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Stochastic Differential Equations and Applications

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Abstract

In this paper, how to obtain stochastic differential equations by using Itô Stochastic integrals is treated. We will refer to stochastic differential equations as SDE. Then, the theory inderlying the Itô calculus is carefully studied and a thorough analysis of the relationship of the class of processes M^2 and the space of integrable functions L^2 is considered. Moreover, under which assumptions a solution of a SDE exists and is unique is provided. Some particular cases of Itô stochastic integrals and SDE are guaranteed throughout a sequence of examples that are linked up with the abstract theory. Finally, the basic ideas and techniques underpinning the simulation of stochastic differential equations are shown. In particular, the Euler-Maruyama method is presented and suitable simulation scenarios are derived from the SDE models developed.

Resum

En aquest treball s'estudia la manera d'obtenir equacions diferencials estocàstiques usant integrals estocàstiques d'Itô. Es tracta doncs la teoria que compren el càlcul estocàstic d'Itô i es fa un profund i rigurós anàlisi de la relació entre la classe de processos estocàstics M^2 i l'espai de funcions integrables L^2. A més, es proveeixen condicions per l'existència i unicitat de solucions d'una equació diferencial estocàstica. Es garanteixen també alguns casos particulars d'integrals estocàstiques d'Itô i d'equacions diferencials estocàstiques, els quals són donats a través de seqüències d'exemples vinculats amb la teoria més abstracta. Finalment, es donen les idees i tècniques bàsiques subjacents a la simulació d'equacions diferencials estocàstiques. En particular, es presenta el mètode d'Euler-Maruyama i es deriven els escenaris necesaris per a realitzar simulacions per cada un dels models d'equacions diferencials estocàstiques desenvolupats.

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M'agradaria donar també les gràcies als meus pares i a la meva germana, per ajudarme en tot el possible, per creure amb mi, i per estar sempre al meu costat malgrat no estiguem sempre l'aprop que voldriem. I finalment, a les matemàtiques, que m'han ensenyat a entendre que els reptes es superen actuant i, sobretot, que m'han permés fer amistats que ja duraran per sempre.

Contents

Chapter 1

Introduction

The history of stochastic differential equations and most of the formal development of this theory began in 1942 when Kiyosi Itô published the paper *On stochastic processes (Infinitely divisible laws of probability)* in the *Japanese Journal of Mathematics*. The tools provided for this work are considered fundamentals today, and it implies that the name of the japanese mathematician K. Itô must be a reference when one is referred to the theory and applications of stochastic differential equations. The importance of K. Itô on the development of stochastic differential equations may be sumarized with the following citation, given in [5], and complemented with a picture in his honour:

In 1942, Dr. Itô began to reconstruct from scratch the concept of stochastic integrals, and its associated theory of analysis. He created the theory of stochastic differential equations, which describe motion due to random events.

Figure 1.1: Kiyosi Itô

Even so, the mathematical notion of stochastic differential equations is meaningless without the notion of stochastic integration. Therefore, digging a little bit deeper into the notion of stochastic integration one must dates back as far as the 19th century. Namely, as far as 1827, when the botanic Robert Brown noticed, under his microscope, the highly irregular random movement of particles within pollen in a water drop. However, such randomness was not formalized until 1905, when Albert Einstein provided a satisfactory explanation of the Brownian motion via kinetic theory. The rigorous mathematical construction of a stochastic process as a model for such motion is due to the mathematician Norbert Weiner; in recognition of his construction, Brownian motion is sometimes referred to as Wiener process. Nevertheless it was not until 1931 when Kolmogorov, with the theory of Markov processes, motivate the beginning of what is known nowadays as the theory of stochastic integration [7, p. 1-2].

The theory developed in this paper concerns stochastic processes, understood in the sense of random evolutions governed by time (continuous or discrete time). A brief outline of the main objectives of this paper could be:

- Firstly, to define the Itô stochastic integral for the yet to be defined class of random step processes M^2_{step} and for a larger class of processes, denoted by M^2.
- Secondly, to derive the notion of stochastic differential equations from the notion of stochastic integral equations and give a result on the existence and uniqueness of a solution for a SDE in the class of the yet to be defined Itô processes.
- Finally, to exemplify the applicability of stochastic differential equations to engineering, in particular, to textile industry.

To start writing the very first part of this paper, I followed the advice of my thesis advisor Dr. Carles Rovira along with the book I started to work with, called *Basic Stochastic Processes*. The outcome of all the above together is a review on Brownian motion, which covers the second chapter of this work. In this part, the stochastic nature of Brownian motion is studied because it is needed to develop the following chapters. Also, two key properties relating to stochastic integration properties are included which are (1) the paths of Brownian motion have non zero finite quadratic variation, such that on an interval (s,t) the quadratic variation is $(t-s)$ and (2) the variation of the paths of a Brownian motion is infinite almost surely.

In the third chapter I get deeper into the study of the Itô calculus, giving firstly an example illustrating that a different integral aside of the Riemann integral needs to be constructed and secondly, a definition of the *Itô stochastic integral* for important and thorough classes of stochastic processes such as M^2_{step} and M^2. Also, it is shown that an Itô stochastic integrals is a martingale with respect to the filtration $\mathcal{F}_t = \sigma\{W(s) : 0 \leq s \leq t\}$ and straightforward results for a stochastic process to belong to M^2_{step} and M^2 are given and proved. Moreover, as an application to illustrate the theory, I have had to adapt some of the examples given in the book *Introduction to Stochastic Integration* of H. Kuo to give some basic examples of Itô Stochastic integrals and finally, I summarize the basic properties of the Itô integral.

In the second part of Chapter 3, a potential tool to solve stochastic differential equations known as Itô formula is studied and the notion of stochastic differential equation is given. Furthermore, as said previously, a result on the existence and uniqueness of solutions for a SDE in the class of the yet to be defined Itô processes. Finally, several examples showing that Itô formula works beautifully to find the solution of a SDE are work out. In this part, I also attempt to shed some light into possible future work, which could be related for instance to explosion time of solutions of SDE or to venture myself into a profound analysis of the Itô formula and its applications.

For the presentation of the fourth chapter, I worked quite hard to find out a suitable SDE model to cover the application part of this project. I finally found in [1] a model for cotton fiber breakage. Then, I searched out for more information and I got amazed of three works: [2], [3] and [16], which are a basic reference for this part. A population model for cotton fiber having different length is then constructed from two different procedures. The first procedure is a natural extension of the procedure used for many years in modeling deterministic dynamical processes, whereas the second procedure is potentially based on determining all the different random changes that may occur in the system. In this part, a complete study of each procedure is done and an attempt to explain their equivalence is made. Furthermore, an approximation model to test SDE models is introduce and simulations are work out.

Chapter 2

Brownian motion

The aim of this chapter is to introduce the one-dimensional Brownian motion. In the first section we focus on defining Brownian motion and consider its basic properties. In particular we look at properties of the increments of a Brownian motion. In the second section, the sample paths properties (i.e those properties which hold with probability one) of such processes are studied. Finally, we give a result that enable us to motivate the definition of the stochastic integrals presented in the next chapter.

Throughout this chapter we have fixed a probability space $(\Omega, \mathcal{F}, P)$ where Ω is a sample space, $\mathcal{F}$ is a set with structure of σ-field and P is a probability measure. The terminology comes from [4], which is a basic reference for this chapter and the next. For finding out more information about probability theory we refer to a basic notes on Stochastic Processes [11].

2.1 Definition and Basic Properties

Definition 2.1. A *stochastic process* is a family of random variables $\xi(t)$ parametrized by $t \in T$, where $T \subset \mathbb{R}$ is called the *parameter set*. The index t represents time, and one shall think of $\xi(t)$ as the *state* of the process at time t. When $T = \{1, 2, \ldots\}$ we shall say that $\xi(t)$ is a stochastic process in *discrete time*. When T is not countable, we shall say that $\xi(t)$ is a stochastic process in *continuous time*. In the latter case the usual example is $T = [0, \infty)$ or $T = [a, b] \subset \mathbb{R}$.
For every $\omega \in \Omega$ the mapping

$$t \mapsto \xi(t, \omega)$$

is called a *path* (or *sample paths*) of $\xi(t)$.

The definition of Brownian motion is given in terms of its increments. Then, let us recall the notion of independent increments of a stochastic process.

Definition 2.2. We say that a stochastic process $\xi(t)$, where $t \in T$, has *independent increments* if

$$\xi(t_1) - \xi(t_0), \ldots, \xi(t_n) - \xi(t_{n-1})$$

are independent for any $t_0 < t_1 < \cdots < t_n$ such that $t_0, t_1, \ldots, t_n \in T$.

Definition 2.3. The Brownian motion is a continuous time stochastic process $\{W(t), t \geq 0\}$ that satisfies the following conditions:

(i) $W(0) = 0$ a.s.;

(ii) the paths $t \longmapsto W(t)$ are continuous a.s.;

(iii) for $0 \leq s < t < \infty$, the increment $W(t) - W(s)$ is independent of $W(s)$;

(iv) for $0 \leq s < t < \infty$, the increment $W(t) - W(s)$ has the normal distribution with mean 0 and variance $t - s$.

We now give some definitions and results concerning the increments of Brownian motion.

Definition 2.4. A stochastic process $\xi(t)$, where $t \in T$, is said to have *stationary increments* if for any $s, t \in T$ with $s \leq t$, the increment $\xi(t) - \xi(s)$ has the same probability distribution as $\xi(t-s) - \xi(0)$.

Proposition 2.1. *Since for any $0 \leq s < t < \infty$, $W(t) - W(s)$ has the normal distribution with mean 0 and variance $t-s$ the Brownian motion $W(t)$ has stationary increments.*

Proposition 2.2. *For any $0 = t_0 \leq t_1 \leq \ldots \leq t_n$ the increments*

$$W(t_1) - W(t_0), \ldots, W(t_n) - W(t_{n-1})$$

are independent random variables.

Remark 2.1. Let $s, t \in [0, \infty)$. Then, from condition (iv) in Definition (2.3) it follows that

$$E\big(|W(t) - W(s)|^2\big) = |t - s|.$$

Definition 2.5. The σ-field $\mathcal{F}_s$ is known as the *filtration* generated by $\{W(r) : 0 \leq r \leq s\}$. In other words, $\mathcal{F}_s$ represents our knowledge at time s and it contains all events $\{W(r), 0 \leq r \leq s\}$ such that at time s it is possible to decide whether $W(r)$ has occurred or not.

Corollary 2.1. *For any $0 \leq s \leq t$ the increment $W(t) - W(s)$ is independent of the σ-field*

$$\mathcal{F}_s = \sigma\{W(r) : 0 \leq r \leq s\}$$

Definition 2.6. A continuous time stochastic process $\{\xi(t), t \geq 0\}$ is *adapted* to a filtration $\{\mathcal{F}_t, t \geq 0\}$ if for all $t \geq 0$ the random variable $\xi(t)$ is measurable with respect the σ-field F_t.

The following result, given without proof, shows that the Brownian motion can also be characterized by its martingale properties. Let us recall the notion of martingale.

Definition 2.7. A stochastic process $\xi(t)$ parametrized by $t \in T$ is called a *martingale* with respect to a filtration $\mathcal{F}_t$ if it holds

(i) $\xi(t)$ is integrable for each $t \in T$;

(ii) $\xi(t)$ is $\mathcal{F}_t$-measurable for each $t \in T$;

(iii) $E(\xi(t)|\mathcal{F}_s) = \xi(s)$ for every $s, t \in T$ such that $s \leq t$.

Theorem 2.1 (Lévy's martingale characterization).
Let $\{W(t), t \geq 0\}$ be a stochastic process and let $\mathcal{F}_t = \sigma\{W(s), s \leq t\}$ be the filtration generated by it. Then $W(t)$ is a Brownian motion if and only if the following conditions hold:

(i) $W(0) = 0$ a.s. ;

(ii) the paths $t \longmapsto W(t)$ are continuous a.s;

(iii) $W(t)$ is a martingale with respect to the filtration $\mathcal{F}_t$;

(iv) $|W(t)|^2 - t$ is a martingale with respect to $\mathcal{F}_t$.

We conclude this chapter by providing a result regarding the highly oscillatory nature of the sample paths of the Brownian motion.

2.2 Sample paths

Let us consider $t_j^n = \frac{jT}{n}$, and define the partition $\Delta_n = \{0 = t_0^n < t_1^n < \cdots < t_n^n = T\}$ of the interval $[0, T]$ into n equal parts. We denote by

$$\Delta_j^n W = W(t_{j+1}^n) - W(t_i^n)$$

the corresponding increments of the Brownian motion $W(t)$.

Definition 2.8. The *variation* of a function $f[0, T] \to \mathbb{R}$ is defined to be

$$\lim_{\Delta t \to 0} \sup \sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j))|$$

where $t = (t_0, t_1, \ldots, t_n)$ is a partition of $[0, T]$, and where

$$\Delta t = \max_{0 \leq i \leq n-1} |t_{j+1} - t_j|.$$

Proposition 2.3. *The sum of the increments of the Brownian motion converges to T in L^2. That is,*

$$\lim_{n \to \infty} \sum_{j=0}^{n-1} (\Delta_j^n W)^2 = T \text{ in } L^2.$$

The limit above is said to be the quadratic variation of the Brownian motion. The fact that Brownian motion has finite quadratic variation is used to proof the result presented below. A proof of this result can be found in [10, Chapter 7, p. 36-37].

Theorem 2.2. *The variation of the paths of a Brownian motion $W(t)$ is infinite a.s.*

This result ensures that the paths of a Brownian motion have not bounded variation with probability one. Therefore, integrals of the form

$$\int_0^t f(s)\, dW(s)$$

cannot be defined pathwise (that is, separately for each $\omega \in \Omega$) as in Lebesgue-Stieltjes[1] integral. For this reason, we will need to take advantage of the fact that Brownian motions are random functions and one can make use of weaker forms of limits. This is the approaching idea of defining such integrals described in the next chapter.

[1]See [17, Chapter 1, p. 1] for the related definition of the Lebesgue-Stieltjes integral.

