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Ulrich bundles and varieties of wild representation type

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Als meus pares.

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J’ai tendu des cordes de clocher à clocher;
des guirlandes de fenêtre à fenêtre;
des chaînes d’or d’étoile à étoile,
et je danse.

Arthur Rimbaud.
Illuminations, 1886.

Who is he that hideth counsel without knowledge?
therefore have I uttered that I understood not;
things too wonderful for me, which I knew not.

Job, 42.3.
The holy Bible, King James version.

Contents

Introduction

The subject of this thesis lies at the junction of mainly three topics: construction of large families of Arithmetically Cohen-Macaulay indecomposable vector bundles on a given projective variety X, the shape (i.e, the graded Betti numbers) of the minimal free resolution of a general set of points on X and the (ir)reducibility of the Hilbert scheme $\mathrm{Hilb}^s(X)$ of zero-dimensional subschemes $Z \subseteq X$ of length s. Let us explain how these topics are intertwined.

Given a projective variety $X \subseteq \mathbb{P}^n$ with coordinate ring R_X, it is usual to try to understand the complexity of X in terms of the associated category of vector bundles that it supports. Since, in general, this category is unwieldy, one usually restricts oneself to the category of (semi)-stable vector bundles, which is known to behave well and, in particular, there exists a nice moduli space parameterizing them. Whereas this approach has been largely and fruitfully exploited, it is also possible to pay attention to another property of a vector bundle $\mathcal{E}$: the fact of having cohomology as simple as possible, i.e., $\mathrm{H}^i(X, \mathcal{E}(l)) = 0$ for all $l \in \mathbb{Z}$ and $i = 1, \dots, \dim(X) - 1$. The vector bundles holding this property are called *Arithmetically Cohen-Macaulay (ACM)* vector bundles. When X is ACM, in terms of the associated R_X-module $E = \mathrm{H}^0_*(\mathcal{E}) := \oplus_l \mathrm{H}^0(X, \mathcal{E}(l))$, they correspond to *Maximally Cohen-Macaulay (MCM)* modules, i.e., modules that verify $\mathrm{depth}(E) = \dim(R_X)$. This correspondence allows us to study the problem alternatively from the algebraic or the geometric point of view. The study of such vector bundles (or modules) has a long and interesting history behind. A seminal result is due to Horrocks which asserts that, on the projective space $\mathbb{P}^n$, any ACM vector bundle splits as a direct sum of line bundles (cf. [Hor64]). Or, in other words, the unique indecomposable ACM vector bundle on $\mathbb{P}^n$, up to twist and isomorphism, is $\mathcal{O}_{\mathbb{P}^n}$. This would correspond with the general philosophy that a "simple" variety should have associated a "simple" category of ACM vector bundles. Following these lines, a cornerstone result was the classification of

ACM varieties of *finite representation type*, namely, those varieties that have only a finite number of indecomposable ACM vector bundles (cf. [BGS87] and [EH88]). It turned out to be that they fall into a very short list: three or less reduced points on $\mathbb{P}^2$, a projective space, a smooth quadric hypersurface $X \subset \mathbb{P}^n$, a cubic scroll in $\mathbb{P}^4$, the Veronese surface in $\mathbb{P}^5$ or a rational normal curve.

For the rest of ACM varieties, it became an interesting problem to give a criterium to split them in a finer classification. An inspiring approach was offered by representation theory, where it was proven that finite-dimensional algebras of infinite type (i.e., having infinitely many indecomposable representations) split into two classes: they are either *tame*, for which indecomposable representations of any fixed dimension form a finite set of at most one-dimensional families; or they are *wild*, for which there exist arbitrarily large families of non-isomorphic indecomposable representations (cf. [Dro86]). An analogous result was also obtained for the category of quivers, where Gabriel obtained a striking classification result: a quiver is of finite representation type exactly when its underlying undirected graph is a union of Dynkin diagrams of type A, D, E (cf. [Gab72]). Also the study of the category of indecomposable Cohen-Macaulay modules over Cohen-Macaulay rings has been a branch of intensive research in the recent years. Therefore, motivated by these results, in [DG01], such a trichotomy (i.e., finite, tame and wild representation type) was proposed for ACM projective varieties (see Definitions 4.2.7 and 4.2.10). For the one-dimensional case, it was proved that such a trichotomy is exhaustive: a smooth projective curve is of finite (resp. tame, wild) if and only if it has genus 0 (resp. 1, ≥ 2). However, it became clear that, for projective varieties, such trichotomy could not be exhaustive. In [CH04], it was shown that the quadric cone $X \subseteq \mathbb{P}^3$ has an infinite discrete set of indecomposable ACM sheaves. Ever since these early results, it has become a challenging problem to decide the representation type of a given ACM variety. It was proved in [CHb] that smooth cubic surfaces are of wild representation type. In [PLT09], it was shown that del Pezzo surfaces of degree ≤ 6 are of wild representation type. In fact, no examples of wild representation type of dimension > 2 were known. Therefore we addressed the following question:

