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Algebraic Groups and Tannakian Categories

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Abstract

The main goal of this memoir is to introduce the notion of an algebraic group and study its properties, generalizing many common notions in group theory, such as representations and actions. In addition, we see that there is a duality between affine algebraic groups and what we call Hopf algebras. Afterwards, we see that we can define a category whose objects are finite representations of affine algebraic groups together with the natural homomorphisms between them. This leads us to the necessity of introducing a more general structure for this kind of categories, which we call tannakian categories. Eventually, we apply the results we obtain with these structures to differential Galois theory.

Resum

L'objectiu principal d'aquesta memòria és introduïr la noció de grup algebraic i estudiar les seves propietats, tot generalitzant nocions comuns a teoria de grups, com ara representacions i accions. A més, veiem que hi ha una dualitat entre els grups afins algebraics i el que anomenem àlgebres de Hopf. Despres d'això, veiem que podem definir una categoria que té per objectes les representacions finites de grups afins algebraics juntament amb els homomorfismes naturals entre aquestes. Això ens porta a la necessitat d'introduïr una estructura més general per a aquest tipus de categories, que anomenem categories tannakianes. Finalment, apliquem els resultats que obtenim amb aquestes estructures a la teoria diferencial de Galois.

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Contents

Introduction

All problems in mathematics are psychological!

Pierre Deligne

The present text is a reflection of a series of interests that I have developed during my BSc in Mathematics. During the last two semesters, the theory of categories gained my attention because of its vast range of applications in mathematics, the way it relates several topics in the field and allows to transfer results from one branch to another in what I believe is a very beautiful manner.

At the beginning of this year, Teresa Crespo talked to me about the theory of tannakian categories, which I immediately accepted as the subject of this thesis. At the beginning, I knew few things about category theory so, in order to distance myself from the most basic definitions of this theory, I had to go through several commutative algebra and algebraic geometry books, which allowed me to see what mathematics looks like after this first stage of undergraduate level, and relate some topics that I was already familiar with to category theory. All in all, I am sure that the concepts I have learnt will allow me to pursue not only a good MSc but also what I expect to be a very interesting PhD thesis.

During this memoir we go through several topics such as affine algebraic schemes, algebraic group theory and category theory. The combination of these three eventually leads us to some very interesting results, such as the fundamental theorem of differential Galois theory using tools from category theory, showing how powerful category theory is when mixed properly with different theories, such as the theory of schemes that we mentioned before.

Since there are different ways of approaching algebraic groups, which are a cornerstone in the development of this memoir, I considered that it is important to show the two most common approaches, via theory of schemes and via category theory. More precisely, an algebraic group can be introduced either by taking an algebraic variety together with a “multiplication” map and an identity and inverse map such that the latter make a couple of diagrams commute or by taking a functor from the category of algebras over an algebraically closed field into the category of groups and asking the composition of the previous with the forgetful functor that forgets the group structure into the category of sets to be representable (in the categorical sense) by some algebra over the field. In Chapter 3 we see these definitions with more detail and we see that, in turn, they are both equivalent.

For this reason, Chapter 1 is entirely dedicated to a short yet precise introduction to the theory of Schemes together with some "flashbacks" of category theory. Furthermore, in this same chapter, we see some of the results that come in handy during the memoir and that allow us to develop the rest of the theory.

In chapter 2 we give an introduction to algebraic group theory, giving some basic results and examples that lead us to the construction of some more complex algebraic groups such as the Frobenius map, as an application of how to use the link between algebraic groups and algebraic schemes.

On the other hand, we introduce in Chapter 3 the notion of Hopf algebras, which are the most common way of dealing with affine algebraic groups, thanks to the duality that exists between them. It is worth mentioning that Hopf Algebras are not only of a big interest in the development of algebraic group theory, but also on the developement of other branches of science such as physics, although we do not deepen much into this latter point. Hence, Hopf algebra theory forms an important branch inside not only mathematics but physics and has a wide variety of applications.

Linear representations are just the tip of the iceberg, and they lead us to the proof of the fundamental theorem of differential Galois theory, so that is why we give a short introduction to them in Chapter 4. Along the chapter, we show the relationship between actions and representation theory which follows naturally in the case of standard group theory. We introduce as well the notion of comodules, which gives a bijective correspondence between linear representation of algebraic groups over vector spaces over a field and the comodule structures on these vector spaces. As an application, we prove the Theorem of decomposition of a representation into characters.

By the time we arrive to Chapter 4, the reader should notice that we no longer use the scheme theoretic definition of an algebraic group but instead, we work all the time with the categorical one. For this same reason, in Chapter 5 we introduce Tannakian categories, which is the last step towards the main theorem of this book, the Tannaka-Krein duality theorem, that we introduce in Chapter 6. These last two chapters show the reader the importance of Tannakian Categories, following a very constructive introduction to these, starting with abelian categories and ending up with the definition of a fiber functor, which leads to the definition of a Tannakian category, along with some important examples. To conclude the introduction to this abstract notion, we show the previously mentioned Tannaka-Krein duality theorem, which states that each tannakian category is equivalent to the category of finite dimensional representations of a certain algebraic group.

This last theorem is the key to the proof of the fundamental theorem of differential Galois theory, so in order to conclude the memoir, we apply all that we have learnt in Chapter 7, giving a short introduction to differential algebra and showing the repeatedly mentioned theorem.

In short, this memoir represents a brief introduction to the theory of algebraic groups and tannakian categories. It goes through several branches of mathematics in order to inspire the reader to apply these theories, which are quite beautiful. I have tried to develop the theory in a way that this is a self-contained memoir and also, in order to introduce this theory to students who are currently finishing a BSc in Mathematics.

Chapter 1

Schemes

1.1 Affine algebraic schemes

Throughout this work, k will denote a field and A a finitely generated k-algebra. In fact, we will simply refer to finitely generated k-algebras as k-algebras. Also, we will denote the category of finitely generated k-algebras as $\mathbf{Alg}_k$.

1.1.1 The Zariski topology

Definition 1.1.1.1. Let X be the set of maximal ideals in A and $\mathfrak{a}$ an ideal in A. We will call the set

$$Z(\mathfrak{a}) = \{\mathfrak{m} \in X : \mathfrak{a} \subset \mathfrak{m}\}$$

zero set of $\mathfrak{a}$.

It is important to notice that if our k-algebra A is $k[X_1, \ldots, X_n]$, this definition coincides with the definition of an algebraic set $V(S)$ from algebraic geometry, that is, the set of common zeroes of a family of polynomials S in A, because if we let $\mathfrak{a}$ be the ideal generated by a set of polynomials in A, $S = \{g_i\}_{i \in I}$, then any element $x \in \mathfrak{a}$ can be written in the form

$$x = \sum_{i \in I} f_i g_i,$$

where each $f_i \in k[X_1, \ldots, X_n]$ and therefore, such sum is zero at every point at which the g_i are all zero. Thus, $V(S) \subseteq V(\mathfrak{a})$. The reverse inclusion is clear, hence the algebraic subsets of k^n are the zero sets of ideals in $k[X_1, \ldots, X_n]$.

Also, according to the following proposition, we can endow X with a certain topology.

Proposition 1.1.1.2. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be a pair of ideals and $\{\mathfrak{a}_i\}_{i \in I}$ a family of ideals. Then,*

1. $Z(0) = X$,
2. $Z(A) = \emptyset$,

3. $Z(\mathfrak{a}\mathfrak{b}) = Z(\mathfrak{a} \cap \mathfrak{b}) = Z(\mathfrak{a}) \cup Z(\mathfrak{b})$ *and*

4. $Z(\sum_{i\in I} \mathfrak{a}_i) = \cap_{i\in I} Z(\mathfrak{a}_i)$.

Proof. All the results follow straightforwardly from the definition. □

Clearly, this shows that the sets $Z(\mathfrak{a})$ are the closed sets for a topology on X, which is called the *Zariski Topology* on X. For every $x \in X$, we will denote $\mathfrak{m}_x$ the point x seen as a maximal ideal. Notice that we can define the residue field at point $x \in X$ as $\kappa(x) := A/\mathfrak{m}_x$. Also, we will denote the set X together with this topology as $\mathrm{spm}(A)$ and $\mathbb{A}^n := \mathrm{spm}(k[X_1, \ldots, X_n])$.

Now, let S be a subset of $\mathrm{spm}(A)$ and let $I(S)$ denote the intersection of all the ideals of S, namely

$$I(S) := \bigcap_{\mathfrak{m}\in S} \mathfrak{m}.$$

We refer to $I(S)$ as the *ideal associated to S*. Also, let us remember the definition of the radical of an ideal $\mathfrak{a}$,

$$\mathrm{rad}(\mathfrak{a}) := \{x \in A : x^r \in \mathfrak{a}, \text{for some } r \in \mathbb{N}\},$$

and recall that every prime ideal is also a radical ideal (this can be shown by induction over $r \in \mathbb{N}$, if $x \in A$, then $x^r \in \mathfrak{a}$ implies $x \in \mathfrak{a}$). Let us remember as well the following result from algebraic geometry (see Theorem 4.3 of [Per]).

Theorem 1.1.1.3. *(Hilbert's Strong Nullstellensatz). For every ideal $\mathfrak{a}$ in A,*

$$I(Z(\mathfrak{a})) := \bigcap_{\mathfrak{m}\in Z(\mathfrak{a})} \mathfrak{m} = \bigcap_{\substack{\mathfrak{m}\in X: \\ \mathfrak{a}\subset\mathfrak{m}}} \mathfrak{m} = \mathrm{rad}(\mathfrak{a}).$$

Clearly, if $\mathfrak{a}$ is a radical ideal, $I(Z(\mathfrak{a})) = \mathfrak{a}$.

Notice that every closed subset of X can be written as $Z(\mathfrak{a})$ for some ideal $\mathfrak{a}$ in A. Therefore, via Hilbert's Nullstellensatz, we have that Z and I define a bijective correspondence between the set of closed sets of $\mathrm{spm}(A)$ and the set of radical ideals of A. In fact, since every prime ideal $\mathfrak{p}$ satisfies $\mathrm{rad}(\mathfrak{p}) = \mathfrak{p}$, we have that prime ideals correspond to irreducible sets of $\mathrm{spm}(A)$.

Now that we have studied the closed sets of the Zariski topology, we can move on to the study of the open sets of that same topology. For each $f \in A$, let $D(f) := \{\mathfrak{m} \in X : f \notin \mathfrak{m}\}$. Clearly,

$$D(f) := \{\mathfrak{m} \in X : f \notin \mathfrak{m}\} = X \setminus \{\mathfrak{m} \in X : f \in \mathfrak{m}\} = X \setminus Z(\langle f\rangle)$$

that is, $D(f)$ is an open set because it is the complementary of a closed set. We will call these sets *basic open subsets* of $\mathrm{spm}(A)$. On the other hand, Atiyah and Macdonald show in Theorem 7.5 of [Ati] the following, well-known theorem.

Theorem 1.1.1.4 (Hilbert's Basis Theorem)**.** *If k is a Noetherian ring, then the ring $k[X_1, \ldots, X_n]$ is also a Noetherian ring.*

And thanks to this theorem, we can show the following proposition, because each k-algebra is Noetherian since all fields are noetherian and a quotient of a noetherian ring is noetherian (if R/I is a quotient of a noetherian ring, any ideal $\overline{J} \subset R/I$ is of the form J/I for some ideal $I \subset J \subset R$).

Proposition 1.1.1.5. *The basic open subsets form a basis for* $\operatorname{spm}(A)$.

Proof. Acording to Hilbert's Basis Theorem, A is noetherian, because it is a finitely generated k-algebra. Therefore, $\mathfrak{a}$ is finitely generated, thus there exist $f_1, \dots, f_m \in A$ such that $\mathfrak{a} = \langle f_1, \dots, f_m \rangle$ and

$$X \setminus Z(\mathfrak{a}) = \{\mathfrak{m} \in X : \mathfrak{a} \not\subset \mathfrak{m}\} = \bigcup_{1 \leq i \leq m} \{\mathfrak{m} \in X : f_i \notin \mathfrak{m}\} = \bigcup_{1 \leq i \leq m} D(f_i),$$

so, since any open set of $\operatorname{spm}(A)$ can be written as the union of basic open subsets, the set of basic open subsets forms a basis of $\operatorname{spm}(A)$. □

It is also interesting to see the following proposition, which gives us some useful properties of the basic open subsets.

Proposition 1.1.1.6. *Let f, g be elements of a k-algebra A, and let $\{D(f_i)\}_{i \in I}$ be a family of basic open subsets of* $\operatorname{spm}(A)$.

1. *$\{D(f_i)\}_{i \in I}$ forms an open covering of $\operatorname{spm}(A)$ if and only if 1 can be written as*

$$1 = \sum_{i \in I} a_i f_i,$$

where $a_i \in A$ for each $i \in I$, with only a finite number of non-zero terms. In fact, $\operatorname{spm}(A)$ is quasi-compact.

2. *$D(f) \cap D(g) = D(fg)$.*

3. *Given $f, g \in A$, $D(g) \subseteq D(f)$ if and only if $g \in \operatorname{rad}(\langle f \rangle)$.*

Proof. Since

$$X \setminus Z\left(\sum_{i \in I} \langle f_i \rangle\right) = \bigcup_{i \in I} D(f_i),$$

we have that the open sets form a covering if and only if the ideal generated by all f_i is the whole ring A, namely $1 \in \sum_{i \in I} \langle f_i \rangle$. In fact, since A is a finitely generated k-algebra, this shows that any covering by basic open sets can be reduced to a finite one and, since these basic open subsets of X form a basis for the Zariski topology, any open cover can be refined to one such that its sets are all basic, thus allowing to find a finite covering. This shows the first point.

In order to show the second point, all we have to notice is that, given $\mathfrak{m} \in X$, $f \notin \mathfrak{m}$ and $g \notin \mathfrak{m}$ if and only if $fg \notin \mathfrak{m}$, because $\mathfrak{m}$ is a maximal ideal.

Finally, remember that the fact that A is a finitely generated k-algebra implies that $\operatorname{rad}(\mathfrak{a}) = \cap_{\mathfrak{a} \subseteq \mathfrak{m}} \mathfrak{m}$, because for a general ring, both Jacobson radical and nilradical are equal if the ring is a finitely generated k-algebra. Therefore, $Z(\mathfrak{b}) \subseteq Z(\mathfrak{a})$ implies that

$$\operatorname{rad}(\mathfrak{a}) = \bigcap_{\mathfrak{m}\in Z(\mathfrak{a})} \mathfrak{m} \subseteq \bigcap_{\mathfrak{m}\in Z(\mathfrak{b})} \mathfrak{m} = \operatorname{rad}(\mathfrak{b}),$$

and if $\mathfrak{m} \in Z(\mathfrak{b})$ and $\operatorname{rad}(\mathfrak{a}) \subseteq \operatorname{rad}(\mathfrak{b})$, then $\mathfrak{a} \subseteq \operatorname{rad}(\mathfrak{a}) \subseteq \operatorname{rad}(\mathfrak{b}) \subseteq \mathfrak{m}$ implies $\mathfrak{m} \in Z(\mathfrak{a})$, therefore we have that $Z(\mathfrak{b}) \subseteq Z(\mathfrak{a})$ if and only if $\operatorname{rad}(\mathfrak{a}) \subseteq \operatorname{rad}(\mathfrak{b})$ and, in particular, that $Z(\mathfrak{a}) = Z(\operatorname{rad}(\mathfrak{a}))$, so in order to show the third point, we have that $D(g) \subseteq D(f)$ is true if and only if $Z(\langle f\rangle) \subseteq Z(\langle g\rangle)$, and by the previous observation, this is true if and only if $\langle g\rangle \subseteq \operatorname{rad}(\langle f\rangle)$, that is, if and only if $g \in \operatorname{rad}(\langle f\rangle)$. □

Let us review now the relationship between $\mathbf{Alg}_k$ and the category of topological spaces, **Top**. In order to do so, we must remember a result from algebraic geometry.

Theorem 1.1.1.7 (Zariski's Lemma)**.** *If B is a finitely generated algebra over a field k and B is a field, then B is a finite field extension of k.*

Proof. See [Ati], Exercise 18 of Chapter 5. □

Proposition 1.1.1.8. *The morphism of categories* $\operatorname{spm} : \mathbf{Alg}_k \to \mathbf{Top}$ *given by* $A \mapsto \operatorname{spm}(A)$ *is a contravariant functor.*

Proof. We have to see that for every $A \in Ob(\mathbf{Alg}_k)$, $\operatorname{spm}(A) \in Ob(\mathbf{Top})$, and that for every $\alpha \in \operatorname{Hom}_{\mathbf{Alg}_k}(A, B)$, $\operatorname{spm}(\alpha) =: \alpha^* \in \operatorname{Hom}_{\mathbf{Top}}(\operatorname{spm}(B), \operatorname{spm}(A))$.

The first point is clear by construction, because we can endow each $A \in Ob(\mathbf{Alg}_k)$ with the Zariski topology, obtaining $\operatorname{spm}(A) \in Ob(\mathbf{Alg}_k)$. Secondly, let $\mathfrak{m}$ be a maximal ideal in B. Since B is a finitely generated k-algebra, so is $B/\mathfrak{m}$. Therefore, since $\mathfrak{m}$ is a maximal ideal, $B/\mathfrak{m}$ is a field and, according to Zariski's Lemma, it is a finite field extension of k. Furthermore, the image of A in $B/\mathfrak{m}$ is an integral domain of finite dimension over k, thus it is a field, and that image is isomorphic to $A/\alpha^{-1}(\mathfrak{m})$, so $\alpha^{-1}(\mathfrak{m})$ is a maximal ideal in A. We only have to see that the map

$$\alpha^* : \operatorname{spm}(B) \to \operatorname{spm}(A)$$

that assigns to each $\mathfrak{m}$ in B $\alpha^{-1}(\mathfrak{m})$ is continous, but that is clear, because for every $f \in A$,

$$(\alpha^*)^{-1}(D(f)) = \{\mathfrak{n} \subseteq B : \alpha^*(\mathfrak{n}) \in D(f)\} = \{\mathfrak{n} \subseteq B : \alpha^{-1}(\mathfrak{n}) \in D(f)\}$$

$$= \{\mathfrak{n} \subseteq B : f \notin \alpha^{-1}(\mathfrak{n})\} = \{\mathfrak{n} \subseteq B : \alpha(f) \notin \mathfrak{n}\} = D(\alpha(f)),$$

that is, the preimage of an open set is an open set. Thus, spm is a contravariant functor. □

1.1.2 Sheaves and (locally) ringed spaces

In order to give a precise definition of the notion of sheaf, we will first give the definition of a sheaf of rings, and then show that the definition can be extended to the definition of sheaf of k-algebras, that is, the notion can be easily translated from rings to k-algebras and, in fact, to many other categories.

Definition 1.1.2.1. Let X be a topological space. A presheaf $\mathcal{O}$ of rings on X consists of the data

1. For every open set $U \subset X$, a ring $\mathcal{O}(U)$. The elements of $\mathcal{O}(U)$ are called *sections* over U. Usually, we denote $\mathcal{O}(U)$ as $\Gamma(U, \mathcal{O})$.
2. $\mathcal{O}(\emptyset)$ is the trivial ring.
3. For each pair of open sets U and V of X, such that $U \subseteq V$, a homomorphism of rings $\rho_U^V : \mathcal{O}(V) \to \mathcal{O}(U)$ called *restriction map* with the conditions
   (a) ρ_U^U is the identity map.
   (b) For each triple of open subsets $U \subseteq V \subseteq W$ of X, $\rho_U^V \circ \rho_V^W = \rho_U^W$.

Furthermore, given a presheaf $\mathcal{O}$ on a topological space X, we will say that $\mathcal{O}$ is a *sheaf* if it satisfies the *sheaf axiom*, that is,

4. if $U = \cup_{i \in I} U_i$ is an open covering of an open set U and $\{f_i\}_{i \in I}$ is a set of elements $f_i \in \mathcal{O}(U_i)$ for all $i \in I$ such that $f_i\big|_{U_i \cap U_j} = f_j\big|_{U_i \cap U_j}$, then there exists a unique $f \in \mathcal{O}(U)$ such that $f\big|_{U_i} = f_i$ for all $i \in I$.

A *s*heaf of k-algebras is a sheaf of rings such that moreover $\mathcal{O}(U)$ is a k-algebra for every U and ρ_U^V is a k-algebra homomorphism for each pair U, V.

Example 1.1.2.2. One of the most common examples that we can observe is the case of the category of manifolds of differentiability class $\mathcal{C}^m$, which will be denoted as $\mathbf{Man}^m$. Let $M \in Ob(\mathbf{Man}^m)$. For each open set U of M, if we let $\mathfrak{F}^m(U)$ denote the set of real-valued $\mathcal{C}^m$ functions on U, we have that under point-wise addition and multiplication, $\mathfrak{F}^m(U)$ is a ring. Also, if $V \subseteq U$ are open subsets of M, the restriction homomorphism is

$$\rho_U^V : \mathfrak{F}^m(U) \to \mathfrak{F}^m(V),$$

given by the restriction of functions. The other points of the definition are easily seen and, therefore, $\mathfrak{F}^m$ is a sheaf of rings on M.

It is also interesting to notice that smooth manifolds can be defined in another way, that is, instead of being defined as topological spaces with a certain open cover satisfying a series of conditions, they can be defined as a topological space together with a sheaf satisfying a certain property.

Definition 1.1.2.3. Let X be a topological space and $\mathcal{O}$ a sheaf of rings on X. We call *stalk* of $\mathcal{O}$ at x

$$\mathcal{O}_x := \varinjlim_{\substack{x\in U\subseteq X\\ U\text{open}}} \mathcal{O}(U) = \left(\coprod_{\substack{x\in U\subseteq X\\ U\text{open}}} \mathcal{O}(U) \right) \Big/ \sim,$$

where for $f \in \mathcal{O}(V)$ and $g \in \mathcal{O}(U)$, $f \sim g$ if and only if there exists $W \subseteq U \cap V$ such that $f|_W = g|_W$.

Definition 1.1.2.4. Let $\mathcal{O}$ and $\mathcal{O}'$ be two sheaves on a topological space X. A morphism of sheaves $\varphi : \mathcal{O} \to \mathcal{O}'$ is a collection of maps $\{\varphi(U) : \mathcal{O}(U) \to \mathcal{O}'(U)\}_{\substack{U\subseteq X\\ U\text{open}}}$ such that for every V open in X with $U \subset V$, the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}(V) & \xrightarrow{\varphi(V)} & \mathcal{O}'(V) \\ {\scriptstyle \rho_U^V}\downarrow & & \downarrow{\scriptstyle (\rho')_U^V} \\ \mathcal{O}(U) & \xrightarrow{\varphi(U)} & \mathcal{O}'(U) \end{array}$$

Also, it is important to see how to create a sheaf from an existing one, for that reason, we give the next definition.

Definition 1.1.2.5.

1. Let $\mathcal{O}$ be a sheaf on a space X and let U be an open subset of X. We can define the restriction sheaf $\mathcal{O}|_U$ on U by taking $\mathcal{O}_U(V) := \mathcal{O}(V)$, for any open subset V of U. Clearly, the restriction sheaf is, indeed, a sheaf.

2. Let X and Y be topological spaces, let $\mathcal{O}_X$ be a sheaf on X and also let $f : X \to Y$ be a continuous function. We can define the pushforward sheaf $f_*\mathcal{O}_X$ on Y as the sheaf given by

$$f_*\mathcal{O}_X(U) := \mathcal{O}_X(f^{-1}(U)),$$

for any open set U in Y, together with the necessary restriction maps. The pushforward sheaf is also a sheaf.

Let us take a moment to study the ring of fractions of A, $S^{-1}A$, where S is a multiplicative set. Firstly, we have that if $S_f := \{1, f, f^2, \dots\}$, clearly

$$A_f := S_f^{-1}A \simeq \frac{A[T]}{\langle 1 - fT\rangle} \simeq A\left[\frac{1}{f}\right].$$

On the other hand, if D is an open subset of X, we can define

$$S_D := A \setminus \bigcup_{\mathfrak{m}\in D} \mathfrak{m},$$

which is a multiplicative set because if $st \notin S_D$, then there exists $\mathfrak{m} \in D$ such that $st \in \mathfrak{m}$. That implies $s \in \mathfrak{m}$ or $t \in \mathfrak{m}$, which is equivalent to $s \notin S_D$ or $t \notin S_D$. In fact, the inclusion $S_f \subset S_{D(f)}$ gives a map

$$S_f^{-1}A \to S_{D(f)}^{-1}A$$

that is an isomorphism. This will be of great use when we endow $\mathrm{spm}(A)$ together with a sheaf of k-algebras. Finally, if $D' \subset D$, $S_{D'} \supset S_D$, thus there exists a canonical map $S_D^{-1}A \to S_{D'}^{-1}A$. Hence, let us see how can we apply all these observations to $\mathrm{spm}(A)$. All we have to do is show the following proposition.

Proposition 1.1.2.6. *There exists a unique sheaf $\mathcal{O}_X$ of k-algebras on $X = \mathrm{spm}(A)$ such that for every basic open subset D of X, $\mathcal{O}_X(D) = S_D^{-1}A$. We will refer to* $\mathrm{spm}(A)$ *together with that sheaf of k-algebras as* $\mathrm{Spm}(A)$*. Also, for any $x \in X$, we have that $\mathcal{O}_{X,x} = A_{\mathfrak{m}_x}$ and $\kappa_X(x) = A_{\mathfrak{m}_x}/\mathfrak{m}_x A_{\mathfrak{m}_x}$.*

Proof. All we have to do is check that the $\mathcal{O}_X$ given by $\mathcal{O}_X(D) := S_D^{-1}A$ is a sheaf of k-algebras. The first two points of the definition can be verified easily, because the ring of fractions of a k-algebra is a ring. For the third point, we know from the observation preceding this proposition that for two given open subsets $D \subset D'$ of X, there exists a canonical homomorphism of rings

$$\rho_D^{D'} : S_D^{-1}A \to S_{D'}^{-1}A$$

and from that follows the transitivity. The fourth point is clear as well. The point regarding the stalks and residue fields follows from the existence of that sheaf of k-algebras, because for each open subset, $D = D(f)$, $S_D^{-1}A \simeq A_f$, hence

$$\mathcal{O}_{X,x} := \varinjlim_{\substack{x \in U \subseteq X \\ U \text{open}}} \mathcal{O}_X(U) = \varinjlim_{\substack{x \in U \subseteq X \\ U \text{open}}} S_U^{-1}A = A_x,$$

and, thanks to Corollary 7.10 of [Ati], we have that $\kappa_X(x) = A_{\mathfrak{m}_x}/\mathfrak{m}_x A_{\mathfrak{m}_x}$. □

Notice that the sheaf from 1.1.2.6 has a very good property, and that is that its stalks at each point $x \in X$ are local rings. We refer to it as the *structure sheaf.*

Definition 1.1.2.7. A *k-ringed space* is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of k-algebras on X. A *morphism* between k-ringed spaces $\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(\phi^*, \phi^\sharp)$ such that $\phi^* : X \to Y$ is a continous map and $\phi^\sharp : \mathcal{O}_Y \to \phi^*_* \mathcal{O}_X$ is a morphism of sheaves.

So we can finally give the definition of affine algebraic scheme over a field k.

Definition 1.1.2.8. An *affine algebraic scheme* over k is a k-ringed space $(X, \mathcal{O}_X)$ isomorphic to $\mathrm{Spm}(A)$ for some k-algebra A, in the sense that there exists a bijective morphism between both k-ringed spaces. We call *morphism* (or *regular map*) of affine algebraic schemes over k any morphism of k-ringed spaces.

Now, let us remember that given two categories $\mathfrak{C}, \mathfrak{D}$ and two functors F and G in $\mathbf{Func}(\mathfrak{C}, \mathfrak{D})$, a *natural transformation* η or *morphism* from F to G is a family of morphisms that satisfy

1. $\forall X \in Ob(\mathfrak{C})$, there exists a morphism $\eta_X : F(X) \to G(X)$ between objects of $\mathfrak{D}$, which is called *component* of η at X and
2. For each $f \in \mathrm{Hom}_{\mathfrak{C}}(X, Y)$, the component morphisms make the following diagram commute

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_Y} \\ G(X) & \xrightarrow{G(f)} & G(Y). \end{array}$$

In case F and G are contravariant, the horizontal arrows in the previous diagram are reversed, and we say that F and G are (naturally) isomorphic if there exists a natural isomorphism from F to G, that is $\forall X \in Ob(\mathfrak{C})$, η_X is an isomorphism in $\mathfrak{D}$. We will denote the set of natural transformations from F to G as $\mathrm{Nat}(F, G)$.

