in G, |A_{ij}| \leq C, \forall\ 1 \leq i,j \leq n$.

Remark. The definition above is equivalent to the notion of topological compactness if we picture the set of $n\times n$ complex matrices as $\mathbb{C}^{n^2}$, in which a compact Lie Group will be a closed bounded subset. In fact, from now on we will define the topology in the set of $n \times n$ complex (real) matrices to be the standard topology in $\mathbb{C}^{n^2}$ ($\mathbb{R}^{n^2}$).

Definition 2.1.3. A matrix Lie Group G is said to be **connected** if given any two matrices A and B in G, there exists a continuous path $\gamma : [a,b] \longrightarrow M_{n\times n}(\mathbb{C})$ lying in G such that $\gamma(a) = A$ and $\gamma(b) = B$.

Remark. The definition above is equivalent to the topological notion of **path-connected**. Recall that every path-connected space is connected. The inverse is not true in general but it will be for matrix Lie Groups.
A non connected or disconnected matrix Lie Group can be represented as the union of two or more disjoint non-empty connected components.

Proposition 2.1.4. *If G is a matrix Lie Group, then the connected component of G containing the identity is a subgroup of G.*

Proof. Let A and B be in the connected component containing the identity. Then, there exist two continuous paths $A(t)$ and $B(t)$ with $A(0) = B(0) = I$, $A(1) = A$ and $B(1) = B$. The path $A(t)B(t)$ is a continuous path starting at I and ending at AB. The path $A(t)^{-1}$ is a continuous path starting at I and ending at A^{-1}. Thus the identity component is a subgroup. $\square$

Definition 2.1.5. A matrix Lie Group G is said to be **simply connected** if it is connected and given any continuous path $A(t)$, $0 \leq t \leq 1$, lying in G with $A(0) = A(1)$, there exists a continuous function $A(s,t)$, $0 \leq s,t \leq 1$, taking values in G which has the following properties:

(1) $A(s,0) = A(s,1)$, for all s such that $0 \leq s \leq 1$.

(2) $A(0,t) = A(t)$, for all t such that $0 \leq t \leq 1$.

(3) $A(1,t) = A(1,0)$, for all t such that $0 \leq t \leq 1$.

Remark. This is equivalent, in topological terms, to the fundamental group of G being trivial.

Finally, we are going to introduce the notion of maps between matrix Lie Groups.

Definition 2.1.6. Let G and H be matrix Lie Groups. A map Φ from G to H is a **Lie group homomorphism** if it is a group homomorphism and if it is continuous. It is called a **Lie Group isomorphism** if it is bijective and the inverse map Φ^{-1} is also continuous. Then G and H are said to be **isomorphic** and we write $G \cong H$.

2.2 Examples of Matrix Lie Groups

The most important examples of matrix Lie Groups are described either by equations on the entries of an $n \times n$ complex or real matrix or as a subgroup of automorphisms of V, $V \cong \mathbb{R}^n$ or $V \cong \mathbb{C}^n$, preserving some structure of it.

We will state some of the most relevant examples.

Example 2.2.1 (The special linear group $\mathsf{SL}(n;\mathbb{R})$). It is the group of $n \times n$ real invertible matrices having determinant one, and therefore the group of automorphisms of $\mathbb{R}^n$ preserving a volume element.
It is a subgroup of $\mathsf{GL}(n;\mathbb{C})$. In fact,

(1) A matrix having determinant one is invertible.

(2) The identity I lies in the subgroup since $detI = 1$.

(3) If $A \in \mathsf{SL}(n;\mathbb{C})$, then $detA = I$. Since $det(A^{-1}) = \frac{1}{detA}$, then $A^{-1} \in \mathsf{SL}(n;\mathbb{R})$.

(4) If $A, B \in \mathsf{SL}(n;\mathbb{C})$, since $det(AB) = det(A)det(B)$, then $AB \in \mathsf{SL}(n,\mathbb{C})$.

Proposition 2.2.2. *The special linear group is a* ***closed subgroup****.*

Proof. If $\{A_m\}_{m\in\mathbb{N}}$ is a convergent sequence of matrices such that $det(A_m) = 1$, for all $m \in \mathbb{N}$, then it will converge to a matrix A such that $detA = 1$ because the determinant is a continuous map.
□

Proposition 2.2.3. *The special linear group is* ***not compact****.*

Proof. Any matrix of this form:

$$A_m = \begin{pmatrix} m & & & & \\ & \frac{1}{m} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, \tag{2.1}$$

with m being as large as we want, will be in $\mathsf{SL}(n;\mathbb{R})$. For any constant C, there exists an m such that $|(A_m)_{11}| > C$. $\square$

Proposition 2.2.4. *The special linear group* $\mathsf{SL}(n;\mathbb{R})$ ***is connected*** *for all* $n \geq 1$.

Proof. For the trivial case $n = 1$, we have $A = [1]$.
For the case $n \geq 1$ we can use the Jordan canonical form. Every $n \times n$ matrix can be written as:

$$A = CBC^{-1},$$

where B is the Jordan canonical form, which is an upper-triangular matrix:

$$\begin{pmatrix} \lambda_1 & & & * \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}. \tag{2.2}$$

Since $A \in \mathsf{SL}(n;\mathbb{R})$ and $detA = detB = 1$, $\lambda_1, ..., \lambda_n$ must be non-zero. Then we can compute $B(t)$ by multiplying the entries of B above the diagonal by $(1-t)$, with $0 \leq t \leq 1$. Let $A(t) = CB(t)C^{-1}$. Then $A(t)$ is a continuous path which starts at A and ends at CDC^{-1} where D is the following diagonal matrix:

$$\begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}. \tag{2.3}$$

The path $A(t)$ lies in $\mathsf{SL}(n;\mathbb{R})$ because $detA(t) = \lambda_1 \cdots \lambda_n = 1$ for all t such that $0 \leq t \leq 1$.
If we consider each $\lambda_i(t)$ to be a continuous path which connects λ_i and 1 in $\mathbb{C}^*$, then we can define

$$A(t) = C \begin{pmatrix} \lambda_1(t) & & & 0 \\ & \lambda_2(t) & & \\ & & \ddots & \\ 0 & & & \lambda_n(t) \end{pmatrix} C^{-1}, \tag{2.4}$$

where $\lambda_n(t) = (\lambda_1(t) \cdots \lambda_{n-1}(t))^{-1}$, which allows us to connect A to the identity in $\mathsf{SL}(n;\mathbb{R})$.
For any two matrices A and B we can connect them by connecting each one to the identity. $\square$

The special linear group is **simply connected** when we consider complex entries but it is not in the real numbers. $\mathsf{SL}(n;\mathbb{R})$ has the same fundamental group as $\mathsf{SO}(n)$, which is $\mathbb{Z}$ for $n=2$ and $\mathbb{Z}_2$ for $n \geq 2$.

Example 2.2.5 (**The general linear group** $\mathsf{GL}(n;\mathbb{R})$)**.** It is the group of all invertible $n \times n$ matrices. It is a matrix Lie Group because it is a closed subgroup of $\mathsf{GL}(n;\mathbb{C})$. The same holds for $\mathsf{GL}(n;\mathbb{C})$.
Neither of them is **compact**, since a sequence of matrices $\{A_m\}_{m\in\mathbb{N}}$ in $\mathsf{GL}(n;\mathbb{R})$ or $\mathsf{GL}(n;\mathbb{C})$ may converge to a non-invertible matrix.
$\mathsf{GL}(n;\mathbb{R})$ is **not connected** but has two components, which are the sets of $n \times n$ matrices with negative and positive determinant respectively. In order to create a continuous path connecting two matrices, one from each component, we would have to include in it a matrix with determinant zero, and hence passing outside $\mathsf{GL}(n;\mathbb{R})$.
On the other hand, $\mathsf{GL}(n;\mathbb{C})$ is both **connected** and **simply connected**.

Example 2.2.6 (**The orthogonal group** $\mathsf{O}(n)$)**.** It is the group of $n \times n$ real invertible matrices in which the column vectors of any matrix A are orthonormal, i.e:

$$\sum_{i=1}^{n} A_{ij}A_{ik} = \delta_{jk}. \tag{2.5}$$

It is also the subgroup of $\mathsf{GL}(n;\mathbb{R})$ which preserves the Euclidean inner product:

$$\langle v,u\rangle = \langle Av, Au\rangle. \tag{2.6}$$

$\mathsf{O}(n)$ can also be defined as the subgroup of $\mathsf{GL}(n;\mathbb{R})$ such that for all $A \in \mathsf{O}(n)$, $A^{Tr}A = I$ and therefore $A^{Tr} = A^{-1}$.
Since $det(A^{Tr}A) = (detA)^2 = detI = 1$, it holds that for all $A \in \mathsf{O}(n), detA = \pm 1$.
The orthogonal matrices form a group, as both the inverse and the product of matrices preserve the inner product.
Geometrically, the elements of $\mathsf{O}(n)$ are either rotations, compositions of rotations and reflections.
If we consider only the matrices with determinant $+1$, then we get a subgroup of $\mathsf{O}(n)$ which is denoted $\mathsf{SO}(n)$, **the special orthogonal group**. It is also a Lie Group and all its elements are rotations.
Both $\mathsf{O}(n)$ and $\mathsf{SO}(n)$ are **compact** Lie Groups. The limit of orthogonal (special orthogonal) matrices is also orthogonal (special orthogonal) since the the relation $A^{Tr}A = I$ is preserved under limits. If A is orthogonal, then its column vectors have norm one and thus $|A_{ij}| \leq 1$, for all $1 \leq i,j \leq n$.
Only $\mathsf{SO}(n)$ is **connected** and neither of them is **simply connected**.

Example 2.2.7 (**The unitary group** $\mathsf{U}(n)$)**.** It is the group of $n \times n$ complex matrices in which the column vectors of any matrix A are orthonormal, i.e:

$$\sum_{i=1}^{n} \overline{A_{ij}}A_{ik} = \delta_{jk}, \tag{2.7}$$

where for $x \in \mathbb{C}$, $\overline{x}$ stands for the complex conjugate of x.
It is also the subgroup of $\mathsf{GL}(n;\mathbb{C})$ which preserves the inner product, defined as:

$$\langle x,y\rangle = \sum_{i} \overline{x_i}y_i. \tag{2.8}$$

It can also be defined as the subgroup $\mathsf{U}(n)$ of $\mathsf{GL}(n;\mathbb{C})$ such that for all $A \in \mathsf{U}(n)$, $A^*A = I$. Therefore $A^* = A^{-1}$.
Since $det(A^*A) = (detA)^2 = detI = 1$, $|detA| = 1$ for all $A \in \mathsf{U}(n)$.
Using the same arguments as we used for the orthogonal group we can state that $\mathsf{U}(n)$ is a group.

If we consider only the matrices with determinant one, then we get a subgroup of $\mathsf{U}(n)$ which is denoted $\mathsf{SU}(n)$, **the special unitary group**. It is also a matrix Lie Group.

Both the unitary and the special unitary groups are **compact**, which can be shown by a similar argument as the one used for orthogonal and special orthogonal groups.
Both are **connected**, but only the special unitary group is **simply connected**. To see that $\mathsf{SU}(2)$ is simply connected we can picture it as a three-dimensional sphere S^3 sitting inside $\mathbb{R}^4$. On the other hand, since $\mathsf{U}(1) \cong S^1$, it is not simply connected (its fundamental group is not trivial).

Example 2.2.8 (**The symplectic groups** $\mathsf{Sp}(n,\mathbb{R}), \mathsf{Sp}(n,\mathbb{C}), \mathsf{Sp}(n)$)**.** Consider a skew-symmetric bilinear form B on $\mathbb{R}^{2n}$ defined as follows:

$$B[x,y] = \sum_{i=1}^{n} x_i y_{n+i} - x_{n+1} y_i. \tag{2.9}$$

Then, the set of all $2n \times 2n$ matrices A which preserve B (i.e. $B[Ax, Ay] = B[x,y]$) define the **real symplectic group** $\mathsf{Sp}(n,\mathbb{R})$. $\mathsf{Sp}(n,\mathbb{R})$ is a subgroup of $\mathsf{GL}(2n;\mathbb{R})$. Equivalently, $A \in \mathsf{Sp}(n,\mathbb{R})$ if, and only if, $A^{Tr}JA = J$, where J is:

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Taking the determinant, $(detA)^2 detJ = detJ$, and since $detJ = 1$ we get $(detA)^2 = 1$. In fact $det(A) = 1$ for all $A \in \mathsf{Sp}(n,\mathbb{R})$. This definition is also valid for **the complex symplectic group** $\mathsf{Sp}(n,\mathbb{C})$ but switching A^{tr} for A^*.

We can define **the compact symplectic group** $\mathsf{Sp}(n)$ as:

$$\mathsf{Sp}(n) = \mathsf{Sp}(n;\mathbb{C}) \cap \mathsf{U}(2n)$$

which is indeed **compact**, while the symplectic groups $\mathsf{Sp}(n,\mathbb{R})$ and $\mathsf{Sp}(n,\mathbb{C})$ are not.
All of them are **connected** and only $\mathsf{Sp}(n,\mathbb{C})$ and $\mathsf{Sp}(n)$ are **simply connected**.

3 Lie Groups

3.1 Definition and main properties

Definition 3.1.1. A **Lie Group** G is a set which is compatible with a group and differentiable manifold structure simultaneously, i.e:

$$\begin{array}{rcl} G \times G & \to & G \\ (x, y) & \to & x \cdot y^{-1} \end{array}$$

is a differential map ($\mathscr{C}^\infty$). This means that the multiplication and the inverse of the group structure are differential maps.

Remark. A differentiable manifold is a topological manifold with a globally defined differentiable structure.

In general, properties of Lie groups refer to one of its structures. For example, *abelian* refers to the group structure and *n-dimensional* or *connected* refer to the manifold structure.

Definition 3.1.2. A **map or morphism** Φ between two Lie Groups G and H is a group homomorphism which is differentiable.

Definition 3.1.3. A **Lie subgroup** (or closed Lie subgroup) H of a Lie Group G is defined to be a subset that is simultaneously a subgroup and a closed submanifold (which inherits the manifold structure from G).

It can be seen that:

Proposition 3.1.4. *Every closed Lie subgroup of a Lie Group is a Lie Group.*

Example 3.1.5. $\mathsf{GL}(n, \mathbb{C})$ is a Lie Group. Its manifold structure is obtained by assigning each matrix entry to a coordinate, so that we create the following embedding:

$$\mathsf{GL}(n, \mathbb{C}) \hookrightarrow \mathbb{C}^{n^2}.$$

Then, $\mathsf{GL}(n, \mathbb{C})$ is an open subset of $\mathbb{C}^{n^2}$ because given an invertible $n \times n$ matrix A, there is a neighbourhood U of A such that every matrix in U is also invertible. As $\mathbb{C}^{n^2}$ is a smooth manifold, $\mathsf{GL}(n, \mathbb{C})$ is also smooth. The matrix product AB is a smooth (polynomial) function of the entries of A and B and by using Kramer's rule, we see that A^{-1} is a smooth function on the entries of A.

Theorem 3.1.6. *Every matrix Lie Group is a Lie Group.*

The matrix Lie Groups introduced in Section 2.1 are closed subgroups of $\mathsf{GL}(n; \mathbb{C})$. Thus they are Lie Groups and they have both the structure of a group and of a differential manifold.

Remark. Not all Lie Groups are matrix Lie Groups. Nevertheless, we will only deal with the most important cases, which are matrix Lie Groups.

As we will see in the following sections, Lie Groups have an important role in the formulation of **quantum mechanics**. The structure of a physical system is described by its symmetries. Mathematically, symmetries are transformations under which the properties of the system remain invariant. In physics, by studying the effect of a set of transformations we can obtain the conservation laws of a given system. In particular, Lie Groups are used to describe continuous transformations.

4 Lie Algebras

4.1 The exponential of a matrix

A **Lie Algebra** is a vector space endowed with an extra operation called Lie bracket. Before introducing it, we will need to define **the exponential of a matrix**.

Definition 4.1.1. Let X be a $n \times n$ real or complex matrix. We define the exponential of X, e^X or $exp(X)$, by the usual power series

$$e^X = \sum_{m \geq 0} \frac{X^m}{m!}. \tag{4.1}$$

Remark. Recall that the **norm** of a matrix X is defined to be:

$$\|X\| = \sup_{x \neq 0} \frac{\|Xx\|}{\|x\|} \tag{4.2}$$

or equivalently, the smallest finite number λ such that $\|Xx\| \leq \lambda \|x\|$, for all $x \in \mathbb{C}^n$.

Proposition 4.1.2. *The exponential matrix is well-defined.*

Proof. By using the following properties:

(1) $\|XY\| \leq \|X\| \|Y\|$

(2) $\|X + Y\| \leq \|X\| + \|Y\|$

we obtain:

$$\|e^X\| = \| \sum_{k \geq 0} \frac{X^k}{k!} \| \leq \sum_{k \geq 0} \frac{\|X^k\|}{k!} \leq \sum_{k \geq 0} \frac{\|X\|^k}{k!} = e^{\|X\|} < \infty, \tag{4.3}$$

and therefore the definition of the exponential matrix is an absolutely convergent sequence of matrices. ☐

Proposition 4.1.3. *Let X and Y be arbitrary $n \times n$ matrices. Then we have the following properties:*

(1) $e^0 = I$,

(2) e^X *is invertible. This means that* $(e^X)^{-1} = e^{-X}$,

(3) $e^{(\alpha+\beta)X} = e^{\alpha X} e^{\beta X}$,

(4) if $XY = YX$, *then* $e^{X+Y} = e^X e^Y = e^Y e^X$,

(5) if C *is invertible, then* $e^{CXC^{-1}} = Ce^X C^{-1}$,

(6) $\|e^X\| \leq e^{\|X\|}$.

The proof can be found in [6], Section 3.3.

Now we will define the **logarithm of a matrix** in a neighbourhood of the identity as an inverse function of the exponential matrix.

Definition 4.1.4. Given a $n \times n$ matrix A, we define the *log* as follows:

$$\log A = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(A-I)^m}{m}. \tag{4.4}$$

Note that it is a well defined function and it is continuous on the set of all $n \times n$ complex matrices A with $\|A - I\| < 1$.
Moreover, $\log A \in \mathbb{R}$ if $A \in \mathbb{R}$.
For all $n \times n$ matrix A such that $\|A - I\| < 1$, it can be shown that ([6], Section 3.3):

$$e^{\log A} = A, \tag{4.5}$$
$$\log(I + A) = A + \mathcal{O}(\|A\|^2). \tag{4.6}$$

Now we will prove some results that will be used later when studying the Lie Algebra of a matrix Lie Group.