Chapter 3

Itô Stochastic Calculus

Our aim in this chapter is to provide a construction of the Itô stochastic integral and to study stochastic differential equations. The construction of the Itô stochastic integral will be made by steps and will resemble the construction of the Riemann integral, where some restrictions will be needed. Once introduced the Itô stochastic integral, several examples will be given and some properties will be presented.
Finally, we are going to look at stochastic differential equations (or SDE for short). Namely, it will be described the class of Itô processes and it will be considered a particular notation of writing such processes, the stochastic differential notation. Then, we will introduce the so-called stochastic differential equations, that is, the equation

$$d\xi(t) = f(\xi(t))\ dt + g(\xi(t))\ dW(t).$$

We will conclude this chapter proving the theorem of existence and unicity of solutions of a SDE and giving some examples to illustrate the theory.
As in the previous chapter, $W(t)$ will denote a Brownian motion adapted to a filtration $\mathcal{F}_t$. The main concepts of this chapter hold relationship with Brownian motion, so it is essential to be confident with its properties.

3.1 Itô Stochastic Integral: Definition

The construction of the Itô integral will be similar to the construction of the Riemann integral. Firstly, we will define the integral for a class of piecewise constant process called random step processes. Then, it will be extended to a larger class by approximation.
The main differences between the Riemann and the Itô integral are the following:

- The type of convergence: The approximations of the Riemann integral converges in $\mathbb{R}$ whereas the sequences of random variables approximating the Itô integral will converge in L^2.
- The definition of the sum approximating the integral: On the one hand, the Riemann sums approximating the integral of the function $f : [0, T] \to \mathbb{R}$ are

of the form

$$\sum_{j=0}^{n-1} f(s_j)(t_{j+1}-t_j),$$

where $0 = t_0 < t_1 < \ldots < t_n = T$ and s_j is an *arbitrary* point in the subinterval $[t_j, t_{j+1}]$ for each j. On the other hand, in the stochastic case the approximating sums have the form

$$\sum_{j=0}^{n-1} f(s_j)\big(W(t_{j+1}) - W(t_j)\big).$$

A problem that arises in the first approximating sums is that the value of the Riemann integral *does not* depend on the choice of the points $s_j \in [t_j, t_{j+1}]$. Nevertheless, the limit of the latter approximation *does* depend on the choice of $s_j \in [t_j, t_{j+1}]$.

The next example shows the ambiguity resulting from different choices of the intermediate points s_j in each subinterval of the partition.

Example 3.1. Let $f(t) = W(t)$ and $0 = t_0^n < t_1^n < \ldots < t_n^n = T$, where $t_j^n = \frac{jT}{n}$, be a partition of the interval $[0, T]$ into n equals parts. Then:

$$L_n = \sum_{j=0}^{n-1} W(t_j^n)\big(W(t_{j+1}^n) - W(t_j^n)\big) \xrightarrow[n\to\infty]{L^2} \frac{1}{2}W(T)^2 - \frac{1}{2}T \tag{3.1}$$

and

$$R_n = \sum_{j=0}^{n-1} W(t_{j+1}^n)\big(W(t_{j+1}^n) - W(t_j^n)\big) \xrightarrow[n\to\infty]{L^2} \frac{1}{2}W(T)^2 + \frac{1}{2}T. \tag{3.2}$$

To show (3.1) and (3.2) we will use the following identities

$$\begin{aligned} a(b-a) &= \frac{1}{2}(b^2-a^2) - \frac{1}{2}(a-b)^2, \\ b(b-a) &= \frac{1}{2}(b^2-a^2) + \frac{1}{2}(a-b)^2. \end{aligned}$$

From the first identity we obtain

$$\begin{aligned} \sum_{j=0}^{n-1} W(t_j^n)\big(W(t_{j+1}^n) - W(t_j^n)\big) = \; & \frac{1}{2}\sum_{j=0}^{n-1}\big(W(t_{j+1}^n)^2 - W(t_j^n)^2\big) \\ & -\frac{1}{2}\sum_{j=0}^{n-1}\big(W(t_{j+1}^n) - W(t_j^n)\big)^2 \\ = \; & \frac{1}{2}W(T)^2 - \frac{1}{2}\sum_{j=0}^{n-1}\big(W(t_{j+1}^n) - W(t_j^n)\big)^2. \end{aligned}$$

Thus, by Preposition 2.3 the limit in L^2 is

$$\lim_{n\to\infty}\sum_{j=0}^{n-1} W(t_j^n)\big(W(t_{j+1}^n)-W(t_j^n)\big)=\frac{1}{2}W(T)^2-\frac{1}{2}T.$$

Straight forward application of the second identity enable us to write

$$\begin{aligned}\sum_{j=0}^{n-1} W(t_{j+1}^n)\big(W(t_{j+1}^n)-W(t_j^n)\big) &= \frac{1}{2}\sum_{j=0}^{n-1}\big(W(t_{j+1}^n)^2-W(t_j^n)^2\big)\\ &\quad+\frac{1}{2}\sum_{j=0}^{n-1}\big(W(t_{j+1}^n)-W(t_j^n)\big)^2\\ &= \frac{1}{2}W(T)^2+\frac{1}{2}\sum_{j=0}^{n-1}\big(W(t_{j+1}^n)-W(t_j^n)\big)^2\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\sum_{j=0}^{n-1} W(t_{j+1}^n)\big(W(t_{j+1}^n)-W(t_j^n)\big)=\frac{1}{2}W(T)^2+\frac{1}{2}T$$

in L^2. In conclusion, by considering two different choices of intermediate points we get different results.

To overcome this issue, we use the following reasoning: the value of the approximation should consist only of random variables adapted to F_j. This amounts to say that the given approximation should depend only by what has happened up to time t. Then, we shall use the left endpoint for the evaluation of the integrand, that is, $s_j = t_j$ for each j.

Let us now define the concept of random step process and its stochastic integral.

Definition 3.1. A *random step process* $f(t)$ is a stochastic process

$$f(t)=\sum_{j=0}^{n-1}\eta_j 1_{[t_j,t_{j+1})}(t), \tag{3.3}$$

where $0=t_0<t_1<\ldots<t_n=1$ is a partition of $[0,1]$ and η_j are square integrable random variables $\mathcal{F}_{t_j}$-measurable for $j=0,1,\ldots,n-1$. We note for M^2_{step} the set of random step processes.

Definition 3.2. The *stochastic integral* of a random step process $f\in M^2_{step}$ of the form (3.3) is defined by

$$I(f)=\sum_{j=0}^{n-1}\eta_j(W(t_{j+1})-W(t_j)). \tag{3.4}$$

Remark 3.1. The map $I : M^2_{step} \to L^2$ is linear. In other words; for any $f, g \in M^2_{step}$ and any $\alpha, \beta \in \mathbb{R}$

$$I(\alpha f + \beta g) = \alpha I(f) + \beta I(g).$$

Proposition 3.1. *The stochastic integral $I(f)$ of a random step process $f \in M^2_{step}$ is a square integrable random variable, i.e $I(f) \in L^2$, such that*

$$E\Big(|I(f)|^2\Big) = E\bigg(\int_0^\infty |f(t)|^2\ dt\bigg).$$

Proof. Similarly as in Section 1.3, let us denote the increment $W(t_{j+1}) - W(t_j)$ by $\Delta_j W$ and $t_{j+1} - t_j$ by $\Delta_j t$ for brevity. Then

$$E\big(\Delta_j W\big) = 0 \quad \text{and} \quad E\big(\Delta_j^2 W\big) = \Delta_j t.$$

Firstly, we compute the expectation of

$$|I(f)|^2 = \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \eta_j \eta_k \Delta_j W \Delta_k W = \sum_{j=0}^{n-1} \eta_j^2 \Delta_j^2 W + 2\sum_{k<j} \eta_j \eta_k \Delta_j W \Delta_k W.$$

Since η_j and $\Delta_j W$ are independent, it holds

$$E(\eta_j^2 \Delta_j^2 W) = E(\eta_j^2) E(\Delta_j^2 W) = E(\eta_j^2)\Delta_j t.$$

Also, if $k < j$, then $\eta_j \eta_k \Delta_k$ and $W \Delta_j$ are independent, so

$$E(\eta_j \eta_k \Delta_k W \Delta_j W) = E(\eta_j \eta_k \Delta_k W) E(\Delta_j W) = 0.$$

Putting all together we have

$$E\bigg(|I(f)|^2\bigg) = \sum_{j=0}^{n-1} E(\eta_j^2)\Delta_j t.$$

Consequently, as $\eta_0, \eta_1, \ldots, \eta_{n-1}$ belong to L^2, it follows that $I(f)$ belongs to L^2.

On the other hand,

$$|f(t)|^2 = \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \eta_j \eta_k 1_{(t_j, t_{j+1}]}(t) 1_{(t_k, t_{k+1}]}(t) = \sum_{j=0}^{n-1} \eta_j^2 1_{(t_j, t_{j+1}]}(t),$$

which implies

$$E\bigg(\int_0^\infty |f(t)|^2 dt\bigg) = \sum_{j=0}^{n-1} E(\eta_j^2)\Delta_j t.$$

This means that

$$E\Big(|I(f)|^2\Big) = E\bigg(\int_0^\infty |f(t)|^2\ dt\bigg)$$

as desired. □

The stochastic integral $I(f)$ has been defined for any random step process $f \in M^2_{step}$ and has been proven that $I(f) \in L^2$. Now, we want to look at a larger class of processes, which is described below, and define the stochastic integral $I(f)$ for such class by approximation.

Definition 3.3. We denote by M^2 the class of stochastic processes $f(t), t \geq 0$ such that

$$E\left(\int_0^\infty |f(t)|^2 dt\right) < \infty$$

and there is a sequence $f_1, f_2, \ldots \in M^2_{step}$ of random step processes satisfying

$$\lim_{n\to\infty} E\left(\int_0^\infty |f(t) - f_n(t)|^2 dt\right) = 0. \tag{3.5}$$

In this case we shall say that f is approximated by the sequence of random step processes $f_1, f_2, \ldots$ in M^2.

Definition 3.4. We call $I(f) \in L^2$ the *Itô stochastic integral* (from 0 to ∞) of $f \in M^2$ if

$$\lim_{n\to\infty} E\Big(|I(f) - I(f_n)|^2\Big) = 0 \tag{3.6}$$

for any sequence $f_1, f_2, \ldots \in M^2_{step}$ of random step processes that approximates $f \in L^2$, i.e which holds (3.5). Indistinctly we shall also write

$$\int_0^\infty f(t)\ dW(t)$$

instead of $I(f)$.

Remark 3.2. By definition of the norm in L^2, we have

$$E\Big(|I(f) - I(f_n)|^2\Big) = ||I(f) - I(f_n)||_{L^2}.$$

Thus, equality (3.6) can be stated as follows

$$\lim_{n\to\infty} ||I(f) - I(f_n)||_{L^2} = 0$$

for any sequence $f_1, f_2, \ldots \in M^2_{step}$ of random step processes that approximates $f \in L^2$. In other words, the processes for which the stochastic integral exists have been defined as those that can be approximated by random step processes.

The following preposition ensures the existence of the Itô stochastic integral for any $f \in M^2$. Furthermore, it gives rise to the strong relationship between the class of stochastic processes M^2 and the space of integrable functions L^2 in terms of their expectations.

Proposition 3.2. *For any $f \in M^2$ the stochastic integral $I(f) \in L^2$ exists, is unique*[1] *and satisfies*

$$E\big(|I(f)|^2\big) = E\bigg(\int_0^\infty |f(t)|^2 dt\bigg). \tag{3.7}$$

Proof. Let us first write the norms of f and η, for any $f \in M^2$ and $\eta \in L^2$. These norms[2] are in M^2 and L^2, respectively, and have the form:

$$||I(f)||_{M^2} = \sqrt{E\bigg(\int_0^\infty |f(t)|^2 dt\bigg)} \quad \text{and} \quad ||\eta||_{L^2} = \sqrt{E(\eta^2)}.$$

On the one hand, we set a sequence of random step processes $f_1, f_2, \ldots \in M^2_{step}$ approximating $f \in M^2$, i.e satisfying equality (3.5), which can also be formulated as

$$\lim_{n\to\infty} ||f - f_n||_{M^2} = 0. \tag{3.8}$$

On the other hand, we claim that $I(f_1), I(f_2), \ldots$ is Cauchy sequence in L^2. Indeed, due to (3.8), for any $\varepsilon > 0$ there is an N such that for all $n \geq N, ||f - f_n||^2_M < \varepsilon$ and for any $m, n > N$

$$\begin{aligned} ||I(f_m) - I(f_n)||_{L^2} &= ||I(f_m - f_n)||_{L^2} \\ &= ||f_m - f_n||_{M^2} \\ &\leq ||f - f_m||_{M^2} + ||f - f_n||_{M^2} \\ &< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

where in the first step we have used the linearity of the stochastic integral (see Remark 3) and in the second, Preposition 3.1. Therefore, $I(f_1), I(f_2), \ldots$ is Cauchy sequence in L^2.

Now, since L^2 with the norm $||\cdot||_{L^2}$ is a complete space, every Cauchy sequence has its limit. It follows that $I(f_1), I(f_2), \ldots$ has a limit in L^2 for any sequence $f_1, f_2, \ldots$ of random step processes approximating f.

It remains to prove the uniqueness of the limit for all sequences. Suppose that f is approximated by two sequences of random variables, namely $f_1, f_2, \ldots$ and $g_1, g_2, \ldots$ Then, the interlaced sequence $f_1, g_1, f_2, g_2 \ldots$ approximates f too and reasoning as above the sequence $I(f_1), I(g_1), I(f_2), I(g_2), \ldots$ has its limit in L^2.

[1]note that $I(f)$ is unique as an element of L^2, meaning uniqueness within equality a.s.

[2]We adapt the wide used notation of identifying these norms with any element of a class of functions as a representative, from M^2 and L^2 respectively, determined by the relation of equality a.s.

