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GRAU DE MATEMÀTIQUES

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AN INTRODUCTION TO STOCHASTIC VOLATILITY MODELS

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Contents

The main goal of this work is to introduce the stochastic volatility models in mathematical finance and to develop a closed-form solution to option pricing in Heston's stochastic volatiltiy model, following the arguments in Heston 1993.

No background in mathematical finance will be assumed, so another main goal of this work is to develop the theory of stochastic integration and to introduce the Black-Scholes market model, the benchmark model in mathematical finance. Standard topics in the framework of market models, such as trading strategies, completeness and replication, and the notion of arbitrage, will also be reviewed.

Chapter 1

Historical Background on Stock Price Models

Louis Bachelier, in his thesis "*Théorie de la Spéculation*", made the first contribution of advanced mathematics to the study of finance in 1900. This thesis was well received by academics, including his supervisor Henry Poincaré, and was published in the prestigious journal *Annales Scientifiques de l'École Normale Supérieure*. In this pioneering work, the Brownian motion is used for the modelling of movements in stock prices. In the words of Louis Bachelier, in Bachelier 1900:

> La détermination des mouvements de la Bourse se subordonne à un nombre infini de facteurs: il est dès lors impossible d'en espérer la prévision mathématique, [...] et la dynamique de la Bourse ne sera jamais une science exacte.
>
> Mais il est possible d'étudier mathématiquement l'état statique du marché à un instant donné, c'est-à-dire d'établir la loi de probabilité des variations de cours qu'admet à cet instant le marché. Si le marché, en effet, ne prévoit pas les mouvements, il les considère comme étant plus ou moins probables, et cette probabilité peut s'évaluer mathématiquement.

Bachelier argued that, over a short time period, fluctuations in price are independent of the current price and the past values of the price, and that these fluctuations follow a zero mean normal distribution with variance proportional to the time difference. He also assumed that the prices are continuous, therefore modelled as a Brownian motion (see Bachelier 2011).

Many years later, in the famous article by Black and Scholes, Black and Scholes 1973, prices are modeled as a *geometric Brownian motion*, whose fluctuations have a lognormal distribution. This model is based on the assumption that the *log returns* of a stock price are independent and normally distributed, with variance proportional to the time difference. The log returns are defined as $log(p_i) - log(p_j)$, where p_i and p_j denote the prices at times i and j, respectively, with $i > j$.

The *returns* of a stock price are defined as the increments

$$r_i = \frac{p_i - p_j}{p_j}$$

Hence, the approxamation

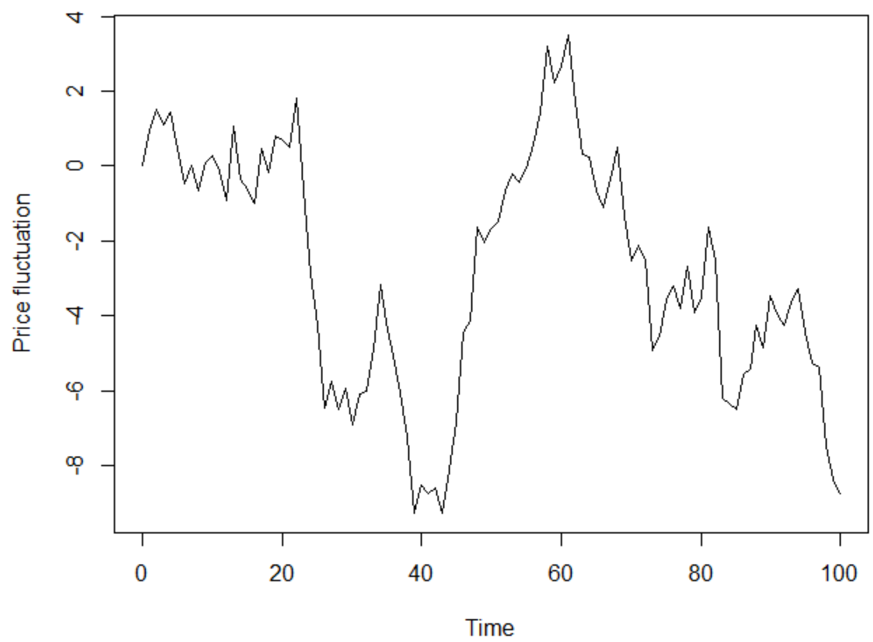

Figure 1.1: Sample path of a standard Brownian motion, used by Bachelier to model fluctuations in stock prices

$$log(1+r) \approx r, \quad \text{when } r \ll 1 \tag{1.1}$$

Gives the approximation

$$\frac{p_i - p_j}{p_j} = r_i \approx log(1+r_i) = log(p_i/p_j) = log(p_i) - log(p_j) \tag{1.2}$$

So that the lognormal distribution for the stock price increments proposed by Black and Scholes obeys to the intuitive idea that the price returns are independent and normaly distributed.

Explicitely, as stated in Black and Scholes 1973, the model proposed by Black and Scholes relies on the following assumptions of an "ideal" market:

(a) The interest rate is known and constant through time

(b) The distribution of stock prices at the end of any finite interval is lognormal

(c) The stock pays no dividends

(d) The variance rate of the return on the stock is constant

(e) The stock price is continuous over time

Empirical observations of stock price distributions have motivated numerous extensions of the Black-Scholes model in which one or more of the previous assumptions are relaxed. One of the main criticisms of the Black-Scholes model is that the normal distribution of stock price returns does not explain the significant presence of outliers in the distribution of returns (see, for example, Mandelbrot 1963). Conveniently, relaxing some of the previous assumptions results in distributions with a higher presence of outliers, or fatter

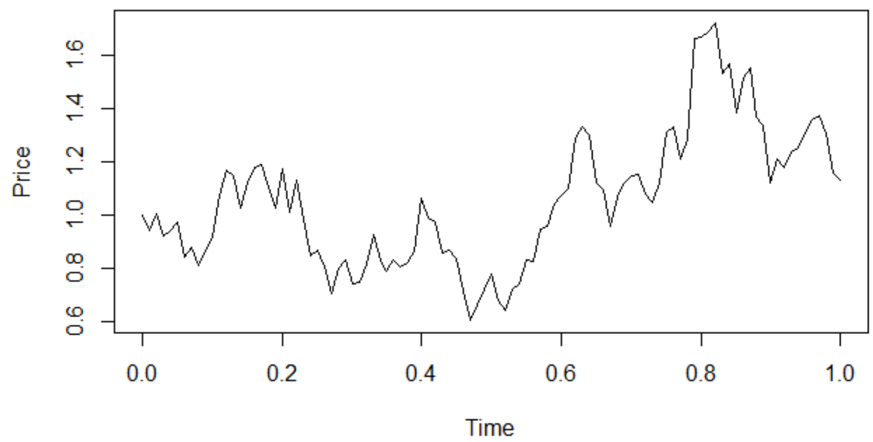

Figure 1.2: Sample path of a Geometric Brownian motion (with $\mu = \sigma = 1$), used by Black and Scholes to model stock prices

tails, which better adjust to the reality of observed prices.

Examples of such extensions include models with dividend payments, stock prices with jumps (non-continuous over time), and models in which the distribution of returns is non-Gaussian, among others. These extensions are widely used by practitioners and are described in most text books on mathematical finance (see, for example, Musiela and Rutkowski 2006).

In this project we will concentrate on the stochastic volatility models, in which the assumption that the variance rate of the return is constant is relaxed.

Chapter 2

Stochastic Integration

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and T an index set. Consider a function $X(t, \omega)$ having two arguments in T and Ω. By fixing t, $\big(X(t, \cdot)\big)_{t\in T}$ is a family of random variables defined in Ω. On the other hand, by fixing ω, $\big(X(\cdot, \omega)\big)_{w\in\Omega}$ can be seen as a family of 'random maps'. We will define a stochastic process following the first point of view and show the equivalence between the two.

Definition 2.1 (Stochastic Process)**.** *Let E be a metric space with the borel σ-field. A stochastic process is a family of E-valued random variables $(X_t)_{t\in T}$ defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$*

In order to define a random map as a function-valued random variable, we must specify the function space and a σ-field in this space. The most natural function space to consider is the space E^T of all maps from T to E. The σ-field $\mathcal{G}$ will be the product σ-field, that is the one generated by cylinder sets of the form $\pi_{t_1}^{-1}(B_1) \cap \ldots \cap \pi_{t_n}^{-1}(B_n)$, with $B_1, \ldots, B_n \in \mathcal{B}(E)$ and π_t being the natural projection defined by the rule $f \mapsto f(t)$.

Let $(X_t)_{t\in T}$ be a stochastic process and define $Y(w)(t) = X_t(w)$. Is the map $Y : (\Omega, \mathcal{F}, \mathbb{P}) \to (E^T, \mathcal{G})$ $\mathcal{G}$-measurable? Indeed, for any cylinder set $C = \pi_{t_1}^{-1}(B_1) \cap \ldots \cap \pi_{t_n}^{-1}(B_n) \in \mathcal{G}$, $Y^{-1}(C) = X_{t_1}^{-1}(B_1) \cap \ldots \cap X_{t_n}^{-1}(B_n) \in \mathcal{F}$. Since $\mathcal{G}$ is generated by the cylinder sets, we conclude that Y is $\mathcal{G}$-measurable. On the other hand, if $Y : (\Omega, \mathcal{F}, \mathbb{P}) \to (E^T, \mathcal{G})$ is $\mathcal{G}$-measurable, for any $t \in T$ and $\omega \in \Omega$ define $X_t(\omega) := Y(\omega)(t)$. Then, for any $B \in \mathcal{B}(E)$, $X_t^{-1}(B) = \{\omega | X_t(\omega) \in B\} = \{\omega | Y(\omega)(t) \in B\} = Y^{-1}(\pi_t^{-1}(B)) \in \mathcal{F}$.

Given the equivalence between both definitions, we will consider E-valued stochastic processes on T as any of the definitions above. A *sample path* of X is a function $x(t) = X(t, w)$. We say that a E-valued stochastic process on T has paths in $U \subset E^T$ if its sample paths are included in U. A stochastic process X is said to be *continuous* if its paths are included in $\mathcal{C}(T)$, the set of continuous functions. The process X is continuous at $t_0 \in T$ if its sample paths $x(t)$ are continuous at t_0 almost surely. Two stochastic processes X and Y are said to be *versions* of each other if $\mathbb{P}(X_t = Y_t) = 1$ for all $t \in T$. The *finite dimensional distributions* of a stochastic process X are the joint distributions of the random vectors $(X_{t_1}, \ldots, X_{t_n})$, $t_1, \ldots, t_n \in T$.

2.1 Martingales and Brownian motion

We will now consider stochastic processes with the index set $T = \mathbb{R}_+$, and with values on $\mathbb{R}$.

Definition 2.2 (Filtration, adapted process)**.** *A filtration $(\mathcal{F}_t)_{t\geq 0}$ is an increasing family of σ-algebras included in the σ-algebra $\mathcal{F}$. A process is said to be adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ if for each t, X_t is $\mathcal{F}_t$ measurable.*

Given a process $(X_t)_{t\geq 0}$, we can define the natural filtration $\mathcal{F}_t = \sigma(\{X_s, s \leq t\})$, for which the process X is adapted to. Moreover, it is frequent to consider the completion of the filtration $(\mathcal{F}_t)_{t\geq 0}$- that is, the filtration in which all the $\mathcal{F}$-negligible sets are $\mathcal{F}_0$-measurable. We will refer to this filtration as the one *generated by* the process X, without making explicit that it is the completion.

Definition 2.3 (Martingale)**.** *A process $(X_t)_{t\geq 0}$ adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$ is a martingale if $\mathbb{E}[|X_t|] < \infty$ $\forall t \geq 0$ and $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$ a.s for all $s < t$.*

The following result will be useful when defining the stochastic integral in the next section:

Theorem 2.4 (Doob inequality)**.** *If $(M_t)_{t\geq 0}$ is a martingale with continuous sample paths, then:*

$$\mathbb{E}\big[\sup_{t\leq T} |M_t|^2\big] \leq 4\mathbb{E}\big(|M_T|^2\big)$$

Proof. Refer to Lamberton and Lapeyre 2007 (page 58, exercise 13). □

We will need some further concepts to define a Brownian motion. A stochastic process is said to have *independent increments* if for all $0 \leq t_1 < \ldots < t_n$, $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent. It is said to have *stationary increments* if for $t_1 < t_2$, $X_{t_2} - X_{t_1} \sim X_{t_2 - t_1} - X_0$.

Definition 2.5 (Brownian motion)**.** *A Brownian motion (also known as Wiener process) is a stochastic process $(W_t)_{t\geq 0}$ with continuous sample paths and independent and stationary increments.*

The following result shows further properties of the Brownian motion which are not explicit in the definition. For a proof of this result, refer to Corcuera n.d.

Proposition 2.6. *Let W_t be a Brownian motion. Then $W_t - W_0 \sim N(rt, \sigma^2 t)$, for some $r \in \mathbb{R}$.*

A *standard Brownian motion* is a Brownian motion such that $W_0 = 0$ *a.e* and $W_t \sim N(0, t)$. From now on, we will usually refer to a standard Brownian motion as a Brownian motion, without specifying that it is standard.

Proposition 2.7 (Martingale property of Brownian motion)**.** *A standard Brownian motion is a martingale.*

Proof. Let X be a Brownian motion and $s < t$.

$$\mathbb{E}[X_t|\mathcal{F}_s] - X_s = \mathbb{E}[X_t - X_s|\mathcal{F}_s] = \mathbb{E}[X_t - X_s] = 0$$

The second equality is due to the fact that $X_t - X_s$ is independent to $\mathcal{F}_s$ due to the independent increments property. □

The following is another important property of the Brownian motion:

Lemma 2.8. *Let $(W_t)_{t\geq 0}$ be a Brownian motion under the probability $\mathbb{P}$. For any fixed $M > 0$,*

$$\mathbb{P}(\max_{t\geq 0} W_t(\omega) \leq M) = 0$$

$$\mathbb{P}(\min_{t\geq 0} W_t(\omega) \geq -M) = 0$$

Proof. This is a consequence of the distribution of the *first passage time* of a Brownian motion. We define the first passage time T_M to a level $M \in \mathbb{R}$ as:

$$T_M(\omega) = \inf\{t \geq 0, W_t(\omega) = M\}$$

It can be proved that the first passage time distribution has the following property (see Karatzas and Shreve 2012, p.80, equation 6.2):

$$\mathbb{P}(T_M < t) = 2\mathbb{P}(W_t > M) \xrightarrow[t\to\infty]{} 1$$

Since W_t is normally distributed. This proves the result, as any level M will be reached almost surely. □

2.2 Integral of mean square integrable processes

The fact that a Brownian motion is almost surely nowhere differentiable - a proof of this result can be found in Karatzas and Shreve 2012-, means that it is not possible to define the integral of a function f over a Brownian motion as $\int_0^T f(s)|W_s'|\mathrm{d}s$, for any fixed $T > 0$. We will construct the stochastic integral by defining it among a class of simple processes and then extend it to a larger class (as in the definition of the Lebesgue Integral). From now on, we will consider processes with an index set $[0, T]$.

Definition 2.9 (Mean Square Integrable Process)**.** *An adapted process $(X_t)_{t\geq 0}$ is said to be mean square integrable if $\mathbb{E}(\int_0^T X_t^2 \mathrm{d}t) < \infty$. We will denote this class of processes as $\mathcal{S}_2$.*

We aim to define the integral for mean square integrable processes. By identifying processes that are versions of one another and defining the natural norm $\|X\|_{\mathcal{S}_2}^2 = \mathbb{E}(\int_0^T X_t^2 \mathrm{d}t)$, it can be shown that $\mathcal{S}_2$ is a Hilbert space, with the scalar product $< X, Y >= \mathbb{E}(\int_0^T X_t Y_t \mathrm{d}t)$.

For constructing the integral, it is necessary to identify the so-called elementary processes for which the integral can be defined trivially, and so that it can be extended to the whole $\mathcal{S}_2$ space.

Definition 2.10 (Simple Process)**.** *A process $(X_t)_{t\leq T} \in \mathcal{S}_2$ is a simple process if there exists a partition $0 = t_0 < \ldots < t_n = T$ of $[0, T]$ for which $X_t = X_{t_i}$ for $t \in [t_i, t_{i+1}[$.*

Definition 2.11. *The integral of a simple process $(X_t)_{0\leq t\leq T}$ from 0 to T is defined as:*

$$I(X)_T = \int_0^T X_t \,\mathrm{d}W_t := \sum_i X_{t_i}(W_{t_{i+1}} - W_{t_i}) \tag{2.1}$$

The integral process is defined as the process $(I(X)_t)_{t\leq T}$, where $I(X)_t = I(X_s 1_{\{s\leq t\}})$. Note that $I(X)_t$ is given by:

$$I(X)_t = \sum_{i=0}^{n-1} X_{t_i}(W_{t_{i+1}\wedge t} - W_{t_i \wedge t})$$

So the integral process is pathwise continuous.