Proposition 4.1.5. *Let X be a $n \times n$ complex matrix. Identify the space of complex matrices with $\mathbb{C}^{n^2}$. Then e^{tX} is a smooth curve in $\mathbb{C}^{n^2}$. Moreover,*

$$\frac{d}{dt} e^{tX} = X e^{tX} = e^{tX} X$$

and thus

$$\left. \frac{d}{dt}(e^{tX}) \right|_{t=0} = X. \tag{4.7}$$

It can be proven by differentiating the power series of the definition of e^{tX} term-by-term ([6], Section 3.3).

Theorem 4.1.6. *(Lie Product formula) Let X and Y be $n \times n$ complex matrices. Then*

$$e^{X+Y} = \lim_{m \to \infty} \left(e^{\frac{X}{m}} e^{\frac{Y}{m}} \right)^m. \tag{4.8}$$

Proof. Recall the definition of the exponential:

$$e^{\frac{X}{m}} = \sum_{k=0}^{\infty} \frac{X^k}{m^n k!}. \tag{4.9}$$

Then,

$$e^{\frac{X}{m}} e^{\frac{Y}{m}} = I + \frac{X}{m} + \frac{Y}{m} + \mathcal{O}(\frac{1}{m^2}). \tag{4.10}$$

For m sufficiently large, $\mathcal{O}(\frac{1}{m^2}) \to 0$ and $\left\| e^{\frac{X}{m}} e^{\frac{Y}{m}} - I \right\| < 1$. Thus, we can perform the logarithm since we are in a neighbourhood of the identity:

$$\begin{aligned} \log(e^{\frac{X}{m}} e^{\frac{Y}{m}}) &= \log(I + \frac{X}{m} + \frac{Y}{m} + \mathcal{O}(\frac{1}{m^2})) \\ &= \frac{X}{m} + \frac{Y}{m} + \mathcal{O}(\frac{1}{m^2}) + \mathcal{O}(\left\|\frac{X}{m} + \frac{Y}{m}\right\|^2) = \frac{X}{m} + \frac{Y}{m} + \mathcal{O}(\frac{1}{m^2}), \end{aligned} \tag{4.11}$$

where we have used the equation (4.6).
If we take the exponentials and powers again,

$$(e^{\frac{X}{m}} e^{\frac{Y}{m}})^m = e^{X+Y+\mathcal{O}(\frac{1}{m})}. \tag{4.12}$$

Finally,

$$\lim_{m\to\infty} (e^{\frac{X}{m}} e^{\frac{Y}{m}})^m = e^{X+Y}. \tag{4.13}$$

□

Theorem 4.1.7. *Let X be an $n \times n$ real or complex matrix. Then,*

$$det\left(e^X\right) = e^{trace(X)}. \tag{4.14}$$

Proof. We will distinguish three cases.
(1) Assume that X is diagonalizable. In this case, there exists a complex invertible matrix C such that:

$$X = C \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} C^{-1}. \tag{4.15}$$

Then,

$$e^X = C \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix} C^{-1}. \tag{4.16}$$

Since for any diagonalizable matrix X, $trace(X) = trace(CDC^{-1}) = trace(D)$, it holds that $det(e^X) = \prod e^{\lambda_i} = e^{\sum \lambda_i} = e^{trace(X)}$.
(2) Let X be a nilpotent matrix. In this case, it can be proved that all roots of the characteristic polynomial must be zero, and thus all eigenvalues are zero. The Jordan canonical form will be strictly upper triangular:

$$X = C \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} C^{-1}. \tag{4.17}$$

Then, it is easy to see that e^X will have the following form:

$$e^X = C \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} C^{-1}. \tag{4.18}$$

Therefore, $trace(X) = 0$ and $det(e^X) = 1$.
(3) Let X be an arbitrary matrix. Any matrix can be written as the sum of two commuting matrices S and N, with S diagonalizable and N nilpotent. Since S and N commute, $e^X = e^S e^N$:

$$det(e^X) = det(e^S)det(e^N) = e^{trace(S)}e^{trace(N)} = e^{trace(X)}. \tag{4.19}$$

□

Definition 4.1.8. A function $A : \mathbb{R} \to \mathsf{GL}(n;\mathbb{C})$ is called a **one-parameter group** if:

(1) it is continuous,

(2) $A(0) = I$,

(3) $A(t+s) = A(t)A(s)$, for all $t, s \in \mathbb{R}$.

Theorem 4.1.9. *(One-parameter Subgroups) If A is a one-parameter group in $\mathsf{GL}(n;\mathbb{C})$, then there exists a unique $n \times n$ complex matrix X such that:*

$$A(t) = e^{tX}. \tag{4.20}$$

A proof of this Theorem can be found in [6], Section 3.4.

4.2 The Lie Algebra of a matrix Lie Group

Definition 4.2.1. Let G be a matrix Lie Group. Then the **Lie algebra** of G, denoted $\mathfrak{g}$, is the set of matrices X such that $e^{tX} \in G$ for all $t \in \mathbb{R}$.

Now we will establish some basic properties of the Lie algebra of a given matrix Lie Group.

Proposition 4.2.2. *Let G be a matrix Lie Group and let X be an element of its Lie Algebra. Then e^X is an element of the identity connected component of G.*

Proof. By definition of the Lie algebra, e^{tX} lies in G for any $t \in \mathbb{R}$. But as t varies from 0 to 1, e^{tX} is a continuous path connecting the identity to e^X. □

Proposition 4.2.3. *Let G be a matrix Lie Group with Lie Algebra $\mathfrak{g}$. Let X be an element of $\mathfrak{g}$ and A an element of G. Then AXA^{-1} is in $\mathfrak{g}$.*

Proof. By Proposition 4.1.3 (5),

$$e^{t(AXA^{-1})} = Ae^{tX}A^{-1}. \tag{4.21}$$

Since $Ae^{tX}A^{-1} \in G$, $AX^{-1}A \in \mathfrak{g}$. □

Theorem 4.2.4. *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$, and let X and Y be elements of $\mathfrak{g}$. Then:*

(1) $sX \in \mathfrak{g}$*, for all* $s \in \mathbb{R}$*,*

(2) $X + Y \in \mathfrak{g}$*,*

(3) $XY - YX \in \mathfrak{g}$*.*

Remark. In particular the Lie Algebra is a vector space over $\mathbb{R}$.

Proof. (1) For any $s \in \mathbb{R}$, we have $e^{(ts)X} \in G$ if, and only if, $X \in \mathfrak{g}$. Then, since $e^{ts(X)} = e^{t(sX)}$, we get $sX \in \mathfrak{g}$.
(2) If X and Y commute, then $e^{t(X+Y)} = e^{tX}e^{tY}$. If they do not commute, by Theorem 4.1.6 (the Lie Product formula),

$$e^{t(X+Y)} = \lim_{m\to\infty} \left(e^{\frac{tX}{m}} e^{\frac{tY}{m}}\right)^m. \tag{4.22}$$

Since $X, Y \in \mathfrak{g}$, $e^{\frac{tX}{m}}, e^{\frac{tY}{m}} \in G$. Thus, by the definition of a Lie Group, $\lim_{m\to\infty} \left(e^{\frac{tX}{m}} e^{\frac{tY}{m}}\right)^m$ is in G. Therefore, $e^{t(X+Y)} \in G$ and then $X + Y \in \mathfrak{g}$.
(3) It follows from Proposition 4.1.5 that $\frac{d}{dt}(e^{tX}Y)\big|_{t=0} = XY$. Hence,

$$\frac{d}{dt}(e^{tX}Ye^{-tX})\bigg|_{t=0} = XY - YX. \tag{4.23}$$

Recall that by Proposition 4.2.3, $e^{tX}Ye^{-tX}$ is an element of $\mathfrak{g}$ for all $t \in \mathbb{R}$. By the statements (1) and (2) that we already proved, $\mathfrak{g}$ is a real vector space. Hence, the derivative of any smooth curve lying in $\mathfrak{g}$ must be again in $\mathfrak{g}$. ☐

Definition 4.2.5. We define the **dimension of a Lie Algebra** as its dimension as a $\mathbb{R}$-vector space.

Definition 4.2.6. Given two $n \times n$ matrices A and B, the **bracket** or **commutator** of A and B is defined to be:

$$[A, B] = AB - BA \tag{4.24}$$

By Theorem 4.2.4 (3), the Lie Algebra of any matrix Lie Group is closed under brackets.

Theorem 4.2.7. *Let* G *and* H *be matrix Lie Groups, with Lie Algebras* $\mathfrak{g}$ *and* $\mathfrak{h}$ *respectively. Suppose that* $\phi : G \to H$ *is a Lie Group homomorphism. Then, there exists a unique real linear map* $\widetilde{\phi} : \mathfrak{g} \to \mathfrak{h}$ *such that:*

$$\phi(e^X) = e^{\tilde{\phi}(X)},$$

for all $X \in \mathfrak{g}$*. The map* $\widetilde{\phi}$ *has the following additional properties:*

(1) $\widetilde{\phi}(AXA^{-1}) = \phi(A)\widetilde{\phi}(X)\phi(A)^{-1}$ *for all* $X \in \mathfrak{g}$ *and all* $A \in G$*,*

(2) $\widetilde{\phi}([X,Y]) = \left[\widetilde{\phi}(X), \widetilde{\phi}(Y)\right]$ *for all* $X, Y \in \mathfrak{g}$*,*

(3) $\widetilde{\phi}(X) = \frac{d}{dt}(\phi(e^{tX}))\Big|_{t=0}$ *for all* $X \in \mathfrak{g}$*.*

If G,H and K are matrix Lie Groups and $\phi : G \to H$*,* $\psi : H \to K$ *are Lie Group homomorphisms, then:*

$$\widetilde{\phi \circ \psi} = \widetilde{\phi} \circ \widetilde{\psi}.$$

Remark. Given a Lie Group homomorphism ϕ we can compute $\widetilde{\phi}$. Since $\widetilde{\phi}$ is linear, it suffices to compute $\widetilde{\phi}$ for a basis of $\mathfrak{g}$. Hence, we can take Theorem 4.2.7 (3) as the standard definition of $\widetilde{\phi}$. Moreover, Theorem 4.2.7 (2) states that $\widetilde{\phi}$ is a Lie Algebra homomorphism. Therefore, every Lie Group homomorphism gives rise in a natural way to a map between the corresponding Lie Algebras which is also an homomorphism. We will see later that the converse is only true when the Lie Groups hold specific conditions.

Proof. Consider the map from $\mathbb{R}$ to $\phi(e^{tX})$ which takes t to $\phi(e^{tX})$ for all $X \in \mathfrak{g}$. Since ϕ is a continuous group homomorphism, this map is a one-parameter subgroup of H. Thus, by Theorem 4.1.9 there exists a unique Z such that:

$$\phi(e^{tX}) = e^{tZ} \tag{4.25}$$

for all $t \in \mathbb{R}$. Then Z must lie in $\mathfrak{h}$ since $e^{tZ} = \phi(e^{tX}) \in H$.
We define $\widetilde{\phi}(X) = Z$. Now we need to check that it has the required properties to be the linear map we want.

(a) From the definition of $\widetilde{\phi}$ at $t = 1$, it follows that $\phi(e^X) = e^{\widetilde{\phi}(X)}$.

(b) If $\phi(e^{tX}) = e^{tZ}$, then $\phi(e^{tsX}) = e^{tsZ}$. Therefore, $\widetilde{\phi}(sX) = s\widetilde{\phi}(X)$ for all $s \in \mathbb{R}$.

(c) We have that:

$$e^{t\widetilde{\phi}(X+Y)} = e^{\widetilde{\phi}(t(X+Y))} = \phi(e^{t(X+Y)}). \tag{4.26}$$

Then, by the Lie Product formula from Theorem 4.1.6:

$$\begin{aligned}\phi(e^{t(X+Y)}) &= \phi\left(\lim_{m\to\infty}\left(e^{\frac{tX}{m}}e^{\frac{tY}{m}}\right)^m\right) = \lim_{m\to\infty}\left(\phi\left(e^{\frac{tX}{m}}\right)\left(e^{\frac{tY}{m}}\right)\right)^m \\ &= \lim_{m\to\infty}\left(e^{\frac{t\widetilde{\phi}(X)}{m}}\right)^m\left(e^{\frac{t\widetilde{\phi}(Y)}{m}}\right)^m = e^{t(\widetilde{\phi}(X)+\widetilde{\phi}(Y))},\end{aligned} \tag{4.27}$$

and therefore,

$$\widetilde{\phi}(X+Y) = \widetilde{\phi}(X) + \widetilde{\phi}(Y). \tag{4.28}$$

Now we need to check that it also verifies the additional properties that we have stated.

(1) By Proposition 4.1.3 (5) we have:

$$\begin{aligned} e^{t\widetilde{\phi}(AXA^{-1})} &= e^{\widetilde{\phi}(tAXA^{-1})} = \phi(e^{tAXA^{-1}}) = \phi(Ae^{tX}A^{-1}) \\ &= \phi(A)\phi(e^{tX})\phi(A)^{-1} = \phi(A)e^{t\widetilde{\phi}(X)}\phi(A)^{-1}. \end{aligned} \tag{4.29}$$

If we differentiate this at $t = 0$ we obtain:

$$\widetilde{\phi}(AXA^{-1}) = \phi(A)\widetilde{\phi}(X)\phi(A)^{-1}. \tag{4.30}$$

(2) From equation (4.23) it follows that:

$$[X, Y] = \frac{d}{dt}(e^{tX}Ye^{-tX})\bigg|_{t=0}. \tag{4.31}$$

Then, since the derivative commutes with the linear transformation,

$$\widetilde{\phi}([X, Y]) = \widetilde{\phi}\left(\frac{d}{dt}(e^{tX}Ye^{-tX})\big|_{t=0}\right) = \frac{d}{dt}\widetilde{\phi}\left(e^{tX}Ye^{-tX}\right)\bigg|_{t=0}. \tag{4.32}$$

Using Theorem 4.2.7 and Proposition 4.1.3 (5),

$$\begin{aligned} \widetilde{\phi}([X, Y]) &= \frac{d}{dt}(\phi(e^{tX})\widetilde{\phi}(Y)\phi(e^{-tX}))\bigg|_{t=0} \\ &= \frac{d}{dt}(e^{t\widetilde{\phi}(X)}\widetilde{\phi}(Y)e^{-t\widetilde{\phi}(X)})\bigg|_{t=0} = \left[\widetilde{\phi}(X), \widetilde{\phi}(Y)\right], \end{aligned} \tag{4.33}$$

and therefore:

$$\widetilde{\phi}([X, Y]) = \left[\widetilde{\phi}(X), \widetilde{\phi}(Y)\right]. \tag{4.34}$$

(3) From the definition of $\widetilde{\phi}$, it follows that $\widetilde{\phi}(X) = \frac{d}{dt}\phi(e^{tX})\Big|_{t=0}$.

In order to prove that $\widetilde{\phi}$ is the unique linear map such that $e^{\widetilde{\phi}(X)} = \phi(e^X)$, let us assume that there exists another such map $\widetilde{\psi}$. Then:

$$e^{t\widetilde{\psi}(X)} = e^{\widetilde{\psi}(tX)} = \phi(e^{tX}), \tag{4.35}$$

and thus,

$$\widetilde{\psi}(X) = \frac{d}{dt}\phi(e^{tX})\bigg|_{t=0} \tag{4.36}$$

which means that $\widetilde{\psi}$ is $\widetilde{\phi}$.
The only thing left to prove is that $\widetilde{\psi \circ \phi} = \widetilde{\psi} \circ \widetilde{\phi}$. For any $X \in \mathfrak{g}$:

$$e^{\widetilde{\psi\circ\phi}(tX)} = \psi \circ \phi(e^{tX}) = \psi(\phi(e^{tX})) = \psi(e^{\widetilde{\phi}(tX)}) = e^{\widetilde{\psi}(\widetilde{\phi}(tX))} = e^{\widetilde{\psi}\circ\widetilde{\phi}(tX)}. \tag{4.37}$$

$\square$

Definition 4.2.8. Let G be a matrix Lie Group and $\mathfrak{g}$ its Lie Algebra. For each $A \in G$, we define **the Adjoint mapping** $Ad(A)$ as the following Lie Group homomorphism $Ad : G \to \mathsf{GL}(\mathfrak{g})$:

$$Ad(A)(X) = AXA^{-1}.$$

Remark. It is an invertible linear transformation of $\mathfrak{g}$ with inverse $Ad(A^{-1})$.

Remark. Note that Proposition 4.2.3 guarantees that $Ad(A)(X) \in \mathfrak{g}$, for all $X \in \mathfrak{g}$.

Proposition 4.2.9. *Let G be a matrix Lie Group and $\mathfrak{g}$ its Lie Algebra. Let $Ad : G \to \mathsf{GL}(\mathfrak{g})$ be the Lie group homomorphism from Definition 4.2.8 and $\widetilde{Ad} : \mathfrak{g} \to \mathsf{gl}(\mathfrak{g})$ the associated Lie Algebra map given by Theorem 4.2.7. Then, for all X and $Y \in \mathfrak{g}$:*

$$\widetilde{Ad}X(Y) = [X, Y]. \tag{4.38}$$

Since $\mathfrak{g}$ is a real vector space, $\mathsf{GL}(\mathfrak{g})$ is equivalent to $\mathsf{GL}(k; \mathbb{R})$ for some $k > 0$, and thus $\mathsf{GL}(\mathfrak{g})$ is a matrix Lie group. Since $Ad : G \to \mathsf{GL}(\mathfrak{g})$ is continuous, it is a Lie group homomorphism.

Remark. We will denote $\widetilde{Ad}$ as ad in future references.

Proof. By Theorem 4.2.7, $\widetilde{Ad}$ can be computed as:

$$\widetilde{Ad}X = \frac{d}{dt}(Ad(e^{tX}))\Big|_{t=0}. \tag{4.39}$$

Therefore, by equation (4.23):

$$\widetilde{Ad}X(Y) = \frac{d}{dt}Ad(e^{tX})(Y)\Big|_{t=0} = \frac{d}{dt}(e^{tX}Ye^{-tX})\Big|_{t=0} = XY - YX = [X, Y]. \tag{4.40}$$

□

4.3 Examples of Lie Algebras

We will introduce the Lie Algebras of some of the matrix Lie Groups we have introduced in Section 2.2.

Example 4.3.1. The **Lie Algebra of the special linear group** $\mathsf{sl}(n; \mathbb{R})$ is the space of all $n \times n$ real matrices with trace zero. In fact, recall from Theorem 4.1.7 that $det(e^X) = e^{traceX}$. Then $det(e^{tX}) = 1$, for all $t \in \mathbb{R}$ if, and only if, $traceX = 0$. Equivalently, $\mathsf{sl}(n; \mathbb{C})$ is the space of all $n \times n$ complex matrices with trace zero.