As a consequence, all subsequence of the latter sequence has the same limit. In particular, the sequences $f_1, f_2, \ldots$ and $g_1, g_2, \ldots$ have the same limit in L^2, which we denote by $I(f)$.
Then, by Remark 3.2,

$$\lim_{n\to\infty} ||I(f) - I(f_n)||_{L^2} = 0 \tag{3.9}$$

for any sequence $f_1, f_2, \ldots \in M^2_{step}$ of random step processes that approximates $f \in L^2$. Moreover, since f_n are random processes for each n, Preposition 3.1 holds and we have

$$||I(f_n)||_{L^2} = ||f_n||_{M^2}.$$

From equalities (3.8) and (3.9), taking the limit as $n \to \infty$ in the last equality we can conclude that

$$||I(f)||_{L^2} = ||f||_{M^2}$$

and the proof is complete. □

Example 3.2. For any random step processes $f, g \in M^2$ it can be seen that

$$E\big(I(f)I(g)\big) = E\bigg(\int_0^\infty f(t)g(t)\ dt\bigg).$$

It is enough to consider suitable scalar products in M^2 and L^2 and to use Proposition 3.2. Let us present the following scalar products in M^2 and L^2:

$$\langle f, g\rangle_{M^2} = E\bigg(\int_0^\infty |f(t)g(t)|^2\ dt\bigg) \quad \text{and} \quad \langle \eta, \zeta\rangle_{L^2} = E(\eta\zeta)$$

for any $f, g \in M^2$ and $\eta, \zeta \in L^2$. They can be expressed in terms of the corresponding norms defined in the proof of Proposition 3.2,

$$\begin{aligned} \langle f, g\rangle_{M^2} &= \frac{1}{4}||f+g||^2_{M^2} - \frac{1}{4}||f-g||^2_{M^2}, \\ \langle \eta, \zeta\rangle_{L^2} &= \frac{1}{4}||\eta+\zeta||^2_{L^2} - \frac{1}{4}||\eta-\zeta||^2_{L^2}. \end{aligned}$$

Finally, applying Proposition 3.2 we have

$$\langle f, g\rangle_{M^2} = \langle \eta, \zeta\rangle_{L^2}$$

which amounts to the equality to be proved.

A question we may wonder is whether we can consider stochastic integrals over any finite time interval $[0, T]$ or not. The answer is yes, and it stems from the fact that we can restrict the class of stochastic processes M^2 to any finite time interval $[0, T]$ via indicator functions of such time interval.

Definition 3.5. For any $T > 0$ we shall denote by M_T^2 the space of all stochastic processes $f(t), t \geq 0$ such that

$$1_{[0,T)} f \in M^2.$$

The Itô stochastic integral (from 0 to T) of $f \in M_T^2$ is defined by

$$I_T(f) = I(1_{[0,T)} f).$$

Indistinctly we shall also write

$$\int_0^T f(t)\, dW(t)$$

instead of $I_T(f)$.

Before we proceed to discuss a straightforward result for a stochastic process to belong to M^2 and M_T^2, let us present a useful proposition.

Proposition 3.3. *Each random step process $f \in M_{step}^2$ belongs to M_t^2 for any $t > 0$ and*

$$I_t(f) = \int_0^t f(s)\, dW(s) \tag{3.10}$$

is a martingale with respect to the filtration $\mathcal{F}_t = \sigma\{W(s) : 0 \leq s \leq t\}$.

Proof. It is easy to see that if $f \in M_{step}^2$ then $1_{[0,t]} f \in M_{step}^2 \subset M^2$ for any $t > 0$. This implies that $f \in M_t^2$ for any $t > 0$.
We now need to verify that $I_t(f)$ is a martingale with respect to the filtration $\mathcal{F}_t$. Let $0 \leq s < t$, by definition of random step process (see Definition 3.3) $f \in M_{step}^2$ can be written as

$$f = \sum_{j=0}^{m-1} \eta_j 1_{(t_j, t_{j+1}]}(t)$$

where

$$0 = t_0 < \ldots < t_k = s < t_{k+1} < \ldots < t_m = t < t_{m+1} < \ldots < t_n$$

is a partition of $[0, t_n]$. We shall denote the increments of the Brownian motion $W(t_{j+1}) - W(t_j)$ by $\Delta_j W$ as in the proof of Proposition 3.1 for brevity. Then

$$1_{[0,t]} f = \sum_{j=0}^{m-1} \eta_j 1_{(t_j, t_{j+1}]}$$

and

$$I_t(f) = I(1_{[0,t]} f) = \sum_{j=0}^{m-1} \eta_j \Delta_j W,$$

where η_j are square integrable random variables $\mathcal{F}_{t_j}$-measurable for $j = 0, \ldots, m-1$. Thus, $I_t(f)$ is adapted to $\mathcal{F}_t$ and square integrable, and so integrable.
Proving that $I_t(f)$ is a martingale with respect to the filtration $\mathcal{F}_t$ amounts to proving that

$$E(I_t(f) |\, \mathcal{F}_s) = I_s(f) \tag{3.11}$$

Let us show (3.11): We have

$$E(I_t(f)|\mathcal{F}_s) = E(I(1_{[0,t]}f)|\mathcal{F}_s) = \sum_{j=0}^{m-1}(\eta_j \Delta_j W|\mathcal{F}_s).$$

If $j < k$, then η_j and $\Delta_j W$ are $\mathcal{F}_s$-measurable and

$$E(\eta_j \Delta_j W) = \eta_j \Delta_j W.$$

If $j \geq k$, then $\mathcal{F}_s \subset \mathcal{F}_{t_j}$ and

$$\begin{aligned} E(\eta_j \Delta_j W|\mathcal{F}_s) &= E(E(\eta_j \Delta_j W|\mathcal{F}_{t_j})|\ \mathcal{F}_s) \\ &= E(\eta_j E(\Delta_j W|\mathcal{F}_{t_j})|\mathcal{F}_s) \\ &= E(\eta_j|\mathcal{F}_s)E(\Delta_j W) = 0, \end{aligned}$$

since η_j is $\mathcal{F}_{t_j}$-measurable and $\Delta_j W$ is independent of $\mathcal{F}_{t_j}$. The above give rise to

$$E(I_t(f)|\mathcal{F}_s) = \sum_{j=0}^{k-1} \eta_j \Delta_j W = I(1_{[0,s]}f) = I_s(f)$$

which is indeed Equation (3.11) as desired. □

For practical purposes, to find a straightforward condition for existence of the Itô stochastic integral is of special importance. That is because is not always easy to find a sequence of random step processes that converges to a stochastic process of the class either M^2 or M_T^2. Hereupon, we present a theorem that provides this straightforward condition.

Theorem 3.1. *Let $f(t), t \geq 0$ be a stochastic process with a.s continuous paths adapted to a filtration $\mathcal{F}_t$. Then*

1) $f \in M^2$, i.e the Itô integral $I(f)$ exists if

$$E\left(\int_0^\infty |f(t)|^2 \; dt\right) < \infty; \tag{3.12}$$

2) $f \in M_T^2$, i.e the Itô integral $I_T(f)$ exists if

$$E\left(\int_0^T |f(t)|^2 \; dt\right) < \infty. \tag{3.13}$$

The notion of Jensen's inequality for integrals and a result that ensures the interchange of the limit and the integral under certain restrictions, i.e the dominated convergence theorem, will be needed to prove the previous Theorem.

Proposition 3.4 (Jensen's Inequality for integrals). *Let (Ω, μ, P) be a probability space such that $\mu(\Omega) = 1$ and $a, b \in \mathbb{R}$. If $f : [a, b] \to \mathbb{R}$ is a non-negative Lebesgue integrable function and $\varphi : \mathbb{R} \to \mathbb{R}$ is a convex*[3] *function, then*

$$\varphi\left(\frac{1}{b-a}\int_a^b f\ d\mu\right) \leq \frac{1}{b-a}\int_a^b \varphi \circ f\ d\mu.$$

Theorem 3.2 (Dominated convergence theorem). *Let $(\xi_n)_n$ be random variables such that $\xi_n \to \xi$ and $|\xi_n| \leq \zeta$ for all n a.s., for some integrable random variable ζ. Then*

$$E(\xi_n) \to E(\xi).$$

Proof of Theorem 3.1.
1) Suppose that $f(t)$ is an adapted process with a.s continuous paths. If (3.12) holds, then

$$f_n(t) = \begin{cases} n\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(s)\ ds & \frac{k}{n} < t \leq \frac{k+1}{n} \text{ for } k = 1, 2, \ldots, n^2 - 1, \\ 0 & \text{otherwise,} \end{cases} \tag{3.14}$$

is a sequence of random step processes in M^2_{step}. Using the sequence above, we observe that for $k = 0, 1, 2, \ldots$

$$\int_{\frac{k}{n}}^{\frac{k+1}{n}} |f_n(t)|^2\ dt = n\left|\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(t)\ dt\right|^2 \leq \int_{\frac{k-1}{n}}^{\frac{k}{n}} |f(t)|^2\ dt \quad \text{a.s.} \tag{3.15}$$

where in the second step, we have used Jensen's Inequality applied to the function $\varphi(t) = |t|^2$.
We need to verify that the sequence approximates f in the sense of Definition 3.3, that is, it holds

$$\lim_{n\to\infty} E\left(\int_0^\infty |f(t) - f_n(t)|^2 dt\right) = 0. \tag{3.16}$$

We are going to proof it by using the dominated convergence theorem and the following result

$$\lim_{n\to\infty} \int_0^\infty |f(t) - f_n(t)|^2\ dt = 0 \quad \text{a.s.} \tag{3.17}$$

[3]We call a function $\varphi : \mathbb{R} \to \mathbb{R}$ *convex* if satisfies $\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y)$ for any $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$.

First let us see (3.17):

$$\begin{aligned}\int_0^\infty |f(t)-f_n(t)|^2\,dt &= \int_0^N |f(t)-f_n(t)|^2\,dt + \int_N^\infty |f(t)-f_n(t)|^2\,dt\\ &\le \int_0^N |f(t)-f_n(t)|^2\,dt + 2\int_N^\infty \left(|f(t)|^2+|f_n(t)|^2\right)\\ &\le \int_0^N |f(t)-f_n(t)|^2\,dt + 4\int_{N-1}^\infty |f(t)|^2\,dt \quad \text{a.s.}\end{aligned}$$

The last inequality holds since by taking the sum from $k = nN$ to ∞ in (3.15), it follows that

$$\int_N^\infty |f_n(t)|^2\,dt \le \int_{N-\frac{1}{n}}^\infty |f(t)|^2\,dt \le \int_{N-1}^\infty |f(t)|^2\,dt \quad \text{a.s.}$$

for any n and N. But we have

$$\lim_{N\to\infty}\int_{N-1}^\infty |f(t)|^2\,dt = 0 \quad \text{a.s.}$$

by (3.12) and

$$\lim_{n\to\infty}\int_0^N |f(t)-f_n(t)|^2\,dt = 0 \quad \text{a.s.}$$

for any fixed N by the continuity of paths of f, completing the proof of (3.17). Now we go back to the proof of (3.16). Note that

$$\begin{aligned}\int_0^\infty |f(t)-f_n(t)|^2\,dt &\le 2\int_0^\infty \left(|f(t)|^2+|f_n(t)|^2\right)\,dt\\ &\le 4\int_0^\infty |f(t)|^2\,dt.\end{aligned}$$

The last inequality yields since

$$\int_0^\infty |f_n(t)|^2\,dt \le \int_0^\infty |f(t)|^2\,dt \quad \text{a.s.}$$

for any n, by taking the sum form $k = 0$ to ∞ in (3.15). Now, by applying the dominated convergence theorem and condition (3.12) it follows that

$$\lim_{n\to\infty} E\left(\int_0^\infty |f(t)-f_n(t)|^2 dt\right) = 0.$$

Therefore, the sequence $f_1, f_2, \ldots \in M^2_{step}$ approximates f in the sense of Definition 3.3, so $f \in M^2$.

2) Proving that f satisfies (3.13) is analogous to prove that $1_{[0,T)}f$ satisfies (3.12). The fact that f is adapted and has continuous a.s. paths implies that $1_{[0,T)}f$ is also adapted and its paths are continuous, except perhaps at T. Nevertheless, condition 1) is not affected by the lack of continuity at the single point T. Thus, $1_{[0,T)}f \in M^2$ and we conclude that $f \in M^2_T$. □

Remark 3.3. The Brownian motion $W(t)$ belongs to M_T^2 for each $T > 0$.

Next theorem provides a characterization of M^2 and M_T^2. This gives rise to necessary and sufficient conditions for a stochastic process, say f, to belong to M^2 and M_T^2. It involves the notion of *progressively measurable* stochastic process.

Definition 3.6. A stochastic process $f(t), t \geq 0$ is called *progressively measurable* if for any $t \geq 0$ the function

$$\begin{aligned} f\colon [0,t] \times \Omega &\to \mathbb{R} \\ (s,w) &\mapsto f(s,w) \end{aligned}$$

is measurable with respect to the σ-field $\mathcal{B}[0,t] \times \mathcal{F}$. Here $\mathcal{B}[0,t] \times \mathcal{F}$ is the product σ-field on $[0,t] \times \Omega$. That is, the smallest σ-field containing all sets of the form $A \times B$, where $A \subset [0,t]$ is a Borel set and $B \in \mathcal{F}$.

Theorem 3.3.