Remember as well that $F : \mathfrak{C} \to \mathfrak{D}$ is a (contravariant) *equivalence of categories* if there exists a (contravariant) functor $G : \mathfrak{D} \to \mathfrak{C}$ such that FG and GF are naturally isomorphic to the identity functors $\mathrm{id}_{\mathfrak{D}}$ and $\mathrm{id}_{\mathfrak{C}}$ respectively. G is called a *quasi-inverse* of F.

The following result will allow us to develop the upcoming theory, as it will give us the relationship between affine algebraic schemes and k-algebras. In turn, we see in Chapter 3 that if we endow affine algebraic schemes with the structure of a "group", we have a strong relation between such structures and Hopf algebras.

Proposition 1.1.2.9. *The morphism* $\mathrm{Spm} : \mathbf{Alg}_k \to \mathbf{AffSch}_k$*, is a contravariant equivalence from* $\mathbf{Alg}_k$ *to the category of affine algebraic schemes over* k*,* $\mathbf{AffSch}_k$ *with quasi-inverse* $\mathbf{AffSch}_k \to \mathbf{Alg}_k$ *given by*

$$(X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X).$$

In fact, we have that, for any $A, B \in Ob(\mathbf{Alg}_k)$*,*

$$\mathrm{Hom}_{\mathbf{Alg}_k}(A, B) \simeq \mathrm{Hom}_{\mathbf{AffSch}_k}(\mathrm{Spm}(B), \mathrm{Spm}(A)).$$

Proof. First, we must see that Spm is a contravariant functor. Clearly, given $A \in Ob(\mathbf{Alg}_k)$ and considering the definition of $\mathrm{Spm}(A)$, we have $\mathrm{Spm}(A) \in Ob(\mathbf{AffSch}_k)$. Secondly, we know that given $\alpha \in \mathrm{Hom}_{\mathbf{Alg}_k}(A, B)$, there exists a continous morphism of topological spaces $\bar{\alpha}^* \in \mathrm{Hom}(\mathrm{spm}(B), \mathrm{spm}(A))$ (cf. prop. 1.1.4). Our goal is to find a regular map of affine schemes $\bar{\alpha} : \mathrm{Spm}(B) \to \mathrm{Spm}(A)$ such that its underlying topological homomorphism is $|\bar{\alpha}| = \bar{\alpha}^*$. We showed in prop. 1.1.5 that given a k-algebra C, there exists a unique sheaf such that for any basic open set D of $\mathrm{spm}(C)$,

$\Gamma(U, \mathcal{O}_{\text{spm}(C)}) = S_D^{-1}C$. Therefore, in order to define $\bar{\alpha}$, it suffices to do it for the sections over basic open sets, because it verifies the sheaf axiom and therefore by a simple patching argument the result follows. Remember that $(\bar{\alpha}^*)^{-1}(D(f)) = D(\bar{\alpha}(f))$, hence we have that

$$\Gamma(D(f), \bar{\alpha}^*_*\mathcal{O}_{\text{spm}(B)}) = \Gamma((\bar{\alpha}^*)^{-1}D(f), \mathcal{O}_{\text{spm}(B)}) = \Gamma(D(\bar{\alpha}(f)), \mathcal{O}_{\text{spm}(B)}) = S^{-1}_{D(\bar{\alpha}(f))}B$$

thus the morphism $\alpha : A \to B$ induces the morphism

$$\bar{\alpha}^\sharp(D(f)) : \Gamma(D(f), \mathcal{O}_{\text{Spm}(A)}) = S_f^{-1}A \to S^{-1}_{D(\bar{\alpha}(f))}B = \Gamma(D(f), \bar{\alpha}^*_*\mathcal{O}_{\text{Spm}(B)}),$$

thus we have defined

$$\bar{\alpha}^\sharp : \mathcal{O}_{\text{spm}(A)} \to \bar{\alpha}^*_*\mathcal{O}_{\text{spm}(B)}$$

Now, since $\Gamma(\text{spm}(A), \mathcal{O}_{\text{spm}(A)}) = A$, we have that if we take $\alpha \in \text{Hom}_{\mathbf{Alg}_k}(A, B)$, there is the induced map $\bar{\alpha}^\sharp : \mathcal{O}_{\text{spm}(A)} \to \bar{\alpha}^*_*\mathcal{O}_{\text{spm}(B)}$ that, by taking sections, satisfies

$$A = \Gamma(\text{spm}(A), \mathcal{O}_{\text{spm}(A)}) \to \Gamma(\text{spm}(A), \bar{\alpha}^*_*\mathcal{O}_{\text{spm}(B)}) = B$$

which is the same as α, therefore we have it. □

Definition 1.1.2.10. Given an affine scheme X over k and A a k-algebra,

$$X(A) := \text{Hom}_{\mathbf{AffSch}_k}(\text{Spm}(A), X),$$

and any homomorphism of $X(A)$ is called an A-point.

1.2 Algebraic Schemes

Let $(X, \mathcal{O}_X)$ be a k-ringed space. An open subset U of X is said to be *affine* if $(U, \mathcal{O}_X|_U)$ is an affine algebraic scheme over k. We sometimes refer to $(U, \mathcal{O}_X|_U)$ as an *open subscheme* as well. Hence, let us give the following definition.

Definition 1.2.0.1. An *algebraic scheme* over k or *algebraic k-scheme* is a k-ringed space $(X, \mathcal{O}_X)$ that admits a finite covering by affine open subsets. A *morphism* of algebraic schemes or *regular map* over k, is a morphism of k-ringed spaces. If $(X, \mathcal{O}_X)$ is an algebraic k-scheme, $\Gamma(X, \mathcal{O}_X)$ is called the *coordinate ring* of X. Also, an open subscheme of X is a pair $(U, \mathcal{O}_X|_U)$. We will denote the category of algebraic schemes as $\mathbf{AlgSch}_k$.

From now on, whenever there is no possible confusion with the notation, we will denote X the algebraic scheme $(X, \mathcal{O}_X)$, and $|X|$ the underlying topological space of X. Furthermore, if the base field k is known, we will simply write algebraic scheme instead of algebraic scheme over k.

Also we say that a regular map $Y \to X$ of algebraic schemes is surjective, injective, open and/or closed whenever the underlying map of topological spaces $|Y| \to |X|$ is surjective, injective, open and/or closed respectively.

Proposition 1.2.0.2. *Let X be an algebraic scheme over k. Then $|X|$ is a noetherian topological space.*

Proof. In the affine case, since A is a finitely generated k-algebra, $\mathrm{spm}(A)$ is noetherian. For the general case, let us write $X = \cup_{i=1}^{n} U_i$, and let $V_j \subseteq V_{j+1}$ be an ascending chain of open subsets of X. Clearly, $V_j = \cup_{i=1}^{n}(U_i \cap V_j)$, and so the $U_i \cap V_j$ forms an ascendent chain of open subsets of U_i, which is noetherian, and therefore, the chain given by the V_j is stationary. □

Remark 1.2.0.3. Let K be a field containing the base field k. Clearly, there exists a functor from $\mathbf{AlgSch}_k \to \mathbf{AlgSch}_K$ that sends any algebraic k-scheme to an algebraic K-scheme. In fact, if $X = \mathrm{Spm}(A)$ for a certain k-algebra A, $X_K = \mathrm{Spm}(K \otimes_k A)$, because the tensorial product turns A into a K-algebra $K \otimes_k A$ if we also endow it with the product by elements of K in the first component.

Notice that given X an algebraic scheme over k and A a k-algebra, Proposition 1.1.2.9 gives us an isomorphism

$$\mathrm{Hom}_{\mathbf{AlgSch}_k}(X, \mathrm{Spm}(A)) \simeq \mathrm{Hom}_{\mathbf{Alg}_k}(A, \mathcal{O}_X(X)).$$

Therefore, it would be interesting to study some of the functorial properties of algebraic schemes. In order to do so, let h^A denote the functor

$$\begin{aligned} h^A\colon \mathbf{Alg}_k &\to \mathbf{Set} \\ R &\longmapsto \mathrm{Hom}_{\mathbf{Alg}_k}(A, R). \end{aligned}$$

We will say that a functor $F \in \mathbf{Func}(\mathbf{Alg}_k, \mathbf{Set})$ is *representable* if it is isomorphic to h^A for some k-algebra A.

We say that for $a \in F(A)$, the pair (A, a) represents F if the natural transformation

$$\begin{aligned} T_a\colon h^A &\to F \\ f &\longmapsto F(f)(a). \end{aligned}$$

is an isomorphism. In that case, we will say that a is *universal.*

Recall now that given a functor $F : \mathfrak{C} \to \mathfrak{D}$, where $\mathfrak{C}$ and $\mathfrak{D}$ are categories, if we define for each $X, Y \in Ob(\mathfrak{C})$ a function

$$F_{X,Y} : \mathrm{Hom}_{\mathfrak{C}}(X, Y) \to \mathrm{Hom}_{\mathfrak{D}}(F(X), F(Y)),$$

we will say that the functor F is

1. Faithful if $F_{X,Y}$ is injective,
2. Full if $F_{X,Y}$ is surjective,
3. Fully faithful if $F_{X,Y}$ is bijective for all $X, Y \in Ob(\mathfrak{C})$ that is, if it is full and faithful.

for each $X, Y \in Ob(\mathfrak{C})$.

Now let us remember Yoneda's lemma. Its proof can be found in [Ma], pages 59-62.

Theorem 1.2.0.4 (Yoneda's Lemma)**.** *If $\mathfrak{C}$ is a category, then the functor*

$$h\colon \mathfrak{C} \to \mathbf{Func}(\mathfrak{C}, \mathbf{Set})$$
$$A \longmapsto h^A.$$

where $h^A(S) := \mathrm{Hom}_{\mathfrak{C}}(A, S)$ is fully faithful, so

$$\mathrm{Hom}_{\mathfrak{C}}(A, B) \simeq \mathrm{Hom}_{\mathbf{Func}(\mathfrak{C},\mathbf{Set})}(h^B, h^A).$$

In fact, if we let $F : \mathfrak{C} \to \mathbf{Set}$ be a functor and $X \in Ob(\mathfrak{C})$, there exists a bijection

$$\mathrm{Hom}_{\mathbf{Func}(\mathfrak{C},\mathbf{Set})}(h^X, F) \stackrel{\sim}{\longleftrightarrow} F(X)$$

given by $\alpha \mapsto \alpha_X(\mathrm{id}_X)$.

This means that for each $B \in Ob(\mathbf{Alg}_k)$, and each functor $F : \mathbf{Alg}_k \to \mathbf{Set}$, there exists $x \in F(B)$ that defines an homomorphism $\mathrm{Hom}(B, R) \to F(R)$, sending $f \in \mathrm{Hom}(B, R)$ to $F(f)(x)$. Also, this homomorphism is natural in R, so there is a map of sets

$$F(B) \to \mathrm{Nat}(h^B, F).$$

Eventually, this is a bijection. Thus, for $F := h^A$,

$$\mathrm{Hom}(A, B) \simeq \mathrm{Nat}(h^B, h^A),$$

so the contravariant functor that sends A to h^A is fully faithful. Let now h_X be the functor $h_X := \mathrm{Hom}_{\mathbf{AlgSch}_k}(-, X)$ from algebraic schemes over k to sets. Yoneda's lemma states that given two algebraic schemes X and Y,

$$\mathrm{Hom}(X, Y) \simeq \mathrm{Nat}(h_X, h_Y).$$

If we let h_X^{aff} denote the functor

$$h_X^{\mathrm{aff}}\colon \mathbf{Alg}_k \to \mathbf{Set}$$
$$A \longmapsto X(A),$$

we have that $h_X^{\mathrm{aff}} = h_X \circ \mathrm{Spm}$, and therefore we can regard it as the restriction of h_X to affine algebraic schemes. Also, since every natural transformation $h_X^{\mathrm{aff}} \to h_Y^{\mathrm{aff}}$ extends uniquely to a natural transformation $h_X \to h_Y$, we have that

$$\mathrm{Nat}(h_X^{\mathrm{aff}}, h_Y^{\mathrm{aff}}) \simeq \mathrm{Nat}(h_X, h_Y) \simeq \mathrm{Hom}(X, Y).$$

The latter isomorphisms show that the functor

$$F\colon \mathbf{AlgSch}_k \to \mathbf{Func}(\mathbf{Alg}_k, \mathbf{Set})$$
$$X \longmapsto h_X^{\mathrm{aff}}$$

is fully faithful, because for each $X, Y \in Ob(\mathbf{AlgSch}_k)$, $F_{X,Y}$ is an isomorphism.

Remark 1.2.0.5. Finally, fix a family $(T_i)_{i\in\mathbb{N}}$, and let $\mathbf{Alg}_k^0$ denote the full subcategory of $\mathbf{Alg}_k$ of objects of the form $k[T_0,\dots,T_n]/\mathfrak{a}$ for some $n\in\mathbb{N}$ and an ideal $\mathfrak{a}$ in $k[T_0,\dots,T_n]$. The inclusion $\mathbf{Alg}_k^0 \hookrightarrow \mathbf{Alg}_k$ is an equivalence of categories, but the objects of $\mathbf{Alg}_k^0$ form a set, and so the set-valued functors on $\mathbf{Alg}_k^0$ form a category. We will call the objects of $\mathbf{Alg}_k^0$ *small* k-algebras. Also, we let $\tilde{X}$ be the functor $\mathbf{Alg}_k^0 \to \mathbf{Set}$ defined by an algebraic scheme. Thanks to Yoneda's lemma, the functor that sends $X \mapsto \tilde{X}$ is fully faithful. Notice that if a functor $F : \mathbf{Alg}_k^0 \to \mathbf{Set}$ is representable by an algebraic scheme X, then X is uniquely determined up to a unique isomorphism, and X extends F to a functor from $\mathbf{Alg}_k$ to $\mathbf{Set}$.

To finish this section, it would be interesting to see an example of an algebraic scheme that is not affine.

Example 1.2.0.6. Let $X = \mathbb{A}_k^2 \setminus \{0\}$. On the one hand, notice that since $\mathbb{A}_k^2 = \mathrm{Spm}(k[X_1,X_2])$, and thanks to 1.1.2.9 we have that the restriction map

$$\Gamma(\mathbb{A}_k^2, \mathcal{O}_{\mathbb{A}_k^2}) = k[X_1,X_2] \to \Gamma(X,\mathcal{O}_X)$$

is bijective. If we take an element of $\Gamma(X,\mathcal{O}_X)$, it is represented by a pair $(f,g) \in \mathcal{O}_X(D(X_1)) \times \mathcal{O}_X(D(X_2)) = k[X_1,X_2]_{X_1} \times k[X_1,X_2]_{X_2}$, where f and g coincide over $D(X_1)\cap D(X_2) = D(X_1X_2)$. By a simple calculation, the latter implies that $f = g$ belongs to $k[X_1,X_2]$. Hence, $\Gamma(X,\mathcal{O}_X) = k[X_1,X_2]$. Therefore, if X was an affine algebraic scheme, there would be a canonical isomorphism

$$X \to \mathrm{Spm}(\Gamma(X,\mathcal{O}_X)) \simeq \mathrm{Spm}(k[X_1,X_2]) = \mathbb{A}_k^2,$$

which is false because $0 \notin X$.

1.2.1 Coherent sheaves and algebraic subschemes

Definition 1.2.1.1. Let $(X,\mathcal{O}_X)$ be a ringed space. A sheaf of $\mathcal{O}_X$-modules or $\mathcal{O}_X$-module is a sheaf $\mathcal{F}$ on X such that

1. For each open set U of X, the abelian group $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module and
2. For each inclusion morphism of open sets $V \subseteq U$, the restriction morphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structures via the ring homomorphism $\rho_U^V : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$.

Also, a *morphism* $\mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules is a morphism of sheaves such that for each open set U of X, the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules. If $\mathcal{F}(U)$ is an ideal in $\mathcal{O}_X(U)$ for all open sets $U \subset X$, we say that $\mathcal{F}$ is a *sheaf of ideals* on X.

One of the most common examples of $\mathcal{O}_X$-modules is the following. Take a morphism of schemes $\phi : X \to Y$, and define

$$\mathcal{J}(U) := \ker(\mathcal{O}_Y(U) \to \phi_*\mathcal{O}_X(U)),$$

for each open subset U of X. Clearly, $\mathcal{J}$ is a sheaf of ideals.

Proposition 1.2.1.2. *Let $X = \mathrm{Spm}(A)$ be an affine scheme and let M be an A-module. There is a unique $\mathcal{O}_X$-module $\tilde{M}$ satisfying*

1. $\tilde{M}(D(f)) = M \otimes_A A_f, \forall f \in A.$
2. *For each $x \in X$, the stalk $\tilde{M}_{\mathfrak{m}_x}$ is isomorphic to the localized module $M_{\mathfrak{m}_x}$.*

Proof. It can be found in [Har], Proposition 5.1 of Chapter II. □

Definition 1.2.1.3. Let X be an algebraic scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ on X is a *quasi-coherent sheaf* if there is an open affine cover $\{U_i = \mathrm{Spm}(A_i)\}_{i\in I}$ of X such that for each $i \in I$ the restriction $\mathcal{F}|_{U_i}$ is isomorphic to an $\mathcal{O}_{U_i}$-module of the form $\tilde{M}_i$, where M_i is an A_i-module. If the previous conditions are given and each M_i is a finitely generated A_i-module, we say that $\mathcal{F}$ is a *coherent sheaf.* Furthermore, $\mathcal{F}$ is *locally free* if the M_i are free A_i-modules.

It is worth mentioning that in Corollary 5.5, Chapter II of [Har], Hartshorne shows that in the affine case, the functor $M \mapsto \tilde{M}$ gives an equivalence of categories between the category of A-modules, $\mathbf{Mod}_A$, and the category of quasi-coherent $\mathcal{O}_X$-modules, that is commonly denoted as $\mathbf{QCoh}(\mathcal{O}_X)$. Finally we can give the definition of a subscheme.

Definition 1.2.1.4. Let X be an algebraic k-scheme and let $\mathcal{J}$ be a coherent sheaf of ideals in $\mathcal{O}_X$. Let Z be the support of the sheaf $\mathcal{O}_X/\mathcal{J}$. We call the algebraic scheme $(Z, (\mathcal{O}_X/\mathcal{J})|_Z)$ *closed subscheme* of X defined by $\mathcal{J}$. Also, a *subscheme* of an algebraic scheme X is a closed subscheme of an open subscheme of X.

Hartshorne shows in Proposition 5.9 of Chapter II in [Har] that Z is a closed subset of X and in Corollary 5.10 that $Z \cap U$ is affine for every open affine U of X.

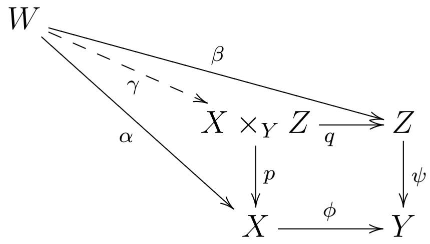
1.3 Algebraic Varieties

First let us give sense to the concept of product of schemes. In order to do so, we must remember first the following definition from category theory.

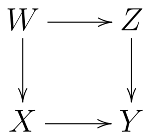
Definition 1.3.0.1. Let $X, Y, Z \in Ob(\mathfrak{C})$, $\phi \in \mathrm{Hom}_{\mathfrak{C}}(X, Y)$ and $\psi \in \mathrm{Hom}_{\mathfrak{C}}(Z, Y)$. A *fibre product* of ϕ and ψ is an object $X \times_Y Z \in Ob(\mathfrak{C})$ together with morphisms $p \in \mathrm{Hom}_{\mathfrak{C}}(X \times_Y Z, X)$ and $q \in \mathrm{Hom}_{\mathfrak{C}}(X \times_Y Z, Z)$ called *projection morphisms* such that the following diagram

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{q} & Z \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \psi} \\ X & \xrightarrow{\phi} & Y \end{array}$$

commutes, and such that the following universal property holds: for any $W \in Ob(\mathfrak{C})$ and morphisms $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(W, X)$ and $\beta \in \mathrm{Hom}_{\mathfrak{C}}(W, Z)$ with $\phi \circ \alpha = \psi \circ \beta$ there is a unique $\gamma \in \mathrm{Hom}_{\mathfrak{C}}(W, X \times_Y Z)$ making the diagram



commute. Also, we say that a commutative diagram



in a category $\mathfrak{C}$ is *cartesian* if W and the homomorphisms $W \to X$ and $W \to Z$ form a fibre product of the morphisms $X \to Y$ and $Z \to Y$.

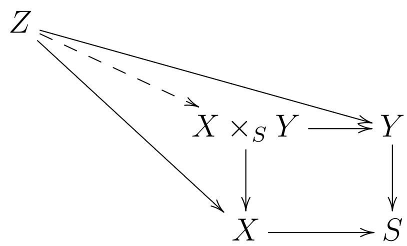
Theorem 1.3.0.2. *Fibre products exist in the category of algebraic k-schemes, and they are unique up to isomorphism.*

Proof. Let X, Y and S be algebraic k-schemes and let $X \to S$ and $Y \to S$ be two given morphisms. All we have to do is prove it for affine algebraic k-schemes and the result will follow using a gluing argument. Let A, B and R be k-algebras such that $X = \mathrm{Spm}(A)$, $Y = \mathrm{Spm}(B)$ and $S = \mathrm{Spm}(R)$. Our goal is to show that $X \times_S Y = \mathrm{Spm}(A \otimes_R B)$. We know that for any scheme Z,

$$\mathrm{Hom}_{\mathbf{AlgSch}_k}(Z, \mathrm{Spm}(A \otimes_R B)) \simeq \mathrm{Hom}_{\mathbf{Alg}_k}(A \otimes_R B, \Gamma(Z, \mathcal{O}_Z)),$$

and also that for any ring T, giving a homomorphism $A \otimes_R B \to T$ is the same as giving two homomorphisms $A \to T$ and $B \to T$ that induce the same homomorphism on R. If we let $T = \Gamma(Z, \mathcal{O}_Z)$, thanks to the previous fact, we have that giving a homomorphism of schemes $Z \to \mathrm{Spm}(A \otimes_R B)$ is the same as giving morphisms $Z \to X$ and $Z \to Y$ that give rise to the same morphism of Z into S. Finally, from the definition of fibred product follows that the previous scheme is unique up to isomorphism, if it exists. The rest is easy to see by following a simple patching argument, glueing sheaves, schemes and morphisms of schemes. □

In other words, given any solid commutative diagram of morphisms of schemes



there exists a unique dotted arrow making the diagram commute. In particular, notice that this means that if X, Y and S are algebraic k-schemes, then $X \times_S Y$ is also an algebraic scheme, and the same happens for affine algebraic k-schemes.

It is important to notice that when ϕ and ψ are the structure maps, that is, when $\phi : X \to \mathrm{Spm}(k)$ and $\psi : Y \to \mathrm{Spm}(k)$, the fibre product becomes the *product*, which we will denote as $X \times Y$, and

$$\mathrm{Hom}_{\mathbf{AlgSch}_k}(T, X \times Y) \simeq \mathrm{Hom}_{\mathbf{AlgSch}_k}(T, X) \times \mathrm{Hom}_{\mathbf{AlgSch}_k}(T, Y).$$

Definition 1.3.0.3. Let $X \in Ob(\mathbf{AlgSch}_k)$. The regular map $\Delta_X : X \to X \times X$ such that its composites with the projection maps equal the identity map of X is called the *diagonal map* of X. We say that X is separated if $\Delta_X(X)$ is closed in $X \times X$.

Proposition 1.3.0.4. *Affine schemes over algebraically closed fields are separated*

Proof. It can be found in [Per], Proposition 2.8 of Chapter VII. □

On the other hand, remember that a ring A is reduced if it has no nonzero nilpotent elements. We say that an algebraic scheme X is *reduced* if the local ring $\mathcal{O}_{X,x}$ is reduced for all $x \in X$. It is crucial to notice that given a k-algebra A, $\mathrm{Spm}(A)$ is reduced if and only if A is reduced.

In addition to this, we say that an algebraic scheme X is *geometrically reduced* if X over $\overline{k}$ is reduced. Throughout this work, we will essentialy focus on the affine case, so we simply mention some of the results that concern us the most in the following proposition.

Proposition 1.3.0.5. *Let A be a k-algebra and let X be an algebraic scheme over k.*

1. $\mathrm{Spm}(A)$ *is geometrically reduced if and only if A is an affine k-algebra, that is, if $\overline{k} \otimes_k A$ is reduced.*
2. *If X is geometrically reduced, for every field K containing k, X is reduced over K.*
3. *If X is geometrically reduced and Y is reduced (resp. geometrically reduced), then $X \times Y$ is reduced (resp. geometrically reduced).*

Notice that since we are supposing that $k = \overline{k}$ most of the time, in the first point of the previous proposition, we have $\overline{k} \otimes_k A \simeq A$, so we simply have to check wether A is reduced or not. Let us give a final definition.

Definition 1.3.0.6. An *algebraic variety* over k is an algebraic scheme over k that is both separated and geometrically reduced.

And so we have that affine schemes over reduced k-algebras are algebraic varieties, so we can refer to them either as affine schemes or as *affine algebraic varieties*.

Chapter 2

Algebraic Groups

2.1 Definition

Throughout this section, k will denote a fixed field, not necessarily algebraically closed. Let us start by giving the definition of algebraic group over k.

Definition 2.1.0.1. Let G be an algebraic scheme over k and let $m : G \times G \to G$ be a regular map. The pair (G, m) is an *algebraic monoid* over k if there exists a regular map $e : \mathrm{Spm}(k) \to G$ such that the following diagrams commute

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{\mathrm{id} \times m} & G \times G \\ {\scriptstyle m \times \mathrm{id}} \downarrow & & \downarrow {\scriptstyle m} \\ G \times G & \xrightarrow[m]{} & G, \end{array} \qquad \begin{array}{ccccc} \mathrm{Spm}(k) \times G & \xrightarrow{e \times \mathrm{id}} & G \times G & \xleftarrow{\mathrm{id} \times e} & G \times \mathrm{Spm}(k) \\ & {\scriptstyle \sim} \searrow & \downarrow {\scriptstyle m} & \swarrow {\scriptstyle \sim} & \\ & & G. & & \end{array}$$

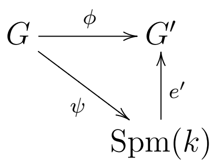
Furthermore, (G, m) is an *algebraic group* if it is an algebraic monoid and there exists a regular map $\mathrm{inv} : G \to G$, such that the following additional diagram commutes

$$\begin{array}{ccccc} G & \xrightarrow{(\mathrm{inv}, \mathrm{id})} & G \times G & \xleftarrow{(\mathrm{id}, \mathrm{inv})} & G \\ \downarrow & & {\scriptstyle m} \downarrow & & \downarrow \\ \mathrm{Spm}(k) & \xrightarrow[e]{} & G & \xleftarrow[e]{} & \mathrm{Spm}(k). \end{array}$$

On the other hand, (G, m) is a *group variety* when G is a variety, and when G is an affine scheme, we will call (G, m) *affine algebraic group.* Also, $\phi : (G, m) \to (G', m')$ is a homomorphism of algebraic groups if $\phi : G \to G'$ is a regular map and the following diagram commutes

$$\begin{array}{ccc} G \times G & \xrightarrow{m} & G \\ {\scriptstyle \phi \times \phi} \downarrow & & \downarrow {\scriptstyle \phi} \\ G' \times G' & \xrightarrow[m']{} & G'. \end{array}$$

In addition, we say that an algebraic group G is *trivial* if $e : \mathrm{Spm}(k) \to G$ is an isomorphism, and a homomorphism $\phi : (G, m) \to (G', m')$ is trivial if it *factors* through $e' : \text{Spm}(k) \to G'$, that is, if there exists a morphism $\psi : G \to \text{Spm}(k)$ such that the following diagram commutes

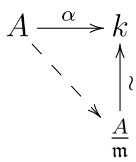


Definition 2.1.0.2. An *algebraic subgroup* of an algebraic group (G, m_G) over k is an algebraic group (H, m_H) over k such that H is a k-subscheme of G and the inclusion map $i : H \hookrightarrow G$ is a homomorphism of algebraic groups, that is $id \circ m = m' \circ (i \times i)$. A *subgroup variety* is an algebraic subgroup that is an algebraic variety.