Example 4.3.2. The **Lie Algebra of the general linear group** $\mathsf{gl}(n; \mathbb{R})$ is the space of all real $n \times n$ matrices. It follows from the fact that for any $n \times n$ real matrix, e^{tX} is invertible and real, and if e^{tX} is real for all $t \in \mathbb{R}$, then $X = \frac{d}{dt}e^{tX}\big|_{t=0}$ is also real.

Equivalently, $\mathsf{gl}(n; \mathbb{C})$ is the space of all $n \times n$ complex matrices.

Example 4.3.3. The **Lie Algebra of the orthogonal group** $\mathsf{o}(n)$ is the space of all $n \times n$ real matrices X such that $X^{Tr} = -X$.
Recall that an $n \times n$ real matrix e^{tX} is orthogonal if, and only if, $(e^{tX})^{Tr} = (e^{tX})^{-1} = e^{-tX}$. This condition holds for all $t \in \mathbb{R}$ if, and only if, $X^{Tr} = -X$.
The identity component of $\mathsf{O}(n)$ is $\mathsf{SO}(n)$. By the definition of Lie Algebra, $\mathsf{so}(n) \subset \mathsf{o}(n)$ and by of Proposition 4.2.2, $\mathsf{o}(n) \subset \mathsf{so}(n)$. Therefore, $\mathsf{so}(n)$ is the same as $\mathsf{o}(n)$.

Example 4.3.4. The **Lie Algebra of the unitary group** $\mathsf{u}(n)$ is the space of all $n \times n$ complex matrices X such that $X^* = -X$.
Recall that a matrix e^{tX} is unitary if, and only if,

$$(e^{tX})^* = (e^{tX})^{-1} = e^{-tX}. \tag{4.41}$$

Then, e^{tX} is unitary for all $t \in \mathbb{R}$ if, and only if, X verifies $X^* = -X$.
By combining the previous conditions, $\mathsf{su}(n)$ is the space of $n \times n$ complex matrices X such that $X^* = -X$ and $trace X = 0$.

Example 4.3.5. The **Lie Algebra of the symplectic group** $\mathsf{sp}(n, \mathbb{R})$ is the space of $2n \times 2n$ real matrices X such that $JX^{Tr}J = X$, with $J \in \mathsf{Sp}(n, \mathbb{R})$.
The Lie algebra of $\mathsf{Sp}(n)$ is $\mathsf{sp}(n) = \mathsf{sp}(n, \mathbb{C}) \cap \mathsf{u}(2n)$.

4.4 The exponential mapping of a matrix Lie group

Definition 4.4.1. If G is a matrix Lie Group with Lie Algebra $\mathfrak{g}$, we define the **exponential mapping** for G as the map $exp : \mathfrak{g} \to G$.

Remark. In general, the exponential mapping is neither injective nor surjective, but it allows us to pass information between the group and the algebra and it is locally bijective.

Theorem 4.4.2. *Let G be a matrix Lie Group with Lie Algebra $\mathfrak{g}$. There exists a neighbourhood U of the zero of $\mathfrak{g}$ and a neighbourhood V of the identity of G such that the exponential mapping from U to V is an homeomorphism.*

The idea behind the proof is to use the Definition 4.1.4 of the matrix logarithm in $\mathsf{GL}(n, \mathbb{C})$ ant the fact that any matrix Lie Group is a subgroup of $\mathsf{GL}(n, \mathbb{C})$.([2], Section I).

Definition 4.4.3. Let U and V be the sets described in Theorem 4.4.2. Then we define the logarithm for G to be the inverse map $exp^{-1} : V \to \mathfrak{g}$.

Proposition 4.4.4. *Let G be a connected matrix Lie Group. Then for all $A \in G$:*

$$A = e^{X_1} e^{X_2} \cdots e^{X_n}$$

with $X_1, \cdots, X_n$ in $\mathfrak{g}$.

A proof of this Proposition can be found in [7].

4.5 General Lie Algebras

In this section, we will introduce the general concept of Lie algebra and some of its most important properties.

Definition 4.5.1. A **finite-dimensional real or complex Lie Algebra** is a finite-dimensional real or complex vector space $\mathfrak{g}$ with a map $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that:

(1) $[,]$ is bilinear,

(2) $[,]$ is antisymmetric, i.e. $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{g}$,

(3) $[,]$ verifies the **Jacobi identity**, $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$, for all $X, Y, Z \in \mathfrak{g}$.

Remark. Notice that condition (2) implies that $[X, X] = 0$ for all $X \in \mathfrak{g}$.

Definition 4.5.2. A **subalgebra** of a real or complex Lie Algebra $\mathfrak{g}$ is a vector subspace $\mathfrak{h}$ of $\mathfrak{g}$ such that it is closed under the brackets:

$$[H_1, H_2] \in \mathfrak{h} \quad \forall H_1, H_2 \in \mathfrak{h}. \tag{4.42}$$

Definition 4.5.3. If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie Algebras, then a Lie Algebra homomorphism is defined as a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ such that $\phi([X, Y]) = [\phi(X), \phi(Y)]$, for all $X, Y \in \mathfrak{g}$.

Note that any subalgebra of a given Lie Algebra $\mathfrak{g}$ is also a Lie Algebra.

Proposition 4.5.4. *The Lie Algebra $\mathfrak{g}$ of a matrix Lie Group G is a Lie Algebra.*

Proof. The space of all $n \times n$ complex matrices $\mathsf{gl}(n; \mathbb{C})$ verifies the three conditions from Definition 4.5.1 and therefore it is a Lie Algebra. Every Lie Algebra of a matrix Lie Group, $\mathfrak{g}$, is a subalgebra of $\mathsf{gl}(n; \mathbb{C})$ and thus it is also a Lie Algebra. □

Theorem 4.5.5. *Every finite-dimensional real Lie Algebra is isomorphic to a subalgebra of $\mathsf{gl}(n; \mathbb{R})$. Every finite-dimensional complex Lie Algebra is isomorphic to a complex subalgebra of $\mathsf{gl}(n; \mathbb{C})$.*

A proof of this Theorem can be found in [9].

It can be proven that every Lie algebra is isomorphic to a Lie algebra of a matrix Lie Group.

Definition 4.5.6. Let $\mathfrak{g}$ be a Lie Algebra. For every $X \in \mathfrak{g}$, let us define a linear map $adX : \mathfrak{g} \to \mathfrak{g}$ such that:

$$adX(Y) = [X, Y]. \tag{4.43}$$

The map $ad : X \to adX$ is a linear map from $\mathfrak{g}$ to $\mathsf{gl}(\mathfrak{g})$, which is the space of linear operators from $\mathfrak{g}$ to $\mathfrak{g}$.

Proposition 4.5.7. *If $\mathfrak{g}$ is a Lie Algebra, then:*

$$ad[X,Y] = adX adY - adY adX = [adX, adY], \tag{4.44}$$

which means that, ad is a Lie Algebra homomorphism.

Proof. Notice that:

$$ad[X,Y](Z) = [[X,Y],Z], \tag{4.45}$$

and

$$[adX, adY](Z) = [X,[Y,Z]] - [Y,[X,Z]]. \tag{4.46}$$

Then, using the Jacobi identity from Definition 4.5.1 (3):

$$[[X,Y],Z] = [X,[Y,Z]] - [Y[X,Z]]. \tag{4.47}$$

□

5 Representation Theory

5.1 Introduction to representations of Lie Groups and Lie Algebras

We can think of a representation as a linear action of a Lie Group or Lie Algebra on a given vector space. Representations of Lie Groups and Lie Algebras allow us to treat them as groups of matrices. Even if they are already groups of matrices, sometimes we are interested in their actions on a given vector space.

Representations naturally arise in many branches of mathematics and physics in order to study symmetry. In quantum mechanics, each system is denoted by a given state, which is a vector in a **Hilbert space**. A Hilbert space is a space that has the structure of an inner product and that is also a complete metric space with respect to the norm. An state ideally contains all the information about the system. Nevertheless, we usually focus on a specific property of the state of the system and we consider only the Hilbert space associated to it. For instance, we could study the spin or the flavour symmetry of a given particle, which correspond to finite-dimensional Hilbert Spaces. Other properties, like position or momentum, are represented on the infinite Hilbert Space of the $\mathsf{L}^2(\mathbb{R})$ functions. Then, we consider representations of Lie Groups and Lie Algebras that act on these states, either transforming them or leaving them invariant.

By studying all the representations of a given group or algebra we can also obtain more information about the group or the algebra itself.

Definition 5.1.1. Let G be a matrix Lie Group. A **finite-dimensional complex representation of** G is a Lie Group homomorphism Π of G into $\mathsf{GL}(V)$, where V is a finite dimensional complex vector space.
A **finite-dimensional real representation of** G is a Lie Group homomorphism Π of G into $\mathsf{GL}(V)$, where V is a finite dimensional real vector space.
Let $\mathfrak{g}$ be a Lie Algebra. A **finite-dimensional real or complex representation of** $\mathfrak{g}$, π, is defined analogously.

Definition 5.1.2. Let Π be a finite-dimensional real or complex representation of a Lie Group (Lie Algebra) acting on a vector space V. A subspace W of V is **invariant** with respect to Π if $\Pi(A)w \in W$, for all $w \in W$ and for all $A \in G$. An invariant subspace is non-trivial if $W \neq 0$ and $W \neq V$.
An **irreducible representation** is a representation which does not have any non-trivial invariant subspace.

Definition 5.1.3. Let G be a Lie Group. Let Π be a representation of G acting on the vector space V and let Σ be another representation of G acting on a vector space W. Then, a linear map Φ from V into W is a **morphism of representations** if:

$$\Phi(\Pi(A)v) = \Sigma(A)\Phi(v), \tag{5.1}$$

for all $A \in G$ and for all $v \in V$.
The definition is analogous for Lie Algebras.

If Φ is invertible then it is an isomorphism between the two representations. In this case we say that the two representations are **equivalent**.

We will be interested in finding all the inequivalent finite-dimensional irreducible representations of a particular Lie Group or Lie Algebra.

Proposition 5.1.4. *Let G be a matrix Lie Group with associated Lie Algebra $\mathfrak{g}$. Let Π be a finite-dimensional real or complex representation of G acting on a vector space V. Then there is a unique representation of $\mathfrak{g}$, π, acting on the same space V such that the following holds:*

$$\Pi(e^X) = e^{\pi(X)}. \tag{5.2}$$

Moreover, for any $X \in \mathfrak{g}$:

$$\pi(X) = \frac{d}{dt}\Pi(e^{tX})\bigg|_{t=0}, \tag{5.3}$$

and

$$\pi(AXA^{-1}) = \Pi(A)\pi(X)\Pi(A)^{-1} \tag{5.4}$$

for all $X \in \mathfrak{g}$.

Proof. By Theorem 4.2.7, for each Lie Group homomorphism $\phi : G \to H$ there exists a unique Lie Algebra homomorphism $\widetilde{\phi} : \mathfrak{g} \to \mathfrak{h}$ such that $\phi(e^X) = e^{\widetilde{\phi}(X)}$. If we take H to be the Lie Group $\mathsf{GL}(V)$ and ϕ to be the representation Π, the associated Lie Algebra homomorphism is the representation π of the Lie Algebra $\mathfrak{g}$. The properties of π follow from the properties of the Lie Algebra homomorphism stated in Theorem 4.2.7. $\square$

5.2 The complexification of a real Lie Algebra

From now on and until the end of the Section we will study the representations of complex Lie Groups and their associated complex Lie Algebras. We need to define the complexification of a real Lie Algebra.

Definition 5.2.1. Let $\mathfrak{g}$ be a finite-dimensional real Lie Algebra. The **complexification of $\mathfrak{g}$** as a real vector space, $\mathfrak{g}_{\mathbb{C}}$, is the space of formal linear combinations $X_1 + iX_2$ with $X_1, X_2 \in \mathfrak{g}$, which becomes a complex vector space if we define the following operation:

$$i(X_1 + iX_2) = -X_2 + iX_1. \tag{5.5}$$

This is the equivalent of considering the space of ordered pairs (X_1, X_2).

Proposition 5.2.2. *Let $\mathfrak{g}$ be a Lie Algebra, and let $[,]$ be the bracket operation in $\mathfrak{g}$. Then, $[,]$ has a unique extension in $\mathfrak{g}_{\mathbb{C}}$ which makes the complexification of $\mathfrak{g}$ a complex Lie Algebra and therefore a subgroup of $\mathfrak{gl}(n;\mathbb{C})$.*

Proof. Take the definition of the bracket operation for two elements $X_1 + iX_2, Y_1 + iY_2 \in \mathfrak{g}_{\mathbb{C}}$ to be:

$$[X_1 + iX_2, Y_1 + iY_2] = [X_1, Y_1] - [X_2, Y_2] + i([X_1, Y_2] + [X_2, Y_1]), \tag{5.6}$$

where $X_1, X_2, Y_1, Y_2 \in \mathfrak{g}$.
It can be easily seen that it is bilinear and skew-symmetric. Since the Jacobi identity holds for the elements of $\mathfrak{g}$, it can be easily seen that it also holds for elements of $\mathfrak{g}_{\mathbb{C}}$. ☐

Example 5.2.3. An interesting equivalence is $\mathsf{su}(2)_{\mathbb{C}} \simeq \mathsf{sl}(2;\mathbb{C})$.
In fact, $\mathsf{sl}(2;\mathbb{C})$ is the space of all 2×2 complex matrices with trace zero. If $X \in \mathsf{sl}(2;\mathbb{C})$, then:

$$X = \frac{X - X^*}{2} + \frac{X + X^*}{2} = \frac{X - X^*}{2} + i\frac{X + X^*}{2i}, \tag{5.7}$$

where both $\frac{X-X^*}{2}$ and $\frac{X+X^*}{2i}$ are in $\mathsf{su}(2)$.
Since this decomposition is unique, $\mathsf{sl}(2;\mathbb{C})$ is isomorphic to $\mathsf{su}(2)_{\mathbb{C}}$ as a vector space.
Other examples of equivalences are:

$$\begin{aligned}
\mathsf{su}(n)_{\mathbb{C}} &\simeq \mathsf{sl}(n;\mathbb{C}) \\
\mathsf{gl}(n;\mathbb{R})_{\mathbb{C}} &\simeq \mathsf{gl}(n;\mathbb{C}) \\
\mathsf{u}(n)_{\mathbb{C}} &\simeq \mathsf{gl}(n;\mathbb{C}) \\
\mathsf{sl}(n;\mathbb{R})_{\mathbb{C}} &\simeq \mathsf{sl}(n;\mathbb{C}) \\
\mathsf{so}(n)_{\mathbb{C}} &\simeq \mathsf{so}(n;\mathbb{C}).
\end{aligned}$$

Note that $\mathsf{gl}(n;\mathbb{R})_{\mathbb{C}} \simeq \mathsf{u}(n)_{\mathbb{C}} \simeq \mathsf{gl}(n;\mathbb{C})$, but $\mathsf{u}(n)$ is not isomorphic to $\mathsf{gl}(n;\mathbb{R})$ unless $n = 1$. The Lie Algebras $\mathsf{gl}(n;\mathbb{R})$ and $\mathsf{u}(n)$ are called real forms of the complex Lie Algebra $\mathsf{gl}(n;\mathbb{C})$. A complex Lie Algebra can have several non-equivalent real forms.

Proposition 5.2.4. *Let $\mathfrak{g}$ be a real Lie Algebra and let $\mathfrak{g}_{\mathbb{C}}$ be its complexification. Then every finite-dimensional complex representation π of $\mathfrak{g}$ has a unique extension to a complex linear representation of $\mathfrak{g}_{\mathbb{C}}$, which we will also denote by π, and it is defined in the following way:*

$$\pi(X + iY) = \pi(X) + i\pi(Y) \quad \forall X, Y \in \mathfrak{g}. \tag{5.8}$$

A proof of Proposition 5.2.4 can be found in [6], Section 3.

5.3 Examples of Representations

In this section, we will introduce three of the main representations of matrix Lie Groups and its associated Lie Algebras.

Example 5.3.1 (The Standard Representation). A matrix Lie Group G is by definition a subset of $\mathsf{GL}(n;\mathbb{C})$. Its standard representation is the inclusion map Π of G into $\mathsf{GL}(n;\mathbb{C})$, by which the group acts in its usual way. The associated Lie Algebra $\mathfrak{g}$ will be a subalgebra of $\mathsf{gl}(n;\mathbb{C})$ and we will define the standard representation of $\mathfrak{g}$ also as the inclusion map π of $\mathfrak{g}$ into $\mathsf{gl}(n,\mathbb{C})$.

Example 5.3.2 (The Trivial Representation). The trivial representation of a matrix Lie Group G is the map:

$$\Pi : G \to \mathsf{GL}(1;\mathbb{C}), \tag{5.9}$$

where $\Pi(A) = I$, for all $A \in G$.
If $\mathfrak{g}$ is the associated Lie Algebra of G, then its trivial representation is the map:

$$\pi : \mathfrak{g} \to \mathsf{gl}(1;\mathbb{C}), \tag{5.10}$$

where $\pi(X) = 0$, for all $X \in \mathfrak{g}$.
Note that both representations are irreducible as $\mathbb{C}$ does not have non-trivial subspaces.

Example 5.3.3 (The Adjoint Representation). Recall the Adjoint Mapping Ad from Definition 4.2.8:

$$Ad : G \to \mathsf{GL}(\mathfrak{g}), \tag{5.11}$$

which is a representation of a matrix Lie Group G that acts on its associated Lie Algebra $\mathfrak{g}$.
Then, the map ad given by Proposition 4.2.9:

$$ad : \mathfrak{g} \to \mathsf{gl}(\mathfrak{g}), \tag{5.12}$$

is the adjoint representation of the associated Lie Algebra $\mathfrak{g}$ of G.

Remark. In some cases, the adjoint and the standard representations are equivalent.

5.4 The representations of $\mathsf{SU}(2)$ and $\mathsf{su}(2)$

Now we will focus our attention on finding all irreducible representations of a specific Lie Group and its associated Lie Algebra. The example of the Lie Algebra $\mathsf{su}(2)$, which is the simplest compact Lie Algebra, will be important to understand the theory that follows. Moreover, $\mathsf{su}(2)$ has an important role in quantum mechanics, since $\mathsf{su}(2) \cong \mathsf{so}(3)$ and $\mathsf{so}(3)$ is related to the notion of the angular momentum.