1) The space M^2 consists of all progressively measurable stochastic processes $f(t)$, $t \geq 0$ such that

$$E\left(\int_0^\infty |f(t)|^2 \, dt\right) < \infty;$$

2) The space M_T^2 consists of all progressively measurable stochastic processes $f(t)$, $t \geq 0$ such that

$$E\left(\int_0^T |f(t)|^2 \, dt\right) < \infty.$$

3.2 Simple Examples of Itô Stochastic Integrals

Example 3.3. In this example we want to proof that the following Itô stochastic integral exists and satisfies

$$\int_0^T W(t)\, dW(t) = \frac{1}{2}W(T)^2 - \frac{1}{2}T. \tag{3.18}$$

Furthermore, we want to see the similarities between the different definitions concerning the Itô stochastic integral appearing throughout Section 2.1. First of all, we have seen in Remark 3.3 that the Brownian motion belongs to M_2^T for any $T > 0$. Therefore the Itô stochastic integral $\int_0^T W(t)\, dW(t)$ exists.
In Example 3.1 we tried to find the limit of the stochastic approximating sums for the Brownian motion $W(t)$. When we used the left endpoint of each subinterval in a partition of $[0,T]$ to evaluate the integrand, we get the sum L_n in Equation (3.1). Now if we take as the integral the limit of L_n as $n \to \infty$ according to Equation (3.1) we have

$$\int_0^T W(t)\, dW(t) = \frac{1}{2}W(T)^2 - \frac{1}{2}T$$

Is this value equal to the integral $\int_0^T W(t)\ dW(t)$ described in Definition 2.5 for $f(t) = W(t)$? Let us consider the partition $\Delta_n = \{0 = t_0^n < t_1^n < \cdots < t_n^n = T\}$ of $[0, T]$ and define the sequence $f_1, f_2, \ldots$ of random step processes given by

$$f_n(t) = W(t_j^n), \quad t_j^n < t \leq t_{j+1}^n.$$

Then by Definition 2.4 we see that the Itô stochastic integral $\int_0^T W(t)\ dW(t)$ is described as follows

$$\int_0^T W(t)\ dW(t) = \lim_{n\to\infty} I(f_n), \qquad \text{in} \quad L^2.$$

On the other hand, by Definition 2.2 the Itô stochastic integral $I(f_n)$ is given by

$$I(f_n) = \sum_{j=0}^{n} W(t_j^n)\big(W(t_{j+1}^n) - W(t_j^n)\big)$$

which is indeed L_n in Equation (2.1). Hence the Itô stochastic integral $\int_0^T W(t)\, dW(t)$ as defined in Section 2.1 has the same value as the one in Equation (3.18).

Example 3.4. We use the same idea as in Example 3.3 to show that

$$\int_0^T W(t)^2\ dW(t) = \frac{1}{3}W(T)^3 - \int_0^T W(t)\ dt \tag{3.19}$$

where the integral in the left-hand side is the Itô stochastic integral for the Brownian motion $W(t)^2$ whereas the integral in the right-hand side is the Riemann integral for the Brownian motion $W(t)$. Let us consider the partition $\Delta_n = \{0 = t_0^n < t_1^n < \cdots < t_n^n = T\}$ of $[0, T]$ where $t_j^n = \frac{jT}{n}$ and define the sequence $f_1, f_2, \ldots$ of random step processes given by

$$f_n(t) = W(t_j^n), \quad t_j^n < t \leq t_{j+1}^n.$$

Then the Itô stochastic integral $\int_0^T W(t)\ dW(t)$ is given by

$$\int_0^T W(t)^2\ dW(t) = \lim_{n\to\infty} \sum_{j=0}^{n-1} W(t_j^n)^2\big(W(t_{j+1}^n) - W(t_j^n)\big), \tag{3.20}$$

where if the series of the right-hand side converges, it does in L^2. Working out the Newton's binomial for $n = 3$ of the series in the right-hand side and rewriting it in a suitable way we get that

$$\begin{aligned} &3\sum_{j=0}^{n-1} W(t_j^n)^2(W(t_{j+1}^n) - W(t_j^n)) \\ &\quad = W(T)^3 - W(0)^3 - \sum_{j=0}^{n-1}(W(t_{j+1}^n) - W(t_j^n))^3 \\ &\quad - 3\sum_{j=0}^{n-1} W(t_j^n)(W(t_{j+1}^n) - W(t_j^n))^2 \end{aligned} \tag{3.21}$$

where by definition, $W(0)^3 = 0$. In order to determine the limit (in L^2) of the series in Equation (3.20) it needs to be checked the corresponding limits of the summations appearing in (3.21).
For the first summation, we use the fact that $E|W(t) - W(s)|^6 = 15|t-s|^3$. Then,

$$\begin{aligned} E\Big|\sum_{j=0}^{n-1}\big(W(t_{j+1}^n) - W(t_j^n)\big)^3\Big|^2 &= 15\sum_{j=0}^{n-1}(t_{j+1}^n - t_j^n)^3 \\ &\leq 15||\Delta_j^n t||^2 T \xrightarrow[n\to\infty]{L^2} 0 \end{aligned} \tag{3.22}$$

since $||\Delta_j^n t|| = \max\limits_{0\leq j\leq n-1}|t_{j+1}^n - t_j^n|$ tends to 0. For the second summation in Equation (3.21), let us denote $W(t_{j+1}^n) - W(t_j^n)$ by $\Delta_j^n W$ and $t_{j+1}^n - t_j^n$ by $\Delta_j^n t$ as earlier in this section. We consider

$$\Phi_n = \sum_{j=0}^{n-1}\big[W(t_j^n)(\Delta_j^n W)^2 - W(t_j^n)(\Delta_j^n t)\big] = \sum_{j=0}^{n-1} X_j \tag{3.23}$$

where $X_j = W(t_j^n)(\Delta_j^n W)^2 - W(t_j^n)(\Delta_j^n t)$. Then,

$$\Phi_n^2 = \sum_{j,k=0}^{n-1} = X_j X_k.$$

For $i \neq j$ we have $E(X_j X_k) = 0$ since $W(t)$ has independent increments and $E(W(t) - W(s))^2 = |t-s|$. On the other hand, $E[W(t) - W(s)]^4 = 3(t-s)^2$ and so for $i = j$ we have

$$\begin{aligned} E(X_j^2) &= E\big[W(t_j^n)^2(\Delta_j^n W)^4 - 2W(t_j^n)^2(\Delta_j^n W)^2(\Delta_j^n t) \\ &\quad + W(t_j^n)^2(\Delta_j^n t)^2\big] \\ &= E\big[W(t_j^n)^2\big]\ E[(\Delta_j^n W)^4 - 2(\Delta_j^n W)^2(\Delta_j^n t) + (\Delta_j^n t)^2\big] \\ &= t_j^n\big[3(\Delta_j^n t)^2 - 2(\Delta_j^n t)^2 + (\Delta_j^n t)^2\big] \\ &= t_j^n\big[2(\Delta_j^n t)^2\big]. \end{aligned}$$

It follows that

$$E|\Phi_n|^2 = E\Big|\sum_{j=0}^{n-1} X_j\Big|^2 = \sum_{j=0}^{n-1} 2t_j^n\big[(\Delta_j^n t)^2\big] \leq \frac{2T^2}{n} \xrightarrow[n\to\infty]{} 0. \tag{3.24}$$

Hence, the convergence in L^2 for the first and the second summations in the right-hand side of Equation (3.21) are 0 and $\int_0^T W(t)\,dt$, respectively. In conclusion, from Equations (3.20) and (3.21) the equality in Equation (3.19) is obtained.

3.3 Properties of the Itô Stochastic Integral

Below we state a theorem that summarizes the basic properties of the Itô integral.

Theorem 3.4. *The following properties hold for any $f, g \in M_t^2$, any $\alpha, \beta \in \mathbb{R}$, and any $0 \leq s < t$:*

1) linearity

$$\int_0^t (\alpha f(r) + \beta g(r))\ dW(r) = \alpha \int_0^t f(r)\ dW(r) + \beta \int_0^t g(r)\ dW(r);$$

2) isometry

$$E\left(\left| \int_0^t f(r)\ dW(r) \right|^2\right) = \left(\int_0^t |f(r)|^2\ dr \right);$$

3) martingale property

$$E\left(\int_0^t f(r)\ dW(r) \middle| \mathcal{F}_s \right) = \int_0^s f(r)\ dW(r).$$

Proof. 1) If f and g belong to M_t^2 then the functions $1_{[0,t)}f$ and $1_{[0,t)}g$ belong to M^2, so that there are sequences $f_1, f_2, \dots$ and $g_1,\ g_2, \dots$ in M_{step}^2 approximating both $1_{[0,t)}f$ and $1_{[0,t)}g$. It follows that $1_{[0,t)}(\alpha f + \beta g)$ can be approximated by $\alpha f_1 + \beta g_1, \alpha f_2 + \beta g_2, \dots$. By linearity od the stochastic integral (see Remark 3.1)

$$I(\alpha f_n + \beta g_n) = \alpha I(f_n) + \beta I(g_n)$$

for each n. Hence, taking the L^2 limit on both sides of the equality above as $n \to \infty$, we obtain

$$I\big(1_{[0,t)}(\alpha f_n + \beta g_n)\big) = \alpha I(1_{[0,t)} f_n) + \beta I(1_{[0,t)} g_n)$$

which is the required assertion.

2) The function $1_{[0,t)}f$ can be approximated by random step processes in M_{step}^2. It enables us to write

$$E\left(\left| \int_0^t f(r)\ dW(r) \right|^2\right) = E\big(|I_t(f)|^2\big) = E\left(\int_0^t |f(r)|^2\ dr \right)$$

where the second equality follows from Preposition 3.1. This proves 2).

3) If f belongs to M_t^2 then $1_{[0,t)}f$ belongs to M^2. Let $f_1, f_2, \dots$ be a sequence of random step processes approximating $1_{[0,t)}f$. By Remark 3.3

$$E(I(1_{[0,t)} f_n)|\mathcal{F}_s) = I(1_{[0,s)} f_n)$$

for each n. Now we observe that the sequences $1_{[0,s)}f_1, 1_{[0,s)}f_2, \dots$ and $1_{[0,t)}f_1, 1_{[0,t)}f_2, \dots$ belongs to M_{step}^2 and approximates $1_{[0,s)}f$ and $1_{[0,t)}f$ respectively. Hereby, we have

$$I(1_{[0,s)}f_n) \xrightarrow[n\to\infty]{L^2} I(1_{[0,t)}f)$$

and

$$I(1_{[0,t)}f_n) \xrightarrow[n\to\infty]{L^2} I(1_{[0,t)}f).$$

The lemma below implies that

$$E(I(1_{[0,t)}f_n)|\mathcal{F}_s) \xrightarrow[n\to\infty]{L^2} E(I(1_{[0,t)}f)|\mathcal{F}_s)$$

completing the proof.

□

Lemma 3.1. *If ξ and $\xi_1, \xi_2, \ldots$ are square integrable random variables such that $\xi_n \to \xi$ in L^2 as $n \to \infty$, then*

$$E(\xi_n|\mathcal{G}) \xrightarrow[n\to\infty]{L^2} E(\xi|\mathcal{G})$$

for any σ-field $\mathcal{G}$ on Ω contained in $\mathcal{F}$.

Proof. By Jensen's Inequality applied to the function $\varphi(x) = |x|^2$ we have

$$|E(\xi_n|\mathcal{G}) - E(\xi|\mathcal{G})|^2 = |E(\xi_n - \xi|\mathcal{G})|^2 \le E\Big(|\xi_n - \xi|^2\Big|\mathcal{G}\Big),$$

which implies that

$$\begin{aligned} E\Big(\big|E(\xi_n|\mathcal{G}) - E(\xi|\mathcal{G})\big|^2\Big) &\le E\Big(E\Big(|\xi_n - \xi|^2|\mathcal{G}\Big)\Big) \\ &= E\Big(|\xi_n - \xi|^2\Big) \to 0 \end{aligned}$$

as $n \to \infty$. □

Next we study the continuity property of the stochastic integral $\int_0^t f(s)\, dW(s)$ as a function of the upper limit t. Note that the stochastic integral is not defined for each fixed $w \in \Omega$ as a Riemann, Riemann-Stieltjes, or even Lebesgue integral. Even for the Brownian integral case, it is not defined this way. Therefore, the continuity of the stochastic process $f \in M_t^2$ is not a trivial fact as in elementary real analysis [17].
Let us first recall the notion of *modification* of a stochastic process.

Definition 3.7. Let $\xi(t)$ and $\zeta(t)$ be stochastic processes defined for $t \in T$, where $T \subset \mathbb{R}$. We say that $\xi(t)$ is a *modification* of $\zeta(t)$ if

$$P\{\xi(t) = \zeta(t)\} = 1 \text{ for all } \; t \in T. \tag{3.25}$$

Remark 3.4. If $T \subset \mathbb{R}$ is a countable set, then (3.25) is equivalent to the condition

$$P\{\xi(t) = \zeta(t) \text{ for all } \ t \in T\} = 1.$$

However, this does not necessarily hold if T is uncountable. In the following example we see that two processes which are modifications of one another may have quite different sample paths.

Example 3.5. Let τ be a nonnegative random variable with continuous distribution function. Set $T = [0, \infty)$. The processes

$$\xi(t) = 0 \quad \text{for all} \ \ t \in T,$$
$$\zeta(t) = \begin{cases} 0 & \text{if} \quad \tau \neq t \\ 1 & \text{if} \quad \tau = t \end{cases}$$

are equivalent but their sample paths are different.

In the next theorem, stated without proof, we consider the stochastic integral $\int_0^t f(s)\, dW(s)$ as a function of the upper limit t. It yields information about the continuous behavior of this function in terms of t.

Theorem 3.5. *Let $f(t)$ be a process belonging to M_t^2 and let*

$$\xi(t) = \int_0^t f(s) dW(s)$$

for every $t \geq 0$. Then there exists an adapted modification $\zeta(t)$ of $\xi(t)$ with a.s continuous paths. This modification is unique up to equality a.s.

From now we shall identify $\int_0^t f(s)\, dW(s)$ with the adapted modification having a.s continuous paths, which works extremely well alongside with Theorem 3.1. Then, it will be helpful to use this convention whenever there is a need to show that a stochastic integral can be used as the integrand of another stochastic integral, i.e belongs to M_T^2 for $T \geq 0$. This is illustrated by the next example.

Example 3.6. The stochastic integral

$$\xi(t) = \int_0^t W(s)\, dW(s)$$

belongs to M_T^2 for any $T \geq 0$. Indeed, by Theorem 3.5 $\xi(t)$ can be identified with an adapted modification having a.s continuous paths. Therefore, it suffices to verify that $\xi(t)$ satisfies condition (3.13) of Theorem 3.1. That is, we shall see that

$$E\left(\int_0^T |\xi(t)|^2 \, dt \right) < \infty.$$

Since the stochastic integral is an isometry,

$$E\left|\int_0^t W(s)\ dW(s)\right|^2 = E\int_0^t |W(s)|^2\ ds = \int_0^t s\ ds = \frac{t^2}{2}.$$

Consequently,

$$E\int_0^T |\xi(t)|^2\ dt = E\int_0^T \left|\int_0^t W(s)\ dW(s)\right|^2\ dt = \int_0^T \frac{t^2}{2}\ dt = \frac{T^3}{6} < \infty$$

i.e $\xi(t)$ satisfies condition (3.13). This means that $\xi(t)$ belongs to M_T^2.