Let $\mathcal{E} \subset \mathcal{S}_2$ be the set of simple processes. Note that the integral of a simple process is a random variable. In addition, it is in L^2, and the integral is an isometry between these two spaces:

Lemma 2.12 (Isometry Property for Simple Processes). *The integral defines an isometry between $\mathcal{E}$ and a subspace of $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$.*

Proof. Let I be the integral operator. We have to prove that:

$$\|I(X)\|_{L^2}^2 = \mathbb{E}\Big[\Big(\int_0^T X_t \,\mathrm{d}W\Big)^2\Big] = \mathbb{E}\Big[\Big(\int_0^T X_t^2 \,\mathrm{d}t\Big)\Big] = \|X\|_{\mathcal{S}_2}^2$$

Now,

$$I(X)^2 = \sum_{i=0}^{n-1} X_{t_i}^2 (W_{t_{i+1}} - W_{t_i})^2 + 2 \sum_{0 \le i < j < n} X_{t_i} X_{t_j} (W_{t_{i+1}} - W_{t_i})(W_{t_{j+1}} - W_{t_j})$$

Where $t_0 = 0$ and $t_n = T$. Note that $W_{t_{j+1}} - W_{t_j}$ is independent of $\mathcal{F}_{t_j}$ and has 0 expectation.

$$\mathbb{E}\big(X_{t_i} X_{t_j} (W_{t_{i+1}} - W_{t_i})(W_{t_{j+1}} - W_{t_j})\big) = \mathbb{E}\big(X_{t_i} X_{t_j} (W_{t_{i+1}} - W_{t_i})\big)\mathbb{E}\big(W_{t_{j+1}} - W_{t_j}\big) = 0$$

Now, applying a similar reasoning,

$$\mathbb{E}[X_{t_i}^2 (W_{t_{i+1}} - W_{t_i})^2] = \mathbb{E}[X_{t_i}^2]\mathbb{E}[(W_{t_{i+1}} - W_{t_i})^2] = \mathbb{E}[X_{t_i}^2](t_{i+1} - t_i)$$

Because $W_{t_{i+1}} - W_{t_i} \sim N(0, t_{i+1} - t_i)$. We obtain:

$$\mathbb{E}[I(X)^2] = \sum_{i=0}^{n-1} \mathbb{E}[X_{t_i}^2](t_{i+1} - t_i)$$

Since X is constant on every interval $[t_i, t_{i+1}[$, the expression above is equal to:

$$\sum_{i=0}^{n-1} \mathbb{E}\Big[\int_i^{i+1} X_t^2 \,\mathrm{d}t\Big] = \mathbb{E}\Big[\int_0^T X_t^2 \,\mathrm{d}t\Big]$$

□

The following result will allow us to extend the integral to the $\mathcal{S}_2$ space:

Lemma 2.13. *The simple processes are dense in $\mathcal{S}_2$*

The proof of this result requires several steps and will be omitted - it can be found in Karatzas and Shreve 2012 (page 134, problem 2.5). However, to give an idea on how it is accomplished, note that any *continuous* process $(X_t)_{0 \le t \le T}$ can be approximated in $\mathcal{S}_2([0,T])$ by simple processes X_t^n defined as:

$$X_t^n = \sum_{i=0}^{T2^n - 1} X(i2^{-n}) 1_{]i/2^n, (i+1)/2^n[}(t)$$

However, it is not trivial to obtain a similar result for more general adapted processes.

To extend the integral, we apply the general result that given two metric spaces X and Y and a dense subset A of X, any uniformly continuous map $f : A \to Y$ can be extended in a unique way to X by defining $\hat{f}(x) = \lim_n f(a_n)$ where a_n is any sequence in A converging to x, and this extension is well defined and continuous. In addition, if Y is a vector space, the extension preserves the supremum norm. In our case, we can extend the integral $I : \mathcal{E} \to L^2(\Omega, \mathcal{F}_T, \mathcal{P})$ to $\hat{I} : \mathcal{S}_2 \to L^2(\Omega, \mathcal{F}_T, \mathcal{P})$ and it remains an isometry.

Given a process $(X_t)_{0 \le t \le T}$, we define the process $\big(I(X)\big)_t$ by $I(X)_t = \int_0^T X_s 1_{[0,t]}(s)\, \mathrm{d}W_s$. This process is a continuous martingale:

Lemma 2.14. *[Martingale Property of the Integral] The process* $\big(I(X)\big)_t$ *is a martingale.*

Proof. Consider a first case in which X is a simple process, and let $s < t$. Then, we can assume $t_n = t$ and $s \in [t_k, t_{k+1}[$.

$$\int_0^t X_s \, \mathrm{d}W = \sum_{i=0}^{n} X_{t_i}(W_{t_{i+1}} - W_{t_i})$$

$$= \sum_{i=0}^{k-1} X_{t_i}(W_{t_{i+1}} - W_{t_i}) + X_{t_k}(W_{t_{k+1}} - W_{t_k}) + \sum_{i=k+1}^{n} X_{t_i}(W_{t_{i+1}} - W_{t_i}) \tag{2.2}$$

Now, the first part of the last equation is an $\mathcal{F}_s$ measurable random variable, so its conditional expectation to $\mathcal{F}_s$ is the same variable.

For the second part, using the martingale property of the Brownian motion,

$$E[X_{t_k}(W_{t_{k+1}} - W_{t_k})|\mathcal{F}_s] = X_{t_k} E[W_{t_{k+1}} - W_{t_k}|\mathcal{F}_s] = X_{t_k}(W_s - W_{t_k})$$

Finally, for $j > k$, and by the tower property of the conditional expectation,

$$E[X_{t_j}(W_{t_{j+1}} - W_{t_j})|\mathcal{F}_s] = E[E[X_{t_j}(W_{t_{j+1}} - W_{t_j})|\mathcal{F}_{t_j}]|\mathcal{F}_s] = E[X_{t_j}(W_{t_j} - W_{t_j})|\mathcal{F}_s] = 0$$

So the last term in the equation is 0. Adding the results, we obtain:

$$E[I(X)_t|\mathcal{F}_s] = \sum_{i=0}^{k-1} X_{t_i}(W_{t_{i+1}} - W_{t_i}) + X_{t_k}(W_s - W_{t_k}) = I(X)_s$$

We have proven that the integral of a simple process is a martingale. To extend this result to the $\mathcal{S}_2$ space, it is sufficient to show that the martingale property is preserved by limits in L^2. This is a consequence of the fact that the conditional expectation is continuous in L^2. To prove this, recall that for X, Y in L^2:

$$\mathbb{E}\left[\Big(\mathbb{E}[X|\mathcal{F}_t] - \mathbb{E}[Y|\mathcal{F}_t]\Big)^2\right] = \mathbb{E}\left[\Big(\mathbb{E}[X - Y|\mathcal{F}_t]\Big)^2\right]$$
$$\le \mathbb{E}\Big[\mathbb{E}[(X-Y)^2|\mathcal{F}_t]\Big] = \mathbb{E}\Big[(X-Y)^2\Big]$$

Where the inequality is a consequence of Jensen's inequality. □

Lemma 2.15 (Continuity of the integral)**.** *The process* $\big(I(X)\big)_t$ *has almost surely continuous sample paths.*

The proof is based on the proof in Lamberton and Lapeyre 2007 (page 38, Proposition 3.4.4).

Proof. Let X^n be a sequence of simple processes converging to X in $\mathcal{S}_2$. Then, by the Doob inequality,

$$\mathbb{E}\Big[\sup_{t\leq T}\big|I(X^{n+p})_t - I(X^n)_t\big|^2\Big] \leq 4\mathbb{E}\Bigg[\Big(\int_0^T X_s^{n+p} - X_s^n \mathrm{d}W_s\Big)^2\Bigg]$$

$$= 4\mathbb{E}\Big[\int_0^T |X_s^{n+p} - X_s^n|^2 \mathrm{d}s\Big] \xrightarrow[n\to\infty]{} 0$$

Thus, $\sup_{t\leq T}\big|I(X^{n+p})_t - I(X^n)_t\big|$ converges to 0 in $L^2(\Omega,\mathcal{F},\mathcal{P})$, so there exists a subsequence $\phi(n)$ such that:

$$\sup_{t\leq T}\big|I(X^{\phi(n+1)})_t - I(X^{\phi(n)})_t\big| \xrightarrow[n\to\infty]{} 0, \quad a.e$$

Hence, taking a subsequence if necessary, the simple (and continuous) processes $I(X^{\phi(n)})$ converge uniformly to $I(X)$ almost surely, so X is almost surely continuous. □

2.3 Extension of the integral

We would like to extend the integral to the bigger space:

$$\mathcal{S} = \{X \text{ adapted process}, \quad \mathcal{P}\big(\int_0^T |X_s|^2 \mathrm{d}s < \infty\big) = 1\}$$

If $\mathbb{E}\Big[\int_0^T |X_s|^2 \mathrm{d}s\Big] < \infty$, then the set $\{\omega, \int_0^T |X_s(\omega)|^2 \mathrm{d}s = \infty\}$ must have measure zero, or otherwise the expectation of $\int_0^T |X_s|^2 \mathrm{d}s$ would be infinity. So there is an inclusion $\mathcal{S}_2 \subseteq \mathcal{S}$.

The extension can be achieved by using a technique called *localization*. For this, we will define the concept of a *stopping time*: a map $\tau : \Omega \to T$ such that $\{\tau \leq s\} \in \mathcal{F}_s$ for all $s \in \mathcal{T}$. Moreover, given a process $X \in \mathcal{S}$, we say that a sequence of stopping times τ_n is *localising* for X in $\mathcal{S}_2$ if:

1. The sequence $(\tau_n)_{n\in\mathbb{N}}$ is increasing
2. $\forall n$, the stopped process $X_t^n = X_t 1_{[t\leq\tau_n]} \in \mathcal{S}_2$
3. $\lim_{n\to\infty} \tau_n = T$, *a.e.*

Let $X \in \mathcal{S}$, and consider the increasing sequence of stopping times defined by:

$$\tau_n = inf\Big\{t \in [0,T] \mid \int_0^t |X_s|^2 \mathrm{d}s \geq n\Big\}$$

Defined with the convention $inf\,\emptyset = T$. These are indeed stopping times. To show this, we begin by observing that $\int_0^t |X_s|^2 \mathrm{d}s$ is measurable:

Lemma 2.16. *Given a process* $X \in \mathcal{S}$, $\int_0^t X_s^2 \mathrm{d}s$ *is* $\mathcal{F}_t$*-measurable.*

Proof. Considering the integral as a Riemann integral, it can be expressed as a limit of partial sums. Then it is clear that $\int_0^t X_s^2 \mathrm{d}s$ is measurable, being a limit of measurable functions almost surely.

□

Now, for any $t \in T$,

$$\{\tau_n > t\} = \{\omega | \forall s \leq t, \int_0^s |X_u(\omega)|^2 \mathrm{d}u < n\} = \{\omega | \int_0^t |X_u(\omega)|^2 \mathrm{d}u < n\}$$

Because $\int_0^s |X_u(\omega)|^2 \mathrm{d}u$ is an increasing function of s. So τ_n is a stopping time.

The sequence τ_n is localising for X in $\mathcal{S}_2$. Indeed, since $X \in \mathcal{S}$,

$$a.e\ \omega \in \Omega, \quad \exists N_\omega \in \mathbb{N}, \quad \int_0^T |X_s(\omega)|^2 \mathrm{d}s < N_\omega$$

So that $\lim_{n\to\infty} \tau_n = T$, *a.e.* Clearly, the process $X_t^n = X_t 1_{\{t \leq \tau_n\}}$ is mean square integrable, so we can define its integral as usual. Now, we would like to define the integral process of X as the following limit:

$$I(X)_t = \int_0^t X_s \mathrm{d}W_s := \lim_{n\to\infty} \int_0^t X_s^n \mathrm{d}W_s, \quad 0 \leq t \leq T \tag{2.3}$$

In order to prove that the integral is well defined, it is necessary to show that the limit exists, almost surely. We state this result in the following theorem:

Theorem 2.17. *For any process $X \in \mathcal{S}$, the integral process of X as defined in 2.3 exists almost surely.*

The proof is based on the one by Lamberton and Lapeyre 2007. We will need the following proposition:

Proposition 2.18. *Let $H \in \mathcal{S}_2$ and τ be an $\mathcal{F}_t$-stopping time. Then,*

$$I(H)_\tau = \int_0^T 1_{\{s \leq \tau\}} H_s \mathrm{d}W_s, \quad a.s.$$

Proof. For any $H \in \mathcal{S}_2$, we define

$$\int_t^T H_s \mathrm{d}W_s := \int_0^T H_s \mathrm{d}W_s - \int_0^t H_s \mathrm{d}W_s$$

Let $A \in \mathcal{F}_t$. The following property holds:

$$\int_0^T 1_A H_s 1_{\{s>t\}} \mathrm{d}W_s = 1_A \int_t^T H_s \mathrm{d}W_s$$

This is clearly true for simple processes, and the property can be extended to mean-square integrable processes by a density argument.

Now, let $\tau = \sum_{i=1}^n t_i 1_{A_i}$, where all A_i are disjoint and $\mathcal{F}_{t_i}$-measurable. In this case, for each i, the process $1_{A_i} 1_{\{s>t_i\}}$ is adapted because 1_{A_i} is $\mathcal{F}_s$ measurable if $s > t_i$, and the process is zero otherwise. Hence,

$$\int_0^T 1_{\{s>\tau\}} H_s \mathrm{d}W_s = \int_0^T (\sum_{i=1}^n 1_{A_i} 1_{\{s>t_i\}}) H_s \mathrm{d}W_s = \sum_{i=1}^n \int_0^T 1_{A_i} 1_{\{s>t_i\}} H_s \mathrm{d}W_s$$

$$= \sum_{i=1}^{n} 1_{A_i} \int_{t_i}^{T} H_s \mathrm{d}W_s = \int_{\tau}^{T} H_s \mathrm{d}W_s$$

It follows that:

$$\int_0^{\tau} H_s \mathrm{d}W_s = \int_0^{T} 1_{\{s \leq \tau\}} H_s \mathrm{d}W_s$$

Now, consider an arbitrary stopping time τ and define a decreasing sequence τ_n by:

$$\tau_n = \sum_{k=0}^{2^n-1} \frac{(k+1)T}{2^n} 1_{\{\frac{kT}{2^n} \leq \tau \leq \frac{(k+1)T}{2^n}\}}$$

Clearly, τ_n converges to τ almost surely. Since the map $t \mapsto \int_0^t H_s \mathrm{d}W_s$ is almost surely continuous, $\int_0^{\tau_n} H_s \mathrm{d}W_s$ converges to $\int_0^{\tau} H_s \mathrm{d}W_s$ almost surely. By the previous discussion, we know that $\int_0^{\tau_n} H_s \mathrm{d}W_s = \int_0^T 1_{\{s \leq \tau_n\}} H_s \mathrm{d}W_s$ for all $n \geq 1$. Now,

$$\mathbb{E}\left(\left| \int_0^T 1_{\{s \leq \tau_n\}} H_s \mathrm{d}W_s - \int_0^T 1_{\{s \leq \tau\}} H_s \mathrm{d}W_s \right|^2 \right) = \mathbb{E}\left(\int_0^T 1_{\{\tau < s \leq \tau_n\}} H_s^2 \mathrm{d}s \right)$$

The last expression converges to 0 by the dominated convergence theorem. Consequently,

$$\int_0^{\tau_n} H_s \mathrm{d}W_s = \int_0^T 1_{\{s \leq \tau_n\}} H_s \mathrm{d}W_s \xrightarrow[n \to \infty]{L^2} \int_0^T 1_{\{s \leq \tau\}} H_s \mathrm{d}W_s$$

In particular, a subsequence of $\int_0^{\tau_n} H_s \mathrm{d}W_s$ converges almost surely to $\int_0^T 1_{\{s \leq \tau\}} H_s \mathrm{d}W_s$. This concludes the proof, because we also know that $\int_0^{\tau_n} H_s \mathrm{d}W_s \xrightarrow[n \to \infty]{} \int_0^{\tau} H_s \mathrm{d}W_s$ almost surely. □

Proof of the Theorem. To see that $\int_0^t X^n \mathrm{d}W$ converges, note that because τ_n are increasing, $X_t^n = 1_{\{t \leq \tau_n\}} X_t^{n+1}$. The previous proposition implies that

$$\int_0^t X_s^n \mathrm{d}W_s = \int_0^{t \wedge \tau_n} X_s^{n+1} \mathrm{d}W_s$$

Hence, on the set $\{\omega, \int_0^T X_s(\omega)^2 \mathrm{d}s < n\}$,

$$I(X^n)_t = I(X^m)_t \quad \text{for all } m \geq n \tag{2.4}$$

Since $\{\omega, \int_0^T X_s(\omega)^2 \mathrm{d}s < \infty\} = \bigcup_{n \in \mathbb{N}} \{\omega, \int_0^T X_s(\omega)^2 \mathrm{d}s < n\}$ and the set has probability 1, the sequence $I(X^n)_t$ converges almost surely. □

By construction, and by 2.4, the extended integral has almost surely continuous sample paths. Note that the extended integral does not necessarily have the martingale property. However, we will show that it is a *local martingale*.