$\mathsf{SU}(2)$ is the special unitary group which contains all 2×2 complex matrices U with determinant one such that $UU^* = U^*U = I$.
Let V_m be the $(m+1)$-dimensional vector space of homogeneous polynomials in $(z_1, z_2) \in \mathbb{C}^2$ with total degree m:

$$f(z_1, z_2) = a_0 z_1^m + a_1 z_1^{m-1} z_2 + \cdots + a_m z_2^m. \tag{5.13}$$

By definition, an element U of $\mathsf{SU}(2)$ is a linear transformation of $\mathbb{C}^2$. Let $\Pi_m(U)$ be a linear transformation of V_m defined as:

$$[\Pi_m(U)f](z_1,z_2) = f(U^{-1}(z_1,z_2)) = \sum_{k=0}^{m} a_k(U_{11}^{-1}z_1 + U_{12}^{-1}z_2)^{m-k}(U_{21}^{-1}z_1 + U_{22}^{-1}z_2)^k. \tag{5.14}$$

Note that this linear transformation maps V_m into V_m. We can also see that:

$$\begin{aligned}\Pi_m(U_1)[\Pi_m(U_2)f](z_1,z_2) &= [\Pi_m(U_2)f](U_1^{-1}(z_1,z_2)) \\ &= f(U_2^{-1}U_1^{-1}(z_1,z_2)) = \Pi_m(U_1U_2)f(z_1,z_2),\end{aligned} \tag{5.15}$$

and therefore,

$$\Pi_m : \mathsf{SU}(2) \to \mathsf{GL}(V_m) \tag{5.16}$$

is an homomorphism. Then $\Pi_m(U)$ **is a finite-dimensional complex representation of $\mathsf{SU}(2)$ acting on V_m.**

It turns out that each representation of Π_m of $\mathsf{SU}(2)$ is irreducible and that any finite-dimensional irreducible representation of $\mathsf{SU}(2)$ is equivalent to one and only one representation Π_m ([3]).

We can use this to compute the corresponding representations π_m of the Lie Algebra $\mathsf{su}(2)$. By Proposition 5.1.4:

$$\pi_m(X) = \frac{d}{dt}\Pi_m(e^{tX})\bigg|_{t=0}. \tag{5.17}$$

Since for all $U \in \mathsf{SU}(2)$, there exists $X \in \mathsf{su}(2)$ such that $U = e^{tX}$:

$$\begin{aligned}(\pi_m(X)f)(z_1,z_2) &= \frac{d}{dt}f(e^{-tX}(z_1,z_2))\bigg|_{t=0} \\ &= -\frac{\partial f}{\partial z_1}(X_{11}z_1 + X_{12}z_2) - \frac{\partial f}{\partial z_2}(X_{21}z_1 + X_{22}z_2).\end{aligned} \tag{5.18}$$

Recall that the complexification of the Lie Algebra $\mathsf{su}(2)$ is isomorphic to $\mathsf{sl}(2;\mathbb{C})$. By Proposition 5.2.4, every finite-dimensional complex representation of the Lie Algebra $\mathsf{su}(2)$, π_m, extends uniquely to a complex linear representation of $\mathsf{sl}(2;\mathbb{C})$, which we will also call π_m. Let H be the following element of $\mathsf{sl}(2;\mathbb{C})$:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.19}$$

Then,

$$\pi_m(H) = -z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}. \tag{5.20}$$

Let $z_1^k z_2^{m-k}$ be any basis vector of V_m. We can see that $z_1^k z_2^{m-k}$ is an eigenvector of π_m with eigenvalue $(m-2k)$. In fact,

$$\pi_m(H)z_1^k z_2^{m-k} = -kz_1^k z_2^{m-k} + (m-k)z_1^k z_2^{m-k} = (m-2k)z_1^k z_2^{m-k}. \tag{5.21}$$

Therefore $\pi_m(H)$ is diagonalizable.

Now let X and Y be the elements of $\mathsf{sl}(2;\mathbb{C})$ of the form:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{5.22}$$

As before,

$$\pi_m(X) = -z_2 \frac{\partial}{\partial z_1}; \quad \pi_m(Y) = -z_1 \frac{\partial}{\partial z_2}, \tag{5.23}$$

and therefore, for any vector of the basis $z_1^k z_2^{m-k}$:

$$\begin{aligned} \pi_m(X) z_1^k z_2^{m-k} &= (-k) z_1^{k-1} z_2^{m-k+1} \\ \pi_m(Y) z_1^k z_2^{m-k} &= (k-m) z_1^{k+1} z_2^{m-k-1}. \end{aligned} \tag{5.24}$$

Now we are ready to prove the following proposition.

Proposition 5.4.1. *The representation π_m from equation (5.17) is an irreducible representation of* $\mathsf{sl}(2;\mathbb{C})$.

Proof. We need to show that if a non-zero subspace of V_m is invariant, then it is V_m itself. Let W be such a subspace, then there is at least one $w \in W$ such that:

$$w = a_0 z_1^m + a_1 z_1^{m-1} z_2 + \cdots + a_m z_2^m, \tag{5.25}$$

where at least one $a_k \neq 0$. Now let k_0 be the largest value of k for which $a_k \neq 0$. Then:

$$\pi_m(X)^{k_0} w = k_0!(-1)^{k_0} a_{k_0} z_2^m, \tag{5.26}$$

since $\pi_m(X)^{k_0}$ will kill all the terms whose power of z_1 is less than k_0. Since $\pi_m(X)^{k_0} w$ is a non-zero multiple of z_2^m and we considered W to be invariant, W must contain z_2^m.
But note that $\pi_m(Y)^k z_2^m$ is a multiple of $z_1^k z_2^{m-k}$ and again, since W is invariant, it must contain all the elements of the basis of V_m. Thus, W is V_m itself. $\square$

Up to now, we have the finite-dimensional irreducible complex representations of the Lie Group $\mathsf{SU}(2)$ acting on V_m and we have related them to given representations of the Lie Algebra $\mathsf{sl}(2;\mathbb{C})$.
Now, we want to compute the finite-dimensional irreducible representations of the Lie Algebra $\mathsf{su}(2)$. We will see that they are equivalent to the representations we have just described.

Proposition 5.4.2. *Let π be a finite-dimensional complex representation of* $\mathsf{su}(2)$. *If we extend it to a finite-dimensional complex linear representation of* $\mathsf{sl}(2;\mathbb{C})$, *$\pi$ is irreducible as a representation of* $\mathsf{sl}(2;\mathbb{C})$ *if and only if it is irreducible as a representation of* $\mathsf{su}(2)$.

Proof. Suppose π is a finite-dimensional complex irreducible representation of $\mathsf{su}(2)$ acting on the vector space V. If $W \subset V$ is an invariant subspace under $\mathsf{sl}(2;\mathbb{C})$, then it is also invariant under $\mathsf{su}(2)$ since $\mathsf{su}(2) \subset \mathsf{sl}(2;\mathbb{C})$. Therefore $W = 0$ or $W = V$. This means that π is an irreducible representation of $\mathsf{sl}(2;\mathbb{C})$. $\square$

Thus, in order to study the irreducible representations of $\mathsf{su}(2)$ we will use its complexification. Let us consider again the following basis of $\mathsf{sl}(2;\mathbb{C})$:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{5.27}$$

which has the following commutation relations:

$$\begin{aligned} [H, X] &= 2X \\ [H, Y] &= -2Y \\ [X, Y] &= H. \end{aligned} \tag{5.28}$$

Now, let us consider the finite-dimensional vector space V and the operators $A, B, C \in \mathsf{gl}(V)$ which verify:

$$\begin{aligned} [A, B] &= 2B \\ [A, C] &= -2C \\ [B, C] &= A. \end{aligned} \tag{5.29}$$

The linear map:

$$\pi : sl(2;\mathbb{C}) \to \mathsf{gl}(V) \tag{5.30}$$

which sends:

$$\pi(H) = A; \quad \pi(X) = B; \quad \pi(Y) = C, \tag{5.31}$$

is a representation of $\mathsf{sl}(2;\mathbb{C})$.
We need to prove some results before being able to find all the irreducible representations of $\mathsf{su}(2)$.

Lemma 5.4.3. *Let $u \in V$ be an eigenvector of $\pi(H)$ with eigenvalue $\alpha \in \mathbb{C}$. Then,*

$$\begin{aligned} \pi(H)\pi(X)u &= (\alpha + 2)\pi(X)u, \\ \pi(H)\pi(Y)u &= (\alpha - 2)\pi(Y)u, \end{aligned} \tag{5.32}$$

and therefore $\pi(X)u$ is either an eigenvector of $\pi(H)$ with eigenvalue $\alpha + 2$ or it is zero and $\pi(Y)u$ is either an eigenvector of $\pi(H)$ with eigenvalue $\alpha - 2$ or it is zero. We will call $\pi(X)$ the ***raising operator*** *and $\pi(Y)$ the* ***lowering operator****.*

Proof. Since $[\pi(H), \pi(X)] = 2\pi(X)$, we have that:

$$\begin{aligned} \pi(H)\pi(X)u &= \pi(X)\pi(H)u + 2\pi(X)u \\ &= \pi(X)(\alpha u) + 2\pi(X)u = (\alpha + 2)\pi(X)u. \end{aligned} \tag{5.33}$$

To see that $\pi(H)\pi(Y)u = (\alpha - 2)\pi(Y)u$, we use the relation $[\pi(H), \pi(Y)] = -2\pi(Y)$. □

Since we are working over the algebraically closed field of $\mathbb{C}$, $\pi(H)$ must have at least one eigenvector u, $u \neq 0$, with some eigenvalue $\alpha \in \mathbb{C}$. Therefore, we can generalize Lemma 5.4.3:

$$\begin{aligned} \pi(H)\pi(X)^n u &= (\alpha + 2n)\pi(X)^n u, \\ \pi(H)\pi(Y)^n u &= (\alpha - 2n)\pi(Y)^n u. \end{aligned} \tag{5.34}$$

Note that an operator on a finite-dimensional vector space cannot have infinite eigenvalues, and therefore $\pi(X)^n u$ cannot be different from zero for all $n \in \mathbb{N}$. Thus, there exists some $N \in \mathbb{N}, N \geq 0$ such that:

$$\begin{aligned} \pi(X)^N u &\neq 0, \\ \pi(X)^{N+1} u &= 0. \end{aligned} \tag{5.35}$$

We will denote the eigenvector $u_0 = \pi(X)^N u$ with eigenvalue $\lambda = \alpha + 2N$. Then, we have:

$$\begin{aligned} \pi(H)u_0 &= \lambda u_0 \\ \pi(X)u_0 &= 0. \end{aligned} \tag{5.36}$$

We can also denote:

$$u_k = \pi(Y)^k u_0, \tag{5.37}$$

for $k \in \mathbb{N}, k > 0$. Note that u_k is an eigenvector of $\pi(H)$ with eigenvalue $\lambda - 2k$.

Lemma 5.4.4. *Keeping the above notations, we have:*

$$\begin{aligned} \pi(X)u_k &= [k\lambda - k(k-1)]u_{k-1} \\ \pi(X)u_0 &= 0, \end{aligned} \tag{5.38}$$

for all $k \in \mathbb{N}, k > 0$.

Proof. For $k = 1$, $u_1 = \pi(Y)u_0$. Since

$$[\pi(X), \pi(Y)] = \pi(X)\pi(Y) - \pi(Y)\pi(X) = \pi(H), \tag{5.39}$$

we have:

$$\pi(X)u_1 = \pi(X)\pi(Y)u_0 = (\pi(Y)\pi(X) + \pi(H))u_0 = \lambda u_0, \tag{5.40}$$

where we have used the fact that $\pi(X)u_0 = 0$ and $\pi(H)u_0 = \lambda u_0$.
Now, by definition, $u_{k+1} = \pi(Y)u_k$. Using equation (5.39) and the induction hypothesis, we get:

$$\begin{aligned} \pi(X)u_{k+1} &= \pi(X)\pi(Y)u_k = (\pi(Y)\pi(X) + \pi(H))u_k \\ &= \pi(Y)[k\lambda - k(k-1)]u_{k-1} + (\lambda - 2k)u_k \\ &= [k\lambda - k(k-1) + (\lambda - 2k)]u_k \\ &= [(k+1)\lambda - (k+1)k]u_k. \end{aligned} \tag{5.41}$$

□

Again, u_k cannot be different from zero for all $k \in \mathbb{N}$ as we have only a finite number of eigenvalues for $\pi_m(H)$. Thus, there exists some $m \in \mathbb{N}$ such that:

$$\begin{aligned} u_k = \pi(Y)^k u_0 &\neq 0 \quad \forall k \in \mathbb{N}, k \leq m, \\ u_{m+1} = \pi(Y)^{m+1} u_0 &= 0. \end{aligned} \tag{5.42}$$

Note that we must have $m = \lambda$. Since $u_{m+1} = 0$, then $\pi(X)u_{m+1} = 0$ and by Lemma 5.4.4, $(m+1)(\lambda - m)u_m = 0$. Since $u_m \neq 0$ and $m + 1 \neq 0$ for all $m \geq 0$, m must be equal to λ.

Proposition 5.4.5. *For any finite-dimensional complex irreducible representation of* $\mathsf{sl}(2;\mathbb{C})$*,* π*, acting on a vector space* V*, there exists an integer* m *such that the following holds:*

$$\begin{aligned}
\pi(H)u_k &= (m-2k)u_k \\
\pi(Y)u_k &= u_{k+1} \quad \forall(k<m) \\
\pi(X)u_k &= [km-k(k-1)]u_{k-1} \quad \forall(k>0) \\
\pi(Y)u_m &= 0 \\
\pi(X)u_0 &= 0.
\end{aligned} \tag{5.43}$$

A detailed proof of Proposition 5.4.5 can be found in [2].

Note that the vectors $u_0, \cdots, u_m$ must be linearly independent, since they are all eigenvectors of $\pi(H)$.
From Proposition 5.4.5, we see that the space spanned by $u_0, \cdots, u_m$ is invariant under $\pi(H)$, $\pi(X)$ and $\pi(Y)$, which are the generators of $\mathsf{sl}(2;\mathbb{C})$. Therefore, this space is invariant under the action of $\pi(Z)$, for all $Z \in \mathsf{sl}(2;\mathbb{C})$.
Since π is defined to be an irreducible representation of $\mathsf{sl}(2,\mathbb{C})$ acting on a $(m+1)$-dimensional vector space V, the vectors $u_0, \cdots, u_m$ must be a basis of V.
Every irreducible representation of $\mathsf{sl}(2;\mathbb{C})$ is of this form, and every set of operators acting on the basis vectors as in Proposition 5.4.5 is an irreducible representation of $\mathsf{sl}(2;\mathbb{C})$.
Note that any two $(m+1)$-dimensional irreducible representations of $\mathsf{sl}(2;\mathbb{C})$ acting on spaces U and V are equivalent as we can define an isomorphism $\phi : U \to V$ which sends each element of one basis u_k to its correspondent v_k.

Finally, each $(m+1)$-dimensional representation of $\mathsf{sl}(2;\mathbb{C})$ is related to an irreducible representation π_m from Proposition 5.4.1 by defining the following basis:

$$u_k = [\pi_m(Y)]^k(z_2^m) = (-1)^k \frac{m!}{(m-k)!} z_1^k z_2^{m-k}. \tag{5.44}$$

The following theorem summarizes all the results we that have stated.

Theorem 5.4.6. *For each integer* m *there is a finite-dimensional complex irreducible representation of* $\mathsf{sl}(2;\mathbb{C})$ *of dimension* $m+1$*. Any two irreducible representations of* $\mathsf{sl}(2;\mathbb{C})$ *with the same dimension are equivalent. Therefore if* π *is an irreducible representation of* $\mathsf{sl}(2;\mathbb{C})$ *of dimension* $m+1$*, then it is equivalent to* π_m *from (5.17).*

Example 5.4.7. Now we have classified all complex finite-dimensional irreducible representations of $\mathsf{su}(2)$ by labelling them with positive integer numbers. We have seen that each irreducible representation of $\mathsf{su}(2)$ is associated to a representation of $\mathsf{SU}(2)$. Recall that the Lie Group $\mathsf{SU}(2)$ is used to describe the **spin** of a particle: the eigenvalues of $\pi_m(H)$ correspond to the values of spin at which we can find the particle. Then, the raising and lowering operators are used to change the spin state of the particle.

The fundamental representation of $\mathsf{SU}(2)$, which has $m = 1$, is related to the electron, which has spin $\frac{1}{2}$. In general, we will see that representations with odd labels are related to **fermions**, which are particles that have a semi-integer spin. If we take $m = 2$, we obtain the adjoint representation, and it is related to a particle having spin 1. In particular, the adjoint representation describes the three-dimensional rotations. Thus, as we will see in the following sections, it is associated to the standard representation of $\mathsf{SO}(3)$.

5.5 Generating new representations

In this section we are going to explain two different ways of combining representations in order to obtain new ones: direct sums and tensor products.

5.5.1 Direct sums

Definition 5.5.1. Let G be a matrix Lie Group and let $\Pi_1, \cdots, \Pi_n$ be representations of G acting on the vector spaces $V_1, V_2, \cdots, V_n$ respectively. The direct sum of $\Pi_1, \cdots, \Pi_n$ is a representation of G acting on the space $V_1 \oplus \cdots \oplus V_n$, and it is defined as follows:

$$[\Pi_1 \oplus \cdots \oplus \Pi_n(A)](v_1, \cdots, v_n) = (\Pi_1(A)v_1, \cdots, \Pi_n(A)v_n), \tag{5.45}$$

for all $A \in G$.
Let $\mathfrak{g}$ be a Lie Algebra and $\pi_1, \cdots, \pi_n$ be representations of $\mathfrak{g}$ acting on the vector spaces $V_1, \cdots, V_n$, respectively. Then, the direct sum of $\pi_1, \cdots, \pi_n$ is defined analogously and it is a representation of $\mathfrak{g}$ acting on the vector space $V_1 \oplus \cdots \oplus V_n$.

Proof. Since $\Pi_1, \cdots, \Pi_n$ are Lie Group homomorphisms then it is easily proven that the representation:

$$\Pi_1 \oplus \cdots \oplus \Pi_n : G \to \mathsf{GL}(V_1 \oplus \cdots \oplus V_n)$$

is a Lie Group homomorphism. This proof works analogous for Lie Algebras. $\square$

Definition 5.5.2. Let G be a Lie Group. Let Π be a finite-dimensional representation of G acting on a vector space V. Then Π is **completely reducible** if given an invariant subspace $W \subset V$ and a second invariant subspace $U \subset W \subset V$, there exists another invariant subspace $\widetilde{U} \subset W$ such that $U \cap \widetilde{U} = 0$ and $U + \widetilde{U} = W$.

This property is defined equivalently for representations of Lie Algebras.