3.4 Stochastic Differential Equations

In this section we first introduce the notion of stochastic differential and Itô formula. Then, we focus on analysing stochastic differential equations and present a theorem regarding the existence and unicity of solutions for a SDE. Several elaborated examples are given throughout the section to illustrate the theory.

3.4.1 Stochastic differential and Itô Fôrmula

Itô's theory of stochastic integration was originally motivated as a direct method to construct diffusion processes (a subclass of Markov processes) as solutions of the yet to be defined stochastic differential equations [4]. Now we are going to briefly introduce a crucial tool for transforming and computing the also yet to be defined stochastic differential established by Itô, which we denote by *Itô formula*.
Let $x(t)$ be a continuously differentiable function such that $x(0) = 0$ satisfying

$$x(T)^2 = 2\int_0^T x(t)\ dx(t) \tag{3.26}$$

$$x(T)^3 = 3\int_0^T x(t)^2\ dx(t), \tag{3.27}$$

where $dx(t)$ shall simple be understood as a shorthand notation for $x'(t)\ dt$ and the integrals on the right-hand side being Riemann integrals. It turns out that from Examples 3.3 and 3.4 we got similar formulae for Brownian motions:

$$W(T)^2 = \int_0^T\ dt + 2\int_0^T W(t)\ dWt \tag{3.28}$$

$$W(T)^3 = \int_0^T W(t)\ dt + 3\int_0^T W(t)^2\ dW(t) \tag{3.29}$$

where this time the integrals of the right-hand side being a Riemann integral and an Itô integral. This leads to a very special class of stochastic processes that are defined as follows.

Definition 3.8. A stochastic process $\xi(t), t \geq 0$ is called an *Itô Process* if it has a.s. continuous paths and can be represented as

$$\xi(T) = \xi(0) + \int_0^T a(t)\ dt + \int_0^T b(t)\ dW(t) \ \text{ a.s.}, \tag{3.30}$$

where $b(t)$ is a process belonging to M_T^2 for all $T > 0$ and $a(t)$ is a process adapted to the filtration $\mathcal{F}_t$ such that

$$\int_0^T |a(t)|\ dt < \infty \ \text{ a.s.} \tag{3.31}$$

for all $T \geq 0$. The class of all adapted processes $a(t)$ satisfying (3.31) for some $T > 0$ will be denoted by $\mathcal{L}_T^1$.

For an Itô process ξ it is costumary to write (3.30) as

$$d\xi(t) = a(t)\ dt + b(t)\ dW(t) \tag{3.32}$$

and to call $d\xi(t)$ the *stochastic differential* of $\xi(t)$. The latter equation is also known as the *Itô differential notation*. Let us emphasize that the stochastic differential $d\xi(t)$ has no well-defined mathematical meaning on its own and should always be rigorously understood in the context of (3.30). Hence, the Itô differential notation is not an attempt to give a precise mathematical meaning to the stochastic differential but merely an efficient way of writing this equation.

The Equations (3.28)and (3.29) resemble the ones for an smooth function $x(t)$, that is, Equations (3.26) and (3.27). However, there exist intriguing terms $\int_0^T dt$ and $3\int_0^T W(t)\ dt$; Such terms are a feature inherent in the Itô formula and referred to as the *Itô correction*. The formulae for $W(T)^2$ and $W(T)^3$ are examples of the *Itô formula*. Below we state a simplified version of the formula, followed by the general theorem. The proofs of these results are rather complicated and can be found in [4, Chapter 7, p. 196-200] and [17, Chapter 7, proof of Theorem 7.1.2], respectively. We consider that a deeper analysis of the Itô formula and its applications is beyond the scope of this paper so that for a detailed treatment on this topic we refer to [17, Chapter 7-8].

Theorem 3.6 (Itô formula, simplified version)**.** *Suppose that $F(t,x)$ is a real-valued function with continuous partial derivatives $F_t'(t,x)$, $F_x'(t,x)$ and $F_{xx}''(t,x)$ for all $t \geq 0$ and $x \in \mathbb{R}$. We also assume that the process $F_t'(t,x)$ belongs to M_T^2 for all $T > 0$. Then $F(t, W(t))$ is an Itô process such that*

$$\begin{aligned} F(T, W(T)) - F(0, W(0)) = & \int_0^T \left(F_t'(t, W(t)) + \frac{1}{2} F_{xx}''(t, W(t)) \right) dt \\ & + \int_0^T F_x'(t, W(t))\ dW(t) \quad a.s. \end{aligned} \tag{3.33}$$

In differential notation this formula can be rewritten as

$$dF(t, W(t)) = \left(F_t'(t, W(t)) + \frac{1}{2} F_{xx}''(t, W(t)) \right) dt + F_x'(t, W(t))\ dW(t). \tag{3.34}$$

Remark 3.5. The latter equation yields similarity with the chain rule

$$dF(t, x(t)) = F'_t(t, x(t))\ dt + F'_x(t, x(t))\ dx(t) \tag{3.35}$$

for a smooth function $x(t)$, where $dx(t) = x'(t)\ dt$. We also note that the additional term $\frac{1}{2}F''_{xx}(t, W(t))\ dt$ in (3.34) is denoted by the *Itô correction*.

The following result will give rise to a general formula of the simplified version just stated. The Brownian motion $W(t)$ will be replaced by an arbitrary Itô process $\xi(t)$ such that

$$d\xi(t) = a(t)\ dt + b(t)\ dW(t), \tag{3.36}$$

where a and b belongs $\mathcal{L}^1_t$ and M^2_t for all $t \geq 0$, respectively.

Theorem 3.7 (Itô formula, general case)**.** *Let $\xi(t)$ be an Itô process satisfying (3.36). Suppose that $F(t, x)$ is a real-valued function with continuous partial derivatives $F'_t(t, x)$, $F'_x(t, x)$ and $F''_{xx}(t, x)$ for all $t \geq 0$ and $x \in \mathbb{R}$. We also assume that the process $b(t)F'_x(t, \xi(t))$ belongs to M^2_T for all $T \geq 0$. Then $F(t, \xi(t))$ is an Itô process such that*

$$\begin{aligned} F(T, \xi(T)) - F(0, \xi(0)) = \int_0^T \Big(& F'_t(t, \xi(t))\ +\ F'_x(t, \xi(t))\ a(t) \\ & + \frac{1}{2}F''_{xx}(t, \xi(t))\ b(t)^2 \Big)\ dt\ +\int_0^T F'_x(t, \xi(t))\ b(t)\ dW(t) \quad a.s. \end{aligned} \tag{3.37}$$

In differential notation this formula can be rewritten as

$$\begin{aligned} dF(t, \xi(t)) = & \Big(F'_t(t, \xi(t))\ +\ F'_x(t, \xi(t))\ a(t)\ +\ \frac{1}{2}F''_{xx}(t, \xi(t))\ b(t)^2 \Big)\ dt \\ & +\ F'_x(t, \xi(t))\ b(t)\ dW(t). \end{aligned} \tag{3.38}$$

Example 3.7. Suppose that $\alpha \in \mathbb{R}$ and $\sigma > 0$. Consider the Itô process $X(t)$ and the Langevin equation

$$dX(t) = -\alpha X(t)\ dt + \sigma\ dW(t),\ \ X(0) = x_0, \tag{3.39}$$

This equation can be interpreted as in Definition 2.8, that is, by the following stochastic integral equation:

$$X(t) = x_0 - \alpha \int_0^t X(s)\ ds + \sigma \int_0^t dW(s).$$

We are going to use the Itô's formula to find the solution of the Langavin equation. Let us consider $F(t, x) = e^{\alpha t}x$, some elementary calculus shows that $F(t, x)$ has continuous partial derivatives such that $F'_t(t, x) = \alpha e^{\alpha t}x, F'_x(t, x) = e^{\alpha t}$ and $F'_{xx}(t, x) = 0$. Since $\sigma e^{\alpha t}$ is bounded on each set of the form $[0, T] \times \mathbb{R}$ we can assume that $\sigma e^{\alpha t}$ belongs to M^2_T for any $T \geq 0$. Then $F(t, X(t)) = e^{\alpha t}X(t)$ is also an Itô process and the hypothesis of Theorem 3.7 are attained. Thus by the Itô formula we have

$$\begin{aligned} e^{\alpha t}X(t) &= x_0 + \int_0^t \Big(\alpha e^{\alpha s}X(s) - \alpha e^{\alpha s}X(s) \Big) dt + \int_0^t \sigma e^{\alpha s}\, dW(s) \\ &= x_0 + \int_0^t \sigma e^{\alpha s}\, dW(s). \end{aligned}$$

Hence, we have the following stochastic integral equation

$$e^{\alpha t}X(t) = x_0 + \int_0^t \sigma e^{\alpha s}\, dW(s).$$

Finally, since the term $e^{\alpha}t \neq 0$ for all $t \geq 0$, the expression above can be divided by $e^{\alpha}t$ and the solution of the Langavin equation $X(t)$ is obtained:

$$X(t) = e^{-\alpha t}x_0 + \sigma e^{-\alpha t} \int_0^t e^{-\alpha s}\, dW(s).$$

3.4.2 Stochastic Differential Equations

Now we are fully prepared to look at the main goal of this section, which is to analyze *stochastic differential equations* of the form

$$d\xi(t) = f(\xi(t))\, dt + g(\xi(t))\, dW(t). \tag{3.40}$$

The term $f(\xi(t))$ is referred to as the *drift coefficient*, while $g(\xi(t))$ is called the *diffusion coefficient*. The solutions will be sought out in the class of Itô processes $\xi(t)$ with a.s. continuous paths and will be called *diffusion processes*. As in the theory of ordinary differential equations, an initial condition

$$\xi(0) = \xi_0 \tag{3.41}$$

needs to be specified. Here ξ_0 can be either a fixed real number or a random variable. Although in general it tends to be a random variable. From the fact that $\xi(t)$ is an Itô process, it must be adapted to the filtration $\mathcal{F}_t$ of $W(t)$, so ξ_0 must be $\mathcal{F}_0$-measurable.

Definition 3.9. An Itô process $\xi(t), t \geq 0$ is called a solution of the initial value problem

$$\begin{aligned} d\xi(t) &= f(\xi(t))\, dt + g(\xi(t))\, dW(t), \\ \xi(0) &= \xi_0 \end{aligned}$$

if ξ_0 is an $\mathcal{F}_0$-measurable random variable, the processes $f(\xi(t))$ and $g(\xi(t))$ belong, respectively, to $\mathcal{L}^1_T$ and M^2_T and

$$\xi(T) = \xi_0 + \int_0^T f(\xi(t))\, dt + \int_0^T g(\xi(t))\, dW(t) \quad \text{a.s.} \tag{3.42}$$

for all $T \geq 0$.

Remark 3.6. Observe that once we have the stochastic differential equation (3.40) with initial condition (3.41), we can immediately convert it into the integral form of Equation (3.42).

Remark 3.7. Stochastic differential equation relies on stochastic differential notation, hence it has no well-defined mathematical meaning. Only *stochastic integrals equations* of the form (3.42) have a rigorous mathematical meaning. Then, why we try to write stochastic integral equations in such way? The answer is because by using stochastic differential equations we are able to draw on the analogy with ordinary differential equations.
This analogy will be employed to solve some stochastic differential equations later on this section. Moreover, it will play a key role in modeling, as it will be seen in the next chapter.

Example 3.8. From Example 3.7, we have that the Langevin equation

$$dX(t) = -\alpha X(t)\; dt + \sigma\; dW(t),\;\; X(0) = x_0,$$

is an example of an inhomogeneous linear stochastic differential equation. The solution with an initial condition $x_0 = 0$ is the Ornstein-Uhlenbeck process

$$X(t) = \sigma e^{-\alpha t} \int_0^t e^{-\alpha s}\; dW(s).$$

Example 3.9. Consider the stochastic differential equation

$$dX_t = X_t^3\; dt + X_t^2\; dW(t),\;\; X_0 = 1. \tag{3.43}$$

We write X_t instead of $X(t)$ for short. Now, by applying the Itô formula to the function $F(t,x) = \frac{1}{x}$ we get

$$\begin{aligned}\frac{1}{X_t} &= 1 + \int_0^t \left(-\frac{1}{X_s^2}\; X_s^3 + \frac{1}{2}\frac{2}{X_s^3}\; X_s^4 \right) ds + \int_0^t -\frac{1}{X_s^2}\; X_s^2\; dW(s).\\ &= 1 - \int_0^t\; dW(s) = 1 - W(t).\end{aligned}$$

Thus the solution of Equation (3.43) is given by

$$X_t = \frac{1}{1 - W(t)}$$

and may explode[4] at the first exit time of the Brownian motion $W(t)$ from the interval $(-\infty, 1)$.

The example above is a simple modification of a well-known example in ordinary differential equations for the Leibniz-Newton calculus. Then, we would expect to encounter similar phenomena in stochastic calculus for the Itô calculus. This means that in order to ensure the existence and unicity of a globally defined solution, we need to impose the Lipschitz condition [17].

[4]Suitable extensions of the Itô formula and the definition of a solution are required to study stochastic differential equations with non-Lipschitz coefficients and explosion time. Therefore, to prevent an explosion of this paper, we refer the interested reader to [18].