Definition 2.19. *An adapted process X_t is a* local martingale *if there exists a sequence $(\tau_n)_{n \in \mathbb{N}}$ of increasing stopping times such that $\mathbb{P}(\lim_n \tau_n = T) = 1$ and such that for each n, the stopped process $X^{\tau_n}(t) := X_{t \wedge \tau_n}$ is a martingale.*

Proposition 2.20. *Let $X \in \mathcal{S}$. Then, the integral process $I(X)_t$ is a local martingale.*

Proof. The proof is based on the one by Capinski, Kopp, and Traple 2012 (p.140, Proposition 4.25).

Consider again the sequence of stopping times

$$\tau_n = inf\Big\{t \in [0,T] \mid \int_0^t |X_s|^2 \mathrm{d}s \geq n\Big\}$$

Recall that $I(X)_t = \lim_{n\to\infty} M_n(t)$, where $M_n(t) := \int_0^t X_s^n \mathrm{d}W_s$ is a martingale and $X_t^n = X_t 1_{\{t\leq\tau_n\}}$. Now,

$$X^{\tau_k}(t) = X(t\wedge\tau_k) = \lim_n M_n(t\wedge\tau_k) = \lim_n \int_0^T 1_{[0,t\wedge\tau_k]} X_s^n \mathrm{d}W_s$$

$$= \lim_n \int_0^t 1_{[0,\tau_k]} 1_{[0,\tau_n]} X_s \mathrm{d}W_s = \int_0^t 1_{[0,\tau_k]} X_s \mathrm{d}W_s = M_k(t)$$

As for $n \geq k$, $\tau_n \geq \tau_k$. □

The following result will be useful in the next sections:

Proposition 2.21. *Let M be a non-negative local martingale. Then, M is a supermartingale. Moreover, if $\mathbb{E}[M_t] = M_0$ is constant, M is a martingale.*

The proof is based on the one in Musiela and Rutkowski 2006 (p. 591, Proposition A.7.1). We will need the following result (see Musiela and Rutkowski 2006, p.580, Lemma A.1.2 for a proof):

Lemma 2.22 (Conditional Form of Fatou's Lemma)**.** *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Suppose that there exists a random variable Z, such that $X_n \geq Z$ for all n, and such that $\mathbb{E}[Z] > -\infty$. Then,*

$$\mathbb{E}[\liminf_n X_n|\mathcal{G}] \leq \liminf_n \mathbb{E}[X_n|\mathcal{G}]$$

Proof. (Proposition)
Since M is a local martingale, there exists an increasing sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ such that $\mathbb{P}(\lim_n \tau_n = T) = 1$ and such that for each n, the stopped process $M_t^{\tau_n}$ is a martingale.
Let $0 \leq s \leq t \leq T$. Because M is non-negative, and by the conditional form of Fatou's lemma,

$$\mathbb{E}[M_t|\mathcal{F}_s] = \mathbb{E}[\liminf_n M_t^{\tau_n}|\mathcal{F}_s] \leq \liminf_n \mathbb{E}[M_t^{\tau_n}|\mathcal{F}_s]$$
$$= \liminf_n M_s^{\tau_n} = M_s$$

So that M is a supermartingale.

Note that in particular, $\mathbb{E}[M_t] \leq M_0 < \infty$ for all t.

Finally, in the case of constant expectation, it is clear that if $\mathbb{E}[M_t|\mathcal{F}_s] \leq M_s$ a.s and $\mathbb{E}[\mathbb{E}[M_t|\mathcal{F}_s]] = \mathbb{E}[M_t] = \mathbb{E}[M_s]$, then $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$, a.s. □

Chapter 3

Fundamental Theorems

The standard results that follow are basic for the development of the mathematical finance theory.

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space, and W_t an $\mathcal{F}_t$-Brownian motion. We begin with the following definition:

Definition 3.1 (Itô Process)**.** *An Itô process is a process that satisfies the equation:*

$$X_t = X_0 + \int_0^t \alpha_s \mathrm{d}s + \int_0^t \beta_s \mathrm{d}W_s$$

Where:

1. *X_0 is $\mathcal{F}_0$-measurable.*
2. *α_t and β_t are $\mathcal{F}_t$-adapted and measurable processes.*
3. *$\int_0^T |\alpha_s| \mathrm{d}s < \infty$, a.s.*
4. *$\int_0^T |\beta_s|^2 \mathrm{d}s < \infty$, a.s.*

Equivalently, we say that the process satisfies the equation (in differential notation):

$$\mathrm{d}X_t = \alpha_t \mathrm{d}t + \beta_t \mathrm{d}W_t$$

Lemma 3.2. *The expression of an Itô process is unique.*

Proof. Refer to Capinski, Kopp, and Traple 2012 (p. 98, Theorem 3.27). □

3.1 Itô's formula

Theorem 3.3 (Itô's formula)**.** *Let $g \in \mathbb{C}^{1,2}([0,t] \times \mathbb{R}, \mathbb{R})$ and (X_t) be an Itô process, with:*

$$\mathrm{d}X_t = \alpha_t \mathrm{d}t + \beta_t \mathrm{d}W_t$$

Then $Y_t = g(t, X_t)$ is also an Itô process, and satisfies the formula:

$$\mathrm{d}Y_t = \big(g_t + g_x \alpha_t + \frac{1}{2} g_{xx} \beta_t^2\big) \mathrm{d}t + g_x \beta_t \mathrm{d}W_t \tag{3.1}$$

Where all partial derivatives of g are evaluated at (t, X_t).

For a proof of Itô's formula, refer to Capinski, Kopp, and Traple 2012 (p. 136, section 4.7). The following example will be particularly relevant:

Example 3.4. Let S_t be an Itô process such that $\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t$, and let $g(t,x) = \log(x)$. Then, if $Y_t = g(t, S_t)$,

$$\mathrm{d}Y_t = (\mu - \frac{1}{2}\sigma^2)\mathrm{d}t + \sigma \mathrm{d}W_t$$

Or, equivalently:

$$S_t = \exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\}$$

We will also require a two-dimensional version of Itô's formula, adapted from Musiela and Rutkowski 2006.

Theorem 3.5 (Itô's formula, two dimensions)**.** *Let (W_t^1), (W_t^2) be Brownian motions, $g \in \mathbb{C}^{1,2}([0,t] \times \mathbb{R}^2, \mathbb{R})$ and (X_t), (Y_t) be Itô processes, with:*

$$\mathrm{d}X_t = a_t \mathrm{d}t + b_t \mathrm{d}W_t^1$$

$$\mathrm{d}Y_t = \alpha_t \mathrm{d}t + \beta_t \mathrm{d}W_t^2$$

Then $Z_t = g(t, X_t, Y_t)$ is also an Itô process, and satisfies the formula:

$$\mathrm{d}Z_t = \big(g_t + g_x a_t + g_y \alpha_t + \frac{1}{2} g_{xx} b_t^2 + \frac{1}{2} g_{yy} \beta_t^2 + g_{xy} b_t \beta_t\big)\mathrm{d}t \tag{3.2}$$

$$+ g_x b_t \mathrm{d}W_t^1 + g_y \beta_t \mathrm{d}W_t^2$$

Where all partial derivatives of g are evaluated at (t, X_t, Y_t).

3.2 Stochastic differential equations

In the next chapter we will frequently consider stochastic differential equations of the form:

$$\mathrm{d}X_t = a(t, X_t)\mathrm{d}t + b(t, X_t)\mathrm{d}W_t \tag{3.3}$$

$$X_0 = x_0$$

In this situation, it is important to know if a solution to the equation exists, and in that case, if it is unique. The following theorem, adapted from Capinski, Kopp, and Traple 2012 (p.160, Theorem 5.8), provides sufficient conditions for this to happen.

Theorem 3.6. *[Existence and uniqueness of stochastic differential equations] Consider the stochastic differential equation 3.3, and assume that the coefficient functions $a(t,x), b(t,x)$ are Lipschitz with respect to x and uniformly continuous with respect to t. Moreover, assume that they have linear growth. This means that there exists a constant $C > 0$ such that:*

$$|a(t,x)| + |b(t,x)| \leq C(1 + |x|), \quad \forall x \in \mathbb{R}, t \in [0, T] \tag{3.4}$$

Then, 3.3 has a unique solution with continuous paths and that is mean square integrable.

Proof. Refer to Capinski, Kopp, and Traple 2012 (p.160, Theorem 5.8). □

The natural extension of this result to multiple dimensions is also valid. That is, consider a multi-dimensional version of 3.3,

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t \tag{3.5}$$
$$X_0 = x_0$$

Where (X_t) is a d-dimensional stochastic process (understood as d components of one dimensional stochastic processes) and W_t is a d-dimensional Brownian motion (understood as d components of one dimensional Brownian motions), and $x_0 \in \mathbb{R}^d$. Then, if the component functions $a(t,x), b(t,x)$ are Lipschitz with respect to x and uniformly continuous with respect to t, a unique regular solution to the stochastic differential equation exists (see Musiela and Rutkowski 2006, p.639, Theorem A.3.1).

An important property that we would like the solutions of stochastic differential equations of the form 3.3 to satisfy is the Markov property.

Definition 3.7 (Markov Property)**.** *A stochastic process (X_t) in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfies the Markov property if, for any bounded measurable function $f : \mathbb{R} \to \mathbb{R}$ and any $s \leq t$,*

$$\mathbb{E}[f(X_t)|\mathcal{F}_s] = \mathbb{E}[f(X_t)|\mathcal{F}_{X_s}] \tag{3.6}$$

Where $(\mathcal{F}_{X_t})$ is the filtration generated by (X_t).

In our case, $(\mathcal{F}_t)$ is the filtration generated by the Brownian motion (W_t). The following result guarantees that under certain conditions on the coefficient functions, this property is satisfied by solutions of stochastic differential equations. It has been adapted from Capinski, Kopp, and Traple 2012 (p.174, Theorem 5.14).

Theorem 3.8. *[Markov Property] Consider a stochastic differential equation of the form 3.3, with coefficients $a(t,x)$ and $b(t,x)$ that are Lipshitz continuous with respect to x, uniformly continuous with respect to t, and satisfy the linear growth condition 3.4. Then, the solution X_t has the Markov property 3.6. That is,*

$$\mathbb{E}[f(X_t)|\mathcal{F}_{W_s}] = \mathbb{E}[f(X_t)|\mathcal{F}_{X_s}]$$

Proof. Refer to Capinski, Kopp, and Traple 2012 (p.174, Theorem 5.14). □

3.3 Girsanov's theorem

Theorem 3.9 (Girsanov, simple version)**.** *Let (W_t) be a standard Brownian Motion and $\gamma \in \mathbb{R}$. Then, $\tilde{W}_t := W_t + \gamma t$ is a standard Brownian motion under the probability $\tilde{\mathbb{P}}$ defined by:*

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = exp\big(- \gamma W_T - \frac{1}{2}\gamma^2 T\big)$$

Proof. The proof has been adapted from Capinski and Kopp 2012.

It is clear that $\tilde{W}$ has continuous sample paths. We will prove directly that $\tilde{W}_t$ has independent and normally distributed increments, by computing the probability $\tilde{\mathbb{P}}(A)$ where $A = \cap_{i=1}^{n} A_i$ and $A_i = \{\tilde{W}_{t_i} - \tilde{W}_{t_{i-1}} \leq a_i\}$, for any given partition $t_0, \ldots, t_n$ of $\mathcal{T}$, and every $a_1, \ldots, a_n \in \mathbb{R}$. By definition,

$$\tilde{\mathbb{P}}(A) = \mathbb{E}_{\tilde{\mathbb{P}}}\left[1_A\right] = \mathbb{E}_{\mathbb{P}}\left[exp\big(-\gamma W_T - \frac{1}{2}\gamma^2 T\big)1_A\right]$$

Now, W_T can be expressed as $\sum_{i=1}^{n} W_{t_i} - W_{t_{i-1}}$. Similarly, $T = \sum_{i=1}^{n}(t_i - t_{i-1})$ and $1_A = \prod_{i=1}^{n} 1_{A_i}$. We obtain:

$$\tilde{\mathbb{P}}(A) = \mathbb{E}_{\mathbb{P}}\left[\prod_{i=1}^{n} exp\big(-\gamma(W_{t_i} - W_{t_{i-1}}) - \frac{1}{2}\gamma^2(t_i - t_{i-1})\big)1_{A_i}\right]$$

The increments $W_{t_i} - W_{t_{i-1}}$ are independent, and the indicators sets A_i can be expressed as:

$$A_i = \{W_{t_i} - W_{t_{i-1}} + \gamma(t_i - t_{i-1}) \leq a_i\}$$

So, because of the independence property,

$$\tilde{\mathbb{P}}(A) = \prod_{i=1}^{n} \mathbb{E}_{\mathbb{P}}\left[exp\big(-\gamma(W_{t_i} - W_{t_{i-1}}) - \frac{1}{2}\gamma^2(t_i - t_{i-1})\big)1_{A_i}\right] = \prod_{i=1}^{n} \tilde{\mathbb{P}}(A_i) \quad (3.7)$$

Now, because the increments $W_{t_i} - W_{t_{i-1}}$ are distributed as $N(0, t_i - t_{i-1})$,

$$\mathbb{P}(A_i) = \mathbb{E}_{\mathbb{P}}\left[exp\big(-\gamma(W_{t_i} - W_{t_{i-1}}) - \frac{1}{2}\gamma^2(t_i - t_{i-1})\big)1_{A_i}\right]$$

$$= \int_{\{-x+\gamma(t_i - t_{i-1}) \leq a_i\}} exp\big(\gamma x - \frac{1}{2}\gamma^2(t_i - t_{i-1})\big)\frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} exp\big(-\frac{x^2}{2(t_i - t_{i-1})}\big)\mathrm{d}x$$

$$= \int_{\{-x+\gamma(t_i - t_{i-1}) \leq a_i\}} \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} exp\Big(-\frac{\big(x - \gamma(t_i - t_{i-1})\big)^2}{2(t_i - t_{i-1})}\Big)\mathrm{d}x$$

$$= \int_{\{z \leq a_i\}} \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} exp\Big(-\frac{z^2}{2(t_i - t_{i-1})}\Big)\mathrm{d}z \quad (3.8)$$

Equation 3.8 proves that $\tilde{W}_{t_i} - \tilde{W}_{t_{i-1}}$ are distributed as $N(0, t_i - t_{i-1})$, and equation 3.7 proves that they are independent. □

A generalized version of Girsanov's theorem will be required in the Stochastic Volatility chapter. The theorem has been adapted from Musiela and Rutkowski 2006 (p.648, Theorem A.15.1).

Theorem 3.10 (Girsanov's theorem, a generalized version)**.** *Let (W_t) be a d-dimensional Brownian motion in the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, and γ be an $\mathbb{R}^d$-valued $\mathbb{F}$-adapted stochastic process, such that:*

$$\mathbb{E}\left[exp\Big(-\int_0^T \gamma_s \mathrm{d}W_s - \frac{1}{2}\int_0^T \gamma_s^2 \mathrm{d}s\Big)\right] = 1 \quad (3.9)$$

Define $\tilde{W}_t := W_t + \int_0^t \gamma_s \mathrm{d}s$, and define the probability $\tilde{\mathbb{P}}$ by:

$$\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = exp\Big(-\int_0^T \gamma_s \mathrm{d}W_s - \frac{1}{2}\int_0^T \gamma_s^2 \mathrm{d}s\Big) \quad (3.10)$$

Then, $(\tilde{W}_t)$ is a Brownian motion under the probability $\tilde{\mathbb{P}}$.