Question. *Given an ACM projective variety $X \subseteq \mathbb{P}^n$, construct large families of indecomposable ACM vector bundles on it in order to prove that it is of wild representation type.*

In **chapter 4**, we make a contribution to this problem by showing that the two following families of ACM varieties are of wild representation type, namely, Fano varieties (i.e., varieties for which the anticanonical divisor is ample) obtained as the blow-up of points on $\mathbb{P}^n$, $n \geq 2$; and general surfaces $X \subseteq \mathbb{P}^3$ of degree $3 \leq d \leq 9$ (see Theorems 4.3.13 and 4.5.8). In general, one of the main difficulties one faces in order to prove wildness is to assure indecomposability of the vector bundles that one constructs. The strategy we followed to overcome this difficulty was to try to prove some stronger property of a vector bundle that would imply indecomposability. In fact, we managed to prove that the vector bundles $\mathcal{E}$ were either simple (i.e.,End($\mathcal{E}$) = k) or, in the best of the cases, stable.

Among other features of a given vector bundle, a very rich one is the fact of being generated by its global sections or, at least, to have a large number of them. The algebraic counterpart had already arisen a lot of interest. In fact, Ulrich proved (cf. [Ulr84]) that for a local (or *local graded) ring R there exists an upper bound for the minimal number of generators of a Maximally Cohen-Macaulay (MCM) R-module M of positive rank. Precisely, if $\mu(M)$ denotes the minimal number of generators of M and $e(R)$ denotes the multiplicity of R, then it always holds that $\mu(M) \leq e(R)\,\mathrm{rk}(M)$. MCM modules attaining this bound have been called (fortunately) *Ulrich modules*. Once again, the existence of such an R-module sheds some light over the structure of R. For instance, if a Cohen-Macaulay ring R supports an Ulrich module M verifying $\mathrm{Ext}^i_R(M,R) = 0$ for $1 \leq i \leq \dim(R)$, then R is Gorenstein (cf. [Ulr84]). Therefore, it became an interesting question to find out which Cohen-Macaulay rings support Ulrich modules. A positive answer to this question is provided, for instance, when $\dim(R) = 1$, when R has minimal multiplicity or when R is a strict complete intersection (i.e., R is local complete intersection such that its associated graded ring is a homogeneous complete intersection ring). These algebraic considerations prompted to define, for a projective variety $X \subseteq \mathbb{P}^n$, a vector bundle $\mathcal{E}$ on X to be *Ulrich* if it is ACM and the associated graded R_X-module $\mathrm{H}^0_*(\mathcal{E})$ is Ulrich. Notice that, when $\mathcal{E}$ is initialized (i.e., $\mathrm{H}^0(X,\mathcal{E}(-1)) = 0$ but $\mathrm{H}^0(X,\mathcal{E}) \neq 0$) then the last condition is equivalent to $\dim_k \mathrm{H}^0(\mathcal{E}) = \deg(X)\,\mathrm{rk}(\mathcal{E})$. For an initialized vector bundle $\mathcal{E}$, the fact of being Ulrich has an interesting interpretation in cohomological terms (it should hold that $\mathrm{H}^i(X,\mathcal{E}(-i)) = 0$ for $i > 0$ and $\mathrm{H}^i(X,\mathcal{E}(-i-1)) = 0$ for $i < \dim(X)$) and in terms of its minimal free $\mathcal{O}_{\mathbb{P}^n}$-resolution, since it has to be linear of length $n - \dim(X)$.