Proposition 5.5.3. *Any finite-dimensional completely reducible representation of a Lie Group or of a Lie Algebra acting on the vector space V is equivalent to a direct sum of one or more irreducible representations.*

Proof. If $dim(V) = 1$ then the representation is irreducible because all invariant subspaces of V are trivial. Therefore, Π is equivalent to the sum of one irreducible

representation which is itself.
Suppose that it holds for all representations of dimension less than n. If $dim(V) = n$, we will consider two cases.
If Π is irreducible, then we are done.
If Π is not irreducible, by definition there exists a non-trivial invariant subspace $U \subset V$. Then, there exists another invariant subspace $\widetilde{U} \subset V$ such that $U \cap \widetilde{U} = 0$ and $U + \widetilde{U} \cong V$. Notice that $U \oplus \widetilde{U} \cong V$ holds not only as vector spaces but as representations because U and $\widetilde{U}$ are invariant subspaces. This means that the action of the representations of the Lie Group or Lie Algebra restricted to U and $\widetilde{U}$ are representations themselves.
Since $dim(U) < dim(V)$ and $dim(\widetilde{U}) < dim(V)$, by induction we know that $\Pi|_U$ and $\Pi|_V$ can be written as a sum of irreducible representations: $\Pi|_U \cong \Pi_1 \oplus \cdots \oplus \Pi_s$ and $\Pi|_{\widetilde{U}} \cong \Pi_{s+1} \oplus \cdots \oplus \Pi_n$. Then $\Pi \cong \Pi_1 \oplus \cdots \oplus \Pi_n$ and we are done. $\square$

Note that if a group has only completely reducible finite-dimensional representations, by knowing only its finite-dimensional irreducible representations we are able to classify all its finite-dimensional representations. Therefore, we are interested in finding the groups for which this happens. In what follows we are going to prove that this condition is verified in the following situations:

- All finite-dimensional representations of a matrix Lie Group which act on a Hilbert Space are completely reducible.
- All finite-dimensional representations of finite group are completely reducible.
- All finite-dimensional representations of a compact matrix Lie Group are completely reducible.

Let us start with the matrix Lie Groups.

Proposition 5.5.4. *Let G be a matrix Lie Group and Π a finite-dimensional representation of G acting on a Hilbert Space V. Then Π is completely reducible.*

Proof. If V is a Hilbert space and, for each $A \in G$, $\Pi(A)$ is a unitary operator, we have an inner product. Then, for each invariant subspaces $W \subset V$ and $U \subset W \subset V$, we can define the following:

$$\widetilde{U} = U^{\perp} \cap W. \tag{5.46}$$

Note that $\widetilde{U} \cap U = 0$ and $\widetilde{U} + U = W$.
To see that Π is completely reducible, we need to see that $\widetilde{U}$ is invariant. Consider a vector $v \in U^{\perp} \cap W$. Since W is invariant, $\Pi(A)v \in W$ for any $A \in G$. We just need to see that $\Pi(A)v \in U^{\perp}$ for any $A \in G$. For any vectors $v \in U^{\perp}$ and $u \in U$:

$$\langle u, \Pi(A)v \rangle = \langle \Pi(A^{-1})u, \Pi(A^{-1})\Pi(A)v \rangle = \langle \Pi(A^{-1})u, v \rangle = 0. \tag{5.47}$$

Since $\Pi(A^{-1})$ is a unitary operator and U is invariant, $\Pi(A^{-1})u \in U$. This means that $\Pi(A)v \in U^{\perp}$ and thus $\widetilde{U}$ is invariant. $\square$

Proposition 5.5.5. *Let G be a finite group. Then every finite-dimensional representation of G is completely reducible.*

Proof. Let Π be a representation of a finite group G acting on a vector space V. Consider the inner product $\langle , \rangle$:

$$\langle v_1, v_2 \rangle_G = \sum_{g \in G} \langle \Pi(g)v_1, \Pi(g)v_2 \rangle. \tag{5.48}$$

We need to see that Π is a unitary representation with respect to this inner product.

$$\begin{aligned} \langle \Pi(h)v_1, \Pi(h)v_2 \rangle_G &= \sum_{g \in G} \langle \Pi(g)\Pi(h)v_1, \Pi(g)\Pi(h)v_2 \rangle \\ &= \sum_{g \in G} \langle \Pi(gh)v_1, \Pi(gh)v_2 \rangle = \langle v_1, v_2 \rangle_G, \end{aligned} \tag{5.49}$$

since gh ranges over G.
By Proposition 5.5.4, this means that Π is a completely reducible representation. □

Proposition 5.5.6. *If G is a compact matrix Lie Group, then every finite-dimensional representation of G is completely reducible.*

Proof. The idea behind the proof is based on the notion of *left Haar measure.*
A left Haar measure is a non-zero measure μ on the Borel σ-algebra in G such that it is locally finite and left-translation invariant.
It can be proven that every matrix Lie Group has a left Haar measure which is unique up to multiplication by constant and that in the case of compact matrix Lie Groups, the left Haar measure is finite.
Therefore, if Π is a representation of a compact matrix Lie Group G acting on the vector space V, the following inner product $\langle , \rangle_G$ is well-defined on V:

$$\langle v_1, v_2 \rangle_G = \int_G \langle \Pi(g)v_1, \Pi(g)v_2 \rangle d\mu(g). \tag{5.50}$$

It is easy to check that it is an inner product. Now we want see that Π is a unitary representation with respect to $\langle , \rangle_G$. For any $h \in G$:

$$\begin{aligned} \langle \Pi(h)v_1, \Pi(h)v_2 \rangle_G &= \int_G \langle \Pi(g)\Pi(h)v_1, \Pi(g)\Pi(h)v_2 \rangle d\mu(g) \\ &= \int_G \langle \Pi(gh)v_1, \Pi(gh)v_2 \rangle d\mu(g) = \langle v_1, v_2 \rangle_G. \end{aligned} \tag{5.51}$$

Hence, by Proposition 5.5.4, it is completely reducible. □

5.5.2 Tensor products

First, we will define the tensor product between vectors spaces, and then apply this notions to Lie Group and Lie Algebra representations.

Definition 5.5.7. Let U and V be finite-dimensional vector spaces. The **tensor product of** U **and** V is a vector space W together with a bilinear map $\phi : U \times V \to W$ such that if ψ is any bilinear map of $U \times V$ into a vector space X, then there exists a unique linear map $\widetilde{\psi}$ of W into X such that $\widetilde{\psi}(\phi(u)) = \psi(u)$, for all $u \in U \times V$, i.e the following diagram commutes:

$$\begin{array}{c} \psi : U \times V \to X \\ \phi \downarrow \quad \nearrow \widetilde{\psi} \\ W \end{array}$$

Theorem 5.5.8. *Let U and V be finite-dimensional vector spaces. Then the tensor product $U \times V$ exists, and we will call it (W, ϕ). It is unique up to canonical isomorphism. That means that if $(W_1, \phi_1), (W_2, \phi_2)$ are two tensor products of U and V, then there exist a unique isomorphism $\Psi : W_1 \to W_2$ such that we have the following commutation diagram:*

$$\begin{array}{c} \phi_1 : U \times V \to W_1 \\ \phi_2 \downarrow \quad \nearrow \psi \\ W_2 \end{array}$$

This is known as the **Universal Property** of the tensor product (a more detailed explanation can be found in [8]). Since the tensor product is unique, we will denote $\phi(u, v)$ as $\mathbf{u} \otimes \mathbf{v}$.

As a consequence of the Universal Property, if $e_1, \cdots, e_n$ and $f_1, \cdots, f_m$ are basis of U and V respectively, then $\{e_i \otimes f_j \mid 0 \leq i \leq n, 0 \leq j \leq m\}$ is a basis for $U \otimes V$. In particular, $dim(U \otimes V) = dim(U)dim(V)$.
When defining a bilinear map ψ from $U \otimes V$ into another space, it suffices to define it over the elements of the form $u \otimes v$ and extend it by linearity to $U \otimes V$, since $\psi(U \otimes V)$ will be bilinear in $U \times V$.

Proposition 5.5.9. *Let U and V be finite-dimensional real or complex vector spaces. Let $A : U \to U$ and $B : V \to V$ be linear maps. Then there exists a unique linear map from $U \otimes V$ to $U \otimes V$ denoted $A \otimes B$ such that:*

$$(A \otimes B)(u \otimes v) = (Au) \otimes (Bv), \tag{5.52}$$

for all $u \in U$ and $v \in V$. Moreover, if A_1, A_2 and B_1, B_2 are linear operators acting on U and V respectively, then:

$$(A_1 \otimes B_1)(A_2 \otimes B_2) = (A_1 A_2) \otimes (B_1 B_2) \tag{5.53}$$

Proof. First, we need to define a map ψ from $U \times V$ into $U \otimes V$ such that:

$$\psi(u, v) = (Au) \otimes (Bv). \tag{5.54}$$

Since A and B are linear operators and the tensor product is bilinear, ψ is a bilinear map. By the Universal Property of the tensor product, there exists a linear map $\widetilde{\psi}$ such that:

$$\widetilde{\psi}(u \otimes v) = \psi(u, v) = (Au) \otimes (Bv). \tag{5.55}$$

Then $\widetilde{\psi}$ is the map we are looking for.
Let A_1, A_2 and B_1, B_2 be linear operators acting on U and V respectively. Then,

$$(A_1 \otimes B_1)(A_2 \otimes B_2)(u \otimes v) = (A_1 \otimes B_1)(A_2 u \otimes B_2 v) = A_1 A_2 u \otimes B_1 B_2 v. \quad (5.56)$$

This holds for elements of the form $u \otimes v$, but since they generate $U \otimes V$ we are done. ☐

Now we are going to apply the concepts we have defined for vector spaces to representations.
Let G be a closed subgroup of $\mathsf{GL}(n, \mathbb{C})$ and H a closed subgroup of $\mathsf{GL}(m, \mathbb{C})$. Then, $G \times H$ is a closed subgroup of $\mathsf{GL}(m + n, \mathbb{C})$. Therefore, if G and H are matrix Lie Groups, $G \times H$ is also a matrix Lie Group.

Definition 5.5.10. Let G and H be matrix Lie Groups. Let Π_1 be a finite-dimensional representation of G acting on a vector space U and Π_2 be a finite-dimensional representation of H acting on a vector space V. The **tensor product of Π_1 and Π_2**, $\Pi_1 \otimes \Pi_2$, is a representation of $G \times H$ acting on $U \otimes V$ defined as follows:

$$(\Pi_1 \otimes \Pi_2)(A, B) = \Pi_1(A) \otimes \Pi_2(B), \quad (5.57)$$

for all $A \in G$, $B \in H$.

Proposition 5.5.11. *Let $\mathfrak{g}$ be the Lie Algebra associated to a matrix Lie Group G and let $\mathfrak{h}$ be the Lie Algebra associated to a matrix Lie Group H. Then, the algebra of the matrix Lie Group $G \times H$ is isomorphic to $\mathfrak{g} \oplus \mathfrak{h}$.*

A proof of Proposition 5.5.11 can be found in [3].

Proposition 5.5.12. *Let G and H be matrix Lie Groups, and let Π_1, Π_2 be representations of G and H respectively. Consider the representation $\Pi_1 \otimes \Pi_2$ of $G \times H$. Then the associated representation $\pi_1 \otimes \pi_2$ of the Lie Algebra $\mathfrak{g} \oplus \mathfrak{h}$, verifies:*

$$(\pi_1 \otimes \pi_2)(X, Y) = \pi_1(X) \otimes I + I \otimes \pi_2(Y). \quad (5.58)$$

Proof. Let $u(t)$ be a smooth curve in U and let $v(t)$ be a smooth curve in V. Since the product rule also holds for tensor products:

$$\frac{d}{dt}(u(t) \otimes v(t)) = \frac{du}{dt} \otimes v(t) + u(t) \otimes \frac{dv}{dt}. \quad (5.59)$$

Then,

$$\begin{aligned}
(\pi_1 \otimes \pi_2)(X, Y)(u \otimes v) &= \frac{d}{dt}\Pi_1 \otimes \Pi_2(e^{tX}, e^{tY})(u \otimes v)\big|_{t=0} \quad (5.60)\\
&= \frac{d}{dt}\Pi_1(e^{tX})u \otimes \Pi_2(e^{tY})v\big|_{t=0}\\
&= (\frac{d}{dt}\Pi_1(e^{tX})u\big|_{t=0} \otimes v) + u \otimes (\Pi_2(e^{tY})v\big|_{t=0}).
\end{aligned}$$

This works for elements of the form $u \otimes v$. Since they generate the $U \otimes V$, we are done. ☐

Now we are ready to define the tensor product between representations of general Lie Algebras.

Definition 5.5.13. Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie Algebras and let π_1 and π_2 be representations of $\mathfrak{g}$ and $\mathfrak{h}$ acting on the vector spaces V and U respectively. The **tensor product of π_1 and π_2**, $\pi_1 \otimes \pi_2$, is a representation of $\mathfrak{g} \oplus \mathfrak{h}$ acting on the vector space $U \otimes V$ which verifies:

$$(\pi_1 \otimes \pi_2)(X, Y) = \pi_1(X) \otimes I + I \otimes \pi_2(Y), \tag{5.61}$$

for all $X \in \mathfrak{g}$ and all $Y \in \mathfrak{h}$.

When we are dealing with two representations of the same Lie Group or Lie Algebra, we use the following definitions.

Definition 5.5.14. Let G be a matrix Lie Group and Π_1, Π_2 two representations of G acting on the vector spaces V_1 and V_2 respectively. The **tensor product of Π_1 and Π_2** is a representation of G acting on the vector space $V_1 \otimes V_2$ such that:

$$(\Pi_1 \otimes \Pi_2)(A) = \Pi_1(A) \otimes \Pi_2(A), \tag{5.62}$$

for all $A \in G$.

Definition 5.5.15. Let $\mathfrak{g}$ be a Lie Algebra and let π_1, π_2 be two representations of $\mathfrak{g}$ acting on the vector spaces V_1 ,V_2 respectively. Then, **the tensor product of π_1 and π_2** is a representation of $\mathfrak{g}$ acting on the vector space $V_1 \otimes V_2$ such that:

$$(\pi_1 \otimes \pi_2)(X) = \pi_1(X) \otimes I + I \otimes \pi_2(X), \tag{5.63}$$

for all $X \in \mathfrak{g}$.

Suppose we have two irreducible representations of a Lie Group G. The tensor product of them might not longer be irreducible. If it is not, the process of decomposing it as a direct sum of irreducible representations is done by using **Clebsch-Gordan theory** ([4], Section 12).

Example 5.5.16. Tensor products of representations of Lie Groups are used in dealing with systems in quantum mechanics which have two or more particles. For example, suppose we want to compute the total spin of a system composed of four light quarks. As we saw in Example 5.4.7, the spin of a particle is defined by a given representation of the Lie Group $\mathsf{SU}(2)$. To obtain the representation of $\mathsf{SU}(2)$ which defines the total spin of the system, we need to compute the tensor product of the four irreducible representations Π_2 with dimension 2 ($m = 1$) of $\mathsf{SU}(2)$, corresponding to particles with spin $\frac{1}{2}$. The result is not an irreducible representation but a completely reducible representation which decomposes as follows:

$$\Pi_2 \otimes \Pi_2 \otimes \Pi_2 \otimes \Pi_2 \cong \Pi_5 \oplus \Pi_3 \oplus \Pi_3 \oplus \Pi_3 \oplus \Pi_1 \oplus \Pi_1. \tag{5.64}$$

Each of the components of the direct sum is related to a possible state is the system with a given symmetry.

5.6 Schur's Lemma

We will state Schur's Lemma for complex Lie Groups, but **it is analogous for complex Lie Algebras**.

Theorem 5.6.1. ***Schur's Lemma.***

(1) Let Π_1 and Π_2 be two irreducible complex representations of a Lie Group G acting on the vector spaces V and W respectively. Let $\phi : V \to W$ be a morphism between the representations. Then either $\phi = 0$ or ϕ is an isomorphism.

(2) Let Π be an irreducible complex representation of a Lie Group G acting on a vector space V. Let $\phi : V \to V$ be a morphism from Π into itself. Then $\phi = \lambda I$ with $\lambda \in \mathbb{C}$.

(3) Let Π_1 and Π_2 be two irreducible complex representations of a Lie Group G acting on the vector spaces V and W respectively. Let $\phi_1, , \phi_2 : V \to W$ be two non-zero morphisms. Then $\phi_1 = \lambda \phi_2$ with $\lambda \in \mathbb{C}$.

Proof. (1) Since ϕ is a morphism, by Definition 5.1.3, $\phi(\Pi_1(A)v) = \Pi_2(A)(\phi(v))$ for all $v \in V$ and $A \in G$.
Assume that $v \in ker(\phi)$ (and thus $\phi(v) = 0$). Then,

$$\phi(\Pi_1(A)v) = \Pi_2(A)\phi(v) = 0, \tag{5.65}$$

and therefore $ker(\phi)$ is an invariant subspace of $\Pi_1(A)$ for all $A \in G$. Since Π_1 is an irreducible representation, we must have $ker(\phi) = 0$ or $\ker(\phi) = V$.
Thus, ϕ is either one-to-one or it is zero.
Let us suppose it is one-to-one. In this case, the image of ϕ is a non-zero subspace of W. It is invariant under the action of $\Pi_2(A)$ for all $A \in G$, since for all $w \in W$ which is in the image of ϕ:

$$\Pi_2(A)w = \Pi_2(A)\phi(V) = \phi(\Pi_1(A)v). \tag{5.66}$$

Since Π_2 is also an irreducible representation, the image of ϕ must be W.
Then ϕ is either zero or it is an isomorphism.
(2) Let V be an irreducible complex representation and $\phi : V \to V$ a morphism of V into itself. This means that $\phi(\Pi(A)) = \Pi(A)(\phi(v))$ for all $A \in G$ and for all $v \in V$. Since we are working over an algebraically complete field (because we have a complex representation), ϕ must have at least one eigenvalue λ with an associated eigenspace U. That means that for all $u \in U$ and for all $A \in G$:

$$\phi(\Pi(A)u) = \Pi(A)\phi(u) = \lambda\Pi(A)u. \tag{5.67}$$

Then, $\Pi(A)u \in U$ and therefore U is invariant. Since Π is an irreducible representation and $U \neq 0$, then we must have $U = V$. Therefore $\phi = \lambda I$.
(3) If $\phi_2 \neq 0$, by (1) it is an isomorphism. Therefore we can compute $\phi_1 \circ \phi_2^{-1}$, which is a morphism of W into itself. By (2), $\phi_1 \circ \phi_2^{-1} = \lambda I$, and then $\phi_1 = \lambda\phi_2$. □

Corollary 5.6.2. *Let* Π *be an irreducible complex representation of a matrix Lie Group* G *acting on a vector space* V*. Let* $A \in G$ *be in the center of the group* $Z(G)$*. Then* $\Pi(A) = \lambda I$*. It is analogous for Lie Algebras.*

Proof. If $A \in Z(G)$, for all $B \in G$ and for all $v \in V$,

$$\Pi(A)\Pi(B)v = \Pi(AB)v = \Pi(BA)v = \Pi(B)\Pi(A)v. \tag{5.68}$$

Therefore, $\Pi(A)$ is a morphism of V into itself. By Schur's Lemma (2), it is a multiple of the identity. □

Corollary 5.6.3. *Any irreducible complex representation of a commutative Lie Group or Lie Algebra is one-dimensional.*

Proof. If G is a commutative Lie Group, then $G = Z(G)$. By the previous corollary, $\Pi(A) = \lambda I$ for any $A \in G$. Then, any subspace of V is invariant. Since Π is an irreducible representation, V cannot have non-trivial subspaces and it must be one-dimensional. □

6 Relation between Lie Group and Lie Algebra representations

Each representation of a Lie Group G defines a representation of its associated Lie algebra $\mathfrak{g}$ (Proposition 5.1.4). The goal of this section is to find out how it works the other way. More precisely, we would like to answer the following question.