3.4 Martingale representation theorem

In this section, we will prove the martingale representation theorem. These results have been adapted from Corcuera n.d., whose notations and arguments will be followed closely. We will need some preliminary results.

Theorem 3.11 (Martingale L^p-convergence). *Let $(M_n)_{n\in\mathbb{N}}$ be a martingale in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_n), \mathbb{P})$. Assume that, for some $p > 1$, $M_n \in L^p = L^p(\Omega, \mathcal{F}, \mathbb{P})$ $\forall n \in \mathbb{N}$, and $\sup_{n\in\mathbb{N}} \|M_n\|_{L^p} < \infty$. Then, for some $M_\infty \in L^p$,*

$$M_n \xrightarrow[n\to\infty]{L^p} M_\infty$$

Proof. Refer to Kallenberg 2006 (p.109, Corollary 6.22). □

Lemma 3.12. *Let $(\mathcal{G}_n)$ be a filtration in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $X \in L^2 = L^2(\Omega, \mathcal{F}, \mathbb{P})$. Then,*

$$\mathbb{E}[X|\mathcal{G}_n] \xrightarrow[n\to\infty]{L^2} \mathbb{E}[X|\mathcal{G}_\infty]$$

Where $\mathcal{G}_\infty = \sigma(\mathcal{G}_n, n \in \mathbb{N})$.

Proof. Define $X_n := \mathbb{E}[X|\mathcal{G}_n]$. Clearly, X_n is a martingale with respect to the filtration $(\mathcal{G}_n)$, and by the properties of the conditional expectation,

$$\sup_{n\in\mathbb{N}} \|X_n\|_{L^2} \leq \|X\|_{L^2} < \infty$$

By theorem 3.11,

$$X_n \xrightarrow[n\to\infty]{L^2} Y$$

For some $Y \in L^2$. It remains to show that $Y = \mathbb{E}[X|\mathcal{G}_\infty] =: X_\infty$. By the continuity of the conditional expectation in L^2 (this was proved in 2.14),

$$\mathbb{E}[Y|\mathcal{G}_n] = \mathbb{E}[\lim_{m\to\infty} X_m|\mathcal{G}_n] = \lim_{m\to\infty} \mathbb{E}[X_m|\mathcal{G}_n] = X_n$$

By the tower property of the conditional expectation, we also have that

$$\mathbb{E}[X_\infty|\mathcal{G}_n] = \mathbb{E}\Big[\mathbb{E}[X|\mathcal{G}_\infty]\big|\mathcal{G}_n\Big] = X_n$$

So that, for all $n \in \mathbb{N}$,

$$\mathbb{E}[Y|\mathcal{G}_n] = \mathbb{E}[X_\infty|\mathcal{G}_n]$$

Hence, for every $G \in \cup_{n\in\mathbb{N}} G_n$,

$$\mathbb{E}[Y 1_G] = \mathbb{E}[X_\infty 1_G] \tag{3.11}$$

Define the collection

$$\mathcal{C} = \{G \in \mathcal{G}_\infty \mid \mathbb{E}[Y 1_G] = \mathbb{E}[X_\infty 1_G]\} \tag{3.12}$$

Clearly, by 3.11,

$$\cup_{n\in\mathbb{N}} G_n \subset \mathcal{C} \subset \mathcal{G}_\infty \tag{3.13}$$

We wish to show that $\mathcal{C} = \mathcal{G}_\infty$.

Now, if we prove that $\mathcal{C}$ is a σ-algebra, we will have, by equation 3.13 and the fact that $\cup_n \mathcal{G}_n$ generates $\mathcal{G}_\infty$, that $\mathcal{C} = \mathcal{G}_\infty$.

Condition (1) is clear since $X \in \cup_{n\in\mathbb{N}} G_n \subset \mathcal{C}$. For condition (2), if $B \subset A \in \mathcal{C}$,

$$\mathbb{E}[Y1_{A\setminus B}] = \mathbb{E}[Y1_A] - \mathbb{E}[Y1_B] = \mathbb{E}[X_\infty 1_A] - \mathbb{E}[Y1_B] = \mathbb{E}[X_\infty 1_{A\setminus B}]$$

So that $A \setminus B \in \mathcal{C}$. For condition (3) note that, if $(A_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{C}$, by the dominated convergence theorem,

$$\begin{aligned}\mathbb{E}[Y1_{\cup_n A_n}] &= \mathbb{E}[\lim_n Y1_{\cup_{m=1}^n A_m}] = \lim_n \mathbb{E}[Y1_{\cup_{m=1}^n A_m}] \\ &= \lim_n \mathbb{E}[X_\infty 1_{\cup_{m=1}^n A_m}] = \mathbb{E}[X_\infty 1_{\cup_n A_n}]\end{aligned}$$

Hence, $\cup_n A_n \in \mathcal{C}$, and $\mathcal{C} = \mathcal{G}_\infty$.

Now, $B_n := \{X_\infty - Y > \frac{1}{n}\} \in \mathcal{G}_\infty$, so

$$\mathbb{E}[X_\infty 1_{B_n}] = \mathbb{E}[Y1_{B_n}]$$

This implies that $\mathbb{P}(B_n) = 0\ \forall n \in \mathbb{N}$, so that $\mathbb{P}(\cup_n B_n) = 0$. Hence $X_\infty \leq Y$ *a.s*, and a similar argument shows that $X_\infty \geq Y$ *a.s.* ☐

Lemma 3.13. *Let (W_t) be a Brownian motion in the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F}$ is generated by (W_t). Consider the set $\mathcal{J}$ of stepwise functions $f : [0, T] \to \mathbb{R}$ of the form:*

$$f = \sum_{i=0}^{n} \lambda_i 1_{]t_{i-1}, t_i]}$$

With $\lambda_i \in \mathbb{R}$ and $0 = t_0 < \ldots < t_n = T$. For each $f \in \mathcal{J}$, define

$$\mathcal{E}_T^f = exp\Big\{\int_0^T f(s)\mathrm{d}W_s - \frac{1}{2}\int_0^T f^2(s)\mathrm{d}s\Big\}$$

Let $Y \in L^2(\mathcal{F}_T, \mathbb{P})$, and assume that Y is orthogonal to $\mathcal{E}_T^f$ for all $f \in \mathcal{J}$. Then, $Y = 0$.

Proof. Let $f \in \mathcal{J}$ and $Y \in L^2(\mathcal{F}_T, \mathbb{P})$ orthogonal to $\mathcal{E}_T^f$. Define $\mathcal{G}_n := \sigma(W_{t_0}, \ldots, W_{t_n})$. By assumption,

$$\mathbb{E}\left(exp\Big\{\sum_{i=1}^{n} \lambda_i (W_{t_i} - W_{t_{i-1}})\Big\} Y\right) = 0$$

Taking the conditional by $\mathcal{G}_n$ we obtain:

$$\mathbb{E}\left(exp\Big\{\sum_{i=1}^{n} \lambda_i (W_{t_i} - W_{t_{i-1}})\Big\}\mathbb{E}[Y|\mathcal{G}_n]\right) = 0$$

Let $X : \Omega \to \mathbb{R}^n$ be defined by:

$$X = (W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}})$$

Decomposing Y as $Y = Y_+ - Y_-$, we get:

$$\mathbb{E}\left(exp\Big\{\sum_{i=1}^{n} \lambda_i (W_{t_i} - W_{t_{i-1}})\Big\}\mathbb{E}[Y_+|\mathcal{G}_n]\right) = \mathbb{E}\left(exp\Big\{\sum_{i=1}^{n} \lambda_i (W_{t_i} - W_{t_{i-1}})\Big\}\mathbb{E}[Y_-|\mathcal{G}_n]\right) \tag{3.14}$$

Applying the pushforward measure theorem to 3.14, and because of the fact that $\mathbb{E}[Y|\mathcal{G}_n] = \mathbb{E}\Big[Y\big|\mathcal{W}_1 = a_1, \ldots, W_n - W_{n-1} = a_n\Big]\Big|_{a_1=W_1,\ldots,a_n=W_n-W_{n-1}}$:

$$\int_{\mathbb{R}^n} exp\Big\{\sum_{i=1}^{n} \lambda_i x_i\Big\}\mathbb{E}[Y_+|\mathcal{G}_n](x_1,\ldots,x_n)\mathrm{d}\mathbb{P}^X(x_1,\ldots,x_n) \tag{3.15}$$

$$= \int_{\mathbb{R}^n} exp\Big\{\sum_{i=1}^{n} \lambda_i x_i\Big\}\mathbb{E}[Y_-|\mathcal{G}_n](x_1,\ldots,x_n)\mathrm{d}\mathbb{P}^X(x_1,\ldots,x_n) \tag{3.16}$$

Note that 3.15 and 3.16 are, respectively, the Laplace transforms of $\mathbb{E}[Y_+|\mathcal{G}_n](x_1,\ldots,x_n)$ and $\mathbb{E}[Y_-|\mathcal{G}_n](x_1,\ldots,x_n)$ with respect to the measure $\mathbb{P}^X$. We admit the result on the uniqueness of the Laplace transform, which in this case implies that:

$$\mathbb{E}[Y_+|\mathcal{G}_n](x_1,\ldots,x_n) = \mathbb{E}[Y_-|\mathcal{G}_n](x_1,\ldots,x_n), \quad \mathbb{P}^X\ a.s \tag{3.17}$$

So that $\mathbb{E}[Y_+|\mathcal{G}_n](x_1,\ldots,x_n) = \mathbb{E}[Y_-|\mathcal{G}_n](x_1,\ldots,x_n)$ for all $(x_1,\ldots,x_n) \in \mathbb{R}^n \setminus A$ for some set $A \subset \mathbb{R}^n$ with $\mathbb{P}^X(A) = 0$. This means that $\mathbb{E}[Y_+|\mathcal{G}_n](\omega) = \mathbb{E}[Y_+|\mathcal{G}_n](\omega), \quad \forall\omega \in X^{-1}(A)$. Finally,

$$\mathbb{E}[Y_+|\mathcal{G}_n] = \mathbb{E}[Y_+|\mathcal{G}_n] \quad \mathbb{P}\ a.s. \tag{3.18}$$

Since this is true for any $\mathcal{G}_n$ as defined above, by 3.12 we have that

$$\mathbb{E}[Y_\pm|\mathcal{G}_n] \xrightarrow[n\to\infty]{} \mathbb{E}[Y_\pm|\sigma(\mathcal{G}_n; n \in \mathbb{N})] = Y_\pm$$

Hence, $Y = 0$.

□

Proposition 3.14. *Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space with $\mathcal{F} = \mathcal{F}_T$, let (W_t) be a Brownian motion and let $F \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$. Then, there exists an adapted, mean square integrable process (Y_t) such that*

$$F = \mathbb{E}[F] + \int_0^T Y_t \mathrm{d}W_t$$

Proof. Consider the Hilbert space $\mathcal{H}$ of centered random variables in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, and its subspace $\mathcal{I}$ consisting of the random variables of the form $\int_0^T Y_t \mathrm{d}W_t$, for some adapted, mean square integrable process (Y_t). Note that proving the proposition is equivalent to proving that $\mathcal{I} = \mathcal{H}$.

If, on the contrary, $\mathcal{I} \subsetneq \mathcal{H}$, then there would exist a centered, non-trivial random variable $Z \in \mathcal{H}$ orthogonal to $\mathcal{I}$. We will prove that this is not possible.

Indeed, suppose that such Z exists. Take $Y_t := \mathcal{E}_t^f$, with $\mathcal{E}_t^f$ as defined in Lemma 3.13. Then,

$$\mathbb{E}\Big[Z \cdot \int_0^T \mathcal{E}_t^f \mathrm{d}W_t\Big] = 0$$

And also

$$\mathbb{E}\Big[Z \cdot (1 + \int_0^T \mathcal{E}_t^f \mathrm{d}W_t)\Big] = 0 \tag{3.19}$$

Now, a similar argument as the one in 3.4 shows that $\mathcal{E}_t^f$ is the solution to the following stochastic differential equation:

$$\mathrm{d}\mathcal{E}_t^f = f(t)\mathcal{E}_t^f \mathrm{d}W_t$$

In particular,

$$\mathcal{E}_T^f = 1 + \int_0^T f(t)\mathcal{E}_t^f \mathrm{d}W_t \tag{3.20}$$

From 3.19 and 3.20 we conclude:

$$\mathbb{E}[Z\mathcal{E}_T^f] = 0 \tag{3.21}$$

Since this is true for any f defined as in 3.13, by 3.13 we conclude that $Z = 0$. □

Theorem 3.15 (Martingale Representation Theorem)**.** *Let (M_t) be a square integrable martingale with respect to the filtration $(\mathcal{F}_t)$, and (W_t) a Brownian motion. Then, there exists an adapted, mean square integrable process (X_t) such that, in differential notation:*

$$\mathrm{d}M_t = X_t \mathrm{d}W_t, \ a.s$$

Proof. Applying the previous proposition applied to M_T, we obtain the existence of a mean square integrable process (Y_t) such that:

$$M_T = \mathbb{E}[M_T] + \int_0^T Y_t \mathrm{d}W_t = M_0 + \int_0^T Y_t \mathrm{d}W_t$$

Since (M_t) is a martingale, for any $t \in [0, T]$ we have:

$$M_t = \mathbb{E}[M_T|\mathcal{F}_t] = \mathbb{E}\Big[M_0 + \int_0^T Y_t \mathrm{d}W_t \big| \mathcal{F}_t\Big] = M_0 + \int_0^t Y_t \mathrm{d}W_t$$

Where we have used the fact that $\big(\int_0^t Y_t \mathrm{d}W_t\big)$ is a martingale (see 2.14). □

Chapter 4

The Black-Scholes Model

The Black-Scholes model consists of two stocks S and b, in a time frame $[0, T]$, where b is a bank account such that $b(t) = e^{rt}$, and the risky stock S satisfies the differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{4.1}$$

Where W_t is a Brownian motion in the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, and $\mathbb{P}$ represents the so-called empirical or physical probability. The discounted stock price is defined as $S_t^* = b_t^{-1} S_t$.

Define $\mathbb{F} = (\mathcal{F}_t)$ to be the filtration generated by the Brownian motion (W_t). We assume that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. The coefficients of the stochastic differential equation 4.1, according to the notations in 3.3, are

$$\begin{aligned} a(t, x) &= \mu x \\ b(t, x) &= \sigma x \end{aligned} \tag{4.2}$$

These functions clearly satisfy the regularity conditions in 3.6, so there exists a unique mean-square integrable solution S_t with continuous sample paths. By theorem 3.8, this solution also satisfies the Markov property.

An explicit solution for this stochastic differential equation was found in 3.4:

$$S_t = S_0 \exp\{\ (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\} \tag{4.3}$$

The following are some implicit assumptions of this model, as argued in Corcuera n.d.:

- it has continuous trajectories
- its returns $\frac{S_t - S_u}{S_u}$ are independent of $\sigma(S_s, 0 \leq s \leq u)$. Indeed,

  $$\frac{S_t - S_u}{S_u} = \frac{S_t}{S_u} - 1 = \exp\{\ (\mu - \frac{1}{2}\sigma^2)(t-u) + \sigma(W_t - W_u)\} - 1$$

  Which is independent of $\sigma(S_s, 0 \leq s \leq u)$.

- its returns are stationary:

  $$\frac{S_t - S_u}{S_u} \sim \frac{S_{t-u} - S_0}{S_0}$$

A *trading strategy* or *strategy* is a random vector $\phi = (\phi^1, \phi^2)$ with values in $\mathbb{R}^2$, adapted to the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$. The value of the strategy is the process $V_\phi(t) := \phi_t \cdot (b(t), S_t)$.
A strategy is *self-financing* if:

$$\mathrm{d}V_\phi(t) = \phi_t^1 \mathrm{d}b(t) + \phi_t^2 \mathrm{d}S_t \tag{4.4}$$

This is a natural extension of the discrete time expression $\Delta_n V_\phi = \phi_n^1 \Delta_n b + \phi_n^2 \Delta_n S$. The discounted value process is defined as $V_\phi^*(t) = \phi_t \cdot (1, S_t^*)$, where $S_t^* = S_t / b(t)$ is the discounted stock price.
For 4.4 to make sense, we need to require that:

1. $\int_0^T |\phi_t^1| \mathrm{d}t < \infty$, a.s
2. $\int_0^T (\phi_t^2)^2 \mathrm{d}t < \infty$, a.s

We will denote be the set of self-financing strategies as Φ. We will discuss further restrictions on the set of strategies considered in the model later on.