Theorem 3.8. *Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous functions, i.e there is a constant $C > 0$ such that for any $x, y \in \mathbb{R}$*

$$|f(x) - f(y)| \leq C|x - y|,$$
$$|g(x) - g(y)| \leq C|x - y|.$$

Moreover, let ξ_0 be an $\mathcal{F}_0$-measurable square integrable random variable. Then the initial value problem

$$d\xi(t) = f(\xi(t))\ dt + g(\xi(t))\ dW(t), \tag{3.44}$$
$$\xi(0) = \ \xi_0 \tag{3.45}$$

has a solution $\xi(t), t \geq 0$ in the class of Itô processes. The solution is unique in the sense that $\eta(t), t \geq 0$ is another Itô process satisfying (3.32) and (3.45), then the two processes are identical a.s., that is,

$$P\{\xi(t) = \eta(t) \ \ \text{for all} \ \ t \geq 0\} = 1.$$

Proof. To obtain the solution to the initial value problem given by Equations (3.44) and (3.45) it suffices to take a modification with a.s. continuous paths of the process $\xi \in M_T^2$ such that

$$\xi(s) = \xi_0 + \int_0^s f(\xi(t))\ dt + \int_0^s g(\xi(t))\ dW(t) \quad \text{a.s.} \tag{3.46}$$

for all $s \in [0, T]$. By Theorem 3.5 the existence of such modification is guaranteed. In order to find out the solution to the stochastic integral Equation (3.46) we are going to use the Banach fixed point theorem in M_T^2 alongside with the norm defined by

$$||\xi||_\lambda^2 = E\int_0^T e^{-\lambda t}|\xi(t)|^2\ dt, \tag{3.47}$$

which turns M_T^2 into a complete normed vector space. As we will see below, the number $\lambda > 0$ shall be taken large enough. To apply the fixed point theorem let us consider $\Phi : M_T^2 \to M_T^2$ defined by

$$\Phi(\xi(s)) = \xi_0 + \int_0^s f(\xi(t))\ dt + \int_0^s g(\xi(t))\ dW(t) \tag{3.48}$$

for any $\xi \in M_T^2$ and $s \in [0, T]$. Now, if we see that Φ is a strict contraction, i.e

$$||\Phi(\xi) - \Phi(\zeta)||_\lambda \leq \alpha||\xi - \zeta||_\lambda \tag{3.49}$$

for some $\alpha < 1$ and all $\xi, \zeta \in M_T^2$, we can apply the fixed point theorem. Thus, by the Banach theorem, Φ will have a unique fixed point $\xi = \Phi(\xi)$. It turns out that this is the desired solution to (3.46) and the proof is complete.
It remains to check that Φ is indeed a strict contraction. By the definition of Φ, it amounts to prove that the maps Φ_1 and Φ_2 described by

$$\Phi_1 = \int_0^s f(\xi(t))\ dt, \quad \Phi_2 = \int_0^s g(\xi(t))\ dW(t),$$

are strict contractions. In both cases this yields due to the Lipschitz continuity of f, although for Φ_2 the isometry of the Itô integral (see Theorem 3.4) is also needed. Let us prove in detail the latter case. For any $\xi, \zeta \in M_T^2$

$$\begin{aligned}
||\Phi_2(\xi) - \Phi_2(\zeta)||_\lambda &= E \int_0^T e^{-\lambda s} \left| \int_0^s [g(\xi(t)) - g(\zeta(t))] \; dW(t) \right|^2 \; ds \\
&= E \int_0^T e^{-\lambda s} \int_0^s |g(\xi(t)) - g(\zeta(t))|^2 \; dt \; ds \\
&\leq C^2 E \int_0^T e^{-\lambda s} \int_0^s |\xi(t)) - \zeta(t)|^2 \; dt \; ds \\
&= C^2 E \int_0^T \left(\int_t^T e^{-\lambda s} e^{\lambda t} \; ds \right) e^{-\lambda t} |\xi(t) - \zeta(t)|^2 \; dt \\
&\leq \frac{C^2}{\lambda} E \int_0^T e^{-\lambda t} |\xi(t) - \zeta(t)|^2 \; dt = \frac{C^2}{\lambda} ||\xi - \zeta||_\lambda,
\end{aligned}$$

where the second step is due to the isometry of the Itô integral and the fifth step holds since $\int_t^T e^{-\lambda s} e^{\lambda t} \; ds = \frac{1}{\lambda}(1 - e^{-\lambda(T-t)}) \leq \frac{1}{\lambda}$. Throughout the above C denotes the Lipschitz constant of g. If we choose $\lambda > 0$ such that $\frac{C^2}{\lambda} < 1$, that is $C^2 < \lambda$, then Φ_2 is a strict condition.
The main idea of the proof is presented above. A slightly more general result on this vein and its proof may be found in [17, Chapter 10, Theorem 10.3.5]. ☐

Let us conclude this section by discussing an example of a linear stochastic differential equation. This example emphasizes that the solution to the initial value problem for any linear stochastic equation can be found by exploiting the analogy with ordinary differential equations.

Example 3.10. Let a and b be real numbers. The aim of this example is to verify that $X(t) = X_0 e^{at+bW(t)}$ is a solution of the linear stochastic differential equation

$$dX(t) = \left(a + \frac{b^2}{2} \right) X(t) \; dt + bX(t) \; dW(t) \tag{3.50}$$

with the initial condition $X(0) = X_0$. To do so we are going to use the Itô formula applied to the function $F(t, x) = e^{at+bx}$ and the Itô process $W(t)$. First of all we need to check the assumptions of Theorem 3.7. We have that $F(t, x)$ has continuous partial derivatives $F_t'(t, x) = ae^{at+bx}$, $F_x'(t, x) = be^{at+bx}$ and $F_{xx}''(t, x) = b^2 e^{at+bx}$. Also, the term $F_{xx}''(t, W(t)) = b^2 e^{at+bW(t)}$ is bounded on each set of the form $[0, T] \times \mathbb{R}$ implying that $b^2 e^{at+bW(t)}$ belongs to M_T^2 for all $T \geq 0$. Thus by the Itô formula we have

$$\begin{aligned}
X_0 e^{at+bW(t)} &= X_0 + \int_0^t X_0 \left(a e^{as+bW(s)} + \frac{1}{2} b^2 e^{as+bW(s)} \right) \; ds \\
&+ \int_0^t X_0 b e^{as+bW(s)} \; dW(s) \\
&= X_0 + \left(a + \frac{b^2}{2} \right) \int_0^t X(s) \; ds + b \int_0^t X(s) \; dW(s).
\end{aligned}$$

Then we can convert this integral stochastic equation into a differential stochastic equation,

$$dX(t) = \left(a + \frac{b^2}{2}\right) X(t)\ dt + bX(t)\ dW(t).$$

Thus, $X(t)$ satisfies the stochastic differential equation (3.50) and it only remains to be checked that $X(0) = X_0$. In fact,

$$X(0) = X_0 e^{a \cdot 0 + b \cdot W(0)} = X_0.$$

Chapter 4

Applications

In thread manufacture, the cotton fiber length distribution determines many of the characteristics of the thread. In particular, fiber length gives information about the spinning efficiency, the yarn strength and the yarn uniformity of the cotton, which are good indicators of the resulting weave quality. It is logical to wonder how fiber length distribution is affected, and it is due to breakage during processing. It appears that the development of a SDE model for fiber length distribution has provided more understanding of the fiber breakage phenomenon and the origination of different fiber length distributions. See [16] for more information about the work dealing with fiber breakage reported in textile industry.

In the first section of this chapter we develop two equivalent stochastic differential equation (SDE) models for cotton fiber length distribution. The two SDE models are equivalent in the sense that they are structurally different yet they have identical probability distribution [3]. In the second section simulations are work out for different number of fibers having different length to indicate the behavior of the SDE model.

4.1 SDE Model Construction

The construction of a discrete stochastic model studying changes in the system components over a small time interval Δt is a natural extension of the procedure used for many years in modeling deterministic dynamical processes in physics and engineering. This procedure results in a differential equation as the time interval approaches zero. However, in the case considered here when the process is stochastic rather than deterministic, a finite Δt produces a discrete stochastic model. The discrete stochastic model then leads to a stochastic differential equation as $\Delta t \to 0$ (see e.g [1, Chapter 5, p. 135-144] for a detailed treatment of the modeling procedure used for solving SDE models, where several examples are given).

In the development of the stochastic model, fibers are grouped by length. In this manner, the cotton fiber distribution can be considered as a population distribution [1, 2]. There appears to be three procedures for developing SDE models for applications in population biology, physics, chemistry and mathematical finance. In this investigation only two of them are discussed. The first procedure is indicated as follows. Firstly, a discrete stochastic model is derived where the breakage phenomenon is carefully studied for a short time interval. Secondly, the expected change and covariance matrix for the change are calculated for the discrete stochastic process. Finally, a system of stochastic differential equations is identified whose probability distribution approximates that of the discrete stochastic model [2]. The idea for the second procedure is to explicitly determine all the different random changes that occur in the system and to include additional Brownian motions as an alternative to the matrix square root of the covariance matrix that arises in the first procedure. Note that it is assumed that probabilities are given to the order of Δt^2.

The development of the SDE model involves d populations, $\{N_k(t)\}_{k=1}^{d}$, of fibers having different lengths considered as a functions of time t. Let us state some terminology associated with the SDE model:

Let $L =$ fiber length, where it is assumed that $0 \leq L \leq L_{max}$.

Let $L_k = kh$ for $k = 0, 1, \dots, d$ where $h = L_{max}/d$.

Let $N_k(t) =$ number of fibers of length L_k for $k = 1, \dots, d$.

Let $q_k \Delta t =$ fraction of fibers of length L_k broken in time Δt. (Note that $q_1 \Delta t = 0$.)

Let $S_{k,l} =$ fraction of fragments of length L_l formed from breakage of fibers of length L_k.

Let $p_{k,l}(t)\, \Delta t = N_k(t) S_{k,l} q_k \Delta t =$ probability of a fragment of length L_l being formed from breakage of fiber length L_k in time t to $t + \Delta t$.

From the above definitions, it follows that $\sum_{l=1}^{k-1} S_{k,l} = 1$, $S_{k,k-l} = S_{k,l}$. Let $(\Delta N)^{k,l}$ be the change to the vector $N(t)$ for a small time interval Δt, with probability $p_{k,l}(t)$ due to breakage of a fiber in group k to produce one fiber each in group l and $k - l$. Then, the expression of the the ith element of the change $(\Delta N)^{k,l}$ is

$$\big(\Delta N\big)_i^{k,l} = \begin{cases} -1, & \text{if } \ i = k \\ 1, & \text{if } \ i = l \ \text{ or } \ i = k - l \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

To develop a SDE model using the first procedure, the expected change $E(\Delta N(t))$ and the covariance in the change

$$\begin{aligned} E\big(\big(\Delta N - E\Delta N\big)\big(\Delta N - E\Delta N\big)^T\big) &= E\big(\Delta N(\Delta N)^T\big) - E(\Delta N)E(\Delta N)^T \\ &= E\big(\Delta N(\Delta N)^T\big) \end{aligned}$$

are calculated for the intervals Δt. Notice that the value of the expected change $E(\Delta N)$ for any small time Δt is computed by summing the products of the changes with the respective probabilities. Thus, the term $E(\Delta N)E(\Delta N)^T$ arising in the covariance in the change is of order Δt^2 and can be removed.

For example, let us consider the case of $d = 7$ types of fibers where a fiber in the 6th group is breaking into two fibers, one in the 2nd group and one in the 4th group. For this special case the change $(\Delta N)^{6,4}$ is given by:

$$(\Delta N)^{6,4} = [0, 1, 0, 1, 0, -1, 0]^T \quad \text{with probability} \quad p_{6,4}(t)\Delta t = N_6(t) S_{6,4} q_6 \Delta t.$$

Now it is useful to calculate the expected change and the covariance matrix for the change ΔN. As said previously, the value of the expected change $E(\Delta N)$ for any small time Δt is computed by summing the products of the changes with the respective probabilities. In general, for any d, it can be shown that the lth component of the vector $E(\Delta N)$ has the form:

$$\begin{aligned} E(\Delta N)_l &= 2 \sum_{k=l+1}^{d} p_{k,l}(t)\Delta t - \sum_{k=1}^{l-1} p_{l,k}(t)\Delta t \\ &= 2 \sum_{k=l+1}^{d} p_{k,l}(t)\Delta t - N_l(t) q_l \Delta t. \end{aligned} \tag{4.2}$$

where the second step above follows from the fact that $p_{l,k}(t)\ \Delta t = N_l(t) S_{l,k} q_l \Delta t$ and $\sum_{k=1}^{l-1} S_{l,k} = 1$. In addition, the covariance matrix has the form

$$E\big(\Delta N (\Delta N)^T\big) = \sum_{k=1}^{d} \sum_{l=1}^{k-1} C^{k,l} p_{k,l}(t) \Delta t \tag{4.3}$$

where $C^{k,l}$ is a $d \times d$ matrix that accounts for a fiber of group k breaking into a fiber of group l and group $k - l$. From the special case where $d = 7$ and a fiber in the 6th group breaks into two fibers, one in group 4 and one in group 2 , the corresponding term produced in the covariance matrix is:

$$C^{6,4} = (\Delta N)^{6,4}((\Delta N)^{6,4})^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Neglecting terms of order Δt^2, it follows that the expected change and the covariance matrix can be defined in terms of the expectation vector μ and the symmetric definite positive V. That is,

$$(\Delta N) = \mu(t, N(t))\Delta t \quad \text{and} \quad E(\Delta N(\Delta N)^T) = V(t, N(t))\Delta t. \tag{4.4}$$

Let us verify that, in fact, the covariance matrix V is positive definite. We must check that $x^T E(\Delta N(\Delta N)^T)x$ for any $x \in \mathbb{R}^d$, $x \neq 0$. To do so, we observe that

$$x^T E(\Delta N(\Delta N)^T)x = E(x^T \Delta N(\Delta N)^T x) = E(x^T \Delta N(x^T \Delta N)^T)$$

where the term $E(x^T \Delta N(x^T \Delta N)^T)$ is basically $\sum_{i=1}^{d} a_i^2$ where $a_i = x_i^T(\Delta N)_i$ for $i = 1, 2, \ldots, d$, which is positive. Thus, V is positive definite and has a positive definite square root. Denote $B = V^{1/2}$. Now, following the same argument as in reference [12], the vector change ΔN can be expressed by the sum of the expected change $\mu(t, N(t))\Delta t$ plus a term that only depends on the increments of Brownian motions: $B(t, N(t))\Delta W(t)$. This heuristic interpretation of the vector change ΔN then leads to a SDE system of the form

$$dN(t) = \mu(t, N(t))\ dt + B(t, N(t))\ dW(t), \tag{4.5}$$

where $N(t) = [N_1(t), \ldots, N_d(t)]^T$ are the fiber populations of each length group and $W(t) = [W_1(t), \ldots, W_d(t)]^T$ is a d-dimensional Brownian motion. In other words, the stochastic differential equation system (4.5) is obtained by letting the expected change divided by Δt be the drift coefficient and the square root of the covariance matrix divided by Δt be the diffusion coefficient.