Question: Is there a representation of the Lie Group associated to each representation of its Lie Algebra?

Theorem 6.0.1.
(1) Let G, H be matrix Lie groups, let ϕ_1, $\phi_2 : G \to H$ be Lie Group homomorphisms, and let $\widetilde{\phi_1}, \widetilde{\phi_2} : \mathfrak{g} \to \mathfrak{h}$ be the associated Lie Algebra homomorphisms. Assume that G is connected and that $\widetilde{\phi_1} = \widetilde{\phi_2}$. Then, $\phi_1 = \phi_2$.
(2) Let G and H be matrix Lie Groups with associated Lie Algebras $\mathfrak{g}$ and $\mathfrak{h}$. Let $\widetilde{\phi} : \mathfrak{g} \to \mathfrak{h}$ be a Lie Algebra homomorphism. If G is connected and simply connected, then there exists a unique Lie Group homomorphism $\phi : G \to H$ such that ϕ and $\widetilde{\phi}$ are related as in Theorem 4.2.7.

Proof. (1) Since G is connected, by Proposition 4.4.4, every element $A \in G$ can be written as $A = e^{X_1}e^{X_2}\cdots e^{X_n}$, where $X_i \in \mathfrak{g}$.
If $\widetilde{\phi_1} \cong \widetilde{\phi_2}$:

$$\phi_1(e^{X_1}\cdots e^{X_n}) = e^{\widetilde{\phi_1}(X_1)}\cdots e^{\widetilde{\phi_1}(X_n)} = e^{\widetilde{\phi_2}(X_1)}\cdots e^{\widetilde{\phi_2}(X_n)} = \phi_2(e^{X_1}\cdots e^{X_n}), \quad (6.1)$$

and thus $\phi_1 \cong \phi_2$.
(2) By Definition 4.1.4, the inverse of the exponential mapping, log, is well-defined in a neighborhood of the identity of G, which we will denote by V. We will define ϕ as:

$$\phi(A) = exp\{\widetilde{\phi}(log(A))\}, \quad (6.2)$$

for all $A \in V$. Note that ϕ maps V into H.
First, we need to prove that ϕ is a local homomorphism, which means that for all $A, B \in V$, if AB is in V, $\phi(AB) = \phi(A)\phi(B)$. To this end, we need to use a Corollary of the Baker-Campbell-Hausdorff formula.
One form of the Baker-Campbell-Hausdorff formula states that for two $n\times n$ complex and sufficiently small matrices X and Y,

$$log(e^X e^Y) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \cdots . \quad (6.3)$$

From this fact it follows that $\widetilde{\phi}$ is a local homomorphism. That is because all terms in (6.3) are in terms of X, Y or brackets of X and Y. Therefore,

$$\begin{aligned}\widetilde{\phi}(log(e^X e^Y)) &= \widetilde{\phi}(X) + \widetilde{\phi}(Y) + \frac{1}{2}[\widetilde{\phi}(X), \widetilde{\phi}(Y)] + \frac{1}{12}[\widetilde{\phi}(X), [\widetilde{\phi}(X), \widetilde{\phi}(Y)]] + \cdots \\ &= log(e^{\widetilde{\phi}(X)} e^{\widetilde{\phi}(Y)}).\end{aligned} \quad (6.4)$$

Then,
$$\phi(e^X e^Y) = e^{log(e^{\widetilde{\phi}(X)}e^{\widetilde{\phi}(Y)})} = e^{\widetilde{\phi}(X)}e^{\widetilde{\phi}(Y)} = \phi(e^X)\phi(e^Y). \tag{6.5}$$
A complete formulation of the Baker-Campbell-Hausdorff formula can be found in [6], Section 4.

Now we will define ϕ along a given path. Since G has the notion of connectedness from Definition 2.1.3, for any $A \in G$ there exists a path $A(t) \in G$ with $A(0) = I$ and $A(1) = A$. Then, there exist numbers $0 = t_0 < t_1 < \cdots < t_n = 1$ such that:
$$A(s)A(t_i)^{-1} \in V, \tag{6.6}$$
for all $s \in [t_i, t_{i+1}]$.
In particular, for $i = 0$ we have $A(s) \in V$ for $0 \leq s \leq t_1$, and thus we can define $\phi(A(s))$ for $s \in [0, t_1]$.
For $s \in [t_1, t_2]$ we can write:
$$A(s) = [A(s)A(t_1)^{-1}]A(t_1), \tag{6.7}$$
where $A(s)A(t_1)^{-1} \in V$. Then:
$$\phi(A(s)) = \phi([A(s)A(t_1)^{-1}]A(t_1)) = \phi(A(s)A(t_1)^{-1})\phi(A(t_1)), \tag{6.8}$$
where $A(t_1)$ has already been defined. Since $\phi(A(s)A(t_1)^{-1}) \in V$, we can define it by (6.2).
Proceeding on the same way, we can define $\phi(A(s))$ on the whole interval $[0, 1]$. In particular, we will have $\phi(A(1)) = \phi(A)$.
In order to use the procedure above as a definition of $\phi(A)$, we need to see that the result is independent of the choice of the path and independent of the choice of the partition $(t_0, t_1, \cdots, t_n)$.
First we are going to see that when we consider the refinement of a particular partition the result does not change. Then we will have independence of partition because for any two partitions we always have a common refinement which is the union of the two.
Suppose we insert an extra partition point s between t_0 and t_1. Under this new partition:
$$\phi(A(t_1)) = \phi([A(t_1)A(s)^{-1}]A(s)) = exp \circ \widetilde{\phi} \circ \log(A(t_1)A(s)^{-1}) exp \circ \widetilde{\phi} \circ \log(A(s)). \tag{6.9}$$
As we have already seen, by the Baker-Campbell-Hausdorff formula, if two elements A and B are close enough to the identity,
$$exp \circ \widetilde{\phi} \circ log(AB) = [exp \circ \widetilde{\phi} \circ log(A)][exp \circ \widetilde{\phi} \circ log(B)]. \tag{6.10}$$
Therefore
$$\phi(A(t_1)) = exp \circ \widetilde{\phi} \circ \log(A(t_1)), \tag{6.11}$$
which corresponds to the old definition.
To see independence of path, we will use the fact that G is simply connected. Then, any two paths $A(t_1)$ and $A(t_2)$ joining the identity to A will be homotopic.

The idea is to deform A_1 into A_2 in series of steps, and in each step we will only change a small time interval of the path $(t, t+\epsilon)$, to see if the result changes. Since we have independence of partition, we will take t and $t+\epsilon$ to be consecutive partition points. Then:

$$\phi(A(t+\epsilon)) = \phi(A(t+\epsilon)A(t)^{-1})\phi(A(t)). \tag{6.12}$$

The value of $\phi(A(t+\epsilon))$ depends only on $A(t)$ and $A(t+\epsilon)$ but not on the path between them. Therefore the result does not change when we deform the path between t and $t+\epsilon$.
If we consider series of small steps as above, we can deform A_1 into A_2 without changing the result. Finally, we need to prove that ϕ is an homomorphism and that is properly related to $\widetilde{\phi}$.
Since G is connected, for any $A \in G$, A can be written as follows:

$$A = C_n C_{n-1} \cdots C_1, \tag{6.13}$$

where $C_i \in V$. Then, we can choose a path and a partition $(t_1, \cdots, t_n)$ such that:

$$A(t_i) = C_i C_{i-1} \cdots C_1. \tag{6.14}$$

Then,

$$\phi(A) = \phi(A(1)A(t_{n-1})^{-1}) \cdots \phi(A(t_1)A(0)) = \phi(C_n)\phi(C_{n-1}) \cdots \phi(C_1), \tag{6.15}$$

since

$$A(t_i)A(t_{i-1})^{-1} = (C_i C_{i-1} \cdots C_1)(C_{i-1} \cdots C_1)^{-1} = C_i. \tag{6.16}$$

To see that ϕ is an homomorphism, let A and B be two elements of G such that:

$$A = C_n C_{n-1} \cdots C_1 \tag{6.17}$$
$$B = D_n D_{n-1} \cdots D_1. \tag{6.18}$$

Then,

$$\phi(AB) = \phi(C_n C_{n-1} \cdots C_1 D_n D_{n-1} \cdots D_1) \tag{6.19}$$
$$= \phi(C_n) \cdots \phi(C_1)\phi(D_n) \cdots \phi(D_1) = \phi(A)\phi(B). \tag{6.20}$$

Finally, we have to check if ϕ is properly related to $\widetilde{\phi}$. Since near the identity $\phi = exp \circ \widetilde{\phi} \circ log$, we get,

$$\frac{d}{dt}\phi(e^{tX})\Big|_{t=0} = \frac{d}{dt}e^{t\widetilde{\phi}(X)}\Big|_{t=0} = \widetilde{\phi}(X). \tag{6.21}$$

□

Now we will state two useful consequences of this Theorem.

Corollary 6.0.2. *Let G and H be two connected and simply connected matrix Lie Groups with associated Lie Algebras $\mathfrak{g}$ and $\mathfrak{h}$. If $\mathfrak{g} \cong \mathfrak{h}$, then $G \cong H$.*

Proof. Let $\widetilde{\phi}$ be the isomorphism between the Lie Algebras $\mathfrak{g}$ and $\mathfrak{h}$. By Theorem 6.0.1, there exists an associated Lie Group homomorphism $\phi : G \to H$.
Now, since $\widetilde{\phi}^{-1} : \mathfrak{h} \to \mathfrak{g}$ is also a Lie Algebra homomorphism, there is a corresponding Lie Group homomorphism $\psi : H \to G$. If we prove that $\psi = \phi^{-1}$, then ϕ is the isomorphism we are looking for.
We know that $\widetilde{\phi \circ \psi} = \widetilde{\phi} \circ \widetilde{\psi} = I_{\mathfrak{h}}$. Then, by Theorem 6.0.1 (1), $\phi \circ \psi = I_H$. Equivalently, we find that $\psi \circ \phi = I_G$. □

Corollary 6.0.3.
(1) Let G be a connected matrix Lie Group and let Π_1 and Π_2 be two representations of G. Let $\mathfrak{g}$ be the associated Lie Algebra of G and let π_1, π_2 be representations of $\mathfrak{g}$ from Proposition 5.1.4. If $\pi_1 \cong \pi_2$ then $\Pi_1 \cong \Pi_2$.

(2) Let G be a connected and simply connected matrix Lie Group. Then if π is a representation of its associated Lie Algebra $\mathfrak{g}$, there exists a representation Π of G acting on the same space, such that π and Π are related as in Proposition 5.1.4.

Remark. Note that Corollary 6.0.3 implies that there is a bijection between isomorphism classes of representations of G and those of $\mathfrak{g}$.

Proof. (1) Let Π_1 and Π_2 be representations acting on the vector spaces V and W respectively. Assume that $\pi_1 \cong \pi_2$. This means that there exist an invertible map $\phi : V \to W$ such that $\phi(\pi_1(X)v) = \pi_2(X)\phi(v)$ for all $X \in \mathfrak{g}$ and all $v \in V$ (i.e. there exists an isomorphism). This is equivalent to saying that $\phi\pi_1(X) = \pi_2(X)\phi$ and therefore $\phi\pi_1(X)\phi^{-1} = \pi_2(X)$.
Let $\Sigma_2 : G \to \mathsf{GL}(W)$ be the following homomorphism:

$$\Sigma_2(A) = \phi\Pi_1(A)\phi^{-1}. \tag{6.22}$$

By Proposition 5.1.4, the associated Lie Algebra representation is:

$$\sigma_2(X) = \phi\pi_1(X)\phi^{-1} = \pi_2(X), \tag{6.23}$$

for all $X \in \mathfrak{g}$. Then by Theorem 6.0.1 (1), $\Sigma_2 = \Pi_2$ and therefore $\phi\Pi_1\phi^{-1} = \Pi_2$, which shows that $\Pi_1 \cong \Pi_2$.
(2) It follows from Theorem 6.0.1, (2) if we let H be $\mathsf{GL(V)}$. □

The results of this section agree with the case of the Lie Group $SU(2)$, which is a simply connected group.

6.1 Covering Groups

Definition 6.1.1. Let G be a connected matrix Lie Group. A **universal covering** of G is a connected, simply connected Lie Group $\widetilde{G}$ together with a surjective Lie Group homomorphism $\phi : \widetilde{G} \to G$ such that there exists a neighborhood U of the identity in $\widetilde{G}$ which maps homeomorphically under ϕ onto a neighbourhood V of the identity in G.

The notion of universal cover will allow us to determine which representations of the associated Lie Algebra correspond to representations of the Lie Group.

Proposition 6.1.2. *If G is any connected matrix Lie Group, then the universal covering $\widetilde{G}$ of G exists and it is unique (up to isomorphism).*

Proof. Since G is a connected matrix Lie Group, it is a manifold. As a manifold, G has a unique topological cover $\widetilde{G}$ which is a connected and simply connected manifold Moreover, there exists a projection map $\phi : \widetilde{G} \to G$ which is a local homeomorphism. It can be proven that since G is a group, then $\widetilde{G}$ is also a group and ϕ is an homomorphism. $\square$

Proposition 6.1.3. *Let G be a connected matrix Lie Group. Let $\widetilde{G}$ be its universal cover and let ϕ be the projection map from $\widetilde{G}$ to G. Assume that $\widetilde{G}$ is a matrix Lie Group with Lie Algebra $\widetilde{\mathfrak{g}}$. Then, the associated Lie Algebra map $\widetilde{\phi}$ from $\widetilde{\mathfrak{g}}$ to $\mathfrak{g}$ is an isomorphism.*

Due to this result we say that G and its universal cover $\widetilde{G}$ have the same Lie Algebra.

Theorem 6.1.4. *Let G and its universal cover $\widetilde{G}$ be matrix Lie Groups and let $\mathfrak{g}$, $\widetilde{\mathfrak{g}}$ be their Lie Algebras. Let H be another matrix Lie Group with Lie Algebra $\mathfrak{h}$, and let $\widetilde{\psi} : \mathfrak{g} \to \mathfrak{h}$ be an homomorphism. Then, there exists a unique Lie Group homomorphism $\psi : \widetilde{G} \to H$ such that ψ and $\widetilde{\psi}$ are related as in Theorem 4.2.7.*

Proof. The universal cover $\widetilde{G}$ of G is a simply connected Lie Group with Lie Algebra $\mathfrak{g}$. Then, if we have an homomorphism $\widetilde{\psi} : \mathfrak{g} \to \mathfrak{h}$, by Theorem 6.0.1, there exist a unique homomorphism from $\widetilde{G}$ to H. $\square$

Corollary 6.1.5. *Let G and its universal cover $\widetilde{G}$ be matrix Lie Groups. Let $\mathfrak{g}$ be the associated Lie Algebra of G and let π be a representation of $\mathfrak{g}$. Then, there exists a unique representation $\widetilde{\Pi}$ of $\widetilde{G}$ such that:*

$$\pi(X) = \frac{d}{dt}\widetilde{\Pi}(e^{tX})\big|_{t=0}, \tag{6.24}$$

for all $X \in \mathfrak{g}$.

Now we will see a few examples to illustrate the above notions.

Example 6.1.6. The universal cover of S^1 is $\mathbb{R}$ and the projection map ϕ is the map which takes x to e^{ix}.

Example 6.1.7. The Lie Group $\mathsf{SO}(3)$ is not simply connected. Its universal cover is $\mathsf{SU}(2)$. The Lie Algebras $\mathsf{su}(2)$ and $\mathsf{so}(3)$ are isomorphic. Therefore, they have the same irreducible representations. We are interested in finding a relation between these representations and the irreducible representations of $\mathsf{SO}(3)$. We will see that it does not exist a relation for all of them, and that is the reason why we need to

define a universal cover for $\mathsf{SO}(3)$.
Consider the following basis for the Lie Algebra $\mathsf{su}(2)$:

$$E_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}; \quad E_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad E_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{6.25}$$

and the following basis for the Lie Algebra $\mathsf{so}(3)$:

$$F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}; \quad F_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}; \quad F_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.26}$$

Then we have the following commutation relations:

$$[E_1, E_2] = E_3; \quad [E_2, E_3] = E_1; \quad [E_3, E_1] = E_2; \tag{6.27}$$

$$[F_1, F_2] = F_3; \quad [F_2, F_3] = F_1; \quad [F_3, F_1] = F_2. \tag{6.28}$$

Note that the map $\phi : \mathsf{so}(3) \to \mathsf{su}(2)$ which takes F_i to E_i is a Lie Algebra isomorphism.
Then, if π is a representation of $\mathsf{su}(2)$, $\pi \circ \phi$ is a representation of $\mathsf{so}(3)$. All irreducible representations of $\mathsf{so}(3)$ are of the form $\sigma_m = \pi_m \circ \phi$ where π_m are the irreducible representations of $\mathsf{su}(2)$ from Theorem 5.4.6.

We need to state a Lemma in order to prove next Proposition. Using the above notation:

Lemma 6.1.8. *There exists a group homomorphism $\Phi : \mathsf{SU}(2) \to \mathsf{SO}(3)$ such that:*

(1) Φ maps $\mathsf{SU}(2)$ onto $\mathsf{SO}(3)$.

(2) $ker\Phi = \{I, -I\}$.

(3) The associated Lie Algebra isomorphism $\widetilde{\Phi} : \mathsf{su}(2) \to \mathsf{so}(3)$ takes E_i to F_i and therefore it is ϕ^{-1}.