The following result will provide us an intuition of algebraic groups.

Proposition 2.1.0.3. *Let X be a algebraic k-scheme. There exists a bijective correspondence between the set of k-points $X(k)$ and the set of scheme-theoretic points $x \in |X|$ such that $\kappa(x) = k$.*

Proof. Since any algebraic k-scheme can be covered by affines, it will suffice to show it for affine algebraic k-schemes. Thus, we can suppose $X = \text{Spm}(A)$. We know that there exists a bijection between $X(k) := \text{Hom}_{\mathbf{AffSch}_k}(\text{Spm}(k), \text{Spm}(A))$ and $\text{Hom}_{\mathbf{Alg}_k}(A, k)$, therefore, for each $\bar{\alpha} \in X(k)$, there exists a morphism of k-algebras $\alpha : A \to k$. For $\mathfrak{m} := \ker(\alpha)$, the diagram



commutes. For the reverse inclusion, notice that if x is a scheme-theoretic point with residue field $\kappa(x) = K$, then it corresponds to a morphism $\text{Spm}(K) \to X$. In particular, if $\kappa(x) = k$, there exists a morphism $\text{Spm}(k) \to X$. □

Thanks to the previous proposition, it is clear that the map

$$m(k) : G(k) \times G(k) \to G(k)$$

makes $(G(k), m(k))$ a group with neutral element $e(k)$ and inverse map $\text{inv}(k)$. Also, when k is algebraically closed, we know all of the maximal ideals of the k-algebra, therefore, $G(k) = |G|$, so (G, m) is a group where the maps $x \mapsto x^{-1}$ and $x \mapsto ax$, for $a \in G(k)$ are automorphisms of $|G|$ as a topological space.

This last point also shows that we can describe a homomorphism of algebraic groups by describing its action on A-points. For instance, if we say that $\text{inv} : G \to G$ is $x \mapsto x^{-1}$, we mean that for all k-algebras A and all $x \in G(A)$, $\text{inv}(A)(x) = x^{-1}$. Another good example is the special linear group functor. If we define

$$\text{SL}(n, \cdot) := \text{Spm}\left(\frac{k[T_{11}, \ldots, T_{nn}]}{\langle \det(T_{ij}) - 1 \rangle}\right),$$

together with the usual matrix multiplication

$$((a_{ij}), (b_{ij})) \mapsto (c_{ij}) = \left(\sum_{l=1}^{n} a_{il} b_{lj} \right),$$

we have that $\mathrm{SL}(n, \cdot)$, that we will usually denote as SL_n, is the algebraic group over k whose A-points are the matrices $M \in \mathcal{M}(n, A)$ with $\det(M) = 1$.

Finally, an algebraic group (G, m) is said to be *commutative* whenever $m \circ t = m$, where t is the transposition map $G \times G \to G \times G$ such that $(x, y) \mapsto (y, x)$.

2.2 Basic properties of algebraic groups

Now let us see some of the functorial properties of algebraic groups. We know that given an algebraic scheme X over k, we can define a functor $\tilde{X} : \mathbf{Alg}_k^0 \to \mathbf{Set}$ such that $A \mapsto X(A)$. We saw in the previous chapter that the functor

$$\mathbf{AlgSch}_k \to \mathbf{Func}(\mathbf{Alg}_k^0, \mathbf{Set})$$

given by $X \mapsto \tilde{X}$ is fully faithful, thanks to Yoneda's Lemma 1.2.0.4. We say that a functor from k-algebras to sets is *representable* if it is of the form $\tilde{X}$. We want to show that giving an algebraic group is equivalent to giving a functor in groups represented by it. In fact, this gives sense to the fact that in some books in the bibliography, affine algebraic groups are defined as functors from the category of k-algebras to the category of groups (for instance, see Chapter 6 of [Sza]).

But that is easy to see, because one immediate consequence of the definitions of the results from the previous section is that if we take (G, m) an algebraic group over k, then the functor

$$\begin{aligned} \tilde{G} \colon \mathbf{Alg}_k &\to \mathbf{Grp} \\ A &\mapsto (G(A), m(A)), \end{aligned}$$

where **Grp** denotes the category of groups, satisfies that every such functor arises from an essentially unique algebraic group, therefore, we can say that giving an algebraic group over k is equivalent to giving a functor $\mathbf{Alg}_k^0 \to \mathbf{Grp}$ such that its composition with the forgetful functor $\mathbf{Grp} \to \mathbf{Set}$ is representable by an algebraic scheme. Thus, we can give an alternative definition of affine algebraic group.

Definition 2.2.0.1. An *affine algebraic group* G over k is a functor $G : \mathbf{Alg}_k \to \mathbf{Grp}$ such that its composition with the forgetful functor $F : \mathbf{Grp} \to \mathbf{Set}$ is representable by some k-algebra A.

It is also important to observe that given G and H two algebraic k-groups, a morphism $\theta : G \to H$ is in fact a natural transformation between the functors $G, H : \mathbf{Alg}_k \to \mathbf{Grp}$. In particular, giving an affine algebraic monoid is the same as giving a functor $\mathbf{Alg}_k \to \mathbf{Mon}$, where **Mon** denotes the category of monoids, such that its composition with the forgetful functor is representable by an affine algebraic scheme.

Proposition 2.2.0.2. *The maps* e *and* inv *from the definition of algebraic group are uniquely determined by* (G, m)*. Also, if* $\phi : (G, m_G) \to (H, m_H)$ *is a homomorphism of algebraic groups, then* $\phi \circ e_G = e_H$ *and* $\phi \circ \text{inv}_G = \text{inv}_H \circ \phi$*.*

Proof. If we show the second statement, we will have the first one, because given an algebraic group (G, m) the fact that the map $\text{id} : (G, m) \to (G, m)$ is a homomorphism of algebraic groups is trivial, thus if there were two identity elements e and e', we would have that $e = id \circ e = e'$. Hence, let A be a k-algebra. Clearly,

$$(G(A), m_G(A)) \to (H(A), m_H(A))$$

is a homomorphism of groups and so it maps the neutral element of $G(A)$ to the neutral element of $H(A)$. Also, it maps the inversion map of $G(A)$ to the one on $H(A)$. Eventually, Yoneda's Lemma shows that the same is true for ϕ. □

Proposition 2.2.0.3. *Let* G *be an algebraic group and let* H *be a subscheme of* G*.* H *is an algebraic subgroup of* G *if and only if* $H(A)$ *is a subgroup of* $G(A)$ *for all* k*-algebras* A*.*

Proof. Notice that if $H(A)$ is a subgroup of $G(A)$ for all A, then Yoneda's Lemma shows that the maps

$$H(A) \times H(A) \to H(A)$$
$$(h, h') \longmapsto hh'$$

arise from a morphism $m_H : H \times H \to H$ such that (H, m_H) is an algebraic subgroup of (G, m_G). Let A be a k-algebra. The reverse implication can be seen by taking the pair formed by $H(A)$ and the morphism $m_H(A) : H(A) \times H(A) \to H(A)$, which forms a subgroup of $G(A)$. □

Proposition 2.2.0.4. *Given any algebraic group* (G, m)*, we have that the algebraic scheme* G *is separated.*

Proof. All we have to do is remember theorem 1.3.2 from the previous section, that is, show that the diagonal $\Delta_G(G)$ in $G \times G$ is closed, but this follows immediately from the fact that the preimage of $e \in G(k)$ under the map $m \circ (\text{id} \times \text{inv}) : (g, h) \mapsto gh^{-1}$ is $\Delta_G(G)$, because $(\text{id} \times \text{inv})^{-1} = \text{id} \times \text{inv}$ therefore it is a closed set and we have it. □

It is important to remark the fact that from now on, we will refer to an algebraic group G over k being irreducible, connected or geometrically conected indistinctly, because all three properties are equivalent.

Lemma 2.2.0.5. Let G be an algebraic group over k. The Zariski closure of a subgroup S of $G(k)$ is a subgroup of $G(k)$.

This can be shown using a few elementary results from a course in Topology. From the lemma follows that

Theorem 2.2.0.6. *Every algebraic subgroup of an algebraic group is closed.*

Proof. Let H be an algebraic subgroup of an algebraic group G. If $H_{\bar{k}}$ is closed in $G_{\bar{k}}$, then H is closed in G, therefore we can suppose that $k = \bar{k}$ without loss of generality. Also, we may suppose that H and G are reduced, because the transition from an algebraic group to its reduced algebraic subgroup does not change the underlying topological space.

Thus, by definition $|H|$ is locally closed or, equivalently, open in its closure, that we will denote as S. Now, S is a subgroup of $|G|$ by the previous lemma and it is a finite disjoint union of cosets of $|H|$. Since each coset is open, it is also closed and therefore H is closed in S from where we get the equality. $\square$

This shows that the algebraic subgroups of an algebraic group satisfy the topological noetherian condition and also that every algebraic subgroup of an affine algebraic group is affine.

Eventually, one can show that given an algebraic group G over k and S a closed subgroup of $G(k)$, there is a unique subgroup variety H of G such that $S = H(k)$, and that the map from the set of subgroup varieties of G onto the set of closed subgroups of $G(k)$ given by $H \mapsto H(k)$ is a bijection.

2.3 Kernels and Group actions

Definition 2.3.0.1. Let G be an algebraic group. We will say that

1. An algebraic subgroup H of G is normal if for any k-algebra A, $H(A) \trianglelefteq G(A)$. We will denote it as $H \trianglelefteq G$.

2. An algebraic subgroup H of G is characteristic if for all k-algebras A and all $\alpha \in \mathrm{Aut}(G(A))$, $\alpha(H(A)) = H(A)$.

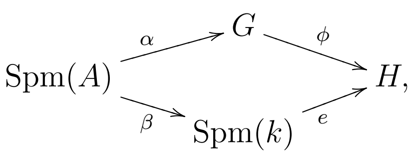
It is important to notice that thanks to Yoneda's lemma, the condition that it must hold for all k-algebras can eventually be restricted to hold for all small k-algebras. Also, it can be shown that the identity component G° of an algebraic group G is characteristic and therefore a normal subgroup of G.

Proposition 2.3.0.2. *Let (G, m) be an algebraic group, let $\phi : G \to H$ be a homomorphism of algebraic groups. We refer to the fiber product $G \times_H \mathrm{Spm}(k)$ as the kernel of ϕ, which we denote as $\mathrm{Ker}(\phi)$. It is an algebraic group, and it satisfies*

$$\mathrm{Ker}(\phi)(A) = \ker(\phi(A)) \qquad \forall A \in Ob(\mathbf{Alg}_k),$$

where ker *has the usual definition of kernel in group theory.*

Proof. Let A be a k-algebra. Notice that giving $\gamma \in (G \times_H \mathrm{Spm}(k))(A)$ is equivalent to giving morphisms $\alpha : \mathrm{Spm}(A) \to G$ and $\beta : \mathrm{Spm}(A) \to \mathrm{Spm}(k)$ so that the diagram



commutes. Then $\alpha \in G(A)$ and $\phi(A)(\alpha) = \phi\circ\alpha = e(\mathrm{Spm}(A))$, that is, $\alpha \in \ker(\phi(A))$. Finally, a subscheme of an affine scheme is affine, therefore we have it. $\square$

Thanks to the first isomorphism theorem, we have that if G and H are affine, since $N := \mathrm{Ker}(\phi)$ is also affine because of the proposition we showed that any subgroup of an affine algebraic group is also affine and that the product of affine algebraic schemes is an affine algebraic scheme,

$$\mathcal{O}(N) = \mathcal{O}(G) \otimes_{\mathcal{O}(H)} k \simeq \frac{\mathcal{O}(G)}{I_H \mathcal{O}(G)},$$

where $I_H = \ker(\mathrm{id} : \mathcal{O}(H) \to k)$ and id is given by $f \mapsto f(e)$. I_H is called the *augmentation ideal* of H.

From now on, we will simply call *functor* any functor from small k-algebras to sets, and *group functor* any functor from small k-algebras to groups.

Definition 2.3.0.3. An *action* of a group functor G on a functor X is a natural transformation $\mu : G \times X \to X$ such that for all k-algebras A, $\mu_A := \mu(A)$ is an action of $G(A)$ on $X(A)$, that is, if

1. $\mu_A(e, x) = x$, for all $x \in X(A)$ and
2. $\mu_A(gh, x) = \mu_A(g, \mu_A(h, x))$, for all $x \in X(A)$ and $g, h \in G(A)$.

On the other hand,

Definition 2.3.0.4. An *action* of an algebraic group G on an algebraic scheme X is a regular map $\mu : G \times X \to X$ such that the following diagrams commute

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{\mathrm{id}\times\mu} & G \times X \\ {\scriptstyle m\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mu} \\ G \times X & \xrightarrow[\mu]{} & X \end{array} \qquad \begin{array}{ccc} \mathrm{Spm}(k) \times X & \hookrightarrow & G \times X \\ & \overset{\sim}{\searrow} & \downarrow{\scriptstyle \mu} \\ & & X. \end{array}$$

Also, for $x \in X(k)$, the *orbit map* $\mu_x : G \to X$ such that $g \mapsto gx$ is $\mu_x := \mu|_{G\times\{x\}\simeq G}$. We will say that G *acts transitively* on X if $G(\bar{k})$ acts transitively on $X(\bar{k})$.

Clearly, because of Yoneda's Lemma, giving the action of an algebraic group on an algebraic scheme is the same as giving an action of $\tilde{G}$ on $\tilde{X}$.

2.4 Affine Algebraic groups

Let (G, m) be an affine algebraic k-group. Remember that, since G is affine, $G = \mathrm{Spm}(\Gamma(G, \mathcal{O}))$. By Proposition 1.1.2.9,

$$\mathrm{Hom}_{\mathbf{AlgSch}_k}(G \times G, G) \simeq \mathrm{Hom}_{\mathbf{Alg}_k}(\Gamma(G, \mathcal{O}), \Gamma(G, \mathcal{O}) \otimes_k \Gamma(G, \mathcal{O})),$$

therefore, the regular map $m : G \times G \to G$ induces a homomorphism of k-rings

$$\Delta : \mathcal{O}(G) \to \mathcal{O}(G \times G) \simeq \mathcal{O}(G) \otimes_k \mathcal{O}(G),$$

sending $f \mapsto \sum g_i \otimes h_i$, where $f(xy) = \sum g_i(x)h_i(y)$. We call this morphism the *comultiplication map*. On the other hand, the identity map $e : \mathrm{Spm}(k) \to G$ induces a morphism $\epsilon : A \to k$, given by $f \mapsto f(e)$, which is called the *coidentity map*. We will study both maps in depth in the next chapter, when we introduce the notion of Hopf algebras.

Example 2.4.0.1. The most commonly known algebraic groups are the following.

1. The *additive group* is the functor

$$\mathbb{G}_a : \mathbf{Alg}_k \to \mathbf{Grp}$$

   such that for any $A \in Ob(\mathbf{Alg}_k)$, $\mathbb{G}_a(A) := (A, +)$. Clearly, $\mathcal{O}(\mathbb{G}_a) = k[X]$, and given $x \in \mathbb{G}_a(A)$, there exists a unique $f \in \mathrm{Hom}_{\mathbf{Alg}_k}(k[X], A)$ such that $\mathbb{G}_a(k[T]) \to \mathbb{G}_a(A)$ sends X to x. From this, we obtain the comultiplication map, $\Delta : k[X] \to k[X] \otimes k[X]$, given by $\Delta(X) = X \otimes 1 + 1 \otimes X$.

2. On the other hand, the *multiplicative group* is the functor $\mathbb{G}_m$ that sends each A to $(A^\times, \cdot)$. Clearly, $\mathcal{O}(\mathbb{G}_m) = k[X,Y]/\langle XY - 1\rangle = k[X, \frac{1}{X}] \subseteq k(X)$. Using the results from the previous chapter, the comultiplication map $\Delta : k[X, \frac{1}{X}] \to k[X, \frac{1}{X}] \otimes k[X, \frac{1}{X}]$ is given by $\Delta(X) = X \otimes X$.

3. Let $n \in \mathbb{N}$. We can define the functor μ_n that sends $A \mapsto \mu_n(A) := (\{x \in A : x^n = 1\}, \cdot)$. Clearly, $\mathcal{O}(\mu_n) = k[X]/\langle X^n - 1\rangle$. It is easy to see that the comultiplication map is induced by that of $\mathbb{G}_m$.

4. Let $m, n \in \mathbb{N}$ and $\mathcal{M}_{m\times n}$ be the functor given by $A \mapsto \mathcal{M}_{m\times n}(A)$, that is, the additive group of $m\times n$ matrices with entries in R. Clearly, it can be represented by $k[X_{11}, \ldots, X_{mn}]$. It is important to notice that if we fix V a k-vector space and denote $\mathrm{End}(V_A)$ as the set of A-linear endomorphisms for a k-algebra A, the functor $\mathrm{End}_V : A \mapsto \mathrm{End}(V_A)$ satisfies that if $\dim V = n$, then fixing a certain basis for V gives an isomorphism between the functors $\mathrm{End}_V \simeq \mathcal{M}_{m\times n}$.

In further chapters, we see more examples of algebraic groups.

2.4.1 The general linear group GL_n

Apart from the previous examples, there is one that concerns us specially, because of its importance regarding representations of affine algebraic groups. That is, the *general linear group*, which is a functor $\mathrm{GL}_n : \mathbf{Alg}_k \to \mathbf{Grp}$ such that

$$A \mapsto \mathrm{GL}_n(A) := \{M \in \mathcal{M}_{n\times n}(A) : \det(M) \in A^\times\}.$$

It is clear that it can be represented by

$$\mathcal{O}(\mathrm{GL}_n) = \frac{k[X_{11}, \ldots, X_{nn}, X]}{\langle \det((X_{ij}))X - 1\rangle}.$$

Also, if we denote $D(X_{ij}) := \det((X_{ij}))$, the comultiplication map is given by the map

$$\Delta : k\left[X_{11},\dots,X_{nn},\frac{1}{D(X_{ij})}\right] \to k\left[X_{11},\dots,X_{nn},\frac{1}{D(X_{ij})}\right] \otimes k\left[X_{11},\dots,X_{nn},\frac{1}{D(X_{ij})}\right]$$

such that for any $1 \le i, j \le n$,

$$\Delta(X_{ij}) = \sum_{k=1}^{n} X_{ik} \otimes X_{kj}.$$

For the arbitrary dimension case, let V denote any k-vector space and define the functor $\mathrm{GL}_V : A \mapsto \mathrm{Aut}(V \otimes_k A)$, that we will study later. Clearly, if $\dim(V) = n$, both functors GL_V and GL_n are isomorphic.

Example 2.4.1.1. To end this section, here are some more examples of algebraic k-subgroups of GL_n.

1. The algebraic group of upper triangular matrices

$$\mathrm{T}_n : A \mapsto \mathrm{T}_n(A) := \{M = (a_{ij}) \in \mathcal{M}_{n\times n}(A) : a_{ij} = 0 \ \ \forall i > j\}.$$

2. The algebraic group of diagonal matrices

$$\mathrm{D}_n : A \mapsto \mathrm{D}_n(A) := \{M = (a_{ij}) \in \mathcal{M}_{n\times n}(A) : a_{ij} = 0 \ \ \forall i \neq j\}.$$

2.5 Homomorphisms and products

Proposition 2.5.0.1. *Let $\{G_i\}_{i=1}^n$ be a finite collection of k-algebraic groups. Then, $G_1 \times G_2 \times \cdots \times G_n$ is an algebraic group. We will refer to it as the product of $\{G_i\}$.*

Proof. All we have to do is notice that the functor

$$\begin{aligned} G_1 \times \cdots \times G_n \colon \mathbf{Alg}_k &\longrightarrow \mathbf{Set} \\ A &\longmapsto G_1(A) \times \cdots \times G_n(A) \end{aligned}$$

is represented by $G_1 \times \cdots \times G_n$. □

If the G_i are all affine, we have previously seen that the product is also affine, hence

$$\mathcal{O}(G_1 \times \cdots \times G_n) \simeq \mathcal{O}(G_1) \otimes \cdots \otimes \mathcal{O}(G_n).$$

In fact, if we let G_1, G_2 and H be algebraic groups together with homomorphisms $\phi_i : G_i \to H$ of algebraic groups, following the same reasoning that we followed in the previous proposition, we can define the *fibre product* of G_1 and G_2 over H as the algebraic group represented by the functor $R \mapsto G_1(A) \times_{H(A)} G_2(A)$. We will denote it as $G_1 \times_H G_2$. Also, in the affine case, that is, when G_1, G_2 and H are all affine, $G_1 \times_H G_2$ is affine as well, hence

$$\mathcal{O}(G_1 \times_H G_2) \simeq \mathcal{O}(G_1) \otimes_{\mathcal{O}(H)} \mathcal{O}(G_2).$$

2.5.1 The Frobenius map

Our goal is to extend the notion of Frobenius map to affine algebraic schemes. In order to do so, let k be a field such that $\mathrm{Char}(k) = p \neq 0$. Also, let A be a k-algebra and

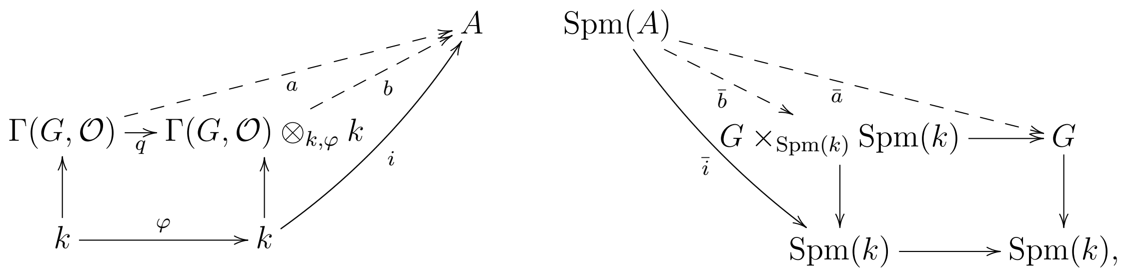
$$\begin{aligned} \varphi_A \colon A &\longrightarrow A \\ a &\longmapsto a^p. \end{aligned}$$

Let $(A, i : k \hookrightarrow A)$ be the k-algebra A together with the k-algebra structure morphism and denote A_φ as the k-algebra $(A, \varphi \circ i)$. On the other hand, let G be an algebraic k-group and let $G^{(p)} : \mathbf{Alg}_k \to \mathbf{Grp}$ be the functor defined by $A \mapsto G(A_\varphi)$.

Proposition 2.5.1.1. *Considering the previous notations and assumptions, if G is an affine algebraic k-group, then*

$$\Gamma(G^{(p)}, \mathcal{O}) = \mathcal{O}(G) \otimes_{k,\varphi} k.$$

Proof. All we have to do is consider the following commutative diagrams



where $a \in G(A_\varphi)$ and $b \in \mathrm{Hom}_{\mathbf{Alg}_k}(\mathcal{O}(G) \otimes_{k,\varphi} k, A)$, and notice that $\bar{a}, \bar{b}$ and $\bar{i}$ are just the associated morphisms of affine algebraic schemes, which are obtained by applying the functor $\mathrm{Spm}(-)$, and we know that the fibre product of affine algebraic schemes is an affine algebraic scheme, which finishes the proof. $\square$

In addition, if G is not affine, we can cover it with open affines, obtaining that it is also an algebraic group. We can summarize everything in the following corollary.

Corollary 2.5.1.2. *$G^{(p)}$ is an algebraic k-group.*

Because of the equivalence between k-algebras and groups, φ_A induces a natural homomorphism of groups $\varphi : A \to A_\varphi$, therefore it arises from a homomorphism of algebraic groups $\Phi : G \to G^{(p)}$. This morphism is called the *Frobenius map.* In fact, if we define Φ^n as the morphism $\Phi^n : G \to G^{(p^n)}$, then,

Proposition 2.5.1.3. *The kernel of $\Phi^n : G \to G^{(p^n)}$ is a characteristic subgroup of G.*

Chapter 3

Hopf Algebras

3.1 Bialgebras and affine monoids

Throughout this section, let k be a field. The notion of k-coalgebra that we are about to define can be seen intuitively as the dual notion of k-algebra. Firstly, though, let us give an alternative definition of k-algebra.

Definition 3.1.0.1. A k-algebra is a triple (A, ∇, η), where A is a k-module, and the maps $\nabla : A \otimes_k A \to A$ and $\eta : k \to A$, called multiplication and identity maps respectively, make the following diagrams commute

$$\begin{array}{ccc} A \otimes_k A \otimes_k A & \xrightarrow{\mathrm{id}_A \otimes \nabla} & A \otimes_k A \\ \downarrow{\scriptstyle \nabla \otimes \mathrm{id}_A} & & \downarrow{\scriptstyle \nabla} \\ A \otimes_k A & \xrightarrow[\nabla]{} & A, \end{array} \qquad \begin{array}{ccccc} A \otimes_k k & \xrightarrow{\mathrm{id} \otimes \eta} & A \otimes_k A & \xleftarrow{\eta \otimes \mathrm{id}} & k \otimes_k A \\ & \searrow{\scriptstyle \sim} & \downarrow{\scriptstyle \nabla} & \swarrow{\scriptstyle \sim} & \\ & & A & & \end{array}$$

Also, if we let (A, ∇, η) and (A', ∇', η') be k-algebras, a k-algebra homomorphism from A to B is a map $\phi : A \to B$ such that is k-linear and

1. $\phi \circ \nabla = \nabla' \circ (\phi \otimes \phi)$ and
2. $\phi \circ \eta = \eta'$.

It can be easily seen that the previous definitions coincide with the usual definition of k-algebra and k-algebra homomorphism.

Definition 3.1.0.2. A *k-coalgebra* is a triple (B, Δ, ϵ), where B is a k-module endowed with maps $\Delta : B \to B \otimes_k B$ and $\epsilon : B \to k$, that we call *comultiplication* and *counit* respectively, making the diagrams commute,

$$\begin{array}{ccc} B & \xrightarrow{\Delta} & B \otimes_k B \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta \otimes \mathrm{id}_B} \\ B \otimes_k B & \xrightarrow[\mathrm{id}_B \otimes \Delta]{} & B \otimes_k B \otimes_k B, \end{array} \qquad \begin{array}{ccccc} & & B & & \\ & \swarrow{\scriptstyle \sim} & \downarrow{\scriptstyle \Delta} & \searrow{\scriptstyle \sim} & \\ B \otimes_k k & \xleftarrow[\mathrm{id} \otimes \epsilon]{} & B \otimes_k B & \xrightarrow[\epsilon \otimes \mathrm{id}]{} & k \otimes_k B. \end{array}$$

Furthermore, if we let $\tau \in \mathrm{Aut}(B \otimes_k B)$ such that $\tau : x \otimes y \mapsto y \otimes x$, k-coalgebra B is said to be *co-commutative* when $\tau \circ \Delta = \Delta$. Let $(B, \Delta_B, \epsilon_B)$ and $(C, \Delta_C, \epsilon_C)$ be k-coalgebras. A map $\phi : B \to C$ is a *k-coalgebra homomorphism* from B to C if it is k-linear and satisfies

1. $(\phi \otimes \phi) \circ \Delta_B = \Delta_C \circ \phi$ and
2. $\epsilon_B = \epsilon_C \circ \phi$.