The formulation of the SDE system (4.5) entails the computation of the square root of the covariance matrix to obtain the diffusion process. However, for a $d \times d$ symmetric positive definite matrix V with $d \geq 3$, there is no explicit formula for $V^{1/2}$ and it must be calculated numerically. Fortunately, there exist suitable numerical procedures for computing $V^{1/2}$ directly (see e.g [6, 9]) although, for a large matrix, it is computationally intensive[1] to accurately compute square roots of matrices. Therefore, it is interesting to give an equivalent SDE model to system (4.5).

In the second procedure the calculation of the square root of the covariance matrix V can be avoided by including additional Brownian motions in the stochastic system (4.5). It is shown in [1, Chapter 5, p. 186-193] and [3] that if

$$V(t, N(t)) = G(t, N(t))G(t, N(t))^T$$

where G is a $d \times m$ matrix, then system (4.5) is equivalent to

$$dN(t) = \mu(t, N(t))\ dt + G(t, N(t))\ dW^*(t), \tag{4.6}$$

[1] To compute $V^{1/2}$, the matrix V is put in canonical form $V = P^T DP$, where $P^T P = I$ and $d_{ii} \geq 0$ for $i = 1, 2, \ldots, d$, then $V^{1/2} = P^T D^{1/2} P$. A issue that arises from this method is that all eigenvalues and eigenvectors of V must be explicitly calculated to determine P and D.

where $W(t) = [W_1(t), \ldots, W_d(t)]^T$, $W^* = [W_1^*(t), \ldots, W_m^*(t)]^T$, and $W_i(t)$ for $i = 1, 2, \ldots, d$, and $W_i^*(t)$ for $i = 1, 2, \ldots, m$ are independent Brownian motions and $m = d(d-1)/2$. We shall recall that the two SDE models are equivalent if they have identical probability distribution. Essentially, this means that system (4.5) and system (4.6) must have identical covariance matrix $V(t, N(t)) = E(\Delta N(\Delta N)^T)/\Delta t$. Then, the entry i, j in the matrix G can be determined[2] in terms of the ith element of change $(\Delta N)^{k,l}$ to the vector $N(t)$ given by (4.1) and the probability $p_{k,l}\Delta t$. Applying the assumptions given above, the SDE system (4.6) can be rewritten in the form

$$dN(t) = \mu(t, N(t))\, dt + \sum_{k=1}^{d}\sum_{l=1}^{k-1}(\Delta N)^{k,l} p_{k,l}(t)^{1/2}\, dW_{k,l}^*(t), \tag{4.7}$$

where $W_{k,l}^*(t)$ for $l = 1, 2, \ldots, k-1$ and $k = 1, 2, \ldots, m$ are m independent Brownian motions.

Before proceeding with the next section, it is interesting to remark that in the first SDE model (4.5) only d Brownian motions are required, whereas for the second $m = d(d-1)/2$ Brownian motions are needed. This brings up to the question of whether the system (4.5) is more complicated than system (4.6). It turns out that system (4.6) is generally easier to solve computationally than (4.5) , as the $d \times d$ matrix B is the square root of V even though G is $d \times m$.

4.2 Computations

Firstly, a short introduction to the approximation model used to test SDE models is made. Then, a possible simulation approach for the first SDE model (4.5) is indicated and simulations for different numbers of fibers having different length are work out for the second SDE model (4.7).

4.2.1 The Euler-Maruyama Method

For the simulation of solutions of SDE models, a numerical method needs to be used. The Euler method is one of the simplest methods used to generate solutions of ordinary differential equations. Here, we use the Euler-Maruyama method, or EM for short. This method is the analogue of the Euler's method for ordinary differential equations but applied to stochastic differential equations.

For computational purposes, to apply the method over the interval $[0, T]$ it is useful to consider the step size $\Delta t = T/N$ for some integer N and the discrete points $t_j = j\Delta t$ for $j = 0, 1, \ldots, N$ of the interval $[0, T]$.

[2]See [3] for a detailed discussion about the equivalence between the two procedures and, in particular, for more information about this procedure for a general discrete stochastic model.

Then, the Euler-Maruyama method is described as follows: Given a stochastic differential equation of the form

$$dX(t) = f(X(t))\ dt + g(X(t))\ dW(t) \tag{4.8}$$

and a step size Δt, we approximate and simulate with

$$X_j = X_{j-1} + f(X_{j-1})\Delta t + g(X_{j-1})\Delta W_j \tag{4.9}$$

where $X_j = X(t_j)$, $\Delta W_j = (W(t_j) - W(t_{j-1}))$ for $j = 1, 2, \ldots, N$. To understand where (4.9) comes from, let us rewrite Equation (4.8) into stochastic form

$$X(T) = X_0 + \int_0^T f(X(t))\ dt + \int_0^T g(X(t))\ dW(t). \tag{4.10}$$

Now, setting $t = t_j$ and $t = t_{j-1}$ in (4.10) we get

$$X(t_j) = X_0 + \int_0^{t_j} f(X(t))\ dt + \int_0^{t_j} g(X(t))\ dW(t) \tag{4.11}$$

and

$$X(t_{j-1}) = X_0 + \int_0^{t_{j-1}} f(X(t))\ dt + \int_0^{t_{j-1}} g(X(t))\ dW(t). \tag{4.12}$$

We see immediately that subtracting the equations above ((4.11) and (4.12)) the following stochastic equation is obtained

$$X(t_j) = X(t_{j-1}) + \int_{t_{j-1}}^{t_j} f(X(t))\ dt + \int_{t_{j-1}}^{t_j} g(X(t))\ dW(t). \tag{4.13}$$

Then, notice that each of the three terms on the right-hand side of (4.9) approximates [3] the corresponding term on the right hand-side of (4.13). That is,

$$\int_{t_{j-1}}^{t_j} f(X(t))\ dt \approx f(X_{j-1})(t_j - t_{j-1}) = f(X_{j-1})\Delta t \tag{4.14}$$

and

$$\int_{t_{j-1}}^{t_j} g(X(t))\ dW(t) \approx g(X_{j-1})\Delta W_{j-1}. \tag{4.15}$$

Hence, the approximations (4.14) and (4.15) alongside with (4.13) gives us Euler-Maruyama formula (4.9). The reader who is interested in the type of convergence of

[3]The first integral (i.e the deterministic) is approximated by using conventional quadrature approximation. For the second integral, we use Itô formula [14].

the EM method and the relationship with others approximation methods is referred to [13].

Finally, it is of importance to point out that the crucial question when simulating stochastic differential equations is how to model the Brownian motion ΔW_j. The answer is given in for example [15]: We define $N(0,1)$ to be the standard random variable that is normally distributed with mean 0 and variance 1. Then, each random number ΔW_j is an independent random variable that can be computed as

$$\Delta W_j = z_j \sqrt{\Delta t_j}$$

where z_j is chosen from $N(0,1)$.

4.2.2 Simulations

After this brief introduction, we focus on the particular case of the stochastic differential equation model for fiber length distribution. Therefore, we aim to solve the systems of SDE given in the previous section. Notice that in these cases the drift coefficient $f(X(t))$ and the diffusion coefficient $g(X(t))$ arising on Equation (4.8) are a vector valued function and a matrix valued function, respectively. Furthermore, W is a multidimensional Brownian motion and the solution X is a vector valued stochastic process.

In the simulation of the SDE systems (4.5) and (4.7), each fiber length is checked for breakage for each small time step. Assumptions for breakage are adapted from [2] and are taken in the way that for each previous breakage, the probability of another breakage is randomly divided. This gives a constant λ that accounts for the rate of fiber breakage fraction of fibers of length k broken in time Δt. Furthermore, it is assumed that the probability for breakage is proportional to the length of the fiber.

Considering this new situation, the fraction $q_k \Delta t$ of fibers of length L_k broken in time Δt can be set up as

$$q_k \Delta t = \lambda \left(\frac{L_k}{L_{max}} \right) \Delta t, \tag{4.16}$$

where λ will be considered to be the unity. In addition,

$$S_{k,j} = \frac{h}{L_k} = \frac{1}{k} \tag{4.17}$$

being $S_{k,j}$ the fraction of fragments of length L_j formed from breakage of fibers of length L_k. Note that the second step in Equality (4.17) follows directly from the fact that $L_k = kh$ for $k = 0, 1, \ldots, d$.

Now, putting (4.16) and (4.17) and together, we have

$$S_{k,j} q_k \Delta t = \frac{\lambda h}{L_{max}} \Delta t. \tag{4.18}$$

As an aside, let us observe that this assumptions may be taken in a different way. For instance, for a slightly different scenario of simulation where it is consider that the probability of another breakage may be either not changed or reduced by a factor of 2 and fiber may break more frequently at certain points than others, see [16].

It turns out that getting back to our specific breakage assumptions, the SDE system (4.5) can be simplified to a SDE system of the form

$$dN(t) = \Theta N(t)\ dt + B(t, N(t))\ dW(t) \tag{4.19}$$

where Θ is a constant $d \times d$ matrix. To understand how the new drift coefficient is determined, let us recall that in system (4.5) the drift coefficient $\mu(t, N(t))$ satisfy the first equality of (4.4), in other words, $\mu(t, N(t)) = E(\Delta N)/\Delta t$. In addition, the lth component of the expected change $E(\Delta N)$ is given by (4.2). In consequence, we can work out a new expression for the lth component of the expectation vector

$$\begin{aligned}
\mu(t, N(t))_l &= E(\Delta N)_l/\Delta t \\
&= 2\sum_{k=l+1}^{d} p_{k,l}(t) - N_l(t) q_l \\
&= 2\sum_{k=l+1}^{d} N_k(t) S_{k,l} q_k - N_l(t) q_l \\
&= 2\sum_{k=l+1}^{d} \frac{\lambda h}{L_{max}} N_k(t) - \frac{\lambda l h}{L_{max}} N_l(t) \\
&= 2\sum_{k=l+1}^{d} \theta_{k,l} N_k(t) - \theta_{l,l} N_l(t)
\end{aligned}$$

where $\theta_{k,l}$ for $k = l, l+1, \ldots, d$ are the entries k, l of the matrix Θ given by

$$\theta_{k,l} = \begin{cases} 2K, & \text{if} \quad k > l \\ -lK, & \text{if} \quad k = l, k \neq 1 \\ 0, & \text{otherwise} \end{cases}$$

and $K = \lambda h / L_{\max}$ is a constant. For example, in the particular case of $d = 7$ fiber groups presented earlier in this chapter, the matrix V is

$$\Theta = K \begin{bmatrix} 0 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & -2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & -3 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & -4 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & -5 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -7 \end{bmatrix}.$$

A similar approach has been adopted to obtain the diffusion coefficient under this breakage assumptions. In this case, we let one more time $B = V^{1/2}$. Then, the expression $V(t) = \sum_{k=1}^{d} \sum_{l=1}^{k-1} C^{k,l} N_k(t) \lambda h / L_{\max}$ for the covariance matrix is derived from the second equality of (4.4) and the definition of covariance matrix, i.e Equation (4.3). Note that for example, if there exist $d = 7$ fiber groups, the term $C^{6,4}$ has the form

$$C^{6,4} = (\Delta N)^{6,4}((\Delta N)^{6,4})^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A simulation of the previous system (4.19) would be potentially based on the computation of the matrix B, which is the square root of the covariance matrix and it has been seen that its calculation may be computationally intensive. Hence, it is important to look out for an alternative (and easier) way of simulation. This alternative scheme of simulation exists and arises from the second procedure shown in section 4.1. Then, let us model the new scenario:
Under the same breakage assumptions, we may substitute the diffusion coefficient $B(t, N(t))$ for the diffusion coefficient arising in the SDE system (4.7). Then an equivalent SDE model is obtained and has the form

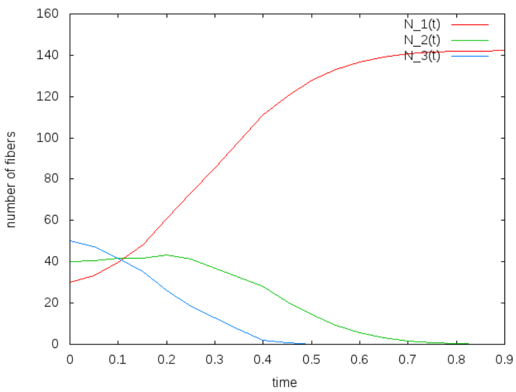
$$dN(t) = \Theta N(t)\ dt + \sum_{k=1}^{d} \sum_{l=1}^{k-1} (\Delta N)^{k,l} p_{k,l}(t)^{1/2}\ dW_{k,l}^*(t). \tag{4.20}$$

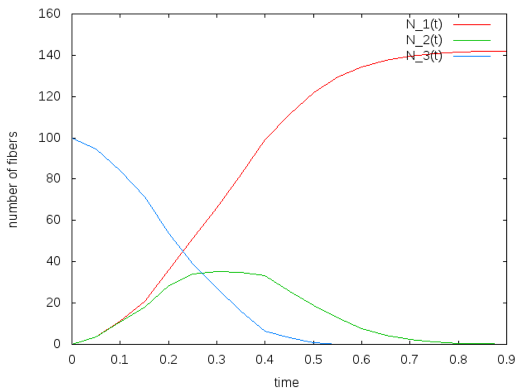
Now, applying the assumptions taken upon $p_{k,l}(t)$ to the latter system, the second term on the right hand side can be rewritten as

$$\sum_{k=1}^{d} \sum_{l=1}^{k-1} (\Delta N)^{k,l} p_{k,l}(t)^{1/2}\ dW_{k,l}^*(t) = K \sum_{k=1}^{d} \sum_{l=1}^{k-1} (\Delta N)^{k,l} N_k(t)^{1/2}\ dW_{k,l}^*(t),$$

where K is a constant that accounts for $\lambda h / L_{\max}$. Henceforth, we are fully prepared to simulate system (4.20) via the Euler-Maruyama method. In the simulation scheme, two situations have been assumed for different groups of fibers having different length. We refer to Appendix A for the implementation of the EM method in C.