Proposition 6.1.9. *Let $\sigma_m = \pi_m \circ \phi$ be the irreducible complex representations from the Lie Algebra $\mathsf{so}(3)$ for $m \geq 0$. Then if m is even, there exists a representation of the group $\mathsf{SO}(3)$, Σ_m, such that σ_m and Σ_m are related as in Proposition 5.1.4. If m is odd, there is no such representation of $\mathsf{SO}(3)$.*

Proof. (1) Assume m is odd. Suppose that there is a representation of $\mathsf{SO}(3)$, Σ_m, such that $\Sigma_m(e^X) = e^{\sigma_m(X)}$ for all $X \in \mathsf{so}(3)$, where σ_m is a representation of the Lie algebra $\mathsf{so}(3)$.
In particular, let us take $X = 2\pi F_1$. Then,

$$e^{2\pi F_1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & cos2\pi & -sin2\pi \\ 0 & sin2\pi & cos2\pi \end{pmatrix} = I. \tag{6.29}$$

We see that $\Sigma_m(e^{2\pi F_1}) = \Sigma_m(I) = I$ but at the same time, $\Sigma_m(e^{2\pi F_1}) = e^{2\pi\sigma_m(F_1)}$. By definition of σ_m , $\sigma_m(F_1) = \pi_m(\phi(F_1)) = \pi_m(E_1) = \pi_m(\frac{i}{2}H)$, where:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.30}$$

Suppose that the representation π_m acts on the vector space V_m. We know that there exists a basis $u_0, u_1, ..., u_m$ for V_m such that u_k is an eigenvector for $\pi_m(H)$ with eigenvalue $m - 2k$. Then, for $k = 1, \cdots, m$, u_k is an eigenvector for $\frac{i}{2}\pi_m(H)$, with eigenvalue $\frac{i}{2}(m - 2k)$. Therefore,

$$\pi_m(\frac{H}{2}) = \begin{pmatrix} \frac{i}{2}m & & & \\ & \frac{i}{2}(m-2) & & \\ & & \ddots & \\ & & & \frac{i}{2}(-m) \end{pmatrix}, \tag{6.31}$$

and,

$$e^{2\pi\sigma_m(F_1)} = \begin{pmatrix} e^{2\pi\frac{i}{2}m} & & & \\ & e^{2\pi\frac{i}{2}(m-2)} & & \\ & & \ddots & \\ & & & e^{2\pi\frac{i}{2}(-m)} \\ . & & & \end{pmatrix} \tag{6.32}$$

Since m is odd, $(m - 2k)$ is odd too, and therefore $e^{2\pi\sigma_m(F_1)} = -I$. This is a contradiction as we had $e^{2\pi\sigma_m(F_1)} = I$.

(2) Assume m is even and consider the irreducible representations Π_m of $\mathsf{SU}(2)$ that we have already described. Since,

$$e^{2\pi E_1} = \begin{pmatrix} e^{\pi i} & 0 \\ 0 & e^{-\pi i} \end{pmatrix} = -I, \tag{6.33}$$

we get $e^{2\pi\pi_m(E_1)} = \Pi_m(e^{2\pi E_1}) = \Pi_m(-I)$.
As in the previous case, suppose that Π_m acts on a vector space V_m. Then, there exists a basis $u_0, u_1, ..., u_m$ for V_m such that:

$$e^{2\pi\pi_m(E_1)} = \begin{pmatrix} e^{2\pi\frac{i}{2}m} & & & \\ & e^{2\pi\frac{i}{2}(m-2)} & & \\ & & \ddots & \\ & & & e^{2\pi\frac{i}{2}(-m)} \end{pmatrix}. \tag{6.34}$$

Since m is even, $e^{2\pi\pi_m(E_1)} = \Pi_m(-I) = I$.
This means that $\Pi_m(-U) = \Pi_m(U)$ for all $U \in \mathsf{SU}(2)$. Then, by Lemma 6.1.8, for any $R \in \mathsf{SO}(3)$ there is a unique pair of elements $\{-U, U\}$ in $\mathsf{SU}(2)$ such that $\Phi(U) = \Phi(-U) = R$. Since $\Pi_m(U) = \Pi_m(-U)$ it makes sense to define $\Sigma_m(R) = \Pi_m(U)$ as a representation of $\mathsf{SO}(3)$.
Then, $\Pi_m = \Sigma_m \circ \Phi$. Using the fact that $\widetilde{\Phi} = \phi^{-1}$ and that $\pi_m = \widetilde{\Sigma}_m \circ \widetilde{\Phi}$, it follows that $\widetilde{\Sigma}_m = \pi_m \circ \phi$.
□

Note that a representation for which m is even has an odd dimension, since $dim(V_m) = m + 1$. Then, only the odd-dimensional representations of $\mathsf{so}(3)$ are related to group representations. In fact, in physics the irreducible representations of $\mathsf{su}(2)$ are labeled by $l = \frac{m}{2}$ instead of m. Thus, a representation of $\mathsf{su}(2)$ is related to a representation of $\mathsf{SO}(3)$ only when l is an integer. That corresponds to **bosons**, i.e particles with integer spin. Particles with half-integer spin, like the electrons, are called **fermions** and have no associated representation in $\mathsf{SO}(3)$.

Example 6.1.10. The universal cover of $\mathsf{SO}(4)$ is isomorphic to $\mathsf{SU}(2) \times \mathsf{SU}(2)$. It turns out that the universal cover of $\mathsf{SO}(n)$ for $n \geq 3$ is a double cover. It is called $Spin(n)$ and it is a matrix Lie Group.

7 Representations of $\mathsf{SU}(3)$

In this section we want to check if any irreducible finite-dimensional representation of $\mathsf{SU}(3)$ can be classified in terms of its highest weight. The notion of highest weight is analogous to the labeling m of the irreducible representations of $\mathsf{SU}(2)$ from Theorem 5.4.6.
Both the Lie Groups $\mathsf{SU}(2)$ and $\mathsf{SU}(3)$ are **connected**, **simply connected** and **compact**.
Although the following statements are true for both $\mathsf{SU}(2)$ and $\mathsf{SU}(3)$, to get a general picture of how to classify the representations we will consider the case of $\mathsf{SU}(3)$. Moreover, $\mathsf{SU}(3)$ is tightly related to physics.

Matter is mostly composed by elementary particles called **quarks**, which are bound together forming **hadrons**. The weak and the strong nuclear forces handle the interactions between these quarks. The properties that model these interactions are flavour and colour. Quarks have three different flavour states: up, down and strange and three different states regarding color: red, green and blue. We use the Lie Group $\mathsf{SU}(3)$ to study the symmetries regarding the flavour and colour states of quarks.

7.1 Roots and weights

Since $\mathsf{SU}(3)$ is connected and simply connected, by Theorem 6.0.1 we know that any irreducible representation of $\mathsf{SU}(3)$ is related to an irreducible representation of $\mathsf{su}(3)$. On the other hand, the representations of $\mathsf{su}(3)$ are in one-to-one correspondence with the representations of the complexified Lie Algebra $\mathsf{su}(3)_{\mathbb{C}}$, which is isomorphic to $\mathsf{sl}(3;\mathbb{C})$. This lets us to pose the following result:

Proposition 7.1.1. *There is a one-to-one correspondence between finite-dimensional complex representations* Π *of* $\mathsf{SU}(3)$ *and finite-dimensional complex representations* π *of* $\mathsf{sl}(3;\mathbb{C})$. *More precisely, for all* $X \in \mathsf{su}(3) \subset \mathsf{sl}(3,\mathbb{C})$, *the following holds:*

$$\Pi(e^X) = e^{\pi(X)}. \tag{7.1}$$

Moreover, the representation Π *is irreducible if, and only if,* π *is irreducible. Therefore, if* Π *and* π *act on a vector space* V, *then a subspace* $W \subset V$ *is invariant for* Π *if, and only if, is invariant for* π.

Since $\mathsf{SU}(3)$ is a compact matrix Lie Group, by Proposition 5.5.6, all its finite-dimensional representation are completely reducible, and it follows that:

Proposition 7.1.2. *Every finite-dimensional complex representation* π *of* $\mathsf{sl}(3;\mathbb{C})$ *is completely reducible and decomposes as a direct sum of irreducible representations.*

From now on, we will consider only finite-dimensional complex linear representations and the following basis for $\mathsf{sl}(3;\mathbb{C})$:

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \tag{7.2}$$

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \quad X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \tag{7.3}$$

$$Y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad Y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad Y_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{7.4}$$

We have the commutation relations given by the following list:

$$\begin{array}{lll}
[H_1, X_1] = 2X_1 & [H_1, Y_1] = -2Y_1 & [X_1, Y_1] = H_1 \\
[H_2, X_2] = 2X_2 & [H_2, Y_2] = -2Y_2 & [X_2, Y_2] = H_2 \\
[H_1, H_2] = 0 & [H_2, X_1] = -X_1 & [H_2, Y_1] = Y_1 \\
[H_1, X_2] = -X_2 & [H_1, Y_2] = Y_2 & [H_1, X_3] = X_3 \\
[H_2, X_3] = X_3 & [H_1, Y_3] = -Y_3 & [H_2, Y_3] = -Y_3 \\
[X_1, X_2] = X_3 & [Y_1, Y_2] = -Y_3 & [X_2, Y_3] = Y_1 \\
[X_1, Y_3] = -Y_2 & [X_3, Y_2] = X_1 & [X_3, Y_1] = -X_2 \\
[X_1, Y_2] = 0 & [X_2, Y_1] = 0 & [X_1, X_3] = 0 \\
[Y_1, Y_3] = 0 & [X_2, X_3] = 0 & [Y_2, Y_3] = 0
\end{array} \tag{7.5}$$

Note that both $\{H_1, X_1, Y_1\}$ and $\{H_2, X_2, Y_2\}$ are subspaces of $\mathsf{sl}(3;\mathbb{C})$ which are isomorphic to $\mathsf{sl}(2;\mathbb{C})$, since the commutation relations correspond to the ones in (5.4). In order to classify the representations of $\mathsf{sl}(3;\mathbb{C})$, we need to diagonalize both $\pi(H_1)$ and $\pi(H_2)$. Since $[H_1, H_2] = 0$, we also have $[\pi(H_1), \pi(H_2)] = 0$ and hence we can find a pair of simultaneous eigenvalues for $\pi(H_1)$ and $\pi(H_2)$.

Definition 7.1.3. Given a representation π of $\mathsf{sl}(3;\mathbb{C})$ acting on the vector space V, we define a **weight** $\mu = (\mu_1, \mu_2) \in \mathbb{C}^2$ of π if there exists a vector $v \in V$, $v \neq 0$, such that:

$$\begin{aligned} \pi(H_1)v &= \mu_1 v, \\ \pi(H_2)v &= \mu_2 v. \end{aligned} \tag{7.6}$$

The vector v is called the **weight vector** corresponding to μ. The space of all vectors corresponding to a given weight is called a **weight space**.

Now we will prove some results related to the notion of weight.

Proposition 7.1.4. *Every representation of* $\mathsf{sl}(3;\mathbb{C})$ *has at least one weight.*

Proof. We know that $\pi(H_1)$ has at least one eigenvalue μ_1. Let $W \subset V$ be the eigenspace for $\pi(H_1)$ with eigenvalue μ_1. Then, for any $w \in W$:

$$\pi(H_1)(\pi(H_2)w) = \pi(H_2)\pi(H_1)w = \pi(H_2)(\mu_1 w) = \mu_1 \pi(H_2)w. \tag{7.7}$$

Therefore, W is invariant under $\pi(H_2)$.
Let us consider $\pi(H_2)$ as an operator on W. Since we are working over $\mathbb{C}$, $\pi(H_2)$ must have at least one eigenvector w with eigenvalue μ_2. Then w will be simultaneous eigenvector for $\pi(H_1)$ and $\pi(H_2)$. $\square$

Proposition 7.1.5. *Let π be a representation of $\mathsf{sl}(2;\mathbb{C})$ acting on the vector space V and consider the basis $\{H, X, Y\}$ for $\mathsf{sl}(2;\mathbb{C})$. Then the eigenvalues of $\pi(H)$ are integers.*

Proof. By Proposition 7.1.2, any representation π of $\mathsf{sl}(2;\mathbb{C})$ decomposes in irreducible representations, which are the ones from the classification in Proposition 5.4.5 . In fact, each of them can be diagonalized and its eigenvalues are integers. Thus, π can be diagonalized and it also has integer eigenvalues. □

By using this Proposition on the restriction of a representation of $\mathsf{sl}(3;\mathbb{C})$, π, to $\{H_1, X_1, Y_1\}$ and $\{H_2, X_2, Y_2\}$ we obtain the following result:

Corollary 7.1.6. *If π is a representation of $\mathsf{sl}(3;\mathbb{C})$, all its weights are of the form $\mu = (m_1, m_2)$ where m_1 and m_2 are integers.*

Definition 7.1.7. An ordered pair $(\alpha_1, \alpha_2) \in \mathbb{C}^2$ is called a **root** if $\alpha_1 \cdot \alpha_2 \neq 0$ and if there exists $Z \in \mathsf{sl}(3, \mathbb{C})$ such that:

$$\begin{aligned} [H_1, Z] &= \alpha_1 Z, \\ [H_2, Z] &= \alpha_2 Z. \end{aligned} \tag{7.8}$$

Then Z is the **root vector** corresponding to the root α.

Z	α	
X_1	$(2, -1)$	$\alpha^{(1)}$
X_2	$(-1, 2)$	$\alpha^{(2)}$
X_3	$(1, 1)$	$\alpha^{(1)} + \alpha^{(2)}$
Y_1	$(-2, 1)$	$-\alpha^{(1)}$
Y_2	$(1, -2)$	$-\alpha^{(2)}$
Y_3	$(-1, -1)$	$-\alpha^{(1)} - \alpha^{(2)}$

Table 7.1: Roots of $\mathsf{sl}(3;\mathbb{C})$.

As we see in Table 7.1, all roots can be expressed as a linear combination of $\alpha^{(1)}$ and $\alpha^{(2)}$, which are the **simple roots** of $\mathsf{sl}(3;\mathbb{C})$.

Lemma 7.1.8. *Let α be a root and let $Z_\alpha \neq 0$ be its root vector in $\mathsf{sl}(3;\mathbb{C})$. Let π be an irreducible representation of $\mathsf{sl}(3;\mathbb{C})$ acting on V and $\mu = (m_1, m_2)$ a weight with weight vector $v \in V$. Then:*

$$\begin{aligned} \pi(H_1)\pi(Z_\alpha)v &= (m_1 + \alpha_1)\pi(Z_\alpha)v \\ \pi(H_2)\pi(Z_\alpha)v &= (m_2 + \alpha_2)\pi(Z_\alpha)v. \end{aligned} \tag{7.9}$$

Therefore, either $\pi(Z_\alpha)v = 0$ or else $\pi(Z_\alpha)v$ is a new weight vector with weight $\mu + \alpha = (m_1 + \alpha_1, m_2 + \alpha_2)$.

Proof. By Definition 7.1.7, we know that $[\Pi_1, Z_\alpha] = \alpha_1 Z_\alpha$. Thus,

$$\begin{aligned}\pi(H_1)\pi(Z_\alpha)v &= (\pi(Z_\alpha)\pi(H_1) + \alpha_1\pi(Z_\alpha))v \\ &= \pi(Z_\alpha)(m_1 v) + \alpha_1\pi(Z_\alpha)v \\ &= (m_1 + \alpha_1)\pi(Z_\alpha)v.\end{aligned}$$

An equivalent argument is used to compute $\pi(H_2)\pi(Z_\alpha)v$. $\square$

7.2 Classification Theorem and highest weight

The root vectors, which in the case of $\mathsf{sl}(3;\mathbb{C})$ are $X_1, X_2, X_3, Y_1, Y_2, Y_3$, are used to obtain new weights within a given representation. When we apply a root vector with root α to a weight vector with weight $\mu = (m_1, m_2)$ we obtain a new weight of the form $\mu+\alpha$. Since we are working with finite-dimensional representations, there are finitely many weights and hence most of the weight vectors that we obtain are zero.

The first step is to single out a highest weight for a given representation. Thus, we need to define the notion of higher.

Definition 7.2.1. Let $\alpha^{(1)} = (2, -1)$ and $\alpha^{(2)} = (-1, 2)$ be the roots defined in Table 7.1. Let μ_1 and μ_2 be two weights. Then we will say that μ_1 is higher than μ_2 if $\mu_1 - \mu_2$ can be written as a linear combination of $\alpha^{(1)}$ and $\alpha^{(2)}$ with coefficients greater than or equal to zero. Then we will write $\mu_1 \succeq \mu_2$.
Given a representation of $\mathsf{sl}(3;\mathbb{C})$, a weight μ_0 is the highest weight if for all weights μ in the representation, $\mu_0 \succeq \mu$.

Remark. Note that a finite set of weights might not have a highest element.

The next step, and the main goal of this section, is to classify all the irreducible representations of $\mathsf{sl}(3;\mathbb{C})$ regarding their highest weights.

Theorem 7.2.2. *(1) Any irreducible representation π of $\mathsf{sl}(3;\mathbb{C})$ is the direct sum of its weight spaces.*

(2) Any irreducible representation π of $\mathsf{sl}(3;\mathbb{C})$ has a unique highest weight μ_0. Two irreducible representations are equivalent if, and only if, they have the same highest weight.

(3) The highest weight of any irreducible representation π of $\mathsf{sl}(3;\mathbb{C})$ is of the form $\mu_0 = (m_1, m_2)$ where m_1 and m_2 are non-negative integers, and for any pair of non-negative integers m_1 and m_2 there exists a unique representation π of $\mathsf{sl}(3;\mathbb{C})$ with highest weight $\mu = (m_1, m_2)$.

Note the parallelism between this result and the one we proved for $\mathsf{sl}(2,\mathbb{C})$.

Theorem 7.2.3. *The dimension of an irreducible representation with highest weight $\mu_0 = (m_1, m_2)$ is computed as follows:*

$$\frac{1}{2}(m_1+1)(m_2+1)(m_1+m_2+2) \tag{7.10}$$

A proof of Theorem 7.2.3 can be found in [3], Section 24.

In order to prove Theorem 7.2.2 we need to state some definitions and results.

Definition 7.2.4. Let π be a representation of $\mathsf{sl}(3;\mathbb{C})$ acting on the vector space V. We will say it is a **highest weight cyclic representation** with weight $\mu_0 = (m_1, m_2)$ if there exists a vector $v \neq 0$ in V such that v is a weight vector with weight μ_0 and it holds that $\pi(X_1)v = \pi(X_2)v = 0$ and that the smallest invariant subspace of V containing v is all of V.
We will say that the vector v is a cyclic vector for π.