Remark 3.1.0.3. It is worth mentioning how to build a coalgebra given a pair of coalgebras. Assume (B, Δ, ϵ) and (B', Δ', ϵ') are k-coalgebras. Since $k \simeq k \otimes_k k$, we can construct a counit by defining the map

$$k \simeq k \otimes_k k \xrightarrow{\epsilon \otimes \epsilon'} B \otimes_k B'.$$

For the comultiplication, if we let $\tau \in \mathrm{Hom}(B \otimes_k B', B' \otimes_k B)$ that sends $b \otimes b' \mapsto b' \otimes b$, then we have

$$\begin{array}{ccc} B \otimes_k B & \xrightarrow{\Delta \otimes \Delta'} & B \otimes_k B \otimes_k B' \otimes_k B' \\ & \overset{\overline{\Delta}}{\searrow} & \downarrow {\scriptstyle \mathrm{id}_B \otimes \tau \otimes \mathrm{id}_{B'}} \\ & & (B \otimes_k B') \otimes_k (B \otimes_k B'), \end{array}$$

where $\overline{\Delta} = (\mathrm{id}_B \otimes \tau \otimes \mathrm{id}_{B'}) \circ (\Delta \otimes \Delta')$. Thus, since $B \otimes_k B'$ is a k-module, we have that the triple $(B \otimes_k B', \overline{\Delta}, \epsilon \otimes \epsilon')$ is a k-coalgebra. We can follow a similar argument in order to obtain a new k-algebra from a pair of k-algebras.

Example 3.1.0.4. Some of the most common examples of coalgebras are the following.

1. The *polynomial coalgebra* over a field k consists on the k-module $k[X]$ and the maps
$$\begin{array}{rcl} \Delta : k[X] & \longrightarrow & k[X]^{\otimes_k 2} \\ X^n & \longmapsto & X^n \otimes X^n \end{array} \qquad \begin{array}{rcl} \epsilon : k[X] & \longrightarrow & k \\ X^n & \longmapsto & 1. \end{array}$$
It is easy to verify that Δ and ϵ make the diagrams from 3.1.0.2 commute. It is also cocommutative.

2. Assume that $\{e_{ij}\}_{1 \leq i,j \leq n}$ is the canonical k-basis for $\mathcal{M}_n(k)$. The *matrix coalgebra* is the coalgebra given by
$$\begin{array}{rcl} \Delta : \mathcal{M}_n(k) & \to & \mathcal{M}_n(k)^{\otimes_k 2} \\ e_{ij} & \mapsto & \sum e_{ik} \otimes e_{kj} \end{array} \qquad \begin{array}{rcl} \epsilon : \mathcal{M}_{n \times n}(k) & \longrightarrow & k \\ e_{ij} & \longmapsto & \delta_{ij}, \end{array}$$
where δ_{ij} denotes the Kronecker delta function. Clearly, it is a k-coalgebra.

It is easy to verify that the previous definitions translate into the following lemma.

Lemma 3.1.0.5. Let B be a k-module endowed with a multiplication $\nabla : B \otimes_k B \to B$, a unit $\eta : k \to B$, a comultiplication $\Delta : B \to B \otimes_k B$ and a counit $\epsilon : B \to k$. Suppose that the triple (B, ∇, η) is a k-algebra and that (B, Δ, ϵ) is a k-coalgebra. Then, it is equivalent to say that

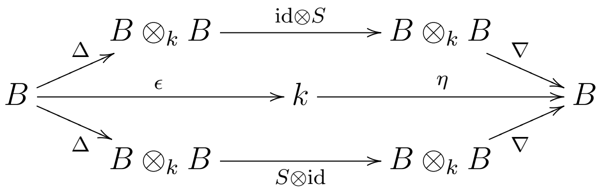
1. Δ and ϵ are morphisms of k-algebras,
2. ∇ and η are morphisms of k-coalgebras.

Finally, we can give the definition of a k-bialgebra.

Definition 3.1.0.6. A k-bialgebra is a 5-tuple $(A, \nabla, \eta, \Delta, \epsilon)$ such that (A, ∇, η) is a k-algebra, (A, Δ, ϵ) is a k-coalgebra, and one of the two equivalent conditions of the previous lemma is satisfied.

3.2 Hopf Algebras

Definition 3.2.0.1. Let $(B, \nabla, \Delta, \eta, \epsilon)$ be a k-bialgebra. An *antipode* is a morphism of k-algebras $S : B \to B$ such that the following diagram commutes



A *Hopf algebra* is a k-bialgebra that admits an antipode. Let A and B be Hopf k-algebras. A *homorphism* of Hopf k-algebras $f : A \to B$ is a homomorphism of k-algebras such that

$$(f \otimes f) \circ \Delta_A = \Delta_B \circ f.$$

Also, we say that a k-Hopf algebra is *commutative* if it is a commutative k-algebra, and *cocommutative* if it is a cocommutative k-coalgebra.

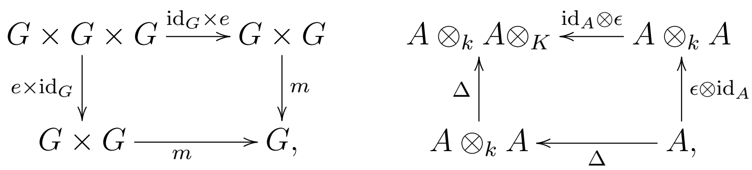
Now, let A be a k-algebra and $\Delta : A \to A \otimes A$ a homomorphism of k-algebras. Taking into account the properties of Spm that we saw in the first chapter, we have that

$$\mathrm{Spm}(\Delta) : \mathrm{Spm}(A \otimes A) \simeq \mathrm{Spm}(A) \times \mathrm{Spm}(A) \to \mathrm{Spm}(A).$$

The following proposition shows the duality between affine algebraic groups and Hopf algebras.

Proposition 3.2.0.2. *Given a k-algebra (A, ∇, η), a 6-tuple $(A, \nabla, \eta, \Delta, \epsilon, S)$ is a Hopf k-algebra if, and only if, the 4-tuple $(\mathrm{Spm}(A), \mathrm{Spm}(\Delta), \mathrm{Spm}(\epsilon), \mathrm{Spm}(S))$ is an algebraic group.*

Proof. Set $G := \mathrm{Spm}(A)$, $m := \mathrm{Spm}(\Delta)$, $e := \mathrm{Spm}(\epsilon)$ and $\mathrm{inv} := \mathrm{Spm}(S)$. Then, the commutativity of the left diagram implies the comutativity of the right one, and viceversa



because of the duality between $\mathbf{Alg}_k$ and $\mathbf{AlgSch}_k$. The rest of the proof proceeds the same way, by comparing the corresponding commutative diagrams. □

Corollary 3.2.0.3. *The functor* Spm *is an equivalence of categories between the category* $\mathbf{Hopf}_k$ *of Hopf* k*-algebras and* $\mathbf{AffAlgGrp}_k$*, with quasi inverse* $(G, m) \mapsto (\Gamma(G, \mathcal{O}), \mathcal{O}(m))$.

The previous corollary gives us a lot of examples of Hopf algebras, all we have to do is take an affine algebraic group and give the morphisms that make its associated k-module A a Hopf k-algebra. Example 2.4.0.1 gives us the most important examples and how to find the Hopf algebra associated to an affine algebraic group. Either way, let us show another interesting example of Hopf algebras.

Example 3.2.0.4. Remember that a Lie algebra $\mathfrak{g}$ is a vector space over a field k with an operation $[-,-] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ which is bilinear, antisymmetric and that satisfies the Jacobi identity, that is, $\forall x, y, z \in \mathfrak{g}$,

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]].$$

Also, remember that since $\mathfrak{g}$ is a k-vector space, we can construct the tensor algebra associated to $\mathfrak{g}$ as

$$T(\mathfrak{g}) := k \oplus \bigoplus_{n\geq 1} \mathfrak{g}^{\otimes_k n}$$

and then, the universal algebra associated to $\mathfrak{g}$, $U(\mathfrak{g})$, is given by taking the quotient

$$U(\mathfrak{g}) := \frac{T(\mathfrak{g})}{I},$$

where $I = \langle a \otimes b - b \otimes a - [a, b] : a, b \in \mathfrak{g}\rangle$. It turns out that the universal enveloping algebra $U(\mathfrak{g})$ is a Hopf algebra when it is endowed with the obvious structure of k-algebra and the k-coalgebra structure given by the coproduct $\Delta(x) = x \otimes 1 + 1 \otimes x$, the counit $\epsilon(x) = 0$ and antipode $S(x) = -x$ where $x \in \mathfrak{g}$ extended to $U(\mathfrak{g})$ in the natural way. It is also clear that it is commutative if and only if $\mathfrak{g}$ is abelian and that it is cocommutative.

3.2.1 Hopf subalgebras

Definition 3.2.1.1. A k-subalgebra B of a Hopf Algebra (A, Δ, ϵ, S) is a *Hopf subalgebra* if $\Delta(B) \subseteq B \otimes_k B$ and $S(B) \subseteq B$.

The following proposition shows some properties of Hopf k-subalgebras.

Proposition 3.2.1.2. *Let* $(A, \Delta_A, \epsilon_A, S_A)$ *and* $(B, \Delta_B, \epsilon_B, S_B)$ *be two Hopf* k*-algebras and* $f : A \to B$ *a homomorphism of Hopf algebras. Then,*

1. *If* C *is a Hopf* k*-subalgebra of* A*, then* $(C, \Delta_A|_C)$ *is a Hopf* k*-algebra.*
2. *The image* $f(A)$ *is a Hopf* k*-subalgebra of* B.

Definition 3.2.1.3. A *Hopf ideal* in a Hopf k-algebra $(A, \Delta, \epsilon, \nabla, \eta, S)$ is an ideal $\mathfrak{a}$ in A such that $\epsilon(\mathfrak{a}) = 0$, $S(\mathfrak{a}) \subseteq \mathfrak{a}$ and

$$\Delta(\mathfrak{a}) \subseteq A \otimes \mathfrak{a} + \mathfrak{a} \otimes A.$$

In fact, the following proposition shows that some results from linear algebra can be extended to Hopf k-algebras.

Proposition 3.2.1.4. *Let A and B be Hopf k-algebras and let $f : A \to B$ be a morphism of Hopf k-algebras. The kernel and the image of a homomorphism of Hopf k-algebras are Hopf ideals. Also,*

1. *If $\mathfrak{a} \subseteq A$ is a Hopf ideal, the quotient vector space $A/\mathfrak{a}$ has a unique Hopf k-algebra structure for which the canonical morphism $\phi : A \to A/\mathfrak{a}$ is a homomorphism of Hopf k-algebras. In addition, if $\mathfrak{a} \subseteq \ker(f)$, f factors uniquely through ϕ.*

2. *The homomorphism f induces an isomorphism of Hopf k-algebras,*

$$\tilde{f} : \frac{A}{\ker(f)} \longrightarrow \operatorname{im}(f).$$

And finally, if we let G be an affine algebraic k-group, we can give the relationship between Hopf k-subalgebras of $\mathcal{O}(G)$ and of algebraic subgroups of G.

Proposition 3.2.1.5. *Let G be an affine algebraic k-group. In the bijective correspondence between closed subschemes of G and ideals in $\mathcal{O}(G)$, algebraic subgroups correspond to Hopf ideals.*

Chapter 4

Linear representations and characters of algebraic groups

4.1 Representations

Henceforth, G will denote an affine algebraic k-group and V a k-module. Also, remember that GL_V denotes the functor

$$\begin{aligned} \mathrm{GL}_V \colon \mathbf{Alg}_k &\longrightarrow \mathbf{Grp} \\ A &\longmapsto \mathrm{Aut}(V \otimes_k A). \end{aligned}$$

If V is a finite dimensional k-vector space, then GL_V is given by

$$A \mapsto \mathrm{GL}(\dim(V), A),$$

and also let us define functor $\mathbb{W}_V : \mathbf{Alg}_k \to \mathbf{Mod}_A$ by $\mathbb{W}_V(A) := V \otimes_k A$.

Definition 4.1.0.1. A *linear representation* of G is a pair (V, Φ), where V is a k-module and Φ is a natural transformation, $\Phi : G \to \mathrm{GL}_V$, such that the component of Φ at each k-algebra A is a homomorphism of groups. We say that Φ is *faithful* if $\Phi(A)$ is injective for any $A \in Ob(\mathbf{Alg}_k)$. Sometimes we will denote linear representations with caligraphic letters, for instance, $\mathcal{M}$ will denote (M, Φ). In addition, if $\mathcal{M}$ and $\mathcal{N}$ are representations of G, a *G-homomorphism* $f : \mathcal{M} \to \mathcal{N}$ consists of a natural transformation $f : \mathbb{W}_M \to \mathbb{W}_N$ such that the diagram

$$\begin{array}{ccc} \mathbb{W}_M(A) & \xrightarrow{\Phi_{\mathcal{M}}(A)(g)} & \mathbb{W}_M(A) \\ {\scriptstyle f(A)}\downarrow & & \downarrow{\scriptstyle f(A)} \\ \mathbb{W}_N(A) & \xrightarrow[\Phi_{\mathcal{N}}(A)(g)]{} & \mathbb{W}_N(A) \end{array}$$

commutes for every $A \in Ob(\mathbf{Alg}_k)$ and $g \in G(A)$. We denote the category of representations of an algebraic group G as $\mathbf{Rep}(G)$ and the set of G-homomorphisms is denoted as $\mathrm{Hom}_G(\mathcal{M}, \mathcal{N})$ or $\mathrm{Hom}_{\mathbf{Rep}(G)}(\mathcal{M}, \mathcal{N})$.

The following proposition gives us the relationship between actions and representations.

Proposition 4.1.0.2. *Given a k-module V, there is a bijective correspondence between the set of actions of an affine algebraic k-group G on $\mathbb{W}_V$ and the set of linear representations of G on V.*

Proof. Suppose that $\Phi : G \to \mathrm{GL}_V$ is a linear representation and let A be a k-algebra. Then, we can define an action $\mu : G \times \mathbb{W}_V \to \mathbb{W}_V$ by setting, for each $x = v \otimes a \in \mathbb{W}_V(A) = V \otimes_k A$ and each $g, h \in G(A)$, $\mu(A)(g, x) := \Phi(A)(g)(v \otimes a)$. Clearly, $\mu(A)(e, v\otimes a) = v \otimes a$ and $\mu(A)(gh, v\otimes a) = \mu(A)(g, \mu(A)(h, v\otimes a))$, hence it is indeed an action. For the reverse implication, if μ is an action of the group functor G on $\mathbb{W}_V$, then we can define, for each $A \in Ob(\mathbf{Alg}_k)$,

$$\Phi(A)(g)(v \otimes a) := \mu(A)(g, v \otimes a), \qquad \forall v \otimes a \in V \otimes_k A, g \in G(A).$$

The result follows from the properties of $\mu(A)$, that is, if we translate the diagrams that commute from the definition of μ, we get the result we want. ☐

This shows that giving a representation (V, Φ) is equivalent to giving an action of G on $\mathbb{W}_V$. When we see the representation as an action, we will refer to (V, Φ) as a *G-module.*

Example 4.1.0.3. Let us show some important examples of representations.

1. Let $V = k$ and let $A \in Ob(\mathbf{Alg}_k)$. Take the morphism

$$\begin{aligned} \Phi(A)\colon G(A) &\longrightarrow \mathrm{GL}_k(A) = A^\times \\ g &\longmapsto \mathbb{1}_A. \end{aligned}$$

   Clearly, this is a representation, and we refer to it as the *trivial representation* of G. In many books, it is written as triv.

2. Let $\mathcal{M}, \mathcal{N} \in Ob(\mathbf{Rep}(G))$. It is easy to see that for any k-algebra A, there is a natural isomorphism $\mathbb{W}_M(A) \otimes_A \mathbb{W}_N(A) \xrightarrow{\sim} \mathbb{W}_{M\otimes_k N}(A)$. Therefore, we can define an action of $G(A)$ on $\mathbb{W}_M(A) \otimes_A \mathbb{W}_N(A)$ by setting $\Phi_{M\otimes_k N}(A)(g) = \Phi_M(A)(g) \otimes_A \Phi_N(A)(g)$. It is also easy to check that this is a representation of G, and we denote it as $\mathcal{M} \otimes_G \mathcal{N} \in Ob(\mathbf{Rep}(G))$. It will be of great use for the next chapter to notice that the natural k-isomorphism $M \otimes_k N \to N \otimes_k M$ gives an isomorphism

$$\beta_{\mathcal{M},\mathcal{N}} : \mathcal{M} \otimes_G \mathcal{N} \xrightarrow{\sim} \mathcal{N} \otimes_G \mathcal{M} \in \mathrm{Hom}_{\mathbf{Rep}(G)}(\mathcal{M} \otimes_G \mathcal{N}, \mathcal{N} \otimes_G \mathcal{M}),$$

   such that $\beta_{\mathcal{N},\mathcal{M}} \circ \beta_{\mathcal{M},\mathcal{N}} = \mathrm{id}_{\mathcal{M}\otimes_G\mathcal{N}}$, and that for a pair of morphisms $F \in \mathrm{Hom}_{\mathbf{Rep}(G)}(\mathcal{M}, \mathcal{M}')$ and $F' \in \mathrm{Hom}_{\mathbf{Rep}(G)}(\mathcal{N}, \mathcal{N}')$, the morphism $F \otimes_G F' \in \mathrm{Hom}_{\mathbf{Rep}(G)}(\mathcal{M} \otimes_G \mathcal{N}, \mathcal{M}' \otimes_G \mathcal{N}')$ is defined naturally. In the next chapter, we see that this translates into the fact that $\mathbf{Rep}(G)$ is a symmetric monoidal category.

3. Let $\mathcal{M} \in Ob(\mathbf{Rep}(G))$. Take the dual of M, $M^\vee := \mathrm{Hom}_{\mathbf{Mod}_k}(M, k)$, and note that for every k-algebra A, there is an isomorphism $\mathrm{Hom}_{\mathbf{Mod}_A}(\mathbb{W}_M(A), A) \to \mathbb{W}_M(A)^\vee$, thus we can define an action of G on $M^\vee$ by taking $g \in G(A)$, $x \in \mathbb{W}_M(A)$ and setting $(g\phi)(x) := \phi(g^{-1}(x))$. This is a representation of G on $M^\vee$, denoted $\mathcal{M}^\vee \in Ob(\mathbf{Rep}(G))$.

4. Finally, for $\mathcal{M}, \mathcal{N} \in Ob(\mathbf{Rep}(G))$, using the previous constructions and the natural isomorphism $M^\vee \otimes_k N \to \mathrm{Hom}_{\mathbf{Mod}_k}(M, N)$ (see [Lan]), we have a representation denoted as $\mathrm{Hom}_k(\mathcal{M}, \mathcal{N})$.

Definition 4.1.0.4. Let G be an affine algebraic k-group. A $\Gamma(G, \mathcal{O})$-*comodule* is a k-linear map $\rho : V \to V \otimes_k \Gamma(G, \mathcal{O})$ such that the following diagrams commute

$$\begin{array}{ccc} V & \xrightarrow{\rho} & V \otimes_k \Gamma(G,\mathcal{O}) \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \mathrm{id}_V \otimes \Delta} \\ V \otimes_k \Gamma(G,\mathcal{O}) & \xrightarrow[\rho \otimes \mathrm{id}_{\Gamma(G,\mathcal{O})}]{} & V \otimes_k \Gamma(G,\mathcal{O}) \otimes \Gamma(G,\mathcal{O}), \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{\rho} & V \otimes_k \Gamma(G,\mathcal{O}) \\ & \searrow\!\!\!= & \downarrow{\scriptstyle \mathrm{id}_V \otimes \epsilon} \\ & & V \simeq V \otimes_k k. \end{array}$$

Furthermore, a $\Gamma(G, \mathcal{O})$-*subcomodule* of the $\Gamma(G, \mathcal{O})$-*comodule* (V, ρ) is a k-subspace $W \subseteq V$ such that

$$\rho(W) \subseteq W \otimes_k \Gamma(G, \mathcal{O}).$$

Notice that if W is a $\Gamma(G, \mathcal{O})$-subcomodule, $(W, \rho|_W)$ is in fact a $\Gamma(G, \mathcal{O})$-comodule, because the fact that ρ makes the previous diagrams commute implies that the restriction of ρ to W makes the diagrams commute as well.

Now let us see that the set of linear representations of an affine algebraic k-group G on V is not only in bijection with the set of actions of G on $\mathbb{W}_V$, but also with the set of $\Gamma(G, \mathcal{O})$-comodule structures on V.

Theorem 4.1.0.5. *Let G be an affine group scheme. The set of linear representations of G on V is in a bijective correspondence with the set of $\Gamma(G, \mathcal{O})$-comodule structures on V.*

Proof. Let $\Phi : G \to \mathrm{GL}_V$ be a representation of G, and suppose that $G = \mathrm{Spm}(A)$. For the general element $\mathrm{id} \in G(A)$, applying the natural transformation Φ, we get an A-linear map $\Phi(\mathrm{id}) : \Phi(A) = V \otimes_k A \to \Phi(A) = V \otimes_k A$ such that it is determined by its restriction to V, that is, we can define the restriction $\rho := \Phi(\mathrm{id})|_{V \otimes_k k} : V \otimes_k k \simeq V \to V \otimes_k A = V \otimes_k \Gamma(G, \mathcal{O})$. On the other hand, thanks to Yoneda's lemma and the fact that Φ is a natural transformation, for each $R \in Ob(\mathbf{Alg}_k)$ and for each $g \in G(R) \simeq \mathrm{Hom}_{\mathbf{Alg}_k}(A, R)$, the following diagram commutes

$$\begin{array}{ccc} V \otimes_k A & \xrightarrow{\Phi(\mathrm{id})} & V \otimes_k A \\ \downarrow{\scriptstyle \mathrm{id}_V \otimes g} & & \downarrow{\scriptstyle \mathrm{id}_V \otimes g} \\ V \otimes_k R & \xrightarrow[\Phi(g)]{} & V \otimes_k R. \end{array}$$

Therefore, since $\Phi(g)$ acts by $(\mathrm{id} \otimes g) \circ \rho$, we have shown that Φ is determined by ρ. For the reverse implication, we know that for any k-linear map $\rho : V \to V \otimes_k A$ we can get a natural map between sets, namely

$$\Phi : G(R) \to \mathrm{End}_V(V \otimes_k R).$$

We must show that the unit in $G(R)$ acts as the identity, that is, for all $R \in Ob(\mathbf{Alg}_k)$, the diagram

$$\begin{array}{ccc} V & \xrightarrow{\ \rho\ } & V \otimes_k A = V \otimes_k \Gamma(G, \mathcal{O}) \\ & \searrow^{\simeq} & \downarrow{\scriptstyle \mathrm{id} \otimes \epsilon} \\ V \otimes_k R & \longleftarrow & V \otimes k \end{array}$$

must commute for all $R \in Ob(\mathbf{Alg}_k)$, which is equivalent to the second diagram in definition 4.1.4. Finally, we must show that $\Phi(g)\Phi(h) = \Phi(gh)$, but since gh is given by

$$A \xrightarrow{\Delta} A \otimes_k A \xrightarrow{(g,h)} R,$$

on V the action $\Phi(g)\Phi(h)$ is given by

$$V \xrightarrow{\rho} V \otimes_k A \xrightarrow{\mathrm{id} \otimes h} V \otimes_k R \xrightarrow{\rho \otimes \mathrm{id}} V \otimes_k A \otimes_k R \xrightarrow{\mathrm{id} \otimes (g, \mathrm{id})} V \otimes_k R$$

which coincides with

$$V \xrightarrow{\rho} V \otimes_k A \xrightarrow{\mathrm{id} \otimes \Delta} V \otimes_k A \otimes_k A \xrightarrow{\mathrm{id} \otimes (g,h)} V \otimes_k R$$

for all g and h if and only the first diagram of definition 4.1.0.4 commutes. Thus, we have that the two diagrams from 4.1.0.4 commute and we have shown the result. □

4.2 Properties of $\Gamma(G, \mathcal{O})$-comodules

Firstly, let $(v_i)_{i \in \mathcal{I}}$ be a basis of V and let $(r_{ij})_{i,j \in \mathcal{I}}$ be a family of elements of $\Gamma(G, \mathcal{O})$. Our first goal is to translate the conditions of a k-linear map $\rho : V \to V \otimes_k \Gamma(G, \mathcal{O})$ into a more explicit expression. Notice that if we want the map

$$\begin{aligned} \rho \colon V &\longrightarrow V \otimes_k \Gamma(G, \mathcal{O}) \\ v_j &\mapsto \sum_{i \in \mathcal{I}} v_i \otimes r_{ij} \end{aligned}$$

to be a $\Gamma(G, \mathcal{O})$-comodule, it must make the diagrams from definition 4.1.0.4 commute. More specifically, we can translate the two commutative diagrams into the following relations

$$(\mathrm{id}_V \otimes \Delta) \circ \rho = (\rho \otimes \mathrm{id}_{\Gamma(G,\mathcal{O})}) \circ \rho \qquad (\mathrm{id}_V \otimes \epsilon) \circ \rho = \mathrm{id}_{V \to V \otimes_k k}.$$

Therefore, for $i \in \mathcal{I}$, the following expressions must be equal

$$(\mathrm{id}_V \otimes \Delta) \circ \rho(v_j) = (\mathrm{id}_V \otimes \Delta) \left(\sum_{i \in \mathcal{I}} v_i \otimes r_{ij} \right) = \sum_{i \in \mathcal{I}} v_i \otimes \Delta(r_{ij})$$

$$(\rho \otimes \mathrm{id}_{\Gamma(G,\mathcal{O})}) \left(\sum_{i\in\mathcal{I}} v_i \otimes r_{ij} \right) = \sum_{i\in\mathcal{I}} \rho(v_i) \otimes r_{ij} = \sum_{i,k\in\mathcal{I}} v_k \otimes r_{ki} \otimes r_{ij},$$

and also the following equality must hold

$$(\mathrm{id}_V \otimes \epsilon) \left(\sum_{i\in\mathcal{I}} v_i \otimes r_{ij} \right) = \sum_{i\in\mathcal{I}} v_i \otimes \epsilon(r_{ij}) = v_j \otimes 1 = \mathrm{id}_{V\to V\otimes_k k}(v_j),$$

which is equivalent to

1. $\Delta(r_{ij}) = \sum_{k\in\mathcal{I}} r_{ik} \otimes r_{kj}$ and
2. $\epsilon(r_{ij}) = \delta_{ij}$.

This construction is of great use, because it makes some of the results ahead more easy to be worked with. For instance, in the next section we see an interesting property of comodules, which is proved using the previous notation.

Now, our next goal is to show that every representation can be seen as a union of finite-dimensional representations. In order to do so, we are going to need the following proposition, which will characterize what we call group stabilizers. The proof is left to the reader, as it is a mere checking.

Proposition 4.2.0.1. *Let $\Phi : G \to \mathrm{GL}_V$ be a finite-dimensional representation of G and let W be a subspace of V. The functor given by*

$$\begin{aligned} \mathrm{Stab}_G(W)\colon \mathbf{Alg}_k &\longrightarrow \mathbf{Grp} \\ A &\mapsto \mathrm{Stab}_G(W)(A), \end{aligned}$$

where $\mathrm{Stab}_G(W)(A) := \{g \in G(A) : g(\mathbb{W}_W(A)) = \mathbb{W}_W(A)\}$*, is represented by an algebraic subgroup* $\mathrm{Stab}_G(W)$ *of G.*

We call the subgroup $\mathrm{Stab}_G(W)$ the *stabilizer* of W in V, and say that an algebraic subgroup H of G *stabilizes* a subspace W of V if it satisfies $H \subset \mathrm{Stab}_G(W)$.

Proposition 4.2.0.2. *A subspace W of a k-vector space V is stable under G if and only if it is a $\Gamma(G,\mathcal{O})$-subcomodule of V*

Proof. Thanks to Theorem 4.1.0.5, we know that there exists a $\Gamma(G,\mathcal{O})$-comodule associated to the representation. All we have to do is notice that the condition $G \subset \mathrm{Stab}_G(W)$ is equivalent to the condition that $\rho(W) \subseteq W \otimes_k \Gamma(G,\mathcal{O})$. □

The following proposition will be the last step towards our initial goal.