The following characterization of self-financed strategies will be useful:

Lemma 4.1. *Let ϕ be a strategy satisfying integrability conditions (1) and (2). Then, $\phi \in \Phi$ iff*

$$\mathrm{d}\tilde{V}_t(\phi) = \phi_t^2 \mathrm{d}\tilde{S}_t$$

Proof. The proof has been adapted from Lamberton and Lapeyre 2007 (p. 65, Proposition 4.1.2).

Suppose that ϕ is self-financing. The Itô formula gives

$$\mathrm{d}\tilde{V}_\phi(t) = -r\tilde{V}_\phi(t)\mathrm{d}t + e^{-rt}\mathrm{d}V_\phi(t)$$

Imposing the self-financing condition, we obtain:

$$\mathrm{d}\tilde{V}_\phi(t) = -re^{-rt}(\phi_t^1 e^{rt} + \phi_t^2 S_t)\mathrm{d}t + e^{-rt}(\phi_t^1 \mathrm{d}b(t) + \phi_t^2 \mathrm{d}S_t)$$

$$= \phi_t^2(-re^{-rt}S_t\mathrm{d}t + e^{-rt}\mathrm{d}S_t) = \phi_t^2 \mathrm{d}\tilde{S}_t$$

The converse can be proved similarly.

□

4.1 Risk-neutral measure

A *risk-neutral measure* is a measure $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ such that the discounted stock price is a martingale under $\tilde{\mathbb{P}}$. The following results guarantee the existence and uniqueness of a risk-neutral measure. We begin with the following lemma:

Lemma 4.2. *Given a Brownian motion (W_t), the process $M_t := S_0^* exp\{-\frac{1}{2}\sigma^2 t + \sigma W_t\}$ is a martingale.*

Proof. For any $s < t$,

$$\mathbb{E}\Big[M_t/M_s|\mathcal{F}_s\Big] = \mathbb{E}\Big[exp\{-\frac{1}{2}\sigma^2(t-s)+\sigma(\tilde{W}_t-\tilde{W}_s)\}|\mathcal{F}_s\Big]$$

$$= \mathbb{E}\Big[exp\{-\frac{1}{2}\sigma^2(t-s)+\sigma(\tilde{W}_t-\tilde{W}_s)\}\Big] \quad \text{(because} \quad \tilde{W}_t-\tilde{W}_s \perp \mathcal{F}_s)$$

Now,

$$\mathbb{E}\Big[exp\{-\frac{1}{2}\sigma^2(t-s)+\sigma(\tilde{W}_t-\tilde{W}_s)\}\Big]$$

$$= \int_{\mathbb{R}} exp\{-\frac{1}{2}\sigma^2(t-s)+\sigma x\}\frac{1}{\sqrt{2\pi(t-s)}}exp\Big\{-\frac{x^2}{2(t-s)}\Big\}\mathrm{d}x$$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(t-s)}}exp\Big\{-\frac{(x-\sigma(t-s))^2}{2(t-s)}\Big\}\mathrm{d}x = 1$$

□

Proposition 4.3. *A risk neutral measure exists, defined by:*

$$\Big(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big)_t = exp\Big(-\frac{\mu-r}{\sigma}W_T^* - \frac{1}{2}\frac{(r-\mu)^2}{\sigma^2}T\Big) \tag{4.5}$$

Where the process $W_t^* := W_t + \frac{\mu-r}{\sigma}t$ *is a Brownian motion under* $\mathbb{P}$.

Proof. In view of Girsanov's theorem, we seek a value γ such that by defining $W_t^* := W_t + \gamma t$, the stock price is a martingale under the probability $\mathcal{Q}$, defined as in Girsanov's theorem. The stock price evolves according to:

$$\mathrm{d}S_t^* = (\mu - r + \gamma\sigma)S_t^*\mathrm{d}t + \sigma S_t^*\mathrm{d}W_t^*$$

It is a martingale under $\mathbb{Q}$ iff the drift term $\mu - r + \gamma\sigma = 0$.

Indeed, let $\rho = \mu - r + \gamma\sigma$. Then,

$$S_t^* = S_0^*\exp\{(\rho - \frac{1}{2}\sigma^2)t + \sigma W_t^*\} = exp\{\rho t\}S_0^* exp\{-\frac{1}{2}\sigma^2 t + \sigma W_t^*\}$$

The previous lemma implies that $exp\{-\frac{1}{2}\sigma^2 t + \sigma W_t^*\}$ is a martingale under $\mathbb{Q}$, which proves the result.

□

Remark. The stock price S_t evolves according to the equation:

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t^* \tag{4.6}$$

Theorem 4.4. *A unique neutral measure* $\mathbb{Q}$ *exists.*

Proof. Refer to Capinski and Kopp 2012 (p. 51, Theorem 3.12) □

4.2 Arbitrage and admissibility

An *arbitrage opportunity* is an opportunity to have positive returns in an investment with no risk. More specifically, an arbitrage strategy is defined as following:

Definition 4.5. *An arbitrage strategy is a self-financing strategy $\phi = (\phi^1, \phi^2)$ such that $V_\phi(0) = 0$, $V_\phi(T) \geq 0$, a.s and $\mathbb{P}(V_\phi(T) > 0) > 0$.*

The following example shows that the market model described above, with the set of self-financing strategies, has arbitrage opportunities. In order to eliminate these opportunities, it will be necessary to restrict the set of so-called *admissible strategies*, introduced below.

Theorem 4.6. *Arbitrage opportunities exist within the class of self-financing strategies.*

This example is based on the suicide strategy described in Capinski and Kopp 2012 (p.24-27).

We begin by considering the strategy ϕ given by:

$$\phi_t^2 = \frac{1}{\sigma \tilde{S}_t \sqrt{T-t}}$$

The risk free component is determined by the self-financing condition and an (arbitrary) initial value. Now, the strategy is not almost surely square integrable, but it will be modified later on. Firstly, we will study its properties.

Lemma 4.7. *For each $M \geq 0$,*

$$\mathbb{P}(\min\{t : \tilde{V}_\phi(t) - \tilde{V}_\phi(0) \geq M\} \leq T) = 1$$

$$\mathbb{P}(\min\{t : \tilde{V}_\phi(t) - \tilde{V}_\phi(0) \leq -M\} \leq T) = 1$$

Proof.

$$\mathrm{d}\tilde{V}_\phi(t) = \phi_t^2 \mathrm{d}\tilde{S}_t = \phi_t^2 \sigma \tilde{S}_t \mathrm{d}W_t^* = \frac{1}{\sqrt{T-t}} \mathrm{d}W_t^*$$

So that:

$$\tilde{V}_\phi(t) - \tilde{V}_\phi(0) = \int_0^t \frac{1}{\sqrt{T-u}} \mathrm{d}W_u^*$$

Now, let

$$\gamma(t) = \int_0^t \frac{1}{\sqrt{T-u}} \mathrm{d}u$$

Then, $\tilde{V}_\phi(t) - \tilde{V}_\phi(0)$ and $W^*(\gamma(t))$ have the same distribution. A proof of this result can be found at Capinski and Kopp 2012 - here we will focus on its consequences. Recall that - see Lemmma 1.8, for any fixed $M > 0$:

$$\mathbb{Q}(\max_{t \geq 0} W_t^* \leq M) = 0$$

$$\mathbb{Q}(\min_{t \geq 0} W_t^* \geq -M) = 0$$

And since $\mathbb{P}$ and $\mathbb{Q}$ are equivalent,

$$\mathbb{P}(\max_{t\geq 0} W_t^* \leq M) = 0$$

$$\mathbb{P}(\min_{t\geq 0} W_t^* \geq -M) = 0$$

Clearly, $\gamma(t) \xrightarrow[t\to T]{} \infty$, so that

$$\mathbb{P}(\max_{t\leq T} \tilde{V}_\phi(t) \leq M) = \mathbb{P}(\max_{t\leq T} W^*(\gamma(t)) \leq M) = \mathbb{P}(\max_{t\geq 0} W^*(t) \leq M) = 0$$

$$\mathbb{P}(\min_{t\leq T} \tilde{V}_\phi(t) \geq -M) = \mathbb{P}(\min_{t\leq T} W^*(\gamma(t)) \geq -M) = \mathbb{P}(\min_{t\geq 0} W^*(t) \geq -M) = 0$$

□

Proof. (Theorem)

Let $\tilde{V}_\phi(0) = 1$ and take $M = 1$, so that:

$$\mathbb{P}(\min\{t : \tilde{V}_\phi(t) \leq 0\} \leq T) = 1$$

Let

$$\tau = \min\{t : \tilde{V}_\phi(t) = 0\} \leq T, \quad a.s$$

Define the self-financing strategy θ as:

$$\theta_t^2 = \phi_t^2 1_{\{t\leq\tau\}}$$

And such that $V_\theta(0) = 1$. Since $\tau \leq T$ almost everywhere, it is clear that

$$\int_0^T (\theta_t^2)^2 \mathrm{d}t < \infty, \quad a.e$$

So that $\theta \in \Phi$. This suicide strategy begins with a positive wealth and ends almost surely in bankrupcy. As we will see, this cannot be admissible. Indeed, we can construct a new strategy from θ that is an arbitrage opportunity. Define:

$$\chi_t^1 = -\theta_t^1 + 1$$
$$\chi_t^2 = -\theta_t^2$$

Then $V_\chi(0) = -V_\theta(0) + 1 = 0$ and $V_\chi(T) = -V_\theta(T) + e^{rT} = e^{rT}$. Clearly $\chi \in \Phi$, because, just like θ, it is self-financing and satisfies integrability conditions. The previous properties of χ show that it is an arbitrage opportunity. □

We will now define which strategies are considered *admissible* in the model. Although several alternatives have been proposed in the literature, here we follow the definition from:

Definition 4.8. *A self-financing strategy ϕ is* admissible *if it is bounded by below. We will denote the set of admissible strategies as $\Phi' \subset \Phi$.*

The next result guarantees that no arbitrage opportunities exist within the admissible strategies:

Proposition 4.9. *No admissible strategy is an arbitrage opportunity.*

Proof. Suppose that $\phi \in \Phi'$ is an admissible strategy and an arbitrage opportunity - we will argue by contradiction. In that case, $V_\phi(t)$ is a local martingale with respect to Q which is bounded by below, so there exists a constant L such that $V_\phi(t) + L$ is non-negative. Hence, Proposition 1.21 implies that $V_\phi(t)$ is a super martingale under $\mathbb{Q}$. In particular, $\mathbb{E}_{\mathbb{Q}}[V_\phi(T)] \leq V_\phi(0) = 0$, which combined with the fact that $\mathbb{P}(V_\phi(T) > 0) > 0$ and that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, implies that $\mathbb{P}(V_\phi(T) < 0) > 0$, which condradicts the assumption $V_\phi(T) \geq 0$. □

Definition 4.10 (European Option)**.** *A European option with maturity T is defined by a non-negative, $\mathcal{F}_T$-measurable random variable $H = h(S_T)$, that expresses its payoff.*

Definition 4.11. *A strategy $\phi \in \Phi'$* replicates *the derivative with payoff H if $H = V_\phi(T)$. The market model is* complete *if every European option can be replicated.*

We are now able to state a fundamental result in the Black-Scholes model, relating the price process of a derivative to the value process of the replicating strategy. The proof of the result will be outlined - however, substantial technical details that have been omitted can be found at Capinski and Kopp 2012.

Theorem 4.12. *Let H be the payoff of a derivative which is replicated by the strategy ϕ. Assuming that the option price is an Itô process, the No Arbitrage Principle implies that the price of the option at time t, V_t, is equal to $V_\phi(t)$ for all $t \in T$.*

Proof. This proof is based on the one in Capinski and Kopp 2012 (p.21, Theorem 2.16). Assume that this is not the case, and let t_0 be any time in which a difference between $V_\phi(t)$ and V_t appears with positive probability. Consider a strategy ψ with zero initial value and that buys the cheaper of the two and sells the most expensive short at time t_0 and invests the remaining money in the bank account. The value $V_\psi(T)$ of this strategy is positive with a positive probability. Indeed, assuming without a loss of generality that $V_\phi(t_0) > V_{t_0}$,

$$V_\psi(T) = \big(V_\phi(t_0) - V_{t_0}\big)e^{r(T-t_0)} - V_\phi(T) + V_T \tag{4.7}$$

$$= \big(V_\phi(t_0) - V_{t_0}\big)e^{r(T-t_0)} \quad \text{(because } V_T = H = V_\phi(T)\text{)} \tag{4.8}$$

Which is greater than zero with positive probability. Now, if we show that ψ is an admissible strategy, this would violate the No Arbitrage Principle.

The strategy is self-financing by construction, and it remains to show that it is bounded by below. The reader can refer to Capinski and Kopp 2012 (p.30, Theorem 2.16) for a proof of this detail. □

4.3 Completeness

We aim to prove that any square integrable European option can be replicated by an admissible strategy. The proof of the following theorem is based on the proof in Capinski and Kopp 2012:

Theorem 4.13 (A Completeness Theorem)**.** *For any square integrable European option H with respect to $\mathbb{Q}$, there exists an admissible strategy ϕ that replicates H.*

Proof. We will see that, in fact, such a ϕ exists that satisfies the additional property that $V^*_\phi(t)$ is a martingale under $\mathbb{Q}$, and

$$V_\phi(t) = b(t)V^*_\phi(t) = b(t)\mathbb{E}_{\mathbb{Q}}\left[H^*(t)|\mathcal{F}_t\right] \tag{4.9}$$

Where $H^*(t) = H/b(t)$. Hence we seek an admissible strategy ϕ with value given by the previous expression. The martingale representation theorem guarantees the existence of an adapted, mean square integrable process $X(t)$ such that:

$$\mathrm{d}\left(\mathbb{E}_{\mathbb{Q}}\left[H^*(t)|\mathcal{F}_t\right]\right) = X(t)\mathrm{d}W^*_t \tag{4.10}$$

Returning to 4.9 and differentiating on both sides, we have:

$$\mathrm{d}V_\phi(t) = \phi^1_t \mathrm{d}b(t) + \phi^2_t \mathrm{d}S_t = \phi^1_t rb(t)\mathrm{d}t + \phi^2_t(rS_t\mathrm{d}t + \sigma S_t \mathrm{d}W^*_t) \quad \text{(Self-financing condition)}$$

$$\mathrm{d}\left(b(t)\mathbb{E}_{\mathbb{Q}}\left[H^*(t)|\mathcal{F}_t\right]\right) = b(t)\mathrm{d}\left(\mathbb{E}_{\mathbb{Q}}\left[H^*(t)|\mathcal{F}_t\right]\right) + \mathbb{E}_{\mathbb{Q}}\left[H^*(t)|\mathcal{F}_t\right]\mathrm{d}b(t) \quad \text{(Itô Formula)}$$

$$= b(t)X(t)\mathrm{d}W^*_t + rb(t)\mathbb{E}_{\mathbb{Q}}\left[H^*(t)|\mathcal{F}_t\right]\mathrm{d}t$$

Equating both sides and rearranging, we obtain:

$$0 = \left[rb(t)\phi^1_t + r\phi^2_t S_t - rb(t)\mathbb{E}_{\mathbb{Q}}\left[H^*(t)|\mathcal{F}_t\right]\right]\mathrm{d}t + \left[\phi^2_t \sigma S_t - b(t)X(t)\right]\mathrm{d}W^*_t$$

By the uniqueness of the expression of an Itô process,

$$rb(t)\phi^1_t + r\phi^2_t S_t - rb(t)\mathbb{E}_{\mathbb{Q}}\left[H^*(t)|\mathcal{F}_t\right] = 0$$

$$\phi^2_t \sigma S_t - b(t)X(t) = 0 \tag{4.11}$$

Isolating ϕ^2_t in the second equality gives:

$$\phi^2_t = \frac{b(t)X(t)}{\sigma S(t)} \tag{4.12}$$

And substituting in the first equality gives:

$$\phi^1_t = \mathbb{E}_{\mathbb{Q}}\left[H^*(t)|\mathcal{F}_t\right] - X(t)/\sigma$$

We have obtained a unique strategy that may attain H - now we need to verify all the conditions in the theorem. Firstly, the following calculation shows that ϕ attains H:

$$V_\phi(t) = \phi_t * (b(t), S_t) = b(t))\mathbb{E}_{\mathbb{Q}}\left[H^*(t)|\mathcal{F}_t\right] - b(t)X(t)/\sigma + b(t)X(t)/\sigma = b(t)\mathbb{E}_{\mathbb{Q}}\left[H^*(t)|\mathcal{F}_t\right]$$

In particular, $V_\phi(T) = H$.