Proposition 7.2.5. *Let π be a highest weight cyclic representation of $\mathsf{sl}(3;\mathbb{C})$ with weight μ_0. Then π has highest weight μ_0 and the weight space corresponding to μ_0 has dimension one.*

A proof of the above Proposition can be found in [6], Section 6.

Proposition 7.2.6. *A finite-dimensional complex representation π of $\mathsf{sl}(3;\mathbb{C})$ is irreducible if and only if it is a highest weight cyclic representation.*

Proof. First, we are going to prove that if we have a finite-dimensional irreducible representation of $\mathsf{sl}(3;\mathbb{C})$, it is a highest weight representation.
Let π be this representation. Since π is finite, it has finitely many weights. Therefore, there must exist one weight μ_0 such that there is no weight $\mu \neq \mu_0$ such that $\mu \succeq \mu_0$. Then, for any non-zero weight vector v with weight μ_0 we have that $\pi(X_1)v = \pi(X_2)v = 0$. Otherwise there would be exist a vector (either $\pi(X_1)v$ or $\pi(X_2)v$) with weight higher than μ_0. Since π is irreducible, the smallest invariant subspace containing v must be the whole space and thus we have a highest weight cyclic representation.

Now we are going to see that every highest weight cyclic representation π acting on the vector space V is irreducible. Let $v \in V$ be its cyclic vector, with highest weight μ_0.
By Proposition 7.1.2, π is completely reducible. Hence,

$$V \cong \bigoplus_i V_i. \tag{7.11}$$

Then, by Theorem 7.2.2 we know that every vector space V_i is the direct sum of its weight spaces. Since μ_0 occurs in some V_i and its weight space has dimension one, V_i must contain the only vector v with weight μ_0. Then, V_i is an invariant subspace containing v and therefore $V_i = V$ and π is irreducible. $\square$

The following example will be useful for the proof of Theorem 7.2.2.

Example 7.2.7. The trivial representation of $\mathsf{sl}(3;\mathbb{C})$ has highest weight $(0,0)$.

The representation with highest weight $(1,0)$ is the standard representation of $\mathsf{sl}(3;\mathbb{C})$. It is the inclusion of $\mathsf{sl}(3;\mathbb{C})$ into $\mathsf{gl}(\mathbb{C})$. The standard basis vectors $\{e_1, e_2, e_3\}$ of $\mathbb{C}$ are weight vectors corresponding to the weights $(1,0)$, $(-1,1)$ and

$(0,-1)$ respectively. This can be proven by applying H_1 and H_2 from (7.2) to $\{e_1, e_2, e_3\}$.

The representation with highest weight $(0,1)$ is the representation such that $\pi(Z) = -Z^{tr}$, for each $Z \in \mathsf{sl}(3;\mathbb{C})$. Then,

$$\pi(H_1) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \pi(H_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.12}$$

The weight vectors are again $\{e_1, e_2, e_3\}$, and their corresponding weights are $(-1,0)$, $(1,-1)$ and $(0,1)$ respectively.

The last two representations are called the fundamental representations of $\mathsf{sl}(3;\mathbb{C})$.

In order to construct a representation with highest weight $(1,1)$, we have to take the tensor product of the fundamental representations and then take the smallest invariant subspace containing the vector of $e_1 \otimes e_3$, since e_1 and e_3 are the highest weight vectors of each representation. The representation with highest weight $(1,1)$ has dimension 8.

Proof of Theorem 7.2.2. (1) First, we want to prove that every irreducible representation π of $\mathsf{sl}(3;\mathbb{C})$ acts on a vector space V which is the direct sum of its weight spaces. In particular, that means that $\pi(H_1)$ and $\pi(H_2)$ are simultaneously diagonalizable.
Let W be the direct sum of the weight spaces in V. Then, any element in W can be written as a linear combination of eigenvectors for $\pi(H_1)$ and $\pi(H_2)$. By Proposition 7.1.4 π has at least one weight, and therefore $W \neq \{0\}$.
By Lemma 7.1.8, W is invariant under the action of all the root vectors X_i and Y_i and by definition it is also invariant under the action of the operators H_i. Then, by irreducibility, $W = V$.

(2) Next we want to prove that any irreducible representation of $\mathsf{sl}(3;\mathbb{C})$ has a unique highest weight. By Proposition 7.2.6, any irreducible representation of $\mathsf{sl}(3;\mathbb{C})$ is a highest weight cyclic representation. By Proposition 7.2.5, a highest weight cyclic representation has a highest weight. The uniqueness of the highest weight is immediate since two different weights can not be the highest.

Let us see that if two irreducible representations of $\mathsf{sl}(3;\mathbb{C})$ have the same highest weight, then they are equivalent. Let π_1 and π_2 be two irreducible representations with highest weight μ_0 acting on the vector spaces V_1 and V_2 respectively. Let $v_1 \in V_1$ and $v_2 \in V_2$ be the cyclic vectors. Now consider the representation $\pi_1 \otimes \pi_2$. Let U be the smallest invariant subspace of $V_1 \times V_2$ which contains the vector (v_1, v_2) and consider the restriction of $\pi_1 \otimes \pi_2$ on U. Then, U is a highest weight cyclic representation and it is irreducible by Proposition 7.2.6.
If we consider the projection maps $P_1 : V_1 \otimes V_2 \to V_1$ and $P_2 : V_1 \otimes V_2 \to V_2$, then $P_1(v_1, v_2) = v_1$ and $P_2(v_1, v_2) = v_2$. It can be seen that P_1 and P_2 are morphisms of representations and thus the restrictions $P_1|_U$ and $P_2|_U$ will also be morphisms. Since the representations acting on U, V_1 and V_2 are irreducible, by Schur's Lemma (Theorem 5.6.1) $P_1|_U$ and $P_2|_U$ are isomorphisms. Thus, $V_1 \cong U \cong V_2$.

It is direct to prove that if two representations are equivalent, they have the same highest weight.

(3)We already know that all weights are of the form (m_1, m_2), where m_1 and m_2 are integers. If μ_0 is the highest weight and $v \neq 0$ is a weight vector with weight μ_0, then $\pi(X_1)v = \pi(X_2)v = 0$. Therefore, by applying the restriction of π to both $\{H_1, X_1, Y_1\}$ and $\{H_2, X_2, Y_2\}$, it follows that m_1 and m_2 are non-negative integers.

Finally, we want to prove that if m_1 and m_2 are non-negative integers, there exists an irreducible representation of $\mathsf{sl}(2;\mathbb{C})$ with highest weight $\mu_0 = (m_1, m_2)$. In order to prove this we need to construct the fundamental representations π_1 and π_2 of $\mathsf{sl}(3;\mathbb{C})$. The representations π_1 and π_2 act on the vector spaces V_1 and V_2 and have highest weights $\mu_1 = (1, 0)$ and $\mu_2 = (0, 1)$ respectively, with corresponding weight vectors $v_1 = (1, 0, 0)$ and $v_2 = (0, 0, 1)$ (see Example 7.2.7).
Now let us consider the representation $\pi_{m_1 m_2}$ acting on the vector space:

$$\underbrace{V_1 \otimes V_1 \otimes}_{m_1} \cdots \underbrace{\otimes V_2 \otimes V_2}_{m_2}. \tag{7.13}$$

The action of an element of $\mathsf{sl}(3;\mathbb{C})$ on any element of this space is computed as follows:

$$(\pi_1(Z) \otimes I \cdots \otimes I) + (I \otimes \pi_1(Z) \otimes \cdots \otimes I) + \cdots + (I \otimes \cdots \otimes I \otimes \pi_2(Z)). \tag{7.14}$$

If we consider the vector $v_{m_1 m_2} = v_1 \otimes v_1 \otimes \cdots \otimes v_2 \otimes \cdots \otimes v_2$ it verifies the following conditions:

$$\begin{aligned}
\pi_{m_1 m_2}(X_1) v_{m_1 m_2} &= 0 \\
\pi_{m_1 m_2}(X_2) v_{m_1 m_2} &= 0 \\
\pi_{m_1 m_2}(H_1) v_{m_1 m_2} &= m_1 v_{m_1 m_2} \\
\pi_{m_1 m_2}(H_2) v_{m_1 m_2} &= m_2 v_{m_1 m_2}.
\end{aligned} \tag{7.15}$$

The representation $\pi_{m_1 m_2}$ is not an irreducible representation unless $(m_1, m_2) = (1, 0)$ or $(m_1, m_2) = (0, 1)$, but if we take W to be the smallest invariant subspace containing $v_{m_1 m_2}$, then W will be a highest weight cyclic representation. By Proposition 7.2.6, W will be irreducible with highest weight (m_1, m_2). $\square$

Example 7.2.8 (The Weyl Group). The set of weights of a given representation has a symmetry associated to it. This is studied in terms of the Weyl Group.

For $\mathsf{sl}(3;\mathbb{C})$, the Weyl Group is defined to be the following subgroup W of $\mathsf{SU}(3)$:

$$w_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad w_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad w_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix};$$

$$w_3 = -\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad w_4 = -\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad w_5 = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Recall that by Definition 4.2.8, for any $A \in \mathsf{SU}(3)$, $Ad(A)(X) = AXA^{-1}$ is in $\mathsf{su}(3)$.

It turns out that for the elements $w \in W$, $Ad(w)^{-1}(H_1)$ and $Ad(w)^{-1}(H_2)$ are linear combinations of H_1 and H_2. Given a representation of $\mathsf{sl}(3;\mathbb{C})$, π, let us consider the new weights $w_i \cdot \mu$ obtained by applying $\pi_{w_i}(H_1) = \pi(Ad(w_i)^{-1}H_1)$ and $\pi_{w_i}(H_2) = \pi(Ad(w_i)^{-1}H_1)$ for all $w_i \in W$ to the weight vectors of π so we have:

$$\begin{aligned} w_0 \cdot (m_1, m_2) &= (m_1, m_2) & w_3 \cdot (m_1, m_2) &= (-m_1, m_1 + m_2) \\ w_1 \cdot (m_1, m_2) &= (-m_1 - m_2, m_1) & w_4 \cdot (m_1, m_2) &= (-m_2, -m_1) \\ w_2 \cdot (m_1, m_2) &= (m_2, -m_1 - m_2) & w_5 \cdot (m_1, m_2) &= (m_1 + m_2, -m_2). \end{aligned} \tag{7.16}$$

Then we get that $\mu = (m_1, m_2)$ is a weight for π if, and only if, $w_i \cdot \mu$ is a weight for π for all $w_i \in W$.
Let us think of the weights $\mu = (m_1, m_2)$ as sitting in $\mathbb{R}^2$. Then, we can think of (7.16) as a finite group of transformations of $\mathbb{R}^2$. We call this group the Weyl Group. Moreover, it can be seen that there exists a unique inner product on $\mathbb{R}^2$ such that the action of W is orthogonal. In particular, we could see that the action of the Weyl group is generated by a rotation $120°$ and a reflection on the y-axis. More on this topic can be found in [1].

8 Classification of complex semisimple Lie Algebras

This section is meant to be a brief introduction to the method used to construct a classification for all the irreducible representations of complex semisimple Lie Algebras. The idea is similar to what we have used for $\mathsf{sl}(3;\mathbb{C})$.

Definition 8.0.1. Let $\mathfrak{g}$ be a Lie Algebra. A subspace $I \subset \mathfrak{g}$ is an **ideal** of $\mathfrak{g}$ if $[X, Y] \in I$ for all $X \in \mathfrak{g}$ and all $Y \in I$.

Definition 8.0.2. A Lie Algebra $\mathfrak{g}$ is said to be **simple** if $dim(\mathfrak{g}) \geq 2$ and $\mathfrak{g}$ has no ideals other than $\{0\}$ and $\mathfrak{g}$. A Lie Algebra $\mathfrak{g}$ is said to be **semisimple** if $\mathfrak{g}$ can be written as a direct sum of simple Lie Algebras.

Example 8.0.3. The Lie Algebras $\mathsf{sl}(n;\mathbb{C})$ and $\mathsf{so}(n;\mathbb{C})$ for $n \geq 3$ are semisimple Lie Algebras over the complex numbers.

A basic result about complex simple Lie Algebras is that they can be classified in terms of the classical Lie Algebras ([3]).

Theorem 8.0.4. *With five exceptions, every complex simple Lie Algebra is isomorphic to either* $\mathsf{sl}(n;\mathbb{C})$*,* $\mathsf{so}(n;\mathbb{C})$ *or* $\mathsf{sp}(2n;\mathbb{C})$ *for some* $n \in \mathbb{N}$*.*

An approach to many theorems regarding simple Lie Algebras is to prove them by verifying explicitly all these cases.

Definition 8.0.5. Let $\mathfrak{g}$ be a complex semisimple Lie Algebra. A subspace $\mathfrak{h}$ of $\mathfrak{g}$ is a **Cartan subalgebra** of $\mathfrak{g}$ if:

(1) $\mathfrak{h}$ is abelian, which means that for all $H_1, H_2 \in \mathfrak{h}$, $[H_1, H_2] = 0$.

(2) $\mathfrak{h}$ is maximal abelian, which means that if $X \in \mathfrak{g}$ and X satisfies $[H, X] = 0$ for all $H \in \mathfrak{h}$, then $X \in \mathfrak{h}$.

(3) For all $H \in \mathfrak{h}$, $adH : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable.

Definition 8.0.6. Let $\mathfrak{g}$ be a complex semisimple Lie Algebra and let $\mathfrak{h}$ be its Cartan subalgebra. Then, an element $\alpha \in \mathfrak{h}^*$ (where $\mathfrak{h}^*$ stands for the dual of $\mathfrak{h}$) is a **root** for $\mathfrak{g}$ with respect to $\mathfrak{h}$ if $\alpha \neq 0$ and there exists $Z \in \mathfrak{g}$ such that $[H, Z] = \alpha(H)Z$ for all $H \in \mathfrak{h}$.
We will call Z a **root vector** of α. The space of all root vectors of a given root α, $\mathfrak{g}^\alpha$, is called **root space**. We will denote the **set of all roots** by Δ.

Theorem 8.0.7. *Let* $\mathfrak{g}$ *be a complex semisimple Lie Algebra. Then, a Cartan subalgebra* $\mathfrak{h}$ *exists. If* $\mathfrak{h}_1$ *and* $\mathfrak{h}_2$ *are two Cartan subalgebras, there is an automorphism of* $\mathfrak{g}$ *which takes* $\mathfrak{h}_1$ *to* $\mathfrak{h}_2$*. In particular, any two Cartan subalgebras have the same dimension.*

This result allows us to introduce the notion of the rank of a complex semisimple Lie Algebra.

Definition 8.0.8. The **rank** of a complex semisimple Lie Algebra is the dimension of its Cartan subalgebras.

Example 8.0.9. The rank of $\mathsf{sl}(n;\mathbb{C})$ is $n-1$. One possible Cartan subalgebra of this Lie Algebra would be the space of all diagonal matrices with trace 0.

Definition 8.0.10. Let π be a finite-dimensional complex linear representation of $\mathfrak{g}$ acting on the vector space V. Then $\mu \in \mathfrak{h}^*$ is a **weight** for π with weight vector $v \in V$ if the following holds:

$$\pi(H)v = \mu(H)v, \tag{8.1}$$

for all $H \in \mathfrak{h}$.

Definition 8.0.11. A set of roots $\{\alpha_1, \cdots, \alpha_l\}$ is called a **simple system** if:

(1) $\{\alpha_1, \cdots, \alpha_l\}$ is basis of the vector space $\mathfrak{h}^*$.

(2) Every root $\alpha \in \Delta$ can be written as a linear combination of elements of $\{\alpha_1, \cdots, \alpha_l\}$, with all coefficients either non-positive (negative root) or non-negative (positive root).

Definition 8.0.12. Let $\{\alpha_1, \cdots, \alpha_l\}$ be a simple system of roots and let μ_1 and μ_2 be two weights. Then, we say that μ_1 **is higher** than μ_2 if $\mu_1 - \mu_2$ can be written as a linear combination of elements of the simple system with non-negative coefficients. We denote it by $\mu_1 \succeq \mu_2$.
A representation π has a highest weight μ_0 if for any other weight μ we have $\mu_0 \succeq \mu$.

Next we will summarize the most important results related to the classification of complex semisimple Lie Algebras. Details of this can be found in [3].

Theorem 8.0.13. *Let $\mathfrak{g}$ be a complex semisimple Lie Algebra, let $\mathfrak{h}$ be a Cartan subalgebra and let Δ be its set of roots. Then:*

(1) For each root $\alpha \in \Delta$, the corresponding weight space $\mathfrak{g}^\alpha$ is one dimensional.

(2) If α is a root, then $-\alpha$ is also a root.

(3) There exists a simple system of roots $\{\alpha_1, \cdots, \alpha_l\}$.

Theorem 8.0.14. *Let $\mathfrak{g}$ be a complex semisimple Lie Algebra and let $\mathfrak{h}$ be a Cartan subalgebra. Let $\{\alpha_1, \cdots, \alpha_l\}$ be a simple system of roots. There exist $X_i \in \mathfrak{g}^{\alpha_i}$ and $Y_i \in \mathfrak{g}^{-\alpha_i}$ such that if $H_i = [X_i, Y_i]$, then:*

(1) $H_i \neq 0$ and $H_i \in \mathfrak{h}$, for each i, $1 \leq i \leq l$.

(2) The span of $\{H_i, X_i, Y_i\}$ is a subalgebra of $\mathfrak{g}$ which is isomorphic to $\mathsf{sl}(2;\mathbb{C})$.

(3) The set $\{H_1, \cdots, H_l\}$ is a basis for $\mathfrak{h}$.

Theorem 8.0.15. *Let $\mathfrak{g}$ be a complex semisimple Lie Algebra, let $\mathfrak{h}$ be a Cartan subalgebra and let $\{\alpha_1, \cdots, \alpha_l\}$ be a simple system of roots. Let $\{H_1, \cdots, H_l\}$ be as in Theorem 8.0.14. Then the following holds:*

(1) For each irreducible representation π of $\mathfrak{g}$, $\pi(H_i)$ for $H_i \in \{H_1, \cdots, H_l\}$ are all simultaneously diagonalizable.

(2) Each irreducible representation of $\mathfrak{g}$ has a unique highest weight.

(3) Two irreducible representations of $\mathfrak{g}$ with the same highest weight are equivalent.

(4) If μ_0 is the highest weight of an irreducible representation of $\mathfrak{g}$, $\mu_0(H_i)$ is a non-negative integer for $i = 1, 2, \cdots, l$.

(5) If $\mu_0 \in \mathfrak{h}^*$ is such that $\mu_0(H_i)$ is a non-negative integer for all $i = 1, 2, \cdots, l$, then there is an irreducible representation of $\mathfrak{g}$ with highest weight μ_0.

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