Proposition 4.2.0.3. *Every $\Gamma(G,\mathcal{O})$-comodule (V,ρ) is a filtered union of its finite-dimensional sub-comodules.*

Proof. It is known that a finite sum of finite-dimensional sub-comodules is a finite-dimensional sub-comodule, hence we only need to show that each element $v \in V$ is contained in a finite-dimensional sub-comodule. Therefore, let $(e_i)_{i\in\mathcal{I}}$ be a basis for $\Gamma(G, \mathcal{O})$ as a k-module and

$$\rho(v) := \sum_{i\in\mathcal{I}} v_i \otimes e_i,$$

where $v_i \in V$ and all but finitely many v_i are zero. Also, let

$$\Delta(e_i) := \sum_{j\in\mathcal{I}} \sum_{l\in\mathcal{I}} r_{ijl}(e_j \otimes e_l),$$

where $r_{ijl} \in k$. our goal is to show that

$$\rho(v_l) = \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{I}} r_{ijl}(v_i \otimes e_j),$$

because from that it follows that the k-submodule W of V spanned by v and the v_i is a subcomodule containing v, that is, we would be showing that $\rho(W) \subset W \otimes_k \Gamma(G, \mathcal{O})$. In order to do so, all we have to do is translate the commutativity of the first diagram in definition 4.1.4 as $(\mathrm{id}_v \otimes \Delta) \circ \rho = (\rho \otimes \mathrm{id}_{\Gamma(G,\mathcal{O})}) \circ \rho$. If we apply this last equality to v, we get that

$$\sum_{i,j,l\in\mathcal{I}} r_{ijl}(v_i \otimes e_j \otimes e_l) = \sum_{l\in\mathcal{I}} \rho(v_l) \otimes e_l \in V \otimes_k \Gamma(G, \mathcal{O}) \otimes_k \Gamma(G, \mathcal{O}),$$

which implies the equality that we are seeking by comparing the coefficients in each of the two expressions. □

Corollary 4.2.0.4. *Every representation of G is a filtered union of its finite-dimensional subrepresentations*

Proof. Let $\Phi : G \to \mathrm{GL}_V$ be a representation of G and $\rho : V \to V \otimes_k \Gamma(G, \mathcal{O})$ its corresponding co-action. All we have to do is apply the previous proposition and we get the result we want.

□

4.3 Characters

The reader may have noticed at this point that most of the notions that we have from group theory can be extended to algebraic groups. Of course, character theory is not an exception. In this section we show an important result regarding algebraic characters, the character decomposition theorem.

Definition 4.3.0.1. Given an affine algebraic group G, a character of G is a homomorphism $\chi : G \to \mathbb{G}_m$.

The following proposition summarizes some of the most common ways of working with characters.

Proposition 4.3.0.2. *Let G be an affine algebraic k-group. Giving a character χ of G is equivalent giving an element $a = a(\chi) \in \Gamma(G, \mathcal{O})$ such that $\Delta(a) = a \otimes a$. Any element satisfying that property is called group-like.*

Proof. The result follows directly from the fact that $\Gamma(\mathbb{G}_m, \mathcal{O})$ and $\Delta(X) = X \otimes X$. □

Definition 4.3.0.3. Let χ be a character of an affine algebraic k-group G. We say that G acts on V through χ if there exists a representation $\mathcal{M} = (M, \Phi_{\mathcal{M}})$ such that

$$\Phi_{\mathcal{M}}(R)(g)(v \otimes r) = \chi(g)(v \otimes r), \qquad \forall g \in G(R), v \otimes r \in \mathbb{W}_V(R).$$

4.3.1 Eigenspaces and the decomposition theorem

The following proposition will introduce us to how we can study eigenspaces of groups with certain characters.

Proposition 4.3.1.1. *Let (V, Φ) be a representation of G and let $\rho := \rho_\Phi$ its corresponding $\Gamma(G, \mathcal{O})$-comodule. For any character $\chi : G \to \mathbb{G}_m$ of G, there is a greatest subspace V_χ of V on which G acts through χ. Furthermore,*

$$V_\chi = \{v \in V : \rho(v) = v \otimes a(\chi)\}.$$

Proof. For the first part of the proof, we say that G acts on a subspace W of V through a character χ if W is stable under G and G acts on W through χ. This translates into the fact that if $w \in W$, then for every $g \in G(k)$, $\Phi(g)w = \lambda w$ for a certain $\lambda \in k$, that is, the elements of W are common eigenvectors for the $g \in G(k)$. If G acts on subspaces W and W' through a character χ, the it acts on $W + W'$ through χ because of the linearity. This shows the first point of the proposition, i.e, there is a greatest subspace V_χ of V on which G acts through χ. For the second point of the proof, the set

$$\{v \in V : \rho(v) = v \otimes a(\chi)\}$$

is a subspace of V on which G acts through χ, and the fact that it contains every such subspace implies the result we want to show. □

The subspace V_χ is called the *eigenspace for G with character χ*. In addition to this, the following lemma will be of great use in order to show the decomposition theorem.

Lemma 4.3.1.2. The group-like elements in a Hopf k-algebra $(A, \Delta, \epsilon, \nabla, \eta, S)$ are linearly independent.

Proof. Firstly, it is important to notice that for a given Hopf algebra A together with a group-like element $a \in A$, translating the commutative diagrams of 3.1.0.2

$$1 \otimes a = \mathrm{id}_{A \to A \otimes_k k}(a) = ((\epsilon \otimes \mathrm{id}_A) \circ \Delta)(a) = (\epsilon \otimes \mathrm{id}_A)(a \otimes a) = \epsilon(a) \otimes a,$$

which implies that $\epsilon(a) = 1$. Secondly, suppose that the group-like elements in A are not linearly independent. Then, it is possible to express one group-like element e as

$$e = \sum_{i \in \mathcal{I}} c_i e_i,$$

where $c_i \in k$ and $e_i \neq e$. Actually, we may even suppose that the e_i are linearly independent, which gives us that the $e_i \otimes e_j$ are also linearly independent. We have

$$\Delta(e) = e \otimes e = \sum_{i,j \in \mathcal{I}} c_i c_j e_i \otimes e_j \qquad \Delta(e) = \sum_{i \in \mathcal{I}} c_i \Delta(e_i) = \sum_{i \in \mathcal{I}} c_i e_i \otimes e_i,$$

where the left equality is clear and the right one follows from the fact that Δ is a k-algebra morphism and the e_i are group-like. Thanks to the previous equalities,

$$c_i c_j = c_i \delta_{ij}$$

for all $i, j \in \mathcal{I}$. Now, thanks to the first observation, since $\epsilon(e) = 1$,

$$1 = \epsilon(e) = \sum_{i \in \mathcal{I}} c_i \epsilon(e_i) = \sum_{i \in \mathcal{I}} c_i,$$

so the c_i form a complete set of orthogonal idempotents in the field k. Hence, one of them must be equal to one and the rest equal to zero, which contradicts our assumption that e is not equal to any of the e_i. □

All we have to do now is prove the theorem, which follows directly from the lemma.

Theorem 4.3.1.3 (Decomposition in characters)**.** *Let $\Phi : G \to \mathrm{GL}_V$ be a representation of an affine algebraic group on a vector space V. If V is a sum of eigenspaces, for instance $V = \sum_{\chi \in \Xi} V_\chi$, where Ξ is a set of characters of G, then*

$$V = \bigoplus_{\chi \in \Xi} V_\chi$$

Proof. If the sum is not direct, there exists a finite set of characters $\{\chi_1, \ldots, \chi_m\}$ and a relation

$$\sum_{i=1}^{m} v_i = 0,$$

where $v_i \in V_{\chi_i}$ and $v_i \neq 0$. When we apply ρ to this relation, the fact that $V_{\chi_i} = \{v \in V : \rho(v) = v \otimes a(\chi_i)\}$ implies that

$$0 = \rho\left(\sum_{i=1}^{m} v_i\right) = \sum_{i=1}^{m} \rho(v_i) = \sum_{i=1}^{m} v_i \otimes a(\chi_i).$$

Also, for every linear map $f : V \to k$, $\sum_{i=1}^{m} f(v_i) a(\chi_i) = 0$, which contradicts the linear independence of the $a(\chi_i)$ and therefore Lemma 4.3.1.2. □

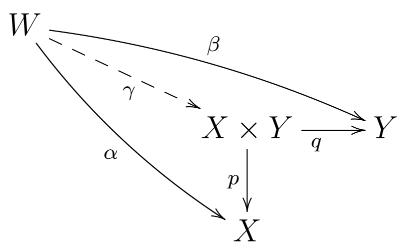
Chapter 5

Tannakian Categories

In this section, our main goal is to introduce the notion of a Tannakian category. It will also serve as a prelude in order to introduce several notions that come in handy throughout this last part of the work.

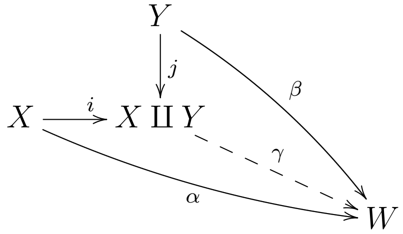
5.1 Abelian Categories

Definition 5.1.0.1. Let $\mathfrak{C}$ be a category and let $X, Y \in Ob(\mathfrak{C})$. We define a *product* of X and Y as an object $X \times Y \in Ob(\mathfrak{C})$ together with morphisms $p \in \mathrm{Hom}_{\mathfrak{C}}(X \times Y, X)$ and $q \in \mathrm{Hom}_{\mathfrak{C}}(X \times Y, Y)$ such that for all $W \in Ob(\mathfrak{C})$ and morphisms $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(W, X)$ and $\beta \in \mathrm{Hom}_{\mathfrak{C}}(W, Y)$ there is a unique $\gamma \in \mathrm{Hom}_{\mathfrak{C}}(W, X \times Y)$ making the next diagram commute.



We say that $\mathfrak{C}$ *has products of pairs of objects* if a product $X \times Y$ exists for any $X, Y \in Ob(\mathfrak{C})$.

Definition 5.1.0.2. Let $\mathfrak{C}$ be a category and let X, Y be a pair of objects in $\mathfrak{C}$. A *coproduct* of X and Y is an object $X \amalg Y \in Ob(\mathfrak{C})$ together with morphisms $i \in \mathrm{Hom}_{\mathfrak{C}}(X, X \amalg Y)$ and $j \in \mathrm{Hom}_{\mathfrak{C}}(Y, X \amalg Y)$ such that for any $W \in Ob(\mathfrak{C})$ and morphisms $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(X, W)$ and $\beta \in \mathrm{Hom}_{\mathfrak{C}}(Y, W)$ there is a unique $\gamma \in \mathrm{Hom}_{\mathfrak{C}}(X \amalg Y, W)$ making the diagram



commute. Also, we say that the category $\mathfrak{C}$ *has coproducts of pairs of objects* if a coproduct $X \amalg Y$ exists for any $X, Y \in Ob(\mathfrak{C})$.

Definition 5.1.0.3. Let $\mathfrak{C}$ be a category. We say that $X \in Ob(\mathfrak{C})$ is an *initial* object if for every object Y of $\mathfrak{C}$ there is exactly one morphism $X \to Y$. On the other hand, X is called a *final* object if for every object $Y \in Ob(\mathfrak{C})$ there is exactly one morphism $Y \to X$.

Definition 5.1.0.4. A category $\mathfrak{A}$ is called *preadditive* if each morphism set $\mathrm{Hom}_{\mathfrak{A}}(X, T)$ is endowed with the structure of an abelian group such that the compositions

$$\mathrm{Hom}_{\mathfrak{A}}(X, Y) \times \mathrm{Hom}_{\mathfrak{A}}(Y, Z) \longrightarrow \mathrm{Hom}_{\mathfrak{A}}(X, Z)$$

are bilinear. A functor $F : \mathfrak{A} \to \mathfrak{B}$ of preadditive categories is called *additive* if and only if $F : \mathrm{Hom}_{\mathfrak{A}}(X, Y) \to \mathrm{Hom}_{\mathfrak{B}}(F(X), F(Y))$ is a homomorphism of abelian groups for all $X, Y \in Ob(\mathfrak{A})$.

Lemma 5.1.0.5. Let $\mathfrak{A}$ be a preadditive category and let X be an object of $\mathfrak{A}$. The following are equivalent

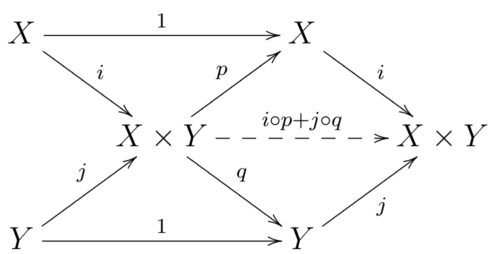
1. X is an initial object,
2. X is a final object, and
3. $\mathrm{id}_X = 0$ in $\mathrm{Hom}_{\mathfrak{A}}(X, X)$.

Any object that is both final and initial is called *zero object*, and is denoted by 0.

The proof of 5.1.0.5 is left as an exercise for the reader, all that needs to be done is apply the fact that each $\mathrm{Hom}_{\mathfrak{A}}(X, Y)$ has the structure of an abelian group and the result easily follows.

Lemma 5.1.0.6. Let $\mathfrak{A}$ be a preadditive category and let $X, Y \in Ob(\mathfrak{A})$. Then, $X \times Y$ exists if and only if $X \amalg Y$ does. In addition, if any of them exists, then $X \times Y \simeq X \amalg Y$.

Proof. Suppose that $X \times Y$ exists, together with projections $p : X \times Y \to X$ and $q : X \times Y \to Y$. Let $i : X \to X \times Y$ be the morphism corresponding to $(1, 0)$ and $j : Y \to X \times Y$ the morphism corresponding to $(0, 1)$. Thus we have the commutative diagram



where the diagonal compositions are zero. It follows that $i \circ p + j \circ q : X \times Y \to X \times Y$ is the identity since it is a morphism such that when it is composed with p it gives p and when it is composed with q gives q. Let $a : X \to W$ and $b : Y \to W$ be two morphisms. Then we can form the map $a \circ p + b \circ q : X \times Y \to W$. In this way we get a bijection $\mathrm{Hom}_{\mathfrak{A}}(X \times Y, W) \simeq \mathrm{Hom}_{\mathfrak{A}}(X, W) \times \mathrm{Hom}_{\mathfrak{A}}(Y, W)$ from which we obtain that that $X \times Y \simeq X \amalg Y$. The other case, that is, the one in which we work with a coproduct, can be easily seen using an analogous reasoning. □

Definition 5.1.0.7. Let $\mathfrak{A}$ be a preadditive category and let $X, Y \in Ob(\mathfrak{A})$. The product $X \times Y$ endowed with the morphisms i, j, p, q from the proof of the previous lemma is called *direct sum* and is denoted as $X \oplus Y$. Furthermore, $\mathfrak{A}$ is called *additive* if it is preadditive and finite products exist, that is, it has a zero object and direct sums.

Definition 5.1.0.8. Let $\mathfrak{A}$ be a preadditive category. Let $f \in \mathrm{Hom}_{\mathfrak{A}}(X, Y)$.

1. A *kernel* of f is a morphism $i : Z \to X$ such that
   (a) $f \circ i = 0$ and
   (b) for any $i' : Z' \to X$ such that $f \circ i' = 0$ there exists a unique morphism $g : Z' \to Z$ such that $i' = i \circ g$.

   If a kernel of f exists, then it is unique, up to a unique isomorphism and we denote it $\mathrm{Ker}(f) \to X$.

2. We define a *cokernel* of f as a morphism $p : Y \to Z$ such that
   (a) $p \circ f = 0$ and
   (b) for any $p' : Y \to Z'$ such that $p' \circ f = 0$ there exists a unique morphism $g : Z \to Z'$ such that $p' = g \circ p$.

   If a cokernel of f exists, then it is unique, up to a unique isomorphism and we denote it $Y \to \mathrm{Coker}(f)$.

3. If a kernel of f exists, then a *coimage of* f is a cokernel for the morphism $\mathrm{Ker}(f) \to X$. If a kernel and a coimage exist then we denote this $X \to \mathrm{Coim}(f)$.

4. If a cokernel of f exists, then the *image of* f is a kernel of the morphism $Y \to \mathrm{Coker}(f)$. If a cokernel and an image of f exist then we denote this $\mathrm{Im}(f) \to Y$.

Lemma 5.1.0.9. Let $\mathfrak{A}$ be a preadditive category, let $X, Y \in Ob(\mathfrak{A})$ and let $f \in \mathrm{Hom}_{\mathfrak{A}}(X, Y)$ such that kernel, cokernel, image and coimage exist. Then, f can be factored uniquely as

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \uparrow \\ \mathrm{Coim}(f) & \longrightarrow & \mathrm{Im}(f). \end{array}$$

Proof. Due to the definition of kernel, we know $\ker(f) \to X \to Y$ is zero, thus we have a canonical morphism $\mathrm{Coim}(f) \to Y$. On the other hand, $\mathrm{Coim}(f) \to Y \to \mathrm{Coker}(f)$ is zero, because it is the unique morphism that gives rise to $X \to Y \to \mathrm{Coker}(f)$, which is zero. Therefore, $\mathrm{Coim}(f) \to Y$ factors uniquely through $\mathrm{Im}(f) \to Y$, therefore we have the map we are looking for. □

Definition 5.1.0.10. A category $\mathfrak{A}$ is *abelian* if it is additive, if all kernels and cokernels exist, and if the natural map from 5.1.0.9, $\mathrm{Coim}(f) \to \mathrm{Im}(f)$, is an isomorphism for all morphisms f of $\mathfrak{A}$.

5.2 Symmetric Monoidal Categories

Definition 5.2.0.1. Let $\mathfrak{C}$ be a category. $\mathfrak{C}$ is a *k-linear category* if every morphism set is a k-module and for every $X, Y, Z \in Ob(\mathfrak{C})$, the composition law

$$\mathrm{Hom}_{\mathfrak{C}}(Y, Z) \times \mathrm{Hom}_{\mathfrak{C}}(X, Y) \to \mathrm{Hom}_{\mathfrak{C}}(X, Z)$$

is k-bilinear.

Definition 5.2.0.2. A monoidal category $\tilde{\mathfrak{C}} = (\mathfrak{C}, \otimes, \alpha, \mathbb{1}, l, r)$ consists on the following data

1. A k-linear category $\mathfrak{C}$,
2. A functor $\otimes : \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$ called the *tensor product* of $\mathfrak{C}$,
3. An isomorphism of functors $\alpha : \otimes \circ (\mathrm{id}_{\mathfrak{C}} \times \otimes) \to \otimes \circ (\otimes \times \mathrm{id}_{\mathfrak{C}})$ and
4. An object $\mathbb{1} \in Ob(\mathfrak{C})$ and isomorphisms $l : \mathrm{id}_{\mathfrak{C}} \to \mathbb{1} \otimes \mathrm{id}_{\mathfrak{C}}$ and $r : \mathrm{id}_{\mathfrak{C}} \to \mathrm{id}_{\mathfrak{C}} \otimes \mathbb{1}$, called *right unit* and *left* unit, respectively,

And such that the previous data satisfies the *pentagon axiom*, that is, for all $X, Y, Z, W \in Ob(\mathfrak{C})$, the following diagram commutes

$$\begin{array}{ccc} & W \otimes (X \otimes (Y \otimes Z)) & \\ {\scriptstyle \mathrm{id}_W \otimes \alpha_{X,Y,Z}} \swarrow & & \searrow {\scriptstyle \alpha_{W,X,Y \otimes Z}} \\ W \otimes ((X \otimes Y) \otimes Z) & & (W \otimes X) \otimes (Y \otimes Z) \\ \downarrow {\scriptstyle \alpha_{W,X \otimes Y,Z}} & & \downarrow {\scriptstyle \alpha_{W \otimes X,Y,Z}} \\ (W \otimes (X \otimes Y)) \otimes Z & \xrightarrow{\alpha_{W,X,Y} \otimes \mathrm{id}_Z} & ((W \otimes X) \otimes Y) \otimes Z. \end{array}$$

Furthermore, if we let $\tilde{\mathfrak{C}} = (\mathfrak{C}, \otimes_{\mathfrak{C}}, \alpha^{\mathfrak{C}}, \mathbb{1}_{\mathfrak{C}}, l^{\mathfrak{C}}, r^{\mathfrak{C}})$ and $\tilde{\mathfrak{D}} = (\mathfrak{D}, \otimes_{\mathfrak{D}}, \alpha^{\mathfrak{D}}, \mathbb{1}_{\mathfrak{D}}, l^{\mathfrak{D}}, r^{\mathfrak{D}})$ be monoidal categories, a monoidal functor $\tilde{\mathfrak{C}} \to \tilde{\mathfrak{D}}$ is a triple (F, J, J_0) such that

1. $F\colon \mathfrak{C} \to \mathfrak{D}$ is a functor,
2. $J : \otimes_{\mathfrak{D}} \circ (F \times F) \to F \circ \otimes_{\mathfrak{C}}$ is a functor isomorphism and

3. $J_0 : F(\mathbb{1}_{\mathfrak{C}}) \to \mathbb{1}_{\mathfrak{D}}$ is an isomorphism

and for every $X, Y, Z \in Ob(\mathfrak{C})$, the following diagrams commute

$$\begin{array}{ccccc} F(X)\otimes_{\mathfrak D}(F(Y)\otimes_{\mathfrak D}F(Z)) & \xrightarrow{\mathrm{id}_{F(X)}\otimes_{\mathfrak D}J_{Y,Z}} & F(X)\otimes_{\mathfrak D}F(Y\otimes_{\mathfrak C}Z) & \xrightarrow{J_{X,Y\otimes_{\mathfrak C}Z}} & F(X\otimes_{\mathfrak C}(Y\otimes_{\mathfrak C}Z)) \\ \downarrow{\scriptstyle \alpha^{\mathfrak D}_{F(X),F(Y),F(Z)}} & & & & \downarrow{\scriptstyle F(\alpha^{\mathfrak C}_{X,Y,Z})} \\ (F(X)\otimes_{\mathfrak D}F(Y))\otimes_{\mathfrak D}F(Z) & \xrightarrow[J_{X,Y}\otimes_{\mathfrak D}\mathrm{id}_{F(Z)}]{} & F(X\otimes_{\mathfrak C} Y)\otimes_{\mathfrak D}F(Z) & \xrightarrow[J_{X\otimes_{\mathfrak C}Y,Z}]{} & F((X\otimes_{\mathfrak C} Y)\otimes_{\mathfrak C} Z). \end{array}$$

$$\begin{array}{ccc} F(\mathbb{1}_{\mathfrak C})\otimes_{\mathfrak D}F(X) & \xrightarrow{J_{\mathbb{1}_{\mathfrak C},X}} & F(\mathbb{1}_{\mathfrak C}\otimes_{\mathfrak C}X) \\ \downarrow{\scriptstyle J_0\otimes\mathrm{id}_{F(X)}} & & \downarrow{\scriptstyle F(l^{\mathfrak C}_X)} \\ \mathbb{1}_{\mathfrak D}\otimes_{\mathfrak D}F(X) & \xrightarrow[l^{\mathfrak D}_{F(X)}]{} & F(X), \end{array} \qquad \begin{array}{ccc} F(X)\otimes_{\mathfrak D}F(\mathbb{1}_{\mathfrak C}) & \xrightarrow{J_{X,\mathbb{1}_{\mathfrak C}}} & F(X\otimes_{\mathfrak C}\mathbb{1}_{\mathfrak C}) \\ \downarrow{\scriptstyle \mathrm{id}_{F(X)}\otimes_{\mathfrak D}J_0} & & \downarrow{\scriptstyle F(r^{\mathfrak C}_X)} \\ F(X)\otimes_{\mathfrak D}\mathbb{1}_{\mathfrak D} & \xrightarrow[r^{\mathfrak D}_{F(X)}]{} & F(X) \end{array}$$

From now on, we will denote the functor $\mathfrak{C} \times \mathfrak{C} \to \mathfrak{C} \times \mathfrak{C}$ that permutes the factors by Π. Notice that we can form the *opposite monoidal category* $\tilde{\mathfrak{C}}^{\mathrm{op}}$ of a monoidal category $\tilde{\mathfrak{C}} = (\mathfrak{C}, \otimes, \alpha, \mathbb{1}, l, r)$ by setting

$$\tilde{\mathfrak{C}}^{\mathrm{op}} := (\mathfrak{C}, \otimes \circ \Pi, \alpha^{-1} \circ \Pi, \mathbb{1}, l, r).$$

Definition 5.2.0.3. Let $\tilde{\mathfrak{C}} = (\mathfrak{C}, \otimes, \alpha, \mathbb{1}, l, r)$ be a monoidal category. A symmetry of $\tilde{\mathfrak{C}}$ is an isomorphism of functors $\beta : \otimes \to \otimes \circ \Pi$ such that $\beta(\Pi) \circ \beta : \otimes \to \otimes$ coincides with the identity on $\otimes$ and for every $X, Y, Z \in Ob(\mathfrak{C})$, the diagram

$$\begin{array}{ccccc} & & (Y\otimes Z)\otimes X & & \\ & \nearrow{\scriptstyle \beta_{Z,Y}\otimes\mathrm{id}_X} & & \searrow{\scriptstyle \beta_{Y\otimes Z,X}} & \\ (Z\otimes Y)\otimes X & & & & X\otimes(Y\otimes Z) \\ \downarrow{\scriptstyle \alpha^{-1}_{Z,Y,X}} & & & & \downarrow{\scriptstyle \alpha_{X,Y,Z}} \\ Z\otimes(Y\otimes X) & & & & (X\otimes Y)\otimes Z \\ & \searrow{\scriptstyle \mathrm{id}_Z\otimes\beta_{Y,X}} & & \nearrow{\scriptstyle \beta_{Z,X\otimes Y}} & \\ & & Z\otimes(X\otimes Y) & & \end{array}$$

commutes. This property is called the *hexagon axiom.* A *symmetric monoidal category* consists of the data $\tilde{\mathfrak{C}} = (\mathfrak{C}, \otimes, \alpha, \mathbb{1}, l, r; \beta)$ of a monoidal category $\tilde{\mathfrak{C}}$ and a fixed symmetry β on $\tilde{\mathfrak{C}}$. In addition, if $\tilde{\mathfrak{C}} = (\mathfrak{C}, \otimes_{\mathfrak{C}}, \alpha^{\mathfrak{C}}, \mathbb{1}_{\mathfrak{C}}, l^{\mathfrak{C}}, r^{\mathfrak{C}}; \beta^{\mathfrak{C}})$ and $\tilde{\mathfrak{D}} = (\mathfrak{D}, \otimes_{\mathfrak{D}}, \alpha^{\mathfrak{D}}, \mathbb{1}_{\mathfrak{D}}, l^{\mathfrak{D}}, r^{\mathfrak{D}}; \beta^{\mathfrak{D}})$ are symmetric monoidal categories, a *symmetric monoidal functor* $\tilde{\mathfrak{C}} \to \tilde{\mathfrak{D}}$ is a monoidal functor (F, J, J_0) such that for every $X, Y \in Ob(\mathfrak{C})$, the following diagram commutes

$$\begin{array}{ccc} F(X)\otimes_{\mathfrak D}F(Y) & \xrightarrow{J_{X,Y}} & F(X\otimes_{\mathfrak C}Y) \\ \downarrow{\scriptstyle \beta^{\mathfrak D}_{F(X),F(Y)}} & & \downarrow{\scriptstyle F(\beta^{\mathfrak C}_{X,Y})} \\ F(Y)\otimes_{\mathfrak D}F(X) & \xrightarrow{J_{Y,X}} & F(Y\otimes_{\mathfrak C}X). \end{array}$$

Remark 5.2.0.4. In many books of the literature, the construction of symmetric monoidal categories is slightly different. For instance, in [Del] and [Mil3], we take a category $\mathfrak{C}$ together with a tensor product $\otimes : \mathfrak{C}^2 \to \mathfrak{C}$, an *associativity constraint* $\phi_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$, which is in fact $\phi = \alpha$ from the definition, a *commutativity constraint*, which is a functorial isomorphism $\psi_{X,Y} : X \otimes Y \to Y \otimes X$ such that $\psi_{Y,X} \circ \psi_{X,Y} : X \otimes Y \to X \otimes Y$ which is clearly equivalent to the symmetry in 5.2.0.3. Instead of using it as a definition, ϕ and ψ are said to be *compatible* if they satisfy the hexagon axiom. Finally, Deligne and Milne define an *identity object* of $(\mathfrak{C}, \otimes)$ as a pair (U, u) such that $U \in Ob(\mathfrak{C})$ is an object satisfying that $X \mapsto U \otimes X$ is an equivalence of categories and $u : U \to U \otimes U$ is an isomorphism, in order to define a *tensor category* as a system $(\mathfrak{C}, \otimes, \phi, \psi)$ where ϕ and ψ are compatible associativity and commutativity constraints respectively and there exists an identity objects. In proposition 1.3 of [Mil3], we see how to construct a symmetric monoidal category using the existence of an identity object.