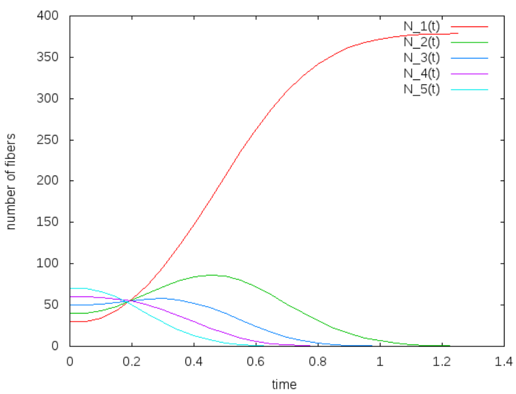
For the following graphics appearing on the left hand side, it has been assumed that the number of steps are $N = 20$ and the fibers are initially distributed as $N_k(0) = 30 + 10j$ for $j = 0, \ldots, d-1$ and $k = 1, \ldots, d$. Whereas, for the graphics appearing on the right and side, it has been assumed that the number of steps are also $N = 20$ and the fibers are initially distributed as $N_k(0) = 0$ for $k = 1, \ldots, d-1$ and $N_d(0) = 100$.

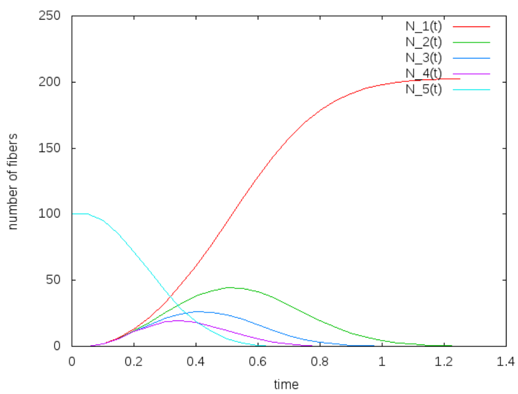




As it can be seen, the first couple of graphics consist in three groups of fibers having different length and the right end of the interval $[0, T]$ is taken as $T = 0.9$. In this manner, we are able to observe that the groups of larger length loses components in favour of the groups of fibers of smaller length. Hence, when the groups consisting of fibers of larger length lose all of their components, the group of smallest fibers does not neither win nor lose any more fibers, but become constant. This implies that there is no need to study to extend the interval chosen. This phenomena is due to the assumption that breakage of fibers of the group of smallest fibers cannot occur, that is, the fraction of fibers of length L_1 broken in time Δt is zero.

Let us now present two graphics consisting of five groups of fibers having different length:





For these graphics, the right end of the interval $[0, T]$ has been chosen as $T = 1.3$. It is noticeable to point out that in the two graphics above there exist a larger number of fibers than in the graphics previous graphics. As a consequence, the group $N_1(t)$ ends up reaching a greater number of fibers (376 and 201, respectively) than in the previous graphics, which were 141. Also, it is seen that in the graphic of the left hand side the numbers of fibers reached for the group $N_1(t)$ is greater than the one on the right hand side. This means that, even having 100 fibers of the group $N_5(t)$, which is the groups of fibers of largest length, the number of fibers in group $N_1(t)$ formed by breakage are less than the number of fibers formed by breakage of fibers of different groups with less components, and components of smaller length.

It will remain to be checked the equivalence of the two SDE models through computations, which shall be left for future work. Even so, we refer to [2, Chapter 4] for more information about computational results showing the agreement between the two procedures.

Chapter 5

Conclusions

The construction of the Itô stochastic integral is build up firstly for the class of random step processes M^2_{step} and thereafter, for a larger class of stochastic processes denoted by M^2. The strong relationship between the class of stochastic processes M^2 and the space of integrable functions L^2 is treated in terms of their expectations and the concept of Itô stochastic integral is extended over a finite time interval $[0, T]$ via indicator functions of the elements of M^2 to any finite time interval $[0, T]$, generating the class of stochastic process denoted by M^2_T. It turns out that trying to find a sequence of random process approximating a stochastic process of the class M^2 or M^2_T could not be an easy task. Therefore, straightforward conditions for a stochastic process to belong to M^2 or M^2_T are given.

Thereupon, the notion of stochastic differential and a crucial tool for transforming and computing the stochastic integral, known as Itô formula, are explained and lead us to the definition of stochastic differential equations. Furthermore, the class of processes for which the solutions of a SDE will be sought is defined and is referred to as Itô processes. Several elaborated examples of stochastic differential equations such as Langevin equation and its solution, the Ornstein-Uhlenbeck process, are also given. Finally, a theorem of existence and uniqueness of a solution for a SDE that resemble the one arising in the theory of ordinary differential equations but for Itô stochastic integral is provided.

In the last part, an application of the theory of stochastic differential equations to textile industry is developed. To generate a stochastic differential equation system, a discrete-time stochastic model is presented which is then approximated by a system of stochastic differential equations. In particular, the two stochastic differential equation systems studied are produced by the first and second procedures briefly explained in the introduction and fully explained in Chapter 4. This last part ends with the presentation of a basic situation of simulation for each of the systems previously formulated and the corresponding simulation for the latter, where the Euler-Maruyama method is used.

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Appendix A

C programme for the simulation of the SDE system (4.20)

```
#include<stdio.h>
#include<stdlib.h>
#include<math.h>


/*
 *This programme simulate the system dN(t) =  Theta N(t) dt + G dW(t)
 *using the E-M method where
 *Theta = constant matrix
 *G = Covariance matrix
 *dW = d x d Lower Triangular matrix of Brownian motions with m >= d
 *and zeros at the diagonal.
 *
 *T = right end of the interval [0,T]
 *N = number of steps
 *dt = T/N
 *
 *Recipe to get dW:
 *Step 1: d random numbers from Normal Distribution are generated
 *Step 2: Each component is set equal to sqrt(dt)*N(0,1)
 */

double **Theta(int d);
double *squareRoot(int d, double *);
double **crea_mat(int n, int m);
double *crea_vect(int n);
double randn (double mu, double sigma);
void prod_mat_vect(double **a, int na, int ma, double *x, double *y);
double *Change(int d,int k,int l);
```

```
int main (void) {

    int i,k,l, cont;
    double d,dt,N,*Neulermethod_temp,*drift,*Neulermethod, **theta,
    **diffusion, **dW, *diffusionVect, *coeff1, *coeff2;
    FILE *f;
    f=fopen("solution.dad", "w");

   /*
    *Number of fibers of different length. This value determines the
    *number of population used in the simulation.
    */
    d = 3;

    theta=crea_mat(d,d);
    Neulermethod_temp=crea_vect(d);
    Neulermethod=crea_vect(d);
    dW=crea_mat(d,d);
    drift=crea_vect(d);
    diffusion = crea_mat(d,d);
    diffusionVect=crea_vect(d);
    coeff1=crea_vect(d);
    coeff2=crea_vect(d);


    theta = Theta(d);
    N=20;
    dt = (double)(1/N);

    k = 0;
    for (i=0; i<d; i++) {
        Neulermethod_temp[i] = 30+k;
        k = k + 10;
    }


    printf("%13.6le,  %le,   %13.6le,  %le,   %13.6le,  %le  \n",
    Neulermethod_temp[0],0.0, Neulermethod_temp[1],0.0,
    Neulermethod_temp[2],0.0);
    fprintf(f,"%13.6le,  %le,   %13.6le,  %le,   %13.6le,  %le  \n",
    Neulermethod_temp[0],0.0, Neulermethod_temp[1],0.0,
    Neulermethod_temp[2],0.0);
```

```
    cont = 0;
    while (dt <= 1) {

        cont++;

        prod_mat_vect(theta,d,d,Neulermethod_temp,drift);

        /*computation of diffusionVect*/

        coeff1=squareRoot(d,Neulermethod_temp);
       /*coeff1 returns the vector of the square rooth of the solution
        *vector evalued at dt
        */
        for (k = 0; k < d; k++) {
            coeff1[k] = coeff1[k]/(double)(d);
        }

       /*A matrix of independent Brownian motions are computed for any
        *time interval dt
        */
        for (k = 0; k < d; k++) {
  for (l = 0; l < k; l++) {
            dW[k][l] = sqrt(dt)*randn(0,1);
  }
}

        for (k = 0; k < d; k++) {
            for (i = 0; i < d; i++) {
            /*the values of are initialized to zero*/
                diffusionVect[i] = 0.;
            }

            for (l = 0; l < k; l++) {
                coeff2 =Change(d,k,l);
               /*For any k and l, coeff2
                *returns the values of the change vector
                */
                for (i = 0; i < d; i++) {
               /*the components of the kth row of G are the
                *components of the change vector
                */
                    diffusionVect[i] = coeff2[i] + diffusionVect[i];
                }
               /*the vector diffusionVect is multiplied
```

```
            *by a Brownian motion
            */
            for (i = 0; i < d; i++) {
                diffusionVect[i]=diffusionVect[i]*dW[k][l];
            }
        }
      /*the vector diffusionVect is multiplied by the vector
       *of the square rooth of the solution vector evalued at
       *dt divided by d
       */
        for (i = 0; i < d; i++) {
            diffusionVect[i]=diffusionVect[i]*coeff1[k];
        }

    }

    /*scheme for one step of E-M method*/
    for (i = 0; i < d; i++) {
        Neulermethod[i] = Neulermethod_temp[i] + drift[i]*dt
        + diffusionVect[i];
    }

    for (i = 0; i < d; i++) {
        if (Neulermethod[i] < 0) {
            Neulermethod[i]=0.;
        }
    }

    printf("%13.6le,  %le,   %13.6le,  %le,   %13.6le,  %le \n",
    Neulermethod[0],dt,Neulermethod[1],dt,Neulermethod[2],dt);

    fprintf(f,"%13.6le,  %le,   %13.6le,  %le,   %13.6le,  %le \n",
    Neulermethod[0],dt,Neulermethod[1],dt,Neulermethod[2],dt);


   /*the values of the solution at the step dt (Neulermethod) are
    *transferred to  Neulermethod_temp
    */
    for (i = 0; i < d; i++) {
        Neulermethod_temp[i] = Neulermethod[i];
    }



    dt = dt + (double)(1/N);
```

```
    }
    fclose(f);



    return 0;

}

/*
 * Function that returns the constant matrix Theta
 */
double **Theta(int d)
{

    int i,j,k,l;
    double **theta; /*It is assumed that lambda = 1, h = L_{max}/d*/

    theta = crea_mat(d,d);

    /*coeff of matrix Theta equals to 0*/
    for (i= 0; i<d; i++)  {
        for (j=0; j<d; j++)  {
            theta[i][j]=0;
        }
    }

    /*matrix Theta*/
    for(k=0; k<d; k++) {
        for(l=k; l<d; l++) {
            if (k == l && k != 0) {
                theta[l][l] = -(l+1)/(double)(d);
            }
            if (k<l) {
                theta[k][l] = 2/(double)(d);

            }
        }
    }

    return theta;

}


/*
```

```
 * Function that returns the vector change
 */
double *Change(int d,int k,int l)
{
    int i;
    double *vector;

    vector=crea_vect(d);

    for (i = 0; i<d; i++) {
        if (i == k-l-1) {
            vector[i]=1;
        }
        if (i == l) {
            vector[i]=1;
        }
        if (i == k) {
            vector[i]=-1;
        }
    }
    return vector;
}


/*
 * Function that returns the square root of the elements of a vector
 */
double *squareRoot(int d, double *v)
{
    int i;
    double *vector1, *vector2;



    vector1=crea_vect(d);
    vector2=crea_vect(d);
    for (i = 0; i<d; i++) {
        vector1[i] = 0.;
    }
    for (i = 0; i<d; i++) {
        vector2[i] = fabs(v[i]);
    }


    for (i = 0; i<d; i++) {
        vector1[i] = sqrt(vector2[i]);
```

```
    }


    return vector1;
}



/*
 * Function that creates a vector
 */

double *crea_vect(int n)
{

    double *vect;

    vect=(double*)malloc(n*sizeof(double));

    if (vect==NULL) {
        return NULL;
    }

    return vect;

}

/*
 * Function that creates a matrix
 */
double **crea_mat(int n, int m)
{

    double **mat;
    int i;
    mat=(double**)malloc(n*sizeof(double*));

    if (mat==NULL) {
        return NULL;
    }

    for(i=0;i<n;i++) {
        mat[i]=(double*)malloc(m*sizeof(double));

        if (mat[i]==NULL) {
            return NULL;
```

```
        }
    }

    return mat;

}


/*
 * The following function uses the function rand() to generate a
 *random number from a Normal distribution of mean mu and standard
 *desviation sigma
 */
double randn (double mu, double sigma)
{
    double U1, U2, W, mult;
    double X1,X2;
    /*static int call = 0;*/

    /*if (call == 1)
     {
     call = !call;
     return (mu + sigma * (double) X2);
     }*/

    do
    {
        U1 =  ((double) rand() / RAND_MAX);
        U2 =  ((double) rand() / RAND_MAX);
        W = pow (U1, 2) + pow (U2, 2);
    }
    while (W >= 1 || W == 0); /*to avoid division by zero*/

    mult = sqrt ((-2 * log (W)) / W);
    X1 = U1 * mult;
   /*normally distributed with mean 0 and
    *standard deviation 1
    */
    X2 = U2 * mult;

    /*call = !call;*/

    return (mu + sigma * (double) X1);
}

/*
```

```
 * Function that derives a vector "y" from the multiplication of the
 *matrix "a" and the vector "x".
 */

void prod_mat_vect(double **a, int na, int ma, double *x, double *y)
{

    int i,j;

    for (i=0; i<na;i++) {
        y[i]=0.0;
        for (j=0;j<ma;j++) {
            y[i] += a[i][j]*x[j];
        }
    }
    return;
}
```