The strategy is admissible because $V^*_\phi(t) = \mathbb{E}_{\mathbb{Q}}\left[H^*|\mathcal{F}_t\right]$ is clearly a martingale.

For the self-financing condition,

$$\mathrm{d}V^*_\phi(t) = X(t)\mathrm{d}W^*_t$$

Note that, from 4.12, we have:

$$X(t) = \sigma\phi_t^2 S_t^*$$

So that:

$$\mathrm{d}V^*_\phi(t) = \sigma\phi_t^2 S_t^* \mathrm{d}W_t^* = \phi_t^2 \mathrm{d}S_t^*$$

□

4.4 Pricing and hedging

We have shown that the Black Scholes model has no arbitrage opportunities and is complete in the sense that every european option that is square integrable with respect to $\mathbb{Q}$ is replicable. We also know that the price of a replicable european option at any time is determined by the conditional expectation of the value of the replicating strategy at the given time, under the risk neutral probability. In this section, we will develop pricing formulas for european options and to obtain the replicating strategy.

Let H be the payoff of a European option with maturity T. We will assume that $H = h(S_T)$ for some function h. For a call option $h(x) = (x - K)^+$ and for a put option $h(x) = (K - x)^+$. Recall that if ϕ is an admissible strategy that replicates H, then the value of the option at time $t < T$ is equal to the value of the strategy ϕ at time t, $V_\phi(t)$ (see Theorem 4.12). In particular, if V_t denotes the value of the option:

$$V_t = e^{rt}\tilde{V}_\phi(t) = e^{rt}\mathbb{E}_\mathbb{Q}[\tilde{V}_\phi(T)|\mathcal{F}_t] = e^{rt}\mathbb{E}_\mathbb{Q}[e^{-rT}H|\mathcal{F}_t] = \mathbb{E}_\mathbb{Q}[e^{-r(T-t)}H|\mathcal{F}_t] \tag{4.13}$$

Note that H should be square-integrable with respect to $\mathbb{Q}$, due to the conditions in Theorem 4.13. In that case, the theorem guarantees the existence of a replicating strategy ϕ, validating the previous argument. Now, if $H = h(S_T)$, with $h : \mathbb{R} \to \mathbb{R}$ being a bounded, measurable function, we can express V_t as a function of S_t and t - the condition that h is bounded will be needed to apply the Markov property. Indeed, following the arguments in Lamberton and Lapeyre 2007 (p. 69, Remark 4.3.3),

$$V_t = \mathbb{E}_\mathbb{Q}[e^{-r(T-t)}H|\mathcal{F}_t] = \mathbb{E}_\mathbb{Q}\left[e^{-r(T-t)}h\left(S_t e^{(r-\sigma^2/2)(T-t)+\sigma(W_T^*-W_t^*)}\right)\Big|\mathcal{F}_t\right]$$

Note that S_t is $\mathcal{F}_t$-measurable, and $W_T^* - W_t^*$ is independent of $\mathcal{F}_t$ under $\mathbb{Q}$. Hence,

$$V_t = \mathbb{E}_\mathbb{Q}\left[e^{-r(T-t)}h\left(S_t e^{(r-\sigma^2/2)(T-t)+\sigma(W_T^*-W_t^*)}\right)\Big|\mathcal{F}_t\right]$$

$$= \mathbb{E}_\mathbb{Q}\left[e^{-r(T-t)}h\left(S_t e^{(r-\sigma^2/2)(T-t)+\sigma(W_T^*-W_t^*)}\right)\Big|S_t\right] \quad \text{(because of the Markov property)}$$

$$= \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}h\left(xe^{(r-\sigma^2/2)(T-t)+\sigma(W_T^*-W_t^*)}\right)\Big|S_t = x\right]_{x=S_t}$$

$$= \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}h\left(xe^{(r-\sigma^2/2)(T-t)+\sigma(W_T^*-W_t^*)}\right)\right]_{x=S_t}$$

Where the last equality is a consequence of the fact that if $W_T^* - W_t^*$ is independent of $\mathcal{F}_t$, then it is independent of S_t. Now, we can express V_t as:

$$V_t = P(t, S_t) \tag{4.14}$$

Where

$$P(t,x) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}h\left(xe^{(r-\sigma^2/2)(T-t)+\sigma(W_T^*-W_t^*)}\right)\right] \tag{4.15}$$

And, since $W_T^* - W_t^*$ is distributed as $\mathcal{N}(0, T-t)$ under $\mathbb{Q}$, the previous expectation is expressed by the following integral:

$$P(t,x) = e^{-r(T-t)}\int_{-\infty}^{\infty} h\left(xe^{(r-\sigma^2/2)(T-t)+\sigma y\sqrt{T-t}}\right)\frac{e^{-y^2/2}}{\sqrt{2\pi}}\mathrm{d}y \tag{4.16}$$

In the case of calls and puts, $P(t,x)$ can be calculated explicitely, giving rise to the Black-Scholes pricing formulas. Firstly, we will need to prove that call and put options are square integrable with respect to $\mathbb{Q}$. The case of a put option with payoff $H = h(S_T) = (K - S_T)^+$, this is clear since the payoff is bounded by K. We admit the following result, which shows that this is also the case for a call option, with payoff $H = h(S_T) = (S_T - K)^+$.

Lemma 4.14. *The payoff $H = h(S_T) = (S_t - K)^+$ of a call option is square integrable with respect to $\mathbb{Q}$.*

Proof. Refer to Capinski and Kopp 2012 (p.55-56, Call options). □

4.4.1 Pricing a put option

We can apply the results obtained in 4.15 to the case of a put option with $h(x) = (K - x)^+$, since the function h is bounded. In this case,

$$P(t,x) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}\left(K - xe^{(r-\sigma^2/2)(T-t)+\sigma(W_T^*-W_t^*)}\right)^+\right]$$

Let $\theta = T - t$ and $z = \frac{W_T^*-W_t^*}{\sqrt{\theta}}$, a standard normal random variable under $\mathbb{Q}$. Then,

$$P(t,x) = \mathbb{E}\left[Ke^{-r\theta} - \left(xe^{\sigma\sqrt{\theta}z-\sigma^2\theta/2}\right)^+\right]$$

Define:

$$d = \frac{\log(K/x) - (r - \sigma^2/2)\theta}{\sigma\sqrt{\theta}} \tag{4.17}$$

Note that $Ke^{-r\theta} - xe^{\sigma\sqrt{\theta}z-\sigma^2\theta/2} \geq 0 \iff z \leq d$. We obtain:

$$F(t,x) = \mathbb{E}\left[\left(Ke^{-r\theta} - xe^{\sigma\sqrt{\theta}z-\sigma^2\theta/2}\right)1_{\{z\leq d\}}\right]$$

$$= \int_{-\infty}^{d} \left(Ke^{-r\theta} - xe^{\sigma\sqrt{\theta}y - \sigma^2\theta/2} \right) \frac{e^{-y^2/2}}{\sqrt{2\pi}} \mathrm{d}y$$

$$= \int_{-\infty}^{d} Ke^{-r\theta} \frac{e^{-y^2/2}}{\sqrt{2\pi}} \mathrm{d}y - \int_{-\infty}^{d} xe^{-\sigma\sqrt{\theta}y - \sigma^2\theta/2} \frac{e^{-y^2/2}}{\sqrt{2\pi}} \mathrm{d}y$$

The value of the first integral is $Ke^{-r\theta}$ times the cumulative distribution function of a standard normal random variable evaluated at d: $N(d)$. For the second integral, the change of variable $t = y + \sigma\sqrt{\theta}$ shows that its value is $xN(d - \sigma\sqrt{\theta})$. Finally,

$$P(t, x) = Ke^{-r(T-t)}N(d) - xN(d - \sigma\sqrt{T-t}) \tag{4.18}$$

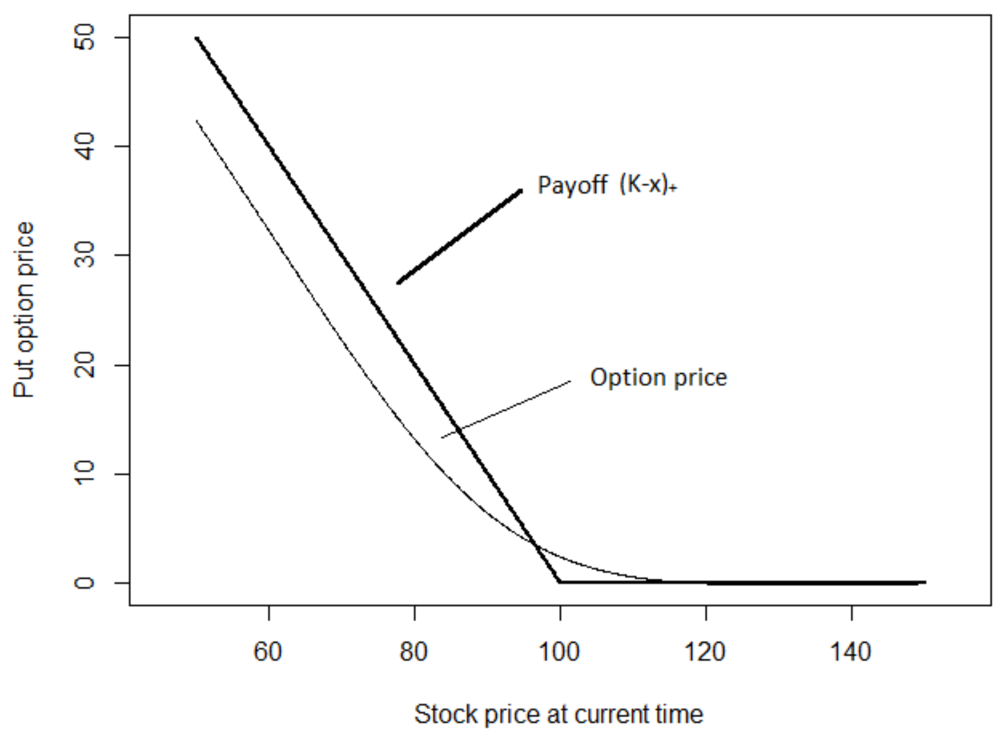

Figure 4.1: Black-Scholes pricing of a put option at time $t = 0$, with $K = 100$, $T = 2$, $r = 0.04$ and $\sigma = 0.1$.

4.4.2 Pricing a call option

A call option has a payoff $H = h(S_T)$, where $h(x) = (x - K)^+$. Because h is not bounded, we can't use the results in 4.15 to obtain a price for the option, as in the case of put options. However, the price will be obtained by means of a relationship between the price of a call option and the price of a put option for a given strike price K, known as the call-put parity. The following theorem has been adapted from Capinski and Kopp 2012 (p.56, Theorem 3.16):

Theorem 4.15 (Call-put parity). *Let C_t, P_t be the price of a call option and a put option, respectively, at time t, both with strike price K and maturity T. Then,*

$$C_t - P_t = S_t - Ke^{-r(T-t)} \tag{4.19}$$

Proof. We follow closely the arguments and notations in Capinski and Kopp 2012 (p.56, Theorem 3.16).

Note that

$$S_T - (S_T - K)^+ + (K - S_T)^+ = K \tag{4.20}$$

This equation can be deduced by separately considering the cases $S_T - K \geq 0$ and $S_T - K \leq 0$. Hence,

$$S_T - C_T + P_T = K \tag{4.21}$$

Multiplying both sides by e^{-rT},

$$S_T^* - e^{-rT} C_T + e^{-rT} P_T = Ke^{-rT} \tag{4.22}$$

Now, (S_t^*), $(e^{-rt} C_t)$, and $(e^{-rt} P_t)$ are $\mathbb{Q}$-martingales (this assertion relies on the fact that call and put options are replicable), so

$$e^{-rT} = \mathbb{E}[e^{-rT} | \mathcal{F}_t] = \mathbb{E}[S_T^* - e^{-rT} C_T + e^{-rT} P_T | \mathcal{F}_t] = S_t^* - e^{-rt} C_t + e^{-rt} P_t \tag{4.23}$$

The result follows by multiplying both sides by e^{rt}.

□

The price of a call option follows from 4.18 and the call-put parity. Indeed,

$$C_t = P_t + S_t - Ke^{-r(T-t)} \quad \text{(by 4.19)}$$

$$= Ke^{-r(T-t)} N(d) - S_t N(d - \sigma\sqrt{T-t}) + S_t - Ke^{-r(T-t)} \quad \text{(by 4.18)}$$

$$= S_t(1 - N(d - \sigma\sqrt{T-t})) - Ke^{-r(T-t)}(1 - N(d))$$

$$= S_t N(-d + \sigma\sqrt{T-t}) - Ke^{-r(T-t)} N(-d) \tag{4.24}$$

In particular, the price of a call option is a function of t and S_t:

$$C_t = C(t, S_t) \tag{4.25}$$

Where

$$C(t, x) = xN(-d + \sigma\sqrt{T-t}) - Ke^{-r(T-t)} N(-d) \tag{4.26}$$

4.4.3 Hedging

Now that the pricing formulas have been justified, we aim to obtain explicit formulas for replicating (or hedging) strategies in the same setting of a European option with an $\mathcal{F}_T$-measurable payoff $H = h(S_T)$ which is square integrable with respect to $\mathbb{Q}$. We follow the arguments in Fouque, Papanicolaou, and Sircar 2000, which begin by recalling that the value of the option is equal to:

$$V_t = P(t, S_t) \tag{4.27}$$

The fact that V_t is a function of t and S_t was proven in 4.15 assuming that h was a bounded function. However, we saw in 4.25 that this is also true for

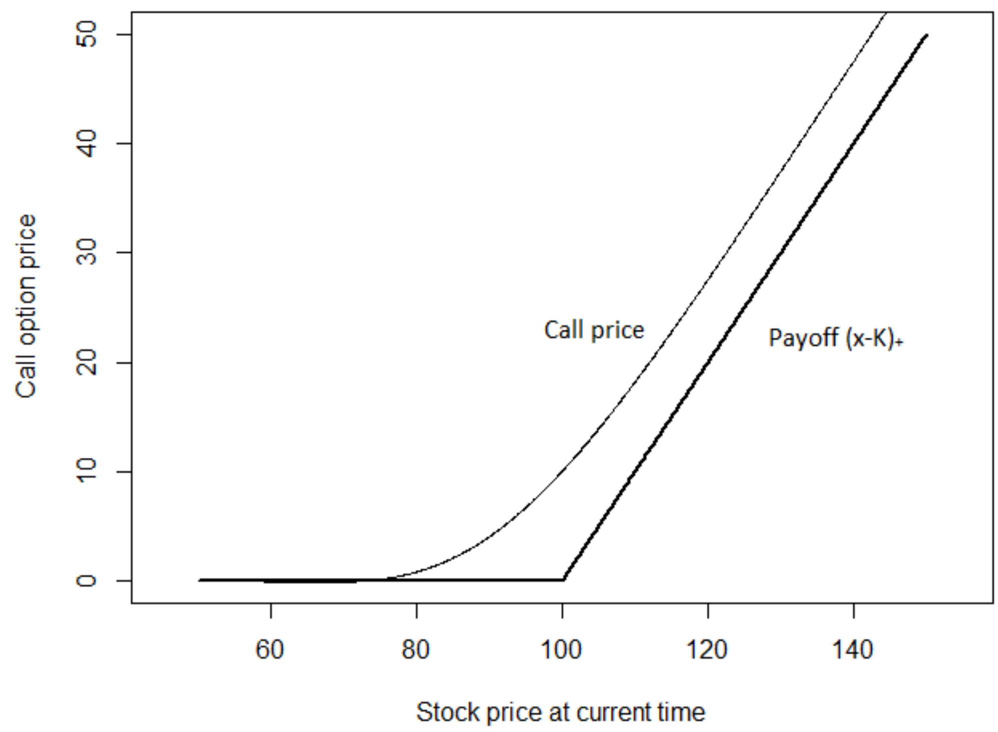

Figure 4.2: Black-Scholes pricing of a call option at time $t = 0$, with $K = 100$, $T = 2$, $r = 0.04$ and $\sigma = 0.1$.

call options. Hence, for the following, we assume that either h is bounded or $h(x) = (x - K)^+$.