5.3 Internal homomorphisms and rigidity

The goal of this section is to give a short introduction to internal homomorphisms and rigidity over symmetric monoidal categories, which allow us to endow such categories with the notion of a ”dual”.

Definition 5.3.0.1. Let $\tilde{\mathfrak{C}}$ be a monoidal category, and let $X, Y \in Ob(\mathfrak{C})$. If

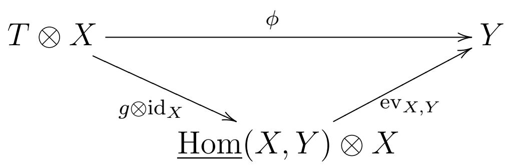
$$\mathrm{Hom}_{\mathfrak{C}}(- \otimes X, Y) : \mathfrak{C}^{\mathrm{opp}} \to \mathbf{Set}$$

is representable, then we denote the representing object by $\underline{\mathrm{Hom}}(X, Y)$ and call it the *internal homomorphism* of X and Y.

Now, assume that $\underline{\mathrm{Hom}}(X, Y)$ exists for a pair of objects $X, Y \in Ob(\mathfrak{C})$. Then, we have a natural isomorphism

$$\mathrm{Hom}_{\mathfrak{C}}(\underline{\mathrm{Hom}}(X, Y) \otimes X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{C}}(\underline{\mathrm{Hom}}(X, Y), \underline{\mathrm{Hom}}(X, Y)),$$

given by the definition of internal homomorphism and what we call the *evaluation map*, $\mathrm{ev}_{X,Y} : \underline{\mathrm{Hom}}(X, Y) \otimes X \to Y$, which corresponds to $\mathrm{id}_{\underline{\mathrm{Hom}}(X,Y)}$ via the previous isomorphism.. Hence, for every $\phi : T \otimes X \to Y$, there exists a unique morphism $\psi : T \to \underline{\mathrm{Hom}}(X, Y)$ such that the following diagram commutes.



In the next remark, we see a property of internal homomorphisms, that will help us find isomorphisms between them.

Remark 5.3.0.2. Assume that internal homomorphisms exists for any pair of objects in $\mathfrak{C}$. Let $X, Y, Z \in Ob(\mathfrak{C})$. Then, for every $T \in Ob(\mathfrak{C})$, we have the following sequence of isomorphisms

$$\mathrm{Hom}_{\mathfrak{C}}(T, \underline{\mathrm{Hom}}(Z, \underline{\mathrm{Hom}}(X, Y)) \simeq \mathrm{Hom}_{\mathfrak{C}}(T \otimes Z, \underline{\mathrm{Hom}}(X, Y)) \simeq \mathrm{Hom}_{\mathfrak{C}}((T \otimes Z) \otimes X, Y)$$

and the latter is isomorphic to $\mathrm{Hom}_{\mathfrak{C}}(T, \underline{\mathrm{Hom}}(Z \otimes X, Y)$, which leads us to the fact that

$$\underline{\mathrm{Hom}}(Z \otimes X, Y) \simeq \underline{\mathrm{Hom}}(Z, \underline{\mathrm{Hom}}(X, Y))$$

Definition 5.3.0.3. Let $\tilde{\mathfrak{C}}$ be a monoidal category and let X be an object of $\mathfrak{C}$. The dual of X is $\underline{\mathrm{Hom}}(X, \mathbb{1})$ if it exists, and we denote it as $X^\vee$.

If $X^\vee$ exists, we can rewrite the evaluation map as $\mathrm{ev}_X : X^\vee \otimes X \to \mathbb{1}$.

Remark 5.3.0.4. Assume that $\tilde{\mathfrak{C}}$ is a symmetric category. Clearly, we have a morphism $\mathrm{ev}_X \circ \beta_{X,X^\vee} : X \otimes X^\vee \to \mathbb{1}$, but since $\mathrm{Hom}_{\mathfrak{C}}(X \otimes X^\vee) \simeq \mathrm{Hom}_{\mathfrak{C}}(X, (X^\vee)^\vee)$, there must exist a unique map $i_X \in \mathrm{Hom}_{\mathfrak{C}}(X, (X^\vee)^\vee)$ that corresponds to $\mathrm{ev}_X \circ \beta_{X,X^\vee}$.

The following definition gives sense to what we meant by the beginning of the subsection when we talked about the notion of a dual.

Definition 5.3.0.5. Let $X \in Ob(\mathfrak{C})$. If the map i_X of the previous definition is an isomorphism, we say that X is *reflexive*.

Of course, it is important to see if we can also endow the duals with a notion of "dual map". We solve the question in the following remark.

Remark 5.3.0.6. Let $X, Y \in Ob(\mathfrak{C})$, and assume that $X^\vee$ and $Y^\vee$ exist. Let $f \in \mathrm{Hom}_{\mathfrak{C}}(X, Y)$. Then, there exists a unique map $f^\vee : Y^\vee \to X^\vee$ such that the diagram

$$\begin{array}{ccc} Y^\vee \otimes X & \xrightarrow{f^\vee \otimes \mathrm{id}_X} & X^\vee \otimes X \\ \downarrow{\scriptstyle \mathrm{id}_{Y^\vee} \otimes f} & & \downarrow{\scriptstyle \mathrm{ev}_X} \\ Y^\vee \otimes Y & \xrightarrow[\mathrm{ev}_Y]{} & \mathbb{1} \end{array}$$

commutes. It is also important to notice that if X, Y are reflexive, $(f^\vee)^\vee = f$.

Finally, we can construct the following natural maps, which will give us a bijective correspondence between the tensor product of internal homomorphisms between the elements of a family and the internal homomorphism of the product of the elements. Let $\{(X_i, Y_i)\}_{1 \leq i \leq n}$ be a family of pairs of $X_i, Y_i \in Ob(\mathfrak{C})$. Then,

$$\bigotimes_{1 \leq i \leq n} \underline{\mathrm{Hom}}(X_i, Y_i) \otimes \bigotimes_{1 \leq i \leq n} X_i \simeq \bigotimes_{1 \leq i \leq n} \underline{\mathrm{Hom}}(X_i, Y_i) \otimes X_i \to \bigotimes_{1 \leq i \leq n} Y_i$$

where the first isomorphism follows from the associativity and commutativity constraints and the second morphism is given by $\mathrm{ev}_{X_1,Y_1} \otimes \cdots \otimes \mathrm{ev}_{X_n,Y_n}$, and this previous map corresponds to

$$\bigotimes_{1 \leq i \leq n} \underline{\mathrm{Hom}}(X_i, Y_i) \to \underline{\mathrm{Hom}}(\bigotimes_{1 \leq i \leq n} X_i, \bigotimes_{1 \leq i \leq n} Y_i).$$

Thus, we can give the following definition.

Definition 5.3.0.7. Let $\tilde{\mathfrak{C}}$ be a symmetric monoidal category. $\tilde{\mathfrak{C}}$ is *rigid* if it satisfies

1. For any $X, Y \in Ob(\mathfrak{C})$, the internal homomorphism $\underline{\mathrm{Hom}}(X, Y)$ exists.
2. Every $X \in Ob(\mathfrak{C})$ is reflexive.
3. For any family $\{(X_i, Y_i)\}_{1 \leq i \leq n}$, where $X_i, Y_i \in Ob(\mathfrak{C})$, the map

$$\bigotimes_{1 \leq i \leq n} \underline{\mathrm{Hom}}(X_i, Y_i) \to \underline{\mathrm{Hom}}(\bigotimes_{1 \leq i \leq n} X_i, \bigotimes_{1 \leq i \leq n} Y_i)$$

is an isomorphism.

And from the definitions it follows immediately that

Proposition 5.3.0.8. *Let $\tilde{\mathfrak{C}}$ be a rigid symmetric monoidal category. Then, for any $X_1, \ldots, X_n, X, Y \in Ob(\mathfrak{C})$,*

1. $X^\vee \otimes Y \simeq \underline{\mathrm{Hom}}(X, Y)$*, and*
2. $\bigotimes_{1 \leq i \leq n} X_i^\vee \simeq \left(\bigotimes_{1 \leq i \leq n} X_i\right)^\vee$*.*

Proof. For the first point, all we have to do is apply the third point in 5.3.0.7, that is,

$$X^\vee \otimes Y \simeq \underline{\mathrm{Hom}}(X, \mathbb{1}) \otimes \underline{\mathrm{Hom}}(\mathbb{1}, Y) \xrightarrow{\sim} \underline{\mathrm{Hom}}(X, Y).$$

For the second point, we apply the same isomorphism,

$$\bigotimes_{1 \leq i \leq n} X_i^\vee = \bigotimes_{1 \leq i \leq n} \underline{\mathrm{Hom}}(X_i, \mathbb{1}) \simeq \underline{\mathrm{Hom}}\left(\bigotimes_{1 \leq i \leq n} X_i, \mathbb{1}\right) = \left(\bigotimes_{1 \leq i \leq n} X_i\right)^\vee.$$

□

One important property of rigid symmetric monoidal categories is the following. Actually, we need it in order to show Theorem 6.2.0.7.

Proposition 5.3.0.9. *Let $\tilde{\mathfrak{C}}$ and $\tilde{\mathfrak{D}}$ be rigid symmetric monoidal categories, and let $F, G : \tilde{\mathfrak{C}} \to \tilde{\mathfrak{D}}$ be monoidal functors. Then, every morphism $\Phi : F \to G$ is an isomorphism of functors.*

Proof. Proposition 1.13 of [Mil3]. □

In the end of the next chapter we see that the following definition is a particular case of a more general definition, although this one is the adequate for the case of affine algebraic groups.

Definition 5.3.0.10. Let $\tilde{\mathfrak{C}} = (\mathfrak{C}, \otimes, \alpha, \mathbb{1}, l, r; \beta)$ be a symmetric abelian monoidal category. A *fiber functor* is an exact and faithful monoidal functor $(F, J, J_0) : \tilde{\mathfrak{C}} \to \mathbf{fVec}_k$, where $\mathbf{fVec}_k$ denotes the category of finite dimensional k-vector spaces. Also, a *Tannakian category* is an abelian, rigid, symmetric monoidal category admitting a fiber functor. Furthermore, we say that a tannakian category is *neutral* if its unit $\mathbb{1}$ satisfies $\mathrm{End}(\mathbb{1}) \simeq k$.

We conclude this section with another important definition. The reason why this definition is relevant is because in many cases we can reduce the study of a given Tannakian category $\tilde{\mathfrak{C}}$ to the study of its full subcategories "tensor generated" by a given element.

Definition 5.3.0.11. Let $\mathfrak{C}$ be a Tannakian category and let $X \in Ob(\mathfrak{C})$. Let $\langle X \rangle_\otimes$ be the full subcategory whose objects are the quotients of elements of the form

$$\left(\bigoplus_{i \in I} X^{\otimes_{\mathfrak{C}} n_i} \otimes_{\mathfrak{C}} (X^\vee)^{\otimes_{\mathfrak{C}} m_i} \right).$$

We call $\langle X \rangle_\otimes$ the category *tensor generated* or *tensor subgenerated* by X.

In fact, in Corollary 6.20 of [Del], Deligne shows the following result.

Proposition 5.3.0.12. *If an abelian, rigid, k-linear symmetric monoidal category $\tilde{\mathfrak{C}}$ admits full subcategory tensor generated by an element of $\mathfrak{C}$ and the latter admits a fibre functor, then $\tilde{\mathfrak{C}}$ admits a fibre functor over a finite convenient extension K of k.*

5.4 Examples of tannakian categories

Let us see a few examples of Tannakian Categories. Here, we see that all of them are equivalent (in the categorical sense) to the category of representations of an affine algebraic groups, which leads us to a natural question: Are all Tannakian categories equivalent to $\mathbf{Rep}(G)$ for a certain affine algebraic group G? In the next chapter we see that the answer is yes.

Proposition 5.4.0.1. *The category of linear representations of an affine algebraic group G, $\mathbf{Rep}(G)$ is a Tannakian Category.*

Proof. From 4.1.0.3, we have that $\mathbf{Rep}(G)$ forms a category. Also, we obtain that the tensor product is $\otimes := \otimes_G$, hence we can define α, l and r easily. From this follows that $\mathbf{Rep}(G)$ is a monoidal category. On the other hand, we also saw in 4.1.0.3 how to build, for any $\mathcal{M}, \mathcal{N} \in Ob(\mathbf{Rep}(G))$, a canonical isomorphism $\mathcal{M} \otimes_G \mathcal{N} \to \mathcal{N} \otimes_G \mathcal{M}$, which defines the symmetry of $\mathbf{Rep}(G)$, therefore we have that it is a symmetric monoidal category.

Now let us see that it is rigid. All we have to do is notice that for any $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3 \in Ob(\mathbf{Rep}(G))$, we have a canonical isomorphism

$$\mathrm{Hom}_{\mathbf{Rep}(G)}(\mathcal{M}_1 \otimes_G \mathcal{M}_2, \mathcal{M}_3) \simeq \mathrm{Hom}_{\mathbf{Rep}(G)}(\mathcal{M}_1, \mathrm{Hom}(\mathcal{M}_2, \mathcal{M}_3)),$$

hence the functor $\mathrm{Hom}_{\mathbf{Rep}(G)}(- \otimes_G \mathcal{M}, \mathcal{N}) : \mathbf{Rep}(G)^{\mathrm{opp}} \to \mathbf{Set}$ is represented by $\mathrm{Hom}(\mathcal{M}, \mathcal{N}) =: \underline{\mathrm{Hom}}(\mathcal{M}, \mathcal{N})$, so we have the existence of internal homomorphisms for each $\mathcal{M}, \mathcal{N} \in Ob(\mathbf{Rep}(G))$. Clearly, all $\mathcal{M} \in Ob(\mathbf{Rep}(G))$ are reflexive, because of the natural isomorphism

$$\underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(X, \mathbb{1}), \mathbb{1}) \simeq X.$$

Thus, the fact that $\mathbf{Rep}(G)$ is rigid follows from the fact that for a family $\{(\mathcal{M}_i, \mathcal{N}_i)\}_{0\leq i\leq n}$ of pairs of $\mathcal{M}_i, \mathcal{N}_i \in \mathbf{Rep}(G)$, $\mathrm{ev}_{\mathcal{M}_1,\mathcal{N}_1} \otimes \cdots \otimes \mathrm{ev}_{\mathcal{M}_n,\mathcal{N}_n}$ always gives an isomorphism between

$$\bigotimes_{1\leq i\leq n} \underline{\mathrm{Hom}}(\mathcal{M}_i, \mathcal{N}_i) \otimes M_i \to \bigotimes_{1\leq i\leq n} \mathcal{N}_i,$$

thus

$$\bigotimes_{1\leq i\leq n} \underline{\mathrm{Hom}}(\mathcal{M}_i, \mathcal{N}_i) \simeq \underline{\mathrm{Hom}}\left(\bigotimes_{1\leq i\leq n} \mathcal{M}_i, \bigotimes_{1\leq i\leq n} \mathcal{N}_i\right).$$

Eventually, in order to find the fiber functor, we take the forgetful functor $F : \mathbf{Rep}(G) \to k-\mathbf{Mod}$, and notice that we can give it a structure of monoidal functor by setting $J_{\mathcal{M},\mathcal{N}} := \mathrm{id}_{\mathcal{M}\otimes\mathcal{N}} : \mathcal{M}\otimes\mathcal{N} \to \mathcal{M}\otimes\mathcal{N}$ and $J_0 := \mathrm{id}_k : k \to k$. Clearly, it is symmetric, exact and faithful. □

Now let's see two other interesting examples. To start, let $\mathbf{GradVec}_k$ denote the category of graded vector spaces. Recall that any object in this category is a family of k-vector spaces $(V^n)_{n\in\mathbb{Z}}$ such that the family has a finite dimensional sum $V := \bigoplus_{n\in\mathbb{Z}} V^n$.

Proposition 5.4.0.2. *The category of graded vector spaces* $\mathbf{GradVec}_k$ *is a Tannakian category.*

Proof. All we have to do is define the tensor product $\otimes : \mathbf{GradVec}_k \times \mathbf{GradVec}_k \to \mathbf{GradVec}_k$ by taking $(V^n)_n, (W^n)_n \in Ob(\mathbf{GradVec}_k)$ and setting

$$(V^n)_n \otimes (W^n)_n := \left(\bigoplus_{i\in\mathbb{Z}} V^i \otimes_k W^{n-i}\right).$$

We can set the unit to be $\mathbb{1} := (k^{\delta_{n,0}})_n \in Ob(\mathbf{GradVec}_k)$. Clearly, this is a symmetric monoidal category. On the other hand, the internal homomorphisms are

$$\underline{\mathrm{Hom}}((V^n)_n, (W^n)_n) = \left(\bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}(V^i, W^{i+n})\right)_n,$$

and if we set the fiber functor to be the forgetful functor $F : \mathbf{GradVec}_k \to k-\mathbf{Mod}$ given by $(V^n)_n \mapsto V$, the rest of points are clear. □

Remark 5.4.0.3. Furthermore, it is interesting to notice that this category is actually equivalent to the category of representations of $\mathbb{G}_m$. All we have to do is notice that each $(V^n)_n$ corresponds to the character χ that sends each λ to λ^n.

One last example is that of Hodge structures. If we let $k = \mathbb{R}$, then a real Hodge structure is a finite dimensional $\mathbb{R}$-vector space V such that it admits a decomposition

$$V \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p,q} V^{p,q},$$

where $V^{p,q}$ and $V^{q,p}$ are complex conjugate subspaces of $V \otimes_{\mathbb{R}} \mathbb{C}$, $\overline{V^{p,q}} = V^{q,p}$. If we let the objects of the category of Hodge structures $\mathfrak{HS}_{\mathbb{R}}$ be the real Hodge structures, then we notice that $\mathfrak{HS}_{\mathbb{R}}$ is a Tannakian category, together with the fiber functor F given by $F : (V, V^{p,q}) \to V$. The proof of this is left to the reader, as it is easy to check, that can be done by following the same steps as in the previous proposition.

Remark 5.4.0.4. It is easy to see that the category of real Hodge structures $\mathfrak{HS}_{\mathbb{R}}$ is equivalent to the category of representations of the real algebraic group, that is, the algebraic group $\mathbb{S}$ represented by the $\mathbb{C}$-algebra $\mathbb{C}[X, 1/X]$, because any real Hodge structure $(V, (V^{p,q}))$ corresponds to the representation of $\mathbb{S}$ on V, by setting that $\lambda \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^{\times}$ acts on each $V^{p,q}$ by $\lambda^{-p}\overline{\lambda}^{-q}$.

Chapter 6

Tannaka-Krein duality

In this chapter we prove Tannaka-Krein's duality theorem and then we see that any tannakian category is equivalent to the category of representations of a certain algebraic group. These two results are the cornerstone of the memoir, because they allow us to relate all the theory we have built up to this point to differential Galois theory, among other branches of mathematics.

6.1 The tannakian reconstruction

Let G be an affine algebraic group. We know that the tannakian category $\mathbf{Rep}(G)$ comes with a fiber functor $\omega : \mathbf{Rep}(G) \to \mathbf{fVec}_k$, that sends $\mathcal{M} \mapsto M$, hence we can extend it, for each $A \in Ob(\mathbf{Alg}_k)$, to a functor $\omega_A : \mathbf{Rep}(G) \to \mathbf{Mod}_A$ given by $\mathcal{M} \mapsto M \otimes_k A$. Let us see that this functor has good properties.

Lemma 6.1.0.1. For each $A \in Ob(\mathbf{Alg}_k)$, ω_A is a monoidal functor.

Proof. All we have to do is notice that $\mathbf{Mod}_A$ is a symmetric monoidal category together with the tensor product $\otimes := \otimes_A$. The diagrams from the definition of monoidal functor commute thanks to the properties of $\mathbf{Rep}(G)$ and $\mathbf{Mod}_A$. □

Clearly, ω_A induces a functor $\mathrm{Aut}^{\otimes}(\omega) : \mathbf{Alg}_k \to \mathbf{Grp}$, defined by

$$\mathrm{Aut}^{\otimes}(\omega)(A) := \{(\lambda_{\mathcal{M}})_{\mathcal{M} \in Ob(\mathbf{Rep}(G))} : \lambda_{\mathcal{M}} \in \mathrm{Aut}_A(M \otimes_k A),$$

$$\forall \alpha \in \mathrm{Hom}_{\mathbf{Rep}(G)}(\mathcal{M}, \mathcal{N}), (\alpha(k) \otimes \mathrm{id}_A) \circ \lambda_{\mathcal{M}} = \lambda_{\mathcal{N}} \circ (\alpha(k) \otimes \mathrm{id}_A)\}.$$

Notice as well that if we take $g \in G(A)$, we can obtain $\tilde{g} \in \mathrm{Aut}^{\otimes}(\omega)(A)$ by setting, for each $\mathcal{M} \in \mathbf{Rep}(G)$, $\tilde{g}_{\mathcal{M}}$ to be the action of g on $\mathbb{W}_M(A)$, which gives us a functorial relationship, hence $G \to \mathrm{Aut}^{\otimes}(\omega)$ is a functor homomorphism.

Our goal now is to show that $G \to \mathrm{Aut}^{\otimes}(\omega)$ is an isomorphism. In order to do so, we need the following lemma.

Lemma 6.1.0.2. Let V be a k-vector space and let $G := \mathrm{GL}_V$. Also, let $I \subset \Gamma(G, \mathcal{O})$ be a Hopf ideal such that the k-algebra $\Gamma(G, \mathcal{O})/I$ defines an algebraic group H embedded in G. Then, there exists a finite dimensional k-vector space W and a line $D \subset W$ such that H is the stabilizer of D on W.

Proof. Let $F \subset \Gamma(G, \mathcal{O})$ be a k-vector subspace of $\Gamma(G, \mathcal{O})$ generated by a finite system of generators of I. Let $\rho : G \times G \to G$ be the action of G on itself that sends each pair (x, y) to yx^{-1}. By following 3.4.2 a) in [Cre], we obtain a finite dimensional k-subspace W of $\Gamma(G, \mathcal{O})$ containing F such that is stable through all $\rho_x : \Gamma(G, \mathcal{O}) \to \Gamma(G, \mathcal{O})$, given by $\rho_x(g)(y) = g(yx)$. Let $M := W \cap I$ so that M generates I. Notice that M is stable under all ρ_x, for all $x \in H$ since $H = \{x \in G : \rho_x I = I\}$ (see [Cre] Lemma 3.4.1). If $\rho_g M = M$, since M generates I, $\rho_x I = I$ so $x \in H$. If we take now $L = \wedge^{dimM} M$, since : $\rho_x M = M$ if and only if $\rho_x L = L$, by Lemma 3.7.1 of [Cre], we have the characterization of H we wanted. □

Theorem 6.1.0.3. *The functor homomorphism $G \to \mathrm{Aut}^{\otimes}(\omega)$ is an isomorphism.*

Proof. Let $\mathcal{M} \in Ob(\mathbf{Rep}(G))$ be a linear representation of G and $\langle \mathcal{M} \rangle_{\otimes}$ be the full subcategory of $\mathbf{Rep}(G)$ that is tensor subgenerated by $\mathcal{M}$. Also, let , $\omega|_{\langle \mathcal{M} \rangle_{\otimes}}$ be the restriction of ω to $\langle \mathcal{M} \rangle_{\otimes}$. In addition, we may also refer to $\mathrm{Aut}(\omega|_{\langle \mathcal{M} \rangle_{\otimes}})$ as a k-group functor. Notice that we have an embedding

$$\mathrm{Aut}^{\otimes}(\omega|_{\langle \mathcal{M} \rangle_{\otimes}}) \to \mathrm{GL}_M = \mathrm{Aut}^{\otimes}(\mathbb{W}_M).$$

Where $\mathbb{W}_M(A) = A \otimes M$, because of the map $\lambda = (\lambda_{\mathcal{N}})_{\mathcal{N} \in Ob(\mathbf{Rep}(G))} \mapsto \lambda_{\mathcal{M}}$, which identifies $\mathrm{Aut}^{\otimes}(\omega|_{\langle \mathcal{M} \rangle_{\otimes}})(A)$ with a subgroup of GL_M. On the other hand, denote by $G_{\mathcal{M}}$ the image of G in GL_M through the previous morphism, which is an algebraic subgroup of GL_M that satisfies

$$G_{\mathcal{M}}(A) \subseteq \mathrm{Aut}^{\otimes}(\omega|_{\langle \mathcal{M} \rangle_{\otimes}})(A) \subseteq \mathrm{GL}_M(A).$$

Our goal now is to see that the image of $\mathrm{Aut}^{\otimes}(\omega|_{\langle \mathcal{M} \rangle_{\otimes}})$ in GL_X coincides with $G_{\mathcal{M}}$.

Thanks to the previous lemma it is enough to check that $\mathrm{Aut}^{\otimes}(\omega|_{\langle \mathcal{M} \rangle_{\otimes}})$ leaves invariant every vector that is invariant under $G_{\mathcal{M}}$. Let $\mathcal{N} \in Ob(\langle \mathcal{M} \rangle_{\otimes})$ and let $v \in V$ be invariant under $G_{\mathcal{M}}$. In particular, the map $\rho : \mathrm{triv} \to \mathcal{N}$ such that $1 \mapsto v$ is G-equiinvariant. Also, since ω is a tensor functor, for every automorphism λ of F and every $A \in Ob(\mathbf{Alg}_k)$,

$$\lambda_V(A)(v \otimes 1_A) = \rho \lambda_{\mathrm{triv}}(A)(1 \otimes 1_A) = v \otimes 1_A,$$

so we have that $G_{\mathcal{M}} = \mathrm{Aut}^{\otimes}(\omega|_{\langle \mathcal{M} \rangle_{\otimes}})$. Now, if $\mathcal{M}_2$ is a subrepresentation of $\mathcal{M}_1$, we have a commutative diagram

$$\begin{array}{ccc} G_{\mathcal{M}_1} & \longrightarrow & \mathrm{Aut}^{\otimes}(\omega|_{\langle \mathcal{M}_1 \rangle_{\otimes}}) \\ \downarrow & & \downarrow \\ G_{\mathcal{M}_2} & \longrightarrow & \mathrm{Aut}^{\otimes}(\omega|_{\langle \mathcal{M}_2 \rangle_{\otimes}}), \end{array}$$

where the vertical maps are given by the restriction. Since the regular representation of G is faithful, we have that

$$G = \varprojlim_{G \in Ob(\mathbf{Rep}(G))} G_{\mathcal{M}}, \qquad \mathrm{Aut}^{\otimes}(\omega) = \varprojlim_{\mathcal{M} \in Ob(\mathbf{Rep}(G))} \mathrm{Aut}^{\otimes}(\omega|_{\langle \mathcal{M} \rangle_{\otimes}}).$$

where the right equality is clear. Hence, $G \simeq \mathrm{Aut}^{\otimes}(\omega)$. □

Before ending this subsection, it would be interesting to see that if we take two affine algebraic groups G and G', under certain hypothesis we can find a "good" relationship between both of them. More specifically, notice that if we take an algebraic k-homomorphism $f : G \to G'$, we can define a tensor functor $\omega^f : \mathbf{Rep}(G') \to \mathbf{Rep}(G)$ such that $\omega^G \circ \omega^f = \omega^{G'}$, using $\mathcal{M} = (M, \Phi_{\mathcal{M}}) \mapsto (M, \Phi_{\mathcal{M}} \circ f)$. Let us go a little bit further.

Corollary 6.1.0.4. *Let G and G' be affine algebraic k-groups and $F : \mathbf{Rep}(G') \to \mathbf{Rep}(G)$ a monoidal functor such that $\omega^G \circ F = \omega^{G'}$. Then, there exists a unique algebraic k-homomorphism $f : G \to G'$ such that $F = \omega^f$.*

Proof. We know that F induces an homomorphism $F^* : \mathrm{Aut}^{\otimes}(\omega^G)(R) \to \mathrm{Aut}^{\otimes}(\omega^{G'})(R)$ via $F^*(\lambda_{\mathcal{M}}) = \lambda_{F(\mathcal{M})}$. Thanks to 6.1.0.3 and Yoneda's lemma, we can identify F^* with an algebraic k-homomorphism $f : G \to G'$, where $F \mapsto F^*$ and $f \mapsto \omega^f$ are clearly the inverse maps. □

6.2 Tannakian Categories and Algebraic Groups

Our goal now is to show that if $\tilde{\mathfrak{C}}$ is an abelian, rigid, symmetric monoidal k-linear category and $\omega : \mathfrak{C} \to \mathbf{fVec}_k$ is a fiber functor for the category,

1. The functor $\mathrm{Aut}^{\otimes}(\omega)$ is an algebraic group.
2. The functor $\tilde{\mathfrak{C}} \to \mathbf{Rep}(\mathrm{Aut}^{\otimes}(\omega))$ induced by F is an equivalence of categories.