Question. *Given an ACM projective variety $X \subseteq \mathbb{P}^n$ and an integer $r \in \mathbb{Z}$, construct Ulrich vector bundles of rank r with support on X.*

Concerning existence results, it is known that an arbitrary curve supports rank one and two Ulrich vector bundles (cf. [ESW03]). In the case of a planar curve, there is a beautiful relation between the existence of such vector bundles and the possibility of writing the equation of the curve as the determinant (resp. the pfaffian) of a matrix (resp. a skew-symmetric matrix) with linear entrances (cf. [Bea00]). As for a general hypersurface $X \subseteq \mathbb{P}^{n+1}$ of degree d, it is known that for $n = 2$, X supports a rank 2 Ulrich vector bundle if and only if $d \leq 15$ and for $n = 3$, it happens if and only if $d \leq 5$ (cf. [Bea00]). For $n \geq 4$ and $d \geq 3$, the general n-dimensional hypersurface does not support a rank 2 Ulrich (cf. [CM05]). On smooth cubic surfaces and threefolds, the existence of Ulrich vector bundles of arbitrary rank has been proved by Casanellas and Hartshorne in [CHb]. In **chapter 4** we face these problems and contribute to them as follows: we construct large families of simple Ulrich vector bundles of arbitrary rank on any del Pezzo surface (see Theorems 4.4.11 and 4.4.19). We also construct large families of simple even rank Ulrich vector bundles on a general surface $X \subseteq \mathbb{P}^3$ of degree $3 \leq d \leq 9$ (see Theorem 4.5.8).

A possible approach to the construction of ACM and Ulrich vector bundles on a given projective variety $X \subseteq \mathbb{P}^n$ is offered by the well-known *Serre correspondence*. For instance in the particular case of a surface X, this correspondence provides a dictionary between rank two vector bundles $\mathcal{E}$ on X with Chern classes $c_1(\mathcal{E})$ and $c_2(\mathcal{E})$ and zero-dimensional locally complete intersection subscheme $Z \subseteq X$ of length $c_2(\mathcal{E})$ such that the couple $(\mathcal{O}_X(K_X + c_1(\mathcal{E})), Z)$ has the Cayley-Bacharach property (cf. [HL97, Theorem 5.1.1]). They are related by the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{Z|X}(c_1(\mathcal{E})) \longrightarrow 0.$$

Moreover, it is possible to translate further information about $\mathcal{E}$ in terms of Z and *vice versa*. For instance, in the previous setting, the vector bundle $\mathcal{E}$ will be ACM if and only if Z is an arithmetically Gorenstein scheme. Since this property can be read out of the minimal free resolution of Z, it is a meaningful problem to find out the shape of the minimal free resolution of the coordinate ring R_Z of a general set of points Z lying on a given variety X. For $X = \mathbb{P}^n$, this is a classical problem that has attired a lot of attention. We know that if Z is a general set of distinct points in $\mathbb{P}^n$ its minimal free resolution has to be of the form:

$$0 \longrightarrow F_n \longrightarrow \dots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow R_Z \longrightarrow 0$$

with $F_0 = R := k[x_0, \dots, x_n]$ and

$$F_i \cong R(-r-i)^{b_{i,r}} \oplus R(-r-i+1)^{b_{i,r-1}}$$

for $i = 1, \dots, n$, where r is the unique nonnegative integer such that

$$\binom{r+n-1}{n} \leq s < \binom{r+n}{n}.$$

Moreover we have:

$$b_{i+1,r-1} - b_{i,r} = \binom{r+i-1}{i}\binom{r+n}{n-i} - s\binom{n}{i}.$$

The *Minimal Resolution Conjecture (MRC)* proposed by Lorenzini (cf. [Lor93]) says that there exist no *ghost* terms in the minimal free resolution of R_Z, i.e., $b_{i+1,r-1}b_{i,r} = 0$ for all i. A lot of work has been devoted to contribute to this conjecture. In particular, the MRC is known to hold for any number of points s in $\mathbb{P}^n$ for $n = 2$ (see [Gae51, p. 912]), $n = 3$ ([BG86]) and $n = 4$ ([Wal95, Theorem 1]). The MRC is known also to hold for large values of s for any n (see [HS96, p. 468]). On the other hand, MRC fails in general for any $n \geq 6$, $n \neq 9$ (see [EPSW02, Theorem 1.1]).