Now, let ϕ be a replicating strategy for H. Such a strategy exists by theorem 4.13. By Theorem 4.12, $P(t, S_t)$ is equal to the value of the strategy $V_\phi(t)$:

$$e^{rt}\phi_t^1 + \phi_t^2 S_t = P(t, S_t) \tag{4.28}$$

Differentiating on both sides, and applying the self-financing condition on the left side, we obtain:

$$(re^{rt}\phi_t^1 + \phi_t^2 \mu S_t)\mathrm{d}t + \phi_t^2 \sigma S_t \mathrm{d}W_t = (P_t + \mu S_t P_x + \frac{1}{2}\sigma^2 S_t^2 P_{xx})\mathrm{d}t + \sigma S_t P_x \mathrm{d}W_t \tag{4.29}$$

Where all the partial derivatives of P are evaluated at (t, S_t) - we will use this abbreviation later on without further mention. By the uniqueness of the expression of an Itô process, we conclude that:

$$\phi_t^2 = P_x(t, S_t) \tag{4.30}$$

And from 4.28 we deduce that:

$$\phi_t^1 = e^{-rt}(P(t, S_t) - P_x(t, S_t)S_t) \tag{4.31}$$

The previous argument also shows the relationship between the pricing function $P(t, x)$ and a certain PDE (the Black-Scholes PDE). Indeed, substituting the value of the hedging strategy in 4.29, we obtain the formula:

$$\big(r(P - P_x S_t) + P_x \mu S_t\big)\mathrm{d}t + P_x \sigma S_t \mathrm{d}W_t \tag{4.32}$$

$$= \left(P_t + P_x \mu S_t + \frac{1}{2}\sigma^2 S_t^2 P_{xx}\right)\mathrm{d}t + \sigma S_t P_x \mathrm{d}W_t$$

Equating the drift terms in both expressions, we obtain the equation:

$$P_t - rP + rP_x S_t + \frac{1}{2}\sigma^2 S_t^2 P_{xx} = 0$$

In particular, the drift terms in equation 4.32 will be equal if P satisfies the following PDE:

$$P_t - rP + rxP_x + \frac{1}{2}\sigma^2 x^2 P_{xx} = 0, \quad \forall\, t, x \in \mathbb{R}_+ \tag{4.33}$$

$$P(t, x) = h(x), \quad \forall\, t, x \in \mathbb{R}_+$$

Where all the partial derivatives of P are evaluated at (t, x).

Chapter 5

Stochastic Volatility

5.1 Empirical motivations

In the Black-Scholes model,

$$\frac{S_t}{S_u} = \exp\{\ (\mu - \frac{1}{2}\sigma^2)(t-u) + \sigma(W_t - W_u)\}$$

So that

$$\frac{S_t - S_u}{S_u} \approx log(\frac{S_t}{S_u}) = (\mu - \frac{1}{2}\sigma^2)(t-u) + \sigma(W_t - W_u)$$

Which is distributed as $N\Big(\mu - \frac{1}{2}\sigma^2)(t-u), \sigma(t-u)\Big)$. Hence, the variance rate of the returns is approximately given by the volatility .
In a Stochastic Volatility framework, the stock price is modelled as:

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}W_t \tag{5.1}$$

Where σ_t is now a process instead of a constant. Since the variance rate of returns in the Black-Scholes model is approximately the volatility σ, the stochastic volatility model is indeed a generalization of the Black-Scholes model in which the variance rate of the returns is no longer assumed to be constant, as argued in Chapter 1.

Moreover, as argued in Gatheral 2011, mixing distributions with different variances produces distributions with higher peaks and fatter tails, which means that the modeling of random volatility produces distributions in returns that are better adjusted to the observed stock price distributions (Mandelbrot 1963). The randomness of the volatility parameter also allows us to incorporate more information into the model, such as correlation between the volatility and the stock price, or diffusion properties of the volatility such as mean reversion, which will be introduced in the Heston model. Further arguments and empirical motivations for the Stochastic Volatility models, such as the observation of implied volatilities, are out of the scope of this project, but can be found in detail in Gatheral 2011 and Fouque, Papanicolaou, and Sircar 2000.

5.2 A general approach for pricing

In this section, we follow closely the arguments and notations in Fouque, Papanicolaou, and Sircar 2000 (p. 46, section 2.5). Consider a stochastic volatility model of the form:

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t \tag{5.2}$$

Where (S_t) represents the stock price, $\sigma_t = f(Y_t)$ for some positive function f, and Y_t is an Ornstein-Uhlenbeck process, defined by:

$$dY_t = \alpha(m - Y_t)dt + \beta_t d\hat{Z}_t \tag{5.3}$$

Where α and m are constants with $\alpha m > 0$, and (β_t) is an adapted process. Intuitively, this process pulls toward or reverts to m, known as the long-run mean level of (Y_t), with a velocity α, known as the rate of mean reversion. We assume that the Brownian motion $(\hat{Z}_t)$ satisfies:

$$\hat{Z}_t = \rho W_t + \sqrt{1-\rho^2} Z_t \tag{5.4}$$

Where (Z_t) is a standard Brownian motion independent of (W_t), and $\rho \in [-1, 1]$. This implies that the correlation between $(\hat{Z}_t)$ and (W_t) is ρ. Indeed,

$$cor(\hat{Z}_t, W_t) = \frac{\mathbb{E}\left[\left(\hat{Z}_t - \mathbb{E}[\hat{Z}_t]\right)\left(W_t - \mathbb{E}[W_t]\right)\right]}{sd(\hat{Z}_t)sd(W_t)} = \frac{\mathbb{E}[\hat{Z}_t W_t]}{\sqrt{t}\sqrt{t}} = \frac{\rho t}{t} = \rho \tag{5.5}$$

Thus, this model allows us not only to incorporate randomness into the volatility, but also to specify the correlation between the volatility and the stock price, or *skewness*, as argued in the previous section.

As usual, the filtered probability space is $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, in this case being $\mathbb{F} = (\mathcal{F}_t)$ the filtration generated by the two Brownian motions. More explicitely, $\mathcal{F}_t$ is generated by the sets $\{w \in \Omega | W_s < a, F_s < b\}$ for $s \leq t$ and the $\mathbb{P}$-null sets. In particular, (β_t) is required to be $\mathbb{F}$-adapted.

In the Black-Scholes model, we were able to obtain a pricing formula for European options with a payoff $H = h(S_T)$ which is square integrable with respect to $\mathbb{Q}$ after proving the existence of the unique risk neutral measure $\mathbb{Q}$. In this case, we will first assume the existence of a certain risk neutral measure $\mathbb{Q}$, under which the discounted stock price S_t^* is a martingale. As in the Black-Scholes model, the No Arbitrage Principle implies that the option price V_t is given by:

$$V_t = \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)} H | \mathcal{F}_t]$$

So, for each risk neutral measure $\mathbb{Q}$, we are able to find a reasonable option price V_t. We will now find a family of equivalent risk neutral measures, by means of the multidimensional Girsanov theorem.

Indeed, let $\theta_t := \frac{\mu - r}{\sigma_t}$, and define:

$$W_t^* = W_t + \int_0^t \theta_s ds$$

And for an arbitrary adapted, square integrable process (γ_t), define:

$$Z_t^* = Z_t + \int_0^t \gamma_s ds$$

Let $\mathbb{Q}_\gamma$ be defined by:

$$\frac{\mathrm{d}\mathbb{Q}_\gamma}{\mathrm{d}\mathbb{P}} = exp\left(- \int_0^T \theta_s \mathrm{d}W_s - \int_0^T \gamma_s \mathrm{d}Z_s - \frac{1}{2}\int_0^T (\theta_s^2 + \gamma_s^2)\mathrm{d}s \right) \tag{5.6}$$

Assuming certain regularity conditions for f - for example, that it is bounded away from 0 - and according to Girsanov's theorem in two dimensions, each measure $\mathbb{Q}_\gamma$ is a risk neutral probability, under which W_t^* and Z_t^* are independent Brownian motions. The process (γ_t) is called the *risk premium factor* or the *market price of volatility risk*, and parametrizes the space of risk neutral measures. The Itô formula shows that the stochastic processes (S_t), (Y_t), and $(\hat{Z}_t^*)$ are driven by the following dynamics under the risk neutral measure $\mathbb{Q}_\gamma$:

$$\mathrm{d}S_t = rS_t\mathrm{d}t + f(Y_t)S_t\mathrm{d}W_t^*, \tag{5.7}$$

$$\mathrm{d}Y_t = \left[\alpha(m - Y_t) - \beta_t(\rho\theta_t + \gamma_t\sqrt{1-\rho^2}\right]\mathrm{d}t + \beta_t\mathrm{d}\hat{Z}_t^*, \tag{5.8}$$

$$\hat{Z}_t^* = \rho W_t^* + \left(\sqrt{1-\rho^2}\right)Z_t^* \tag{5.9}$$

Now, assume that $\gamma_t = \gamma(t, S_t, Y_t)$, $\beta_t = \beta(t, S_t, Y_t)$. Then, the option price $V_t = \mathbb{E}_{\mathbb{Q}_\gamma}[e^{-r(T-t)}H|\mathcal{F}_t]$ is also a function of t, S_t and Y_t, because of the Markov property. Indeed,

$$V_t = \mathbb{E}_{\mathbb{Q}_\gamma}\left[e^{-r(T-t)}h\Big(S_T\Big)\Big|\mathcal{F}_t\right]$$

$$= \mathbb{E}_{\mathbb{Q}_\gamma}\left[e^{-r(T-t)}h\Big(S_T\Big)\Big|\mathcal{F}_{S_t,Y_t}\right] \quad \text{(Because of the Markov property)}$$

$$= \mathbb{E}_{\mathbb{Q}_\gamma}\left[e^{-r(T-t)}h\Big(S_T\Big)\Big|\mathcal{S}_t = s, Y_t = y\right]_{s=S_t, y=Y_t}$$

Under these assumptions, we can proceed as in the previous section to show that the function P defined by $V_t = P(t, S_t, Y_t)$ follows a certain PDE. Indeed,

$$\mathrm{d}(e^{-rt}P(t, S_t, Y_t)) = -re^{-rt}P(t, S_t, Y_t)) + e^{-rt}\mathrm{d}P(t, S_t, Y_t)$$

And the two dimensional Itô formula gives:

$$\mathrm{d}\big(e^{-rt}P(t, S_t, Y_t)\big) = \tag{5.10}$$

$$= e^{-rt}\Bigg[P_t - rP + rS_tP_s + \left[\alpha(m - Y_t) - \beta_t\left(\rho\theta_t + \gamma_t\sqrt{1-\rho^2}\right)\right]P_y$$

$$+\frac{1}{2}f(Y_t)^2S_t^2P_{ss} + \frac{1}{2}\beta_t^2P_{yy} + \rho f(Y_t)S_t\beta_tP_{sy}\Bigg]\mathrm{d}t$$

$$+e^{-rt}\Big(f(Y_t)S_tP_s\Big)\mathrm{d}W_t^* + e^{-rt}\Big(\nu_y\beta_tP_y\Big)\mathrm{d}\hat{Z}_t^*$$

Note: we frequently abbreviate $P(t, s, y)$ as P. We will not abbreviate in cases where confusion may arise or to clarify the dependence on t, s, y.

In particular, $e^{-rt}P(t, S_t, Y_t)$ will be a local $\mathbb{Q}_\gamma$-martingale if $P(t, s, y)$ is a solution of the following PDE:

$$P_t - rP + rsP_s + \Big[\alpha(m-y) - \beta(t,s,y)\Big(\rho\frac{\mu - r}{f(y)} + \gamma(t,s,y)\sqrt{1-\rho^2}\Big)\Big]P_y \quad (5.11)$$

$$+\frac{1}{2}f(y)^2 s^2 P_{ss} + \frac{1}{2}\beta(t,s,y)^2 P_{yy} + \rho\beta(t,s,y)sf(y)P_{sy} = 0$$

5.3 Heston's model

Heston's model, as described in Heston 1993, is a particular case of the general model described above, in which $f(y) = \sqrt{y}$ and $\beta(t,s,y) = \sigma\sqrt{y}$. Hence, the dynamics of the processes (S_t) and (Y_t) are give by:

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sqrt{Y_t}S_t \mathrm{d}W_t \quad (5.12)$$

$$\mathrm{d}Y_t = \alpha(m - Y_t)\mathrm{d}t + \sigma\sqrt{Y_t}\mathrm{d}\hat{Z}_t \quad (5.13)$$

Where, as before, $\hat{Z}_t$ is defined as:

$$\hat{Z}_t = \rho W_t + \sqrt{1-\rho^2}Z_t \quad (5.14)$$

Note that 5.12 and 5.13 can be expressed in the form 3.5, with:

$$a(t,s,y) = (\mu s, \alpha(m-y)) \quad (5.15)$$
$$b(t,s,y) = (\sqrt{y}s, \sigma\sqrt{y})$$

In this case, the coefficients are not Lipschitz. However, 5.13 can be rewritten in the form (see Heston 1993):

$$\mathrm{d}\sqrt{Y_t} = -\beta\sqrt{Y_t}\mathrm{d}t + \delta \mathrm{d}\hat{Z}_t \quad (5.16)$$

Indeed, applying Itô's formula one can obtain an expression or Y_t similar to 5.13. In this form, the coefficients of the stochastic differential equation satisfy the regularity conditions in 3.5, so that a unique solution exists and satisfies the Markov property.

In this section, we wish to obtain a pricing function $P(t, S_t, Y_t)$ for a European call option with maturity T and payoff $H = h(S_t) = (S_t - K)^+$, where S_t is modelled according to Heston's model. Steven L. Heston, in his article Heston 1993, derives a closed-form solution for the call option price $C(t,s,y)$ in this model. We will follow closely its arguments to achieve the desired formula.

Inspired by the results obtained in the previous section, we require that $C(t,s,y)$ satisfies the following PDE:

$$C_t - rC + rsC_s + \big(\alpha(m-y) - \lambda(t,s,y)\big)C_y \quad (5.17)$$

$$+\frac{1}{2}ys^2 C_{ss} + \frac{1}{2}\sigma^2 y C_{yy} + \rho\sigma ys C_{sy} = 0$$

Where

$$\lambda(t,s,y) = \sigma\Big(\rho(\mu - r) + \sqrt{y}\gamma(t,s,y)\sqrt{1-\rho^2}\Big) \quad (5.18)$$

And with boundary condition $C(T,s,y) = (s-K)^+$.

The price $C(t,S_t,Y_t)$ can also be expressed as follows:

$$C(t,S_t,Y_t) = \mathbb{E}_{\mathbb{Q}_\gamma}\left[e^{-r(T-t)}\left(S_T - K\right)_+ \middle| \mathcal{F}_t\right] \tag{5.19}$$

$$= \mathbb{E}_{\mathbb{Q}_\gamma}\left[e^{-r(T-t)} S_T 1_{\{S_T \geq K\}} \middle| \mathcal{F}_t\right] - e^{-r(T-t)} K \mathbb{E}_{\mathbb{Q}_\gamma}\left[1_{\{S_T \geq K\}} \middle| \mathcal{F}_t\right] \tag{5.20}$$

$$= S_t \mathbb{E}_{\mathbb{Q}_\gamma}\left[\frac{e^{-r(T-t)} S_T}{S_t} 1_{\{S_T \geq K\}} \middle| \mathcal{F}_t\right] - e^{-r(T-t)} K \mathbb{E}_{\mathbb{Q}_\gamma}\left[1_{\{S_T \geq K\}} \middle| \mathcal{F}_t\right] \tag{5.21}$$

$$= S_t P_1 - K P(t,T) P_2 \tag{5.22}$$

Where $P(t,T) = e^{-r(T-t)}$, $P_1 = \mathbb{E}_{\mathbb{Q}_\gamma}\left[\frac{e^{-r(T-t)} S_T}{S_t} 1_{\{S_T \geq K\}} \middle| \mathcal{F}_t\right]$,
and $P_2 = \mathbb{E}_{\mathbb{Q}_\gamma}\left[1_{\{S_T \geq K\}} \middle| \mathcal{F}_t\right]$.

Now, $P(t,T)P_2$ is the price at time t of an option with payoff $H = 1_{\{S_T \geq K\}}$. Indeed,

$$P(t,T)P_2 = \mathbb{E}_{\mathbb{Q}_\gamma}\left[e^{-r(T-t)} 1_{\{S_T \geq K\}} \middle| \mathcal{F}_t\right] \tag{5.23}$$

Thus, the equation for $P(t,T)P_2$ is exactly the same as 5.17. Consider the change of variables $x = ln(s)$, so that:

$$\frac{\partial}{\partial s} = \frac{\partial}{\partial x}\frac{1}{s},$$

$$\frac{\partial^2}{\partial s^2} = \frac{\partial^2}{\partial x^2}\frac{1}{s^2} - \frac{1}{s^2}\frac{\partial}{\partial x}$$

Which gives the following PDE for P_2:

$$\frac{\partial P_2}{\partial t} + \left(r - \frac{1}{2}y\right)\frac{\partial P_2}{\partial x} + \left(\alpha(m-y) - \lambda y\right)\frac{\partial P_2}{\partial y} + \frac{1}{2}y\frac{\partial^2 P_2}{\partial x^2} \tag{5.24}$$

$$+\frac{1}{2}\sigma^2 y \frac{\partial^2 P_2}{\partial y^2} + \rho\sigma y \frac{\partial^2 P_2}{\partial x \partial y} = 0$$

With boundary condition $P_2(T,x,y) = 1_{\{x \geq log(k)\}}$.