In order to proceed with the proof, we need a few previous results. For the first proposition, remember that a skeleton for a category $\mathfrak{C}$ is an equivalent category $\mathfrak{D}$ in which no two distinct objects are isomorphic. A more precise definition can be found in [Ma]. Also, from now on, $\tilde{\mathfrak{C}}$ will denote an abelian rigid symmetric monoidal k-linear category.

Proposition 6.2.0.1. *There exists a functor $\boxtimes : \mathbf{Mod}_k \times \mathfrak{C} \to \mathfrak{C}$ such that for every $X, Y \in Ob(\mathfrak{C})$ and $V \in Ob(\mathbf{fVec}_k)$,*

1. $\mathrm{Hom}_{\mathfrak{C}}(X, V \boxtimes Y) \simeq V \otimes_k \mathrm{Hom}_{\mathfrak{C}}(X, Y)$ *and* $\mathrm{Hom}_{\mathfrak{C}}(V \boxtimes Y, X) \simeq V \otimes_k \mathrm{Hom}_{\mathfrak{C}}(Y, X)$.
2. *For any k-linear functor $F : \mathfrak{C} \to \mathbf{fVec}_k$, $F(V \boxtimes X) \simeq V \otimes_k F(X)$.*

Proof. First of all, we pick the standard skeleton of $\mathbf{fVec}_k$, $\mathbf{fVec}_k^s$. Remember that the objects of $\mathbf{fVec}_k^s$ are the V of the form k^n for a certain $n \in \mathbb{N} \cup \{0\}$. Also, for every finite dimensional k-module $V \in Ob(\mathbf{fVec})$ we can define an isomorphism $\phi_V : k^{\dim(V)} \to V$. Therefore, following [Ma] there exists a unique equivalence of categories $\Gamma : \mathbf{fVec}_k \to \mathbf{fVec}_k^s$ such that its quasi-inverse is the inclusion $i : \mathbf{Mod}_k^s \to \mathbf{fVec}_k$. Because of the latter observation, we have a natural isomorphism $\delta : \Gamma \circ i \to \mathrm{id}_{\mathbf{fVec}}$ given by the existence of the quasiinverse. Now we can define $\boxtimes$ easily - for each $n \in \mathbb{N} \cup \{0\}$ and $X \in Ob(\mathfrak{C})$, $k^n \boxtimes X := X^{\oplus^n}$, and for each $V \in Ob(\mathbf{fVec}_k)$, $V \boxtimes X := \Gamma(V) \boxtimes X$. Thus, all that we have to do is verify that the isomorphisms hold. The first one follows directly from the fact that there exists an $n \in \mathbb{N} \cup \{0\}$ such that

$$\mathrm{Hom}_{\mathfrak{C}}(X, V \boxtimes Y) \simeq \mathrm{Hom}_{\mathfrak{C}}(X, \Gamma(V) \boxtimes Y) = \mathrm{Hom}_{\mathfrak{C}}(X, Y^{\oplus^n}) \simeq k^n \otimes_k \mathrm{Hom}_{\mathfrak{C}}(X, Y),$$

and for the second point, $F(V \boxtimes X) = F(X^{\oplus^n}) = k^n \otimes_k F(X)$. Hence, we are done with the proof. □

Now, for a k-module V and an object $X \in Ob(\mathfrak{C})$, we define $\mathfrak{Hom}(V, X) := V^\vee \boxtimes X$. In some books of the literature, for instance [Mil3], $\mathfrak{Hom}$ is written as $\underline{\mathrm{Hom}}$, but we prefer to write it in the first way in order to avoid confusion with the internal homomorphisms. An important question that raises naturally is: how can we define a subspace of $\mathfrak{Hom}(V, X)$ for any k-submodule W of V and any subobject $Y \subseteq X$?. The following definition answers the previous question.

Definition 6.2.0.2. Taking the previous notations, we define the *transporter of W to Y* as

$$(Y : W) := \ker(\mathfrak{Hom}(V, X) \to \mathfrak{Hom}(W, X/Y)).$$

Notice that if $F : \mathfrak{C} \to \mathbf{Mod}_k$ is an exact k-linear functor, thanks to 6.2.0.1, $F(\mathfrak{Hom}(V, X)) = F(V^\vee \boxtimes X) \simeq \mathrm{Hom}_{\mathbf{Mod}_k}(V, F(X))$, hence

$$F(Y : W) = (F(X) : W) = \{f \in \mathrm{Hom}_{\mathbf{Mod}_k}(V, F(X)) : f(W) \subseteq F(Y)\}.$$

In order to show the result of the beginning of the subsection, we will follow the next plan. Just like we saw in 6.1.0.3, we want to restrict $\tilde{\mathfrak{C}}$ to a subcategory tensor generated by an element $X \in Ob(\mathfrak{C})$, $\langle X \rangle_\otimes$. More precisely, we want to show that $\langle X \rangle_\otimes$ is equivalent to the category of comodules over a coalgebra. Using inverse limits, we will see that the entire category $\tilde{\mathfrak{C}}$ is equivalent to the category of comodules over another coalgebra, which turns out to be a Hopf algebra.

In order to see that $\langle X \rangle_\otimes$ is equivalent to the category of comodules over a coalgebra C, denoted as $\mathbf{CoMod}_C$, we see first that $\omega|_{\tilde{\langle X \rangle}_\otimes}$ identifies $\tilde{\mathfrak{C}}_X$ with $\mathbf{CoMod}_C$. Notice that if C is finite dimensional, then there exists an equivalence of categories

$$\mathbf{CoMod}_C \xrightarrow{\sim} \mathbf{Mod}_{C^\vee}.$$

Hence, all we have to do is see that $\langle X \rangle_\otimes$ is equivalent to the category of modules over a certain algebra R.

Lemma 6.2.0.3. For any $X \in Ob(\mathfrak{C})$, the following two subobjects of $\mathfrak{Hom}(\omega(X), X)$ are equal.

1. The largest subobject $P \subseteq \mathfrak{Hom}(\omega(X), X)$ whose image in $\mathfrak{Hom}(\omega(X)^{\oplus n}, X^{\oplus n})$ is contained in $(Y : \omega(Y))$ for any $Y \subseteq X^{\oplus n}$ and $n \in \mathbb{N}$.
2. The smallest subobject $P' \subseteq \mathfrak{Hom}(\omega(X), X)$ such that $\omega(P') \subseteq \omega(\mathfrak{Hom}(\omega(X), X)) = \mathrm{Hom}_{\mathbf{Mod}_k}(\omega(X), \omega(X))$ contains $\mathrm{id}_{\omega(X)}$.

Proof. First, notice that the existence of the fiber functor $\omega : \mathfrak{C} \to \mathbf{fVec}_k$ implies that $\mathfrak{C}$ is artinian and noetherian. Therefore, both P and P' are well-defined subobjects of $\mathfrak{Hom}(\omega(X), X)$. Now, by definition,

$$P = \bigcap_{n\geq 0} \bigcap_{Y\subseteq X^{\oplus n}} \mathfrak{Hom}(\omega(X), X) \cap (Y : \omega(Y)),$$

$$\omega(P) = \bigcap_{n\geq 0} \bigcap_{Y\subseteq X^{\oplus n}} \mathrm{End}_{\mathbf{fVec}_k}(\omega(X)) \cap (F(Y) : F(Y)).$$

This shows that $\omega(P)$ is the largest subring of $\mathrm{End}_{\mathbf{fVec}_k}(\omega(X))$ stabilizing all $F(Y)$, $Y \subseteq X^{\oplus n}$. Hence, $\mathrm{id}_X \in \omega(P)$ and $P' \subseteq P$. For the reverse inclusion, notice that if we take a subobject $Y \subseteq \mathfrak{Hom}(\omega(X), X)$, by definition of P, left multiplication by $\omega(P) \subseteq \mathrm{End}_{\mathbf{fVec}_k}(\omega(X))$ stabilizes $\omega(Y) \subseteq \mathrm{End}_{\mathbf{fVec}_k}(\omega(X))$. Hence, that fact that $\mathrm{id}_{\omega(X)} \in \omega(P')$ implies that $\omega(P) \subseteq \omega(P')$, that is, $P \subseteq P'$. $\square$

After all these preliminaires, we can continue with the construction of a coalgebra. Let $X \in Ob(\mathfrak{C})$ and let $P_X \subseteq \mathfrak{Hom}(\omega(X), X)$ be the object defined in 6.2.0.3. Also, let $\langle X\rangle \subseteq \mathfrak{C}$ be the subcategory whose objects are subobjects of quotients of $X^{\oplus^n}$. By definition, the functor $\omega|_{\langle X\rangle} : \langle X\rangle \to \mathbf{fVec}_k$ factors through $\mathbf{fVec}_{\omega(P_X)}$. From now on, let us denote $A_X := \omega(P_X)$. Then,

Proposition 6.2.0.4. *For any $Y \in Ob(\langle X\rangle)$, there is a natural action of A_X on $\omega(Y)$. Furthermore, $\omega|_{\langle X\rangle} : \langle X\rangle \to \mathbf{fVec}_{A_X}$ is an equivalence of categories sending $\omega|_{\langle X\rangle}$ to the forgetful functor, and $A_X = \mathrm{End}(\omega|_{\langle X\rangle})$.*

Proof. The first point follows from applying the second point of 6.2.0.1 with $F = \omega$. From that, we obtain $A_X = \omega(P_X) \subset \omega(X)^\vee \otimes \omega(X)$, which is isomorphic to $\mathrm{Hom}_{\mathbf{fVec}_k}(\omega(X), \omega(Y))$ and therefore A_X operates on $\omega(X)$, and this action is naturally extended to $\omega(Y)$ for any $Y \in Ob(\langle X\rangle)$. On the other hand, notice that we have an action of $A_X \subseteq \mathrm{End}(\omega(X)^\vee)$ on $\mathfrak{Hom}(\omega(X), X)$ and it is clear that this action stabilizes P_X. Now, if M is a right A-module, then we get two maps $((M \otimes A_X) \boxtimes P_X) \to M \boxtimes P_X$, one by considering the action of A_X on M and the other by considering the action on P_X. We define $M \otimes_{A_X} P_X$ to be the equalizer of these maps. By definition, $M \otimes_{A_X} P_X \in Ob(\langle X\rangle)$ so we have $\omega(M \otimes_{A_X} P_X) = M \otimes_{A_X} \omega(P_X) = M$, that is, ω is essentially surjective. Now, if $f : M \to N$ is an A_X-module map, then we may define a map

$$M \otimes_{A_X} P_X \to N \otimes_{A_X} P_X$$

that shows that ω is full. Finally, the faithfulness of ω follows by hypothesis. Hence, we have shown that ω is a category equivalence. The last point is clear. $\square$

So now, we can let $C_X := A_X^\vee$, so that $\langle X\rangle$ is equivalent to the category of C-comodules. Thus, we have

Corollary 6.2.0.5. *Let $H := \varinjlim \mathrm{End}(F|_{\langle X\rangle})^\vee$. Then, ω factors through $\mathbf{CoMod}_H$ and moreover, it is an equivalence between $\mathfrak{C}$ and the category of $\mathbf{CoMod}_H$ sending ω into the forgetful functor.*

Now, our goal is to define a commutative Hopf algebra structure on H. We will use the fact that $\tilde{\mathfrak{C}}$ is symmetric. In order to do so, let $B \in Ob(\mathbf{CoAlg}_k)$ and consider $F : \mathbf{CoMod}_B \to \mathbf{fVec}_k$. It is clear that

$$B = \varinjlim_{X \in Ob(\mathfrak{C})} \operatorname{End}(F|_{\langle X \rangle})^{\vee}.$$

If we also observe that for a finite dimensional algebra A, there is an isomorphism $A \simeq \operatorname{End}(F_A)$, we have that any functor $\mathbf{CoMod}_B \to \mathbf{CoMod}_{B'}$ that carries F to itself arises from a unique coalgebra homomorphism $B \to B'$.
The last step before we conclude the proof is the following lemma. Notice that if we have a coalgebra homomorphism $B \otimes_k B \to B$, we can define a functor $\mathbf{CoMod}_B \times \mathbf{CoMod}_B \to \mathbf{Comod}_B$ via $(X, Y) \mapsto X \otimes_k Y$, with comodule structure defined by the the coalgebra homomorphism.

Lemma 6.2.0.6. The previously defined map defines a bijective correspondence between the set of coalgebra homomorphisms of the form $B \otimes_k B \to B$ and the set of functors of the form $\mathbf{CoMod}_B \times \mathbf{CoMod}_B \to \mathbf{Comod}_B$. The product induced by the coalgebra homomorphism is associative (resp. commutative) if and only if the natural associativity (resp. commutativity) constraint on $\mathbf{fVec}_k$ indues a similar constraint on $\mathbf{CoMod}_B$. The product induced by that same morphism has a unit if and only if $\mathbf{CoMod}_B$ has a unit object with underlying k-module k.

So we can finally show the main result of this section.

Theorem 6.2.0.7. *Let $\tilde{\mathfrak{C}}$ be an abelian, rigid, symmetric monoidal k-linear category and $\omega : \mathfrak{C} \to \mathbf{fVec}_k$ a fiber functor for $\tilde{\mathfrak{C}}$. Then,*

1. *The functor $\operatorname{Aut}^{\otimes}(\omega)$ is an algebraic group, called the tannakian fundamental group of $(\tilde{\mathfrak{C}}, \omega)$.*

2. *The functor $\tilde{\mathfrak{C}} \to \mathbf{Rep}(\operatorname{Aut}^{\otimes}(\omega))$ induced by F is an equivalence of categories.*

Proof. Taking the notations that we have followed throughout this subsection, 6.2.0.6 shows that $H := \varinjlim \operatorname{End}(\omega|_{\langle X \rangle})^{\vee}$ is a commutative algebra with identity. Notice that $G := \operatorname{Spm}(H)$ is an affine algebraic monoid. Just like in 6.1, we have that $G \simeq \operatorname{End}^{\otimes}(\omega)$. Since both $\tilde{\mathfrak{C}}$ and $\mathbf{fVec}_k$ are rigid, proposition 5.3.0.9 shows that $\operatorname{End}^{\otimes}(\omega) = \operatorname{Aut}^{\otimes}(\omega)$, so G is in fact an affine algebraic group and thanks to 3.2.0.2, H is a Hopf algebra. □

6.3 A note on fibre functors

We begin with the definition we mentioned before 5.3.0.10.

Definition 6.3.0.1. Let $\tilde{\mathfrak{C}}$ be a symmetric monoidal category and let X be an algebraic k-scheme. A *fibre functor* of $\tilde{\mathfrak{C}}$ on S is an exact, faithful and k-linear symmetric monoidal functor

$$\omega : \tilde{\mathfrak{C}} \to \mathbf{QCoh}(\mathcal{O}_S),$$

where $\mathcal{O}_S$ is the sheaf associated to the algebraic scheme S.

From the definition of a symmetric monoidal category and the properties of the previously defined functor follows that ω is in fact a functor from $\tilde{\mathfrak{C}}$ to the category of locally free sheaves of finite rank on $\mathcal{O}_S$. In fact, since the case that concerns us the most is affine algebraic schemes, notice that if $S = \mathrm{Spm}(A)$, then ω can be identified with a functor $\tilde{\mathfrak{C}} \to \mathbf{fProjMod}_A$, where $\mathbf{fProjMod}_A$ denotes the category of finite projective modules over A. Notice that $\mathbf{fProjod}_A$ is a symmetric monoidal category, and this can be shown taking into account the fact that $\mathbf{Mod}_A$ is also a symmetric monoidal category.

Definition 6.3.0.2. If S is an affine algebraic k-scheme, then the previous functor $\overline{\omega} : \tilde{\mathfrak{C}} \to \mathbf{fProjMod}_{\Gamma(S,\mathcal{O}_S)}$ is called a *fibre functor on* $\Gamma(S, \mathcal{O}_S)$.

So let us start giving a proposition which, after a few observations, shows us that Definition 6.3.0.2 is equivalent to 5.3.0.10.

Definition 6.3.0.3. Let A be a k-algebra and let $X \in Ob(\mathbf{Mod}_A)$. If $X^\vee$ exists, then X is a projective A-module of finite rank.

Proof. Let $Y \in Ob(\mathbf{Mod}_A)$. Thanks to 5.3.0.8, we have that $X^\vee \otimes Y \simeq \underline{\mathrm{Hom}}(X, Y)$. Thanks to the properties of $\mathbf{Mod}_A$, we have that $\underline{\mathrm{Hom}}(X, Y) = \mathrm{Hom}_{\mathbf{Mod}_A}(X, Y)$. From this, if we let $Y = A$, then we have that $X^\vee$ is the dual of X. Now, let $Y = X$. Using the previous isomorphism, we have an isomorphism

$$X^\vee \otimes X \to \mathrm{Hom}_{\mathbf{Mod}_A}(X, X).$$

If the image throught the previous isomorphism of $\sum \alpha_k \otimes x_k$ is id_X, then the α_k and x_k define a factorisation of the identity

$$X \xrightarrow{\alpha} A^n \xrightarrow{x} X.$$

Hence, X is a direct factor of a free module of finite rank. □

Notice that if we are given a symmetric monoidal functor $F : \tilde{\mathfrak{C}} \to \tilde{\mathfrak{D}}$ and $X \in Ob(\mathfrak{C})$ admits a dual $X^\vee$, $F(X^\vee)$ is a dual of $F(X)$ thanks to 5.3.0.9. Hence, let V be a finite dimensional k-vector space and let X be an object of an additive k-linear category $\mathfrak{A}$. Thanks to 6.2.0.1, we have the existence of a tensor product $\boxtimes$ that satisfies

$$\mathrm{Hom}(Y, V \boxtimes X) = V \otimes_k \mathrm{Hom}(Y, X).$$

If we take a look at the proof of 6.2.0.1, we can identify $V \otimes X$ with $X^{\dim(V)}$. Hence, the full-subcategory of $\mathfrak{C}$ formed by the multiples $\mathbb{1}^n$ of $\mathbb{1}$ is equivalent to the category of finite dimensional k-vector spaces via the morphism $V \mapsto V \boxtimes \mathbb{1}$. Thanks to Proposition 1.17 of [Mil3], this equivalence is stable for subquotients, and therefore we have shown the equivalence between 6.3.0.2 and 5.3.0.10.

Before we conclude this section, it is also important to mention a generalization of the affine algebraic group $\mathrm{Aut}^{\otimes}(\omega)$ that we introduced in the beginning of 6.1.

Remark 6.3.0.4. In this case, let S denote an algebraic k-scheme and let $\omega_1, \omega_2 : \mathfrak{C} \to \mathbf{QCoh}(\mathcal{O}_S)$ be two fiber functors on S. In [Del], Deligne defines an $\otimes$-isomorphism as a natural transformation $u : \omega_1 \to \omega_2$ that makes the following diagram commute

$$\begin{array}{ccc} \omega_1(X) \otimes \omega_1(Y) & \xrightarrow{\sim} & \omega_1(X \otimes Y) \\ {\scriptstyle u\otimes u}\downarrow & & \downarrow{\scriptstyle u} \\ \omega_2(X) \otimes \omega_2(Y) & \xrightarrow[\sim]{} & \omega_2(X \otimes Y) \end{array}$$

and $u(\mathbb{1}) : \omega_1(\mathbb{1}) \to \omega_2(\mathbb{1})$ is the identity isomorphism of $\mathcal{O}_S$, and later defines $\mathrm{Isom}^{\otimes}_S(\omega_1, \omega_2)$ as the functor that adjoints to any $u : T \to S$ all the fibre functor $\otimes$-isomorphisms from $u^*\omega_1$ to $u^*\omega_2$, where u^* is the dual of $u : T \to S$, namely $u^* : \mathbf{QCoh}(\mathcal{O}_S) \to \mathbf{QCoh}(\mathcal{O}_T)$ which allows us to define

$$\mathrm{Isom}^{\otimes}_k(\omega_1, \omega_2) := \mathrm{Isom}^{\otimes}_{S_1 \times S_2}(\pi_1^*\omega_1, \pi_2^*\omega_2),$$

$$\mathrm{Aut}^{\otimes}_k(\omega) := \mathrm{Isom}^{\otimes}_k(\omega, \omega).$$

where π_i are the natural projections and $\pi_i^*\omega_i : \mathfrak{C} \to \mathbf{QCoh}(\mathcal{O}_{S_i})$, because $\omega_i : \mathfrak{C} \to \mathbf{QCoh}(\mathcal{O}_{S_1 \times S_2})$. This allows Deligne to show that, by taking the definitions in point 1.5 of [Del], if ω is a fibre functor on a non-empty algebraic k-scheme S of a k-tensor category $\tilde{\mathfrak{C}}$,

1. ω induces an equivalence between $\tilde{\mathfrak{C}}$ and the category of representations of $\mathrm{Aut}^{\otimes}_k$, $\mathbf{Rep}(S : \mathrm{Aut}^{\otimes}_k(\omega))$.
2. In particular, there is an isomorphism $G \xrightarrow{\sim} \mathrm{Aut}^{\otimes}_k(\omega)$.

Chapter 7

Application: Differential Galois Theory

7.1 Motivation

We are about to study three important examples of tannakian categories: the category of complex local systems on a connected and locally simply connected topological space X, denoted as $\mathbf{LS}_X$, the category of holomorphic connections on a connected open set $D \subseteq \mathbb{C}$, $\mathbf{Conn}_D$, and the category of differential modules over the field $\mathbb{C}(t)$, $\mathbf{DiffMod}_{\mathbb{C}(t)}$. In fact, we will see that there is an equivalence between the three categories.

7.1.1 $\mathbf{LS}_X$ and $\mathbf{Conn}_D$

Let us start with the following definition.

Definition 7.1.1.1. Let X be a connected and locally simply connected topological space. A *complex local system* on X is a locally constant sheaf of finite dimensional complex vector spaces.

It is clear that if we endow it with the usual tensor product and dual space, we see that it is a $\mathbb{C}$-linear abelian, rigid symmetric monoidal category. Now let us construct the category of holomorphic connections.

Definition 7.1.1.2. A *holomorphic connection* on a connected open set $D \subseteq \mathbb{C}$ is a pair $(\mathcal{E}, \nabla)$, where $\mathcal{E}$ is a locally free sheaf on D and a *connection map* $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}} \Omega^1(D)$ is a morphism of sheaves of $\mathbb{C}$-vector spaces satisfying

$$\nabla(fs) = df \otimes s + f\nabla(s)$$

for all open subsets $U \subseteq D$, $f \in \Gamma(U, \mathcal{O})$ and $s \in \Gamma(U, \mathcal{E})$, where $\Omega^1(D)$ is the sheaf of 1-forms over D, d is the usual derivation in $\mathbb{C}$ and the tensor product $\mathcal{E} \otimes_{\mathcal{O}} \Omega^1(D)$ is defined by the rule

$$U \mapsto \Gamma(U, \mathcal{E}) \otimes_{\Gamma(U,\mathcal{O})} \Gamma(U, \Omega^1(D)).$$

Also, a *morphism* between $(\mathcal{E}, \nabla)$ and $(\mathcal{E}', \nabla')$ is a morphism of $\mathcal{O}$-modules $\phi : \mathcal{E} \to \mathcal{E}$ making the following diagram commute

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\nabla} & \mathcal{E} \otimes_{\mathcal{O}} \Omega^1(D) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi \otimes \mathrm{id}} \\ \mathcal{E}' & \xrightarrow[\nabla']{} & \mathcal{E}' \otimes_{\mathcal{O}} \Omega^1(D). \end{array}$$

Furthermore, given a section $s \in \Gamma(U, \mathcal{E})$ of a connection $(\mathcal{E}, \nabla)$, we say that s is *horizontal* if it satisfies $\nabla(s) = 0$. Horizontal sections form a subsheaf of $\mathbb{C}$-vector spaces $\mathcal{E}^{\nabla} \subseteq \mathcal{E}$.

Clearly, we can define the category of holomorphic connections on D, $\mathbf{Conn}_D$. Let us define a tensor product of connections. In fact, we are going to see that we can endow it with a neutral tannakian category structure. Let $(\mathcal{E}, \nabla), (\mathcal{E}', \nabla') \in Ob(\mathbf{Conn}_D)$. We define the tensor product of $(\mathcal{E}, \nabla)$ and $(\mathcal{E}', \nabla')$ as $(\mathcal{E} \otimes_{\mathcal{O}} \mathcal{E}', \nabla \otimes \nabla') \in Ob(\mathbf{Conn}_D)$ where $\nabla \otimes \nabla' : \mathcal{E} \otimes_{\mathcal{O}} \mathcal{E}' \to \mathcal{E} \otimes_{\mathcal{O}} \mathcal{E}' \otimes_{\mathcal{O}} \Omega^1(D)$ is given by

$$(\nabla \otimes \nabla')(s \otimes s') = \nabla(s) \otimes s' + s \otimes \nabla'(s').$$

This gives $\mathbf{Conn}_D$ the structure of a symmetric monoidal category. Let us see that it is rigid. All we have to do is define the dual connection to $(\mathcal{E}, \nabla)$, $(\mathcal{E}^*, \nabla^*)$, where $\mathcal{E}^*$ is the locally free sheaf given by $U \mapsto \mathrm{Hom}(\mathcal{E}|_U, \mathcal{O}|_U)$ and ∇^* over $\phi \in \mathrm{Hom}(\mathcal{E}|_U, \mathcal{O}|_U)$ via

$$\nabla^*(\phi)(s) = 1 \otimes d\phi(s) - (\phi \otimes \mathrm{id}_{\Omega^1(D)})(\nabla(s)),$$

so all that we need is a fiber functor, but we can define it naturally via

$$\omega^{\mathbf{Conn}_D} : \mathbf{Conn}_D \to \mathbf{Mod}_k$$
$$(\mathcal{E}, \nabla) \longmapsto \mathcal{E}_x^{\nabla},$$

where $x \in D$ is a previously fixed point. Finally, we see that $\mathbf{Conn}_D$ is a $\mathbb{C}$-linear, abelian category, because in [Sza] (Prop. 2.7.5) it is shown that there is and equivalence between $\mathbf{Conn}_D$ and $\mathbf{LS}_D$, compatible with the monoidal category structure and fiber functors, where the fiber functor of $\mathbf{LS}_X$, for a connected and locally simply connected topological space X is given by taking the stalk of a local system at a fixed point $x \in X$, which shows that $\mathbf{LS}_X$ is a neutral tannakian category. With this, we have shown that

Proposition 7.1.1.3. *The category of holomorphic connections* $\mathbf{Conn}_D$ *endowed with the previous tensor products, duals and fiber functor is a tannakian category.*

7.1.2 $\mathbf{DiffMod}_{\mathbb{C}(t)}$

Now, let $(\mathbb{C}, \frac{d}{dt})$ denote the pair formed by the field $\mathbb{C}(t)$ together with its usual derivation $\frac{d}{dt}$. Although we define this notion in a more general way in the next section, a *differential module* over $\mathbb{C}(t)$ is a pair (V, ∇), where $V \in Ob(\mathbf{Vec}_{\mathbb{C}(t)})$

and $\nabla : V \to V$ is a $\mathbb{C}$-linear map satisfying $\nabla(fv) = \frac{df}{dt}v + f\nabla v$. Clearly, given two differential modules over $\mathbb{C}(t)$, we can define a morphism between them as a morphism of $\mathbb{C}(t)$-vector spaces that is compatible with ∇, which gives us the arrows in the category $\mathbf{DiffMod}_{\mathbb{C}(t)}$, whose objects are the differential modules. This category is clearly abelian, because we can define differential submodules as $\mathbb{C}(t)$-subspaces of a given differential module (V, ∇) that are also compatible with ∇. Next, we endow the category with a tensor product. Let $(V, \nabla), (V', \nabla') \in Ob(\mathbf{DiffMod}_{\mathbb{C}(t)})$. We denote $V \otimes V'$ as the usual tensor product of $\mathbb{C}(t)$-vector spaces and $\nabla \otimes \nabla'$ by

$$(\nabla \otimes \nabla')(s \otimes s') := \nabla(s) \otimes s' + s \otimes \nabla'(s'), \qquad \forall s \otimes s' \in V \otimes V',$$

and ∇^* on $V^* = \mathrm{Hom}_{\mathbf{Vec}_{\mathbb{C}(t)}}(V, \mathbb{C}(t))$ by the rule $\nabla^*(\phi)(v) = \frac{d}{dt}\phi(v) - \phi(\nabla(v))$.

Proposition 7.1.2.1. *The category $\mathbf{DiffMod}_{\mathbb{C}(t)}$ together with the previous tensor product and duals, is a $\mathbb{C}$-linear rigid tensor abelian category*

Now let us see how we can relate the category of differential modules to the category of connections. Notice that if we take an n-dimensional $\mathbb{C}(t)$-vector space V and identify it with $\mathbb{C}(t)^n$, we can define a derivation D,

$$D(f_1, \dots, f_n) := \left(\frac{df_1}{dt}, \dots, \frac{df_n}{dt}\right), \qquad \forall f_1, \cdots, f_n \in \mathbb{C}(t)$$

so that (V, d) is a differential module. Notice that thanks to the definition of a differential module, more precisely, the definition of ∇, we have that $\forall f, f_1, \dots, f_n \in \mathbb{C}(t)$,

$$(\nabla - D)(f(f_1, \dots, f_n)) = \frac{df}{dt}(f_1, \dots, f_n) + f\nabla(f_1, \dots, f_n) -$$
$$-\left(\frac{df}{dt}f_1 + f\frac{df_1}{dt}, \dots, \frac{df}{dt}f_n + f\frac{df_n}{dt}\right) = f(\nabla - D)(f_1, \dots, f_n),$$

and $(\nabla - D)((f_1, \dots, f_n) + (g_1, \dots, g_n)) = (\nabla - D)(f_1, \dots, f_n) + (\nabla - D)(f_1, \dots, f_n)$, hence $\nabla - D$ is $\mathbb{C}(t)$-linear and therefore it has an associated matrix $C \in \mathcal{M}_{n \times n}(\mathbb{C}(t))$ called the *connection matrix* of ∇. This is the last step before we show the following result.

Proposition 7.1.2.2. $\mathbf{DiffMod}_{\mathbb{C}(t)}$ *endowed with the operations from 7.1.2.1 is a tannakian category.*

Proof. For this proof, we need Prop. 2.7.7 of [Sza]. Let $D \subset \mathbb{C}$ be an open connected and locally simply connected set such that its complement is finite. Our initial goal is showing that there is a categoric equivalence between the full subcategory tensor generated by a connection $(\mathcal{E}, \nabla)$ of $\mathbf{Conn}_D$ and the full subcategory tensor generated by (V, ∇), a differential $\mathbb{C}(t)$-module The first result listed above states that for an holomorphic connection $(\mathcal{E}, \nabla) \in Ob(\mathbf{Conn}_D)$, there is a connection $(\overline{\mathcal{E}}, \overline{\nabla})$ on $\mathbb{P}^1_{\mathbb{C}}$ with simple poles outside D. This shows that the elements of the connection matrix of $\overline{\nabla}$ are elements of $\mathbb{C}(t)$, so we can associate the differential module $(V, \nabla) \in \mathbf{DiffMod}_{\mathbb{C}(t)}$ to $(\mathcal{E}, \nabla)$ with the very same connection matrix. This leads to our initial goal, and also shows the proposition, because the fact that there is a fiber functor on $\mathbf{Conn}_D$ and the latter equivalence of subcategories shows that we can endow $\mathbf{DiffMod}_{\mathbb{C}(t)}$ with a fiber functor. □

7.2 Differential Galois theory

We give a quick introduction to some of the basic tools that give rise to what is known as differential Galois theory. Afterwards, we will relate all those tools to the theory of neutral tannakian categories, and see how it turns out to be a very powerful tool that eventually leads us to the proof of the main theorem of the theory of differential Galois theory.

7.2.1 Basic definitions

From this point to the end of this chapter, we assume that every field is of characteristic 0. Most of the notions we introduce here can be defined in a more general way. For that reason, we recommend Part 3 of [Cre].

Definition 7.2.1.1. Let K be a field. A *derivation* on K is an additive map $\partial : K \to K$ such that the Leibniz rule holds,

$$\partial(fg) = \partial(f)g + f\partial(g), \qquad \forall f, g \in K.$$

We call the pair (K, ∂) a *differential field*. Let (K, ∂) and (K', ∂') be differential fields. A *differential morphism* is a field homomorphism $\phi : K \to K'$ such that

$$\partial'(\phi(f)) = \phi(\partial(f)), \qquad \forall f \in K$$

Also, an *extension of differential fields* $(L, \partial')|(K, \partial)$ is a field extension $L|K$ such that ∂' extends ∂.

The most simple example we have of a differential field is given by the *trivial derivation*, given by $\partial(f) = 0$, for all $f \in K$. On the other hand, if (K, ∂) is a differential field, the kernel of ∂, which is a subfield of K, is called the *constant field* of (K, ∂). It is commonly denoted as k. Now let us generalize the definition of differential module we give in the previous section.

Definition 7.2.1.2. Let (K, ∂) be a differential field. A *differential module* over (K, ∂) is a pair (V, ∇) where V is a K-module and ∇ is a k-linear map $\nabla : V \to V$ satisfying

$$\nabla(fv) = \partial(f)v + f\nabla(v), \qquad \forall f \in K, v \in V.$$

Also, the elements of $V^{\nabla} := \mathrm{Ker}(\nabla)$ form a k-subspace of V and are called *horizontal vectors.*

Corollary 7.2.1.3. *The category* $\mathbf{DiffMod}_K$ *of differential modules over a differential field* (K, ∂) *is a tannakian category together with the fibre functor given by the forgetful functor.*

Proof. It is a consequence of Proposition 7.1.2.2. □

Notice that if we let $(L, \partial')|(K, \partial)$ be an extension of differential fields, and a differential module $(V, \nabla) \in Ob(\mathbf{DiffMod}_K)$, we can extend (V, ∇) to an element of $\mathbf{DiffMod}_L$ taking $(V \otimes_K L, \nabla_L)$, where ∇_L has the same connection matrix as ∇. We can now give the definition of a Picard-Vessiot extension.

Definition 7.2.1.4. Let (V, ∇) be a differential K-module. A *Picard-Vessiot extension* for (V, ∇) is an extension of differential fields $(L, \partial')|(K, \partial)$ such that

1. The constant field of both differential fields coincide, $k_{(K,\partial)} = k_{(L,\partial')}$.
2. $V \otimes_K L$ is generated as an L-vector space by its horizontal vectors.
3. The coordinates of the elements of $(V \otimes_K L)^{\nabla_L}$ in any L-basis of $V \otimes_K L$ coming from a K-basis of V generate the field extension $L|K$.

Notice that the second condition is stable under tensor products and subquotients. In fact, if it holds for a vector subspace W of V, there is an isomorphism

$$(W \otimes_K L)^{\nabla} \otimes_k L \xrightarrow{\sim} W \otimes_K L,$$

where k is the constant field of L. Then, if $L|K$ is a Picard-Vessiot extension of K for (V, ∇), the functor $\omega_L : \langle (V, \nabla) \rangle_{\otimes} \to \mathbf{fVec}_k$ that sends each V to $(V \otimes_K L)^{\nabla}$ is a fibre functor. Also, thanks to the previous isomorphism, we have an algebraic group homomorhpism

$$\mathrm{Spm}(L) \to \mathrm{Isom}^{\otimes}_K(\omega_L \otimes_k K, \omega|_{\langle (V,\nabla) \rangle_{\otimes}}).$$

This argument eventually leads us to the proof that if the constant field is algebraically closed then there exists a Picard-Vessiot extension L for (V, ∇). In order to do so, we must introduce another common notion in algebraic group theory.

Definition 7.2.1.5. Let G be a linear algebraic group over an algebraically closed field of characteristic 0. A *G-torsor* Z over a field $K \supset k$ is an affine algebraic variety over K with a G-action, that is, a morphism $G_K \times_K Z \to Z$ that sends (g, z) to zg and such that for all $x \in Z(\overline{K})$ and $g, h \in G(\overline{K})$, $z1 = z$, $z(gh) = (zg)h$ and the morphism $G_K \times_K Z \to Z \times_K Z$ given by $(g, z) \mapsto (zg, z)$ is an isomorphism.

Remember that the field of rational functions of a torsor P is defined as the field of fractions of $\Gamma(P, \mathcal{O})$, and also that if P is irreducible, the field of rational functions is integral.

Proposition 7.2.1.6. *The isomorphism $(W \otimes_K L)^{\nabla} \otimes_k L \xrightarrow{\sim} W \otimes_K L$ of the previous observation identifies L with the field of rational functions of* $\mathrm{Isom}^{\otimes}_K(\omega_{L \otimes_k K}, \omega|_{\langle (V,\nabla) \rangle_{\otimes}})$.

Proof. Let P be the G_K-torsor $\mathrm{Isom}^{\otimes}_K(\omega_{L \otimes_k K}, \omega|_{\langle (V,\nabla) \rangle_{\otimes}})$. The morphism

$$\mathrm{Spm}(L) \to \mathrm{Isom}^{\otimes}_K(\omega_{L \otimes_k K}, \omega|_{\langle (V,\nabla) \rangle_{\otimes}})$$

can be translated in terms of k-algebras to

$$\Gamma(P, \mathcal{O}) \to L.$$

The kernel I of the morphism is a ∇-stable ideal, because the kernel of a differential morphism is ∇-stable. Thanks to 9.3.2 in [Del], $I = 0$ and so the previous morphism is injective. The G-torsor P is hence contained in the $\mathrm{GL}(\omega_L(X))$-torsor $\mathrm{Isom}^{\otimes}(\omega_{L \otimes_k K}, \omega|_{\langle (V,\nabla) \rangle_{\otimes}})$, and the fact that $\Gamma(P, \mathcal{O})$ generates L gives us the second point in 7.2.1.4. □

As a result, we obtain the following corollary.

Corollary 7.2.1.7. *If the constant field k is algebraically closed, then there exists a Picard-Vessiot extension $L|K$ for (V, ∇). Furthermore, it is unique up to K-differential isomorphism.*

Proof. Since a fiber functor $\omega_0 : \langle (V, \nabla) \rangle_{\otimes} \to \mathbf{fVec}_k$ exists thanks to 5.3.0.12, the field of rational functions of the previously introduced torsor P, that we denote as L, is a Picard-Vessiot extension of K, because for $(W, \nabla) \in Ob(\langle (V, \nabla) \rangle_{\otimes})$ and $a \in \omega_0(X)$, since $p(a)$ is a horizontal section of W on P, we have an isomorphism between $\omega_0(W) \xrightarrow{\sim} (W \otimes_K L)^{\nabla}$ via $a \mapsto p(a)$, and also $\omega_0(W) \otimes_k L \xrightarrow{\sim} W \otimes_K L$. In the case when $W = V$, point 1. in 7.2.1.4 is clear, point 2. of the same definition follows from the inclusion $P \hookrightarrow \mathrm{Isom}^{\otimes}(\omega_0(X), X \otimes K)$ and point 3. follows from point (ii) in 9.3 of [Del]. Thanks to the first isomorphism given by $a \mapsto p(a)$, ω_0 is canonically isomorphic to ω_L, and we have proven the result. $\square$

7.2.2 The differential Galois algebraic group

First, let us generalize Proposition 7.1.2.1, so that we can apply everything we know to the category $\mathbf{DiffMod}_K$.

Corollary 7.2.2.1. *The category $\mathbf{DiffMod}_K$ of differential modules over K is an abelian, rigid symmetric monoidal k-linear category.*

Proof. All we have to do is define the same tensor product, that is, for every (V, ∇) and (V', ∇') in $Ob(\mathbf{DiffMod}_K)$, we define the tensor product of (V, ∇) and (V', ∇') as $(V \otimes V', \nabla \otimes \nabla')$, where $\nabla \otimes \nabla'$ is given by the same rule in 7.1.2.1,

$$(\nabla \otimes \nabla')(s \otimes s') := \nabla(s) \otimes s' + s \otimes \nabla'(s'), \qquad \forall s \otimes s' \in V \otimes V'.$$

The notion of the dual of (V, ∇), ∇^* on $V^* = \mathrm{Hom}_{\mathbf{Mod}_K}(V, K)$, is given by the rule $\nabla^*(\phi)(v) = \partial\phi(v) - \phi(\nabla(v))$. Following the same reasoning, we obtain the result. $\square$

Notice that if we fix a base differential field (K, ∂) and L a field that satisfies point 2. in Definition 7.2.1.4, each object of the full subcategory $\langle (V, \nabla) \rangle_{\otimes}$ satisfies the same condition, thanks to the abelian and tensor structure on $\mathbf{DiffMod}_K$. Hence, each L as the latter defines a fibre functor ω_L on $\langle (V, \nabla) \rangle_{\otimes}$ given by $(W, \nabla) \mapsto (W \otimes_K L)^{\nabla_L}$. In fact, thanks to the existence and uniquenes (up to isomorphism) of Picard-Vessiot extensions for differential modules and the isomorphism $G \xrightarrow{\sim} \mathrm{Aut}^{\otimes}(\omega)$ show that Picard-Vessiot extensions for a differential module (V, ∇) correspond bijectively to neutral fibre functors on $\langle (V, \nabla) \rangle_{\otimes}$.

Furthermore, thanks to 5.3.0.12, we see that over an algebraically closed field k, on an abelian, rigid symmetric monoidal k-linear category tensor generated by a single element, a neutral fiber functor always exists into $\mathbf{fVec}_K$ for an extension $K|k$. Notice that this shows that Picard-Vessiot extensions always exist for differential modules over differential fields with algebraically closed constant field.

Definition 7.2.2.2. Let (L, ∂) be a Picard-Vessiot extension for a differential module (V, ∇). We define the *differential Galois algebraic group* as

$$\mathrm{Gal}(V, \nabla) := \mathrm{Aut}^{\otimes}(\omega_L),$$

where ω_L is the fiber functor of $\langle (V, \nabla) \rangle_{\otimes}$ given by $(W, \nabla) \mapsto (W \otimes_K L)^{\nabla_L}$.

Thanks to theorem 6.2.0.7, we have that $\mathrm{Gal}(V, \nabla)$ is an affine algebraic k-group. The reader who is familiarized with differential Galois theory should notice at this point that this is not exactly the same definition that can be found in any classical approach, which is the following.

Definition 7.2.2.3. If $L|K$ is a differential extension of fields, the group of differential K-automorphisms of L is called differential Galois group and we denote it as $\mathrm{Gal}_{\partial}(L|K)$.

Our goal now is to prove that we can endow the latter group with the structure of an affine algebraic group and also that it is isomorphic to $\mathrm{Gal}(V, \nabla)$ for a certain differential module (V, ∇). So let us start by solving the first problem we mentioned. Let $A \in Ob(\mathbf{Alg}_k)$. We can extend the derivation ∂ of L by setting

$$\begin{aligned} \partial' : L \otimes_k A &\longrightarrow L \otimes_k A \\ f \otimes r &\longmapsto \partial f \otimes r \end{aligned}$$

So we can define a group functor by setting $\mathrm{Gal}_{\partial}(L|K) := \mathrm{Aut}^{\partial}_{\mathbb{W}_K}(\mathbb{W}_L)$, where $\mathrm{Aut}^{\partial}_{K \otimes_k A}(L \otimes_k A)$ is the group of $K \otimes_k A$-algebra automorphisms of $K \otimes_k A$ commuting with ∂. When the differential field extension $L|K$ is known, we will simply denote it as Gal_{∂}. On the other hand, for any $f \in \mathrm{Hom}_{\mathbf{Alg}_k}(A, B)$, set $\mathrm{Gal}_{\partial}(L|K)(f) \in \mathrm{Hom}_{\mathbf{Alg}_k}(\mathrm{Gal}_{\partial}(A), \mathrm{Gal}_{\partial}(B))$ as the map that sends each automorphism of $L \otimes_k A$ to the automorphism of $L \otimes_k B$ induced by base change via f. Hence, we have defined a group functor $\mathrm{Gal}_{\partial} : \mathbf{Alg}_k \to \mathbf{Grp}$.

Proposition 7.2.2.4. *Let $L|K$ be a Picard-Vessiot extension for a differential module (V, ∇). The previously defined functor $\mathrm{Gal}_{\partial}(L|K)$ defines an affine algebraic k-group.*

Proof. In order to show this, we must find $R \in Ob(\mathbf{Alg}_k)$ such that $\mathrm{Gal}_{\partial}(L|K) = \mathrm{Spm}(R)$. We see that A is in fact a quotient of $\Gamma(\mathrm{GL}_{[L:K]}, \mathcal{O})$. Let $n := [L : K]$, and fix a K-basis of V just like in point 3. of 7.2.1.4, so that we can write the coordinates of a k-basis of horizontal vectors in a matrix $(f_{ij}) \in \mathrm{GL}_n(k)$. Notice that for any $\sigma \in \mathrm{Gal}_{\partial}(k)$, σ multiplies (f_{ij}) by a matrix $M_{\sigma} \in \mathrm{GL}_n(k)$. Since f_{ij} generate L over K, this determines σ and, analogously, for any $\sigma_A \in \mathrm{Gal}_{\partial}(A)$, where $A \in Ob(\mathbf{Alg}_k)$, we can find a matrix $M_{\sigma_A} \in \mathrm{GL}_n(A)$ giving a correspondence between elements of $Gal_{\partial}(A)$ and $\mathrm{GL}_n(A)$. On the other hand, there is a K-algebra exhaustive morphism

$$\phi : B \otimes_k K \to L$$

induced by the map that sends $x_{ij} \in \Gamma(\mathrm{GL}_n, \mathcal{O})$ to f_{ij} and that is also compatible with the derivations of B and L. Its kernel, $P := \mathrm{Ker}(\phi) \subseteq B \otimes_k K$ is a maximal ideal and each $L \otimes_k A$-automorphism $\sigma_A \in \mathrm{Gal}_{\partial}(A)$ canbe lifted to a morphism

$\widetilde{\sigma}_A : B \otimes_k K \otimes_k A \to L \otimes_k A$ such that $\widetilde{\sigma_A}(P_A) = 0$, where P_A is the ideal generated by P in $B \otimes_k K \otimes_k A$. Now, let $\{e_\lambda\}_{\lambda \in \Lambda}$ be a k-basis of L. We can write

$$\widetilde{\sigma_A}(b \otimes r \otimes a) = \sum_{\lambda \in \Lambda} \alpha_\lambda^{\widetilde{\sigma_A}}(b \otimes r \otimes a) e_\lambda, \qquad \forall b \otimes r \otimes a \in B \otimes_k K \otimes_k A.$$

The sum is finite and each $\alpha_\lambda^{\widetilde{\sigma_A}}(b \otimes r \otimes a) \in A$, for all $b \otimes r \otimes a \in B \otimes_k K \otimes_k A$. Hence, the condition we must impose is that

$$\alpha_\lambda^{\widetilde{\sigma_A}}(b \otimes r \otimes a) = 0, \qquad \forall b \otimes r \otimes a \in P_A.$$

It suffices to show it for a finite system of generators of P, $\{p_1, \ldots, p_m\}$. Let $A = B$, and let σ_B be the automorphism corresponding to M_{σ_B}. Also, let

$$R := \frac{B}{\langle \alpha_\lambda^{\widetilde{\sigma_B}}(p_l) : 1 \leq l \leq m, \lambda \in \Lambda \rangle}.$$

For a given $A \in Ob(\mathbf{Alg}_K)$ and σ_A corresponding to M_{σ_A}, let $B \to A$ be the morphism that sends x_{ij} to $(M_{\sigma_R})_{ij}$. Since σ_A comes from a $\widetilde{\sigma_A}$ satisfying $\widetilde{\sigma_A}(P_A) = 0$, $B \to A$ factors through R and therefore we have that $\mathrm{Gal}_\partial = \mathrm{Spm}(R)$. □

Now, in order to achieve our second goal, that is, seeing that $\mathrm{Gal}(V, \nabla)$ is isomorphic to $\mathrm{Gal}_\partial(L|K)$, we must construct a functor between $\langle (V, \nabla) \rangle_\otimes$ and $\mathbf{Rep}(\mathrm{Gal}_\partial(L|K))$. In order to do so, let $(V, \nabla) \in Ob(\mathbf{DiffMod}_K)$ and let $L|K$ be a Picard-Vessiot extension for (V, ∇). We can extend any $\sigma \in \mathrm{Gal}_\partial(L|K)(k)$, i.e, any K-automorphism of L $\sigma : L \to L$ as $\tilde{\sigma} : V \otimes_K L \to V \otimes_K L$ and it will satisfy $\tilde{\sigma}\left((V \otimes_K L)^{\nabla_L}\right) = (V \otimes_K L)^{\nabla_L}$, which shows that $\omega_L : \mathbf{DiffMod}_K \to \mathbf{fVec}_k$ can be regarded as a functor on $\mathbf{DiffMod}_K$ into the category of finite dimensional representations of $\mathrm{Gal}_\partial(L|K)(k)$. On the other hand, for each $(W, \nabla) \in Ob(\langle (V, \nabla) \rangle_\otimes)$, we can define a k-algebra functor $A \mapsto (W \otimes_K L \otimes_K A)^{\nabla_{L \otimes_K R}}$, gives us a functor

$$\Omega : \langle (V, \nabla) \rangle_\otimes \to \mathbf{Rep}(\mathrm{Gal}_\partial(L|K))$$

In order to prove the next proposition, we must see a previous result.

Lemma 7.2.2.5. Let (K, ∂) be a differential field with constant field $k = \overline{k}$ and $L|K$ a Picard-Vessiot extension for (V, ∇). Then, the differential field of $\mathrm{Gal}_\partial(L|L)(k)$-invariant elements of L is K.

Proof. Firstly, thanks to the proof of Proposition 7.2.2.4, we know that L is the field of fractions of $R := \Gamma(\mathrm{GL}_{[L:K]}, \mathcal{O})/P$, where P is a maximal differential ideal. Hence, let $\frac{b}{c} \in L \setminus k$ with $b, c \in R$ and $d := b \otimes c - c \otimes b \in R \otimes_k R$. Thanks to point A.15 in [P], we have $d \neq 0$, and by A.16 from the same book, $R \otimes_k R$ has no nilpotent elements because $\mathrm{Char}(k) = 0$. Let J be a maximal diferential ideal in $(R \otimes_k R)[1/d]$. Also, set a derivation on the latter differential ring via

$$D(r \otimes r') := \partial r \otimes r' + r \otimes \partial r'.$$

Let $\phi_i : R \to N := \frac{(R \otimes_k R)[1/d]}{J}$ given by the tensor product. The images of each ϕ_i are generated by fundamental matrices of the same matrix associated to the differential equation that the latter extension defines. Therefore, both images are equal to a certain $S \subset N$ and $\phi_i : R \to S$ are in fact isomorphism. This shows the existence of an element $\sigma \in \mathrm{Gal}_\partial(L|K)(k)$ satisfying $\phi_1 = \phi_2 \circ \sigma$. The image of d in N is equal to $\phi_1(b)\phi_2(c) - \phi_1(c)\phi_2(b)$, and since the image of d in N is non-zero by previous observations, we have $\phi_1(b)\phi_2(c) \neq \phi_1(c)\phi_2(b)$. Hence, $\phi_2((\sigma(c))b) \neq \phi_2((\sigma(b))c)$ so $(\sigma(b))c \neq (\sigma(c))b$, which shows that $\sigma(b/c) \neq b/c$. □

So now we have all the tools we need in order to show the following proposition, which gives us the relationship we were looking for.

Proposition 7.2.2.6. *Let (V, ∇) be a differential module and let $L|K$ be a Picard-Vessiot extension for (V, ∇). If the constant field k is algebraically closed, then the functor $\Omega : \langle V, \nabla\rangle_\otimes \to \mathbf{Rep}(\mathrm{Gal}_\partial(L|K))$ defines an equivalence of neutral Tannakian categories.*

Proof. Our goal is to show that Ω is fully faithful. Let $(W, \nabla), (W', \nabla) \in Ob(\langle V, \nabla\rangle_\otimes)$. Thanks to the rigidity of $\langle V, \nabla\rangle_\otimes$, we have an isomorphism

$$\mathrm{Hom}_{\mathbf{DiffMod}_K}((W, \nabla), (W', \nabla)) \simeq \mathrm{Hom}_{\mathbf{DiffMod}_K}(K, (W^\vee \otimes_K W', \nabla)).$$

Analogously, for the corresponding representations of Gal_∂ we may assume that (W, ∇) is the field K endowed with the trivial connection. We know that the elements of $\mathrm{Hom}_{\mathbf{DiffMod}_K}(K, (W', \nabla))$ correspond bijectively to W'^∇. Since $k = \overline{k}$, the elements of $\omega_L(\mathrm{Hom}_{\mathbf{DiffMod}_K}(K, (W', \nabla)))$ correspond to elements of $(W' \otimes_K L)^{\nabla_L}$ invariant by $\mathrm{Gal}_\partial(L|K)(k)$. Using the previous lemma, the $\mathrm{Gal}_\partial(L|K)(k)$-invariant elements in $W' \otimes_K L$ are exactly the elements of W', hence we obtain the horizontal elements of W'. Finally, in order to see that it is exhaustive, taking into account point 3 of 7.2.1.4, ω_L maps (V, ∇) to a faithful representation of $\mathrm{Gal}_\partial(L|K)$, or in other words, $\mathrm{Gal}_\partial(L|K) \to \mathrm{GL}_n(k)$ is injective. Using Lemma 6.5.16 of [Sza], the result follows. □

So we conclude this chapter by proving the following classical theorem.

Theorem 7.2.2.7. *Let (K, ∂) be a differential field with constant field $k = \overline{k}$ and let $L|K$ be a Picard-Vessiot extension for the differential module (V, ∇) over K. The map*

$$H(k) \mapsto L^{H(k)}$$

induces a bijection between closed subgroups of $\mathrm{Gal}_\partial(L|K)(k)$ and differential subfields of L containing K. Closed normal subgroups correspond to Picard-Vessiot extensions of K and the associated differential Galois group is $(\mathrm{Gal}_\partial(L|K)/H)(k)$.

Proof. Let M be an intermediate differential field between K and L. Then, L is a Picard-Vessiot extension for the differential module obtained by the change of basis, (V_M, ∇_M). This identifies $\mathrm{Gal}_\partial(L|M)(k)$ with a closed subgroup $H(k)$ of $\mathrm{Gal}_\partial(L|K)(k)$, just like it is closed in $\mathrm{GL}_n(k)$ with the same GL_n as for $\mathrm{Gal}_\partial(L|K)(k)$. Using Lemma 7.2.2.5, we have $L^{H(k)} = M$. Analogously, a closed subgroup $H(k) \subset \mathrm{Gal}_\partial(L|K)(k)$ fixes some M with $\mathrm{Gal}(V_M, \nabla_M)(k) \simeq H(k)$. On the other hand, given a closed normal algebraic subgroup $H \subset \mathrm{Gal}_\partial(L|K)$, let $\mathcal{M} \in \mathbf{Rep}(\mathrm{Gal}_\partial(L|K))$ with kernel H. The full subcateogry $\langle \mathcal{M} \rangle_\otimes$ is equivalent to $\mathbf{Rep}(\mathrm{Gal}_\partial(L|K)(k))$ by Lemma 6.5.16 of [Sza] and on the other hand to $\langle W, \nabla \rangle_\otimes$ for some $(W, \nabla) \in Ob(\langle V, \nabla \rangle_\otimes)$. Since L satisfies points 1. and 2. of 7.2.1.4 for (W, ∇), and by 3. of the same definition. it contains a Picard-Vessiot extension M for (W, ∇). By construction, we have $\mathrm{Gal}(W, \nabla) \simeq \mathrm{Gal}(V, \nabla)/H$ and $\mathrm{Gal}_\partial(L|M)(k) \simeq H(k)$ by the first part of the proof. $\square$

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