Now, substituting $C(t,s,y)$ in 5.17 and using the previous result, we obtain a PDE for P_1:

$$s\frac{\partial P_1}{\partial t} - rsP_1 + rs\left(\frac{\partial P_1}{\partial x} + P_1\right) + s\left(\alpha(m-y) - \lambda y\right)\frac{\partial P_1}{\partial y}$$

$$\frac{1}{2}ys\left(\frac{\partial^2 P_1}{\partial x^2} + 2\frac{\partial P_1}{\partial x} + P_1\right) - \frac{1}{2}ys\left(\frac{\partial P_1}{\partial x} + P_1\right)$$

$$\frac{1}{2}\sigma^2 ys \frac{\partial^2 P_1}{\partial y^2} + \rho\sigma ys\left(\frac{\partial^2 P_1}{\partial x \partial y} + \frac{\partial P_1}{\partial y}\right) = 0$$

Reordering this expression, we obtain the following PDE for P_1:

$$\frac{\partial P_1}{\partial t} + \left(r + \frac{1}{2}y\right)\frac{\partial P_1}{\partial x} + \left(\alpha(m-y) - \lambda y + \rho\sigma y\right)\frac{\partial P_1}{\partial y} \tag{5.25}$$

$$+\frac{1}{2}y\frac{\partial^2 P_1}{\partial x^2} + \frac{1}{2}\sigma^2 y\frac{\partial^2 P_1}{\partial y^2} + \rho\sigma y\frac{\partial^2 P_1}{\partial x \partial y} = 0$$

With the boundary condition:

$$P_1(T,x,y) = 1_{\{x \geq log(K)\}}$$

We can summarize 5.24 and 5.25 in the expression:

$$\frac{\partial P_j}{\partial t} + \left(r + u_j y\right)\frac{\partial P_j}{\partial x} + \left(a_j - b_j y\right)\frac{\partial P_j}{\partial y} + \frac{1}{2}y\frac{\partial^2 P_j}{\partial x^2} + \frac{1}{2}\sigma^2 y\frac{\partial^2 P_j}{\partial y^2} + \rho\sigma y\frac{\partial^2 P_j}{\partial x \partial y} = 0, \tag{5.26}$$

$$P_j(T,x,y) = 1_{\{x \geq log(K)\}}$$

Where

$$\begin{aligned} u_1 &= \frac{1}{2} \\ u_2 &= -\frac{1}{2} \\ a_j &= \alpha m, \quad j = 1,2 \\ b_1 &= \alpha + \lambda - \rho\sigma \\ b_2 &= \alpha + \lambda \end{aligned}$$

Now, continuing with the same notations, assume that x_j, $j = 1,2$, is a stochastic process that follows the stochastic differential equation:

$$\mathrm{d}x_j(t) = \left(r + u_j y_j(t)\right)\mathrm{d}t + \sqrt{y_j(t)}\mathrm{d}W_t \tag{5.27}$$

$$\mathrm{d}y_j(t) = \left(a_j - b_j y_j(t)\right)\mathrm{d}t + \sigma\sqrt{y_j(t)}\mathrm{d}\hat{Z}_t$$

As argued in Heston 1993, P_j is the conditional probability that the option expires in-the-money:

$$P_j(t,x,y) = \mathbb{P}\Big(x_j(T) \geq log(K) \big| x_j(t) = x, y_j(t) = y\Big) \tag{5.28}$$

To prove this, define

$$f_j(t,x,y) = \mathbb{E}\Big[1_{\{x_j(T) \geq log(K)\}} \big| x_j(t) = x, y_j(t) = y\Big]$$

Assume that $f_j(t) = f_j(t, x_j(t), y_j(t))$ is sufficiently regular to apply the Itô formula. In this case,

$$\begin{aligned} \mathrm{d}f_j(t) = \Big(&\frac{\partial f_j}{\partial t} + \left(r + u_j y_j(t)\right)\frac{\partial f_j}{\partial x} + \left(a_j - b_j y_j(t)\right)\frac{\partial f_j}{\partial y} \\ &+ \frac{1}{2}y_j(t)\frac{\partial^2 f_j}{\partial x^2} + \frac{1}{2}\sigma^2 y_j(t)\frac{\partial^2 f_j}{\partial y^2} + \rho\sigma y_j(t)\frac{\partial^2 f_j}{\partial x \partial y}\Big)\mathrm{d}t \\ &+ \Big(\left(r + u_j y_j(t)\right)\frac{\partial f_j}{\partial x}\Big)\mathrm{d}W_t + \Big(\left(a_j - b_j y_j(t)\right)\frac{\partial f_j}{\partial y}\Big)\mathrm{d}\hat{Z}_t \end{aligned} \tag{5.29}$$

Where all the partial derivatives of f are evaluated at $(t, x_j(t), y_j(t))$.

Now, f_j is a martingale, because due to the tower property of conditional expectations and the Markov property,

$$\mathbb{E}\left[f_j(t)\middle|\mathcal{F}_s\right] = \mathbb{E}\Big[\mathbb{E}\left[1_{\{x_j(T) \geq log(K)\}}\middle|x_j(t) = x, y_j(t) = y\right]_{x = x_j(t), y = y_j(t)}\Big|\mathcal{F}_s\Big]$$

$$\mathbb{E}\Big[\mathbb{E}\big[1_{\{x_j(T)\geq log(K)\}}\big|\mathcal{F}_t\big]\big|\mathcal{F}_s\Big] = f_j(s)$$

Thus, the drift term in 5.29 must vanish. We obtain the PDE:

$$\frac{\partial f_j}{\partial t} + \big(r + u_j y_j(t)\big)\frac{\partial f_j}{\partial x} + \big(a_j - b_j y_j(t)\big)\frac{\partial f_j}{\partial y} \tag{5.30}$$

$$+\frac{1}{2}y_j(t)\frac{\partial^2 f_j}{\partial x^2} + \frac{1}{2}\sigma^2 y_j(t)\frac{\partial^2 f_j}{\partial y^2} + \rho\sigma y_j(t)\frac{\partial^2 f_j}{\partial x \partial y} = 0$$

Clearly, f_j must satisfy the boundary condition

$$f_j(T, x, y) = 1_{\{x \geq log(K)\}}$$

Hence, assuming regularity conditions on f_j,

$$P_j(t, x, y) = \mathbb{E}\Big[1_{\{x_j(T)\geq log(K)\}}\big|x_j(t) = x, y_j(t) = y\Big] \tag{5.31}$$

$$= \mathbb{P}\Big(x_j(T) \geq log(K)\big|x_j(t) = x, y_j(t) = y\Big)$$

Now, applying the Itô formula to $P_j(t, x, y)$ we obtain that it is a martingale because the drift term vanishes. Since P_j and f_j satisfy the same boundary conditions and are both martingales, $P_j = f_j$ and the regularity assumption on f_j is validated.

The probabilities 5.31 are not easily obtained in a closed form. However, consider the characteristic function g_j defined as:

$$g_j(t, x, y; \phi) = \mathbb{E}\Big[e^{i\phi x_j(T)}\big|x_j(t) = x, y_j(t) = y\Big]$$

We admit (see Heston 1993, Appendix p.341) that g_j can be expressed as:

$$g_j(t, x, y; \phi) = e^{C(T-t:\phi)+D(T-t:\phi)y+i\phi x} \tag{5.32}$$

Where

$$C_j(\tau; \phi) = r\phi i\tau + \frac{a}{\sigma^2}\Big((b_j - \rho\sigma\phi i + d_j)\tau - 2log(\frac{1 - c_j e^{d\tau}}{1 - c_j})\Big) \tag{5.33}$$

$$D_j(\tau; \phi) = \frac{b_j - \rho\sigma\phi i + d_j}{\sigma^2}(\frac{1 - e^{d_j\tau}}{1 - ce^{d_j\tau}}) \tag{5.34}$$

and

$$c_j = \frac{b_j - \rho\sigma\phi i + d}{b_j - \rho\sigma\phi i - d} \tag{5.35}$$

$$d_j = \sqrt{(\rho\sigma\phi i - b_j)^2 - \sigma^2(2u_j\phi i - \phi^2)} \tag{5.36}$$

To obtain the desired result, the characteristic functions can be inverted in the following way:

$$P_j(t, x, y) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\Big[\frac{e^{-i\phi log(K)}g_j(t, x, y; \phi)}{i\phi}\Big]\mathrm{d}\phi \tag{5.37}$$

The following are illustrative plots of the call option price in Heston's model, according to the obtained closed-form solution. The underlying code has been obtained from Roberts n.d.

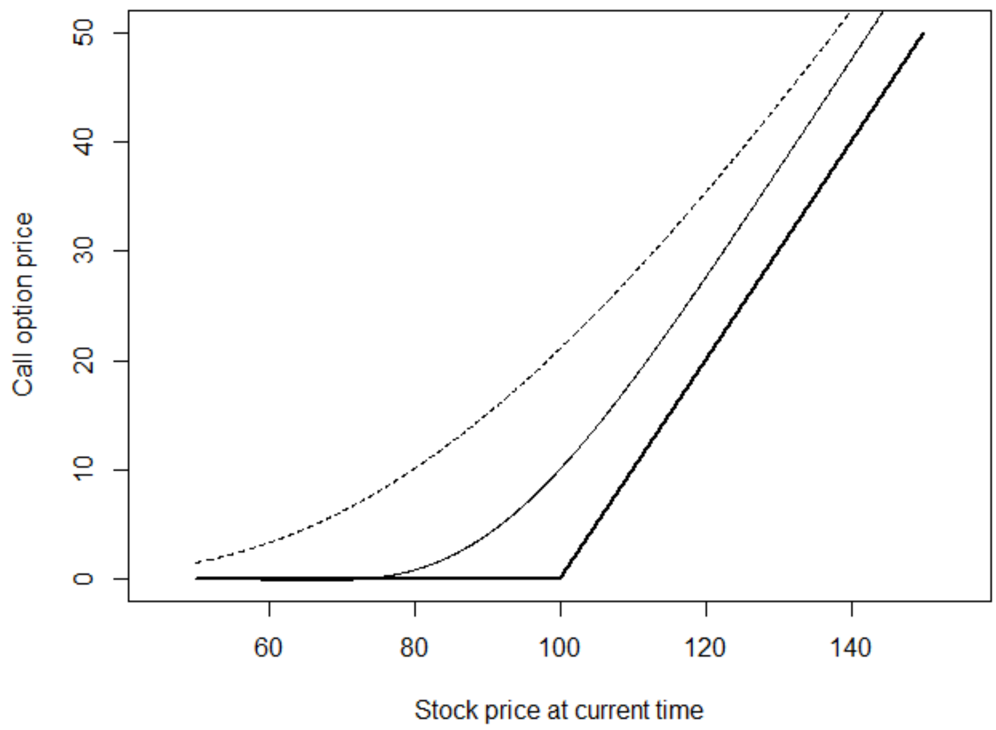

Figure 5.1: Heston pricing of a call option according to the closed-form solution (the dashed line) at time $t = 0$, with $K = 100$, $T = 2$, $r = 0.04$, $\alpha = 0.5$, $\rho = 0.5$, $Y_0 = 0.1$, $m = 0.1$, $\sigma = 0.1$ compared to the Black-Scholes call price (the regular line) at time $t = 0$, with $K = 100$, $T = 2$, $r = 0.04$ and $\sigma = 0.1$, and the payoff function $(x - K)^+$ (the thick line).

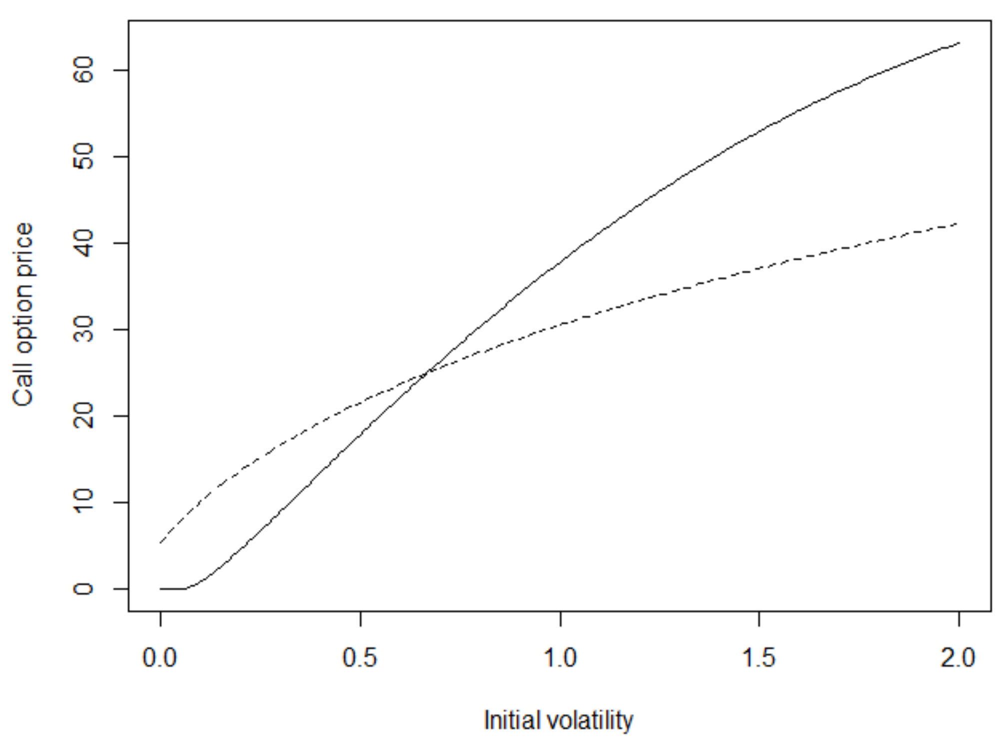

Figure 5.2: Heston pricing of a call option according to the closed-form solution (the dashed line) at time $t = 0$, with $X_0 = 80$, $K = 100$, $T = 2$, $r = 0.04$, $\alpha = 1$, $\rho = 0.5$, $m = 0.1$, $\sigma = 0.1$ compared to the Black-Scholes call price (the regular line) at time $t = 0$, with $K = 100$, $T = 2$, $r = 0.04$ and $S_0 = 80$.

Chapter 6

Conclusions

The Black-Scholes model, which is the benchmark model in continuous-time market models, relies on a series of assumptions that were introduced in Chapter 1: the interest rate is known and constant through time, the distribution of stock prices at the end of any finite interval is lognormal, the stock pays no dividends, the variance rate of the return on the stock is constant and the stock price is continuous over time. In this project we have mathematically formulated the Black-Scholes model, introducing the theory of stochastic integration and the fundamental results on stochastic processes as a necessary background, and taking particular care in examining the underlying assumptions of the model, as well as covering the topics of trading strategies, completeness, and the notion of arbitrage.

The model has been extended to incorporate stochastic volatility, particularly in the setting of Heston's stochastic volatility model. Stochastic volatility models are one of the many extensions of the Black-Scholes model that are frequently employed by practitioners in financial markets. Other models that incorporate jumps, or non-gaussian distributions, among others, are not in the scope of this project but are equally relevant, and can be combined with the stochastic volatility framework adding more precision. We have shown that the stochastic volatility framework not only reduces the assumptions and limitations of the Black-Scholes model, but also allows to capture additional information in the model, such as correlation between the stock price and the volatility, and a mean-reverting structure for the volatility. Moreover, this solution captures some aspects of the empirical distribution of stock prices, such as fat tails and higher peaks, more accurately than the Black-Scholes model. This precision of the stochastic volatility models in capturing stock price distributions results in option valuations that are more reliable than the ones obtained from the Black-Scholes model, as argued in Gatheral 2011. Finally, despite the fact that the incorporation of stochastic volatility adds complexity to the model, we obtained a closed-form solution for pricing options in Heston's model. This closed-form solution reduces the complexity of pricing options within Heston's model and is the cause of the high popularity of this model.

Chapter 7

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