It was also possible to pay attention just to the initial and ending terms of the minimal free resolution of R_Z and therefore two weaker conjectures were proposed: the *Ideal Generation Conjecture (IGC)* which says that the minimal number of generators of the ideal of a general set of points will be as small as possible. In terms of the Betti numbers, it simply says that $b_{1,r}(Z)b_{2,r-1}(Z) = 0$. On the other hand, the *Cohen-Macaulay type Conjecture (CMC)* affirms that the canonical module $K_Z = \operatorname{Ext}^n(R/I_Z, R(-n-1))$ has as few generators as possible. Since the dual of the minimal resolution of R_Z provides a (twisted) resolution of K_Z this conjecture can also been translated in terms of Betti numbers: $b_{n-1,r}(Z)b_{n,r-1}(Z) = 0$. Regarding these two conjectures, CMC has been proved in full generality in the case of the projective space $X = \mathbb{P}^n$, for any n (see [Tru89, p. 112]). It is also known that the IGC holds for large set of points on curves of degree $d \geq 2g$ (see [FMP03, Theorem 2.2].

More recently Mustaţă extended the previous results about the shape of minimal free resolutions of general set of points $Z \subseteq X$ for the case $X = \mathbb{P}^n$ to an arbitrary projective variety $X \subseteq \mathbb{P}^n$ (cf. [Mus98]). He proved that the first rows of the Betti diagram of a general set of distinct points Z in a projective variety X coincide with the Betti diagram of X and that there are two extra nontrivial rows at the bottom. He also gave lower bounds for the Betti numbers in these last two rows. In other words, if we let

$$0 \to F_n \to F_{n-1} \to \cdots \to F_2 \to F_1 \to R \to R_X \to 0$$

be a minimal free R-resolution of R_X, then for a general set of points $Z \subseteq X$ such that $P_X(r-1) \leq |Z| < P_X(r)$ for some $r \geq \operatorname{reg}(X) + 1$ (where P_X denotes the Hilbert polynomial of X), R_Z has a minimal free R-resolution of the following type

$$0 \longrightarrow F_n \oplus R(-r-n+1)^{b_{n,r-1}(Z)} \oplus R(-r-n)^{b_{n,r}(Z)} \longrightarrow \cdots$$
$$\longrightarrow F_2 \oplus R(-r-1)^{b_{2,r-1}(Z)} \oplus R(-r-2)^{b_{2,r}(Z)} \longrightarrow$$
$$\longrightarrow F_1 \oplus R(-r)^{b_{1,r-1}(Z)} \oplus R(-r-1)^{b_{1,r}(Z)} \longrightarrow R \longrightarrow R_Z \longrightarrow 0.$$

Like for the general case, he stated the Minimal Resolution Conjecture in this setting asserting that the graded Betti numbers are as small as possible: $b_{i+1,r-1}(Z)b_{i,r}(Z) = 0$ for all i. This version of the conjecture has already been studied in some interesting cases. For instance, MRC holds for any number of general points on a smooth quadric surface in $\mathbb{P}^3$ (cf. [GMR96]) and for some special cardinalities of sets of general points on a smooth cubic surface. The study of the MRC for curves was pursued in [FMP03], where it was shown that the conjecture holds for large cardinalities of general points on canonical curves $C \subseteq \mathbb{P}^n$ (i.e., curves embedded in $\mathbb{P}^n$ by its canonical divisor). Nevertheless, oppositely to the case of the projective space, the MRC fails for sets of points of arbitrarily large length on curves of high degree.

In **chapter 2** we focus on the three aforementioned conjectures in the case of general set of points on (non necessarily smooth) *ACM quasi-minimal surfaces*, which are defined as nondegenerates ACM varieties $X \subseteq \mathbb{P}^d$ such that $\deg(X) = \operatorname{codim}(X) + 2$. Recall that given a nondegenerate projective variety $X \subseteq \mathbb{P}^d$ it always holds that $\deg(X) \geq \operatorname{codim}(X) + 1$. *Minimal varieties*, i.e., varieties for which there is equality in the previous expression have been classically classified. The next best case, that of quasi-minimal varieties, has been the center of intense research recently. A good classification of such varieties has been obtained by Fujita (cf. [Fuj90]), related to his theory of Δ-genus. In the two dimensional case, the family of strong del Pezzo surfaces is a particular meaningful case of ACM quasi-minimal surfaces. In [Hoa93], an important contribution was made to the understanding of quasi-minimal varieties, in particular to the structure of the singularities that they can support. Among other things, the minimal free resolution of the coordinate ring of an ACM quasi-minimal surface $X \subseteq \mathbb{P}^d$ was found:

$$0 \longrightarrow R(-d) \longrightarrow R(-d+2)^{\alpha_{d-3}} \longrightarrow \ldots \longrightarrow R(-2)^{\alpha_1} \longrightarrow R \longrightarrow R_X \longrightarrow 0$$

where

$$\alpha_i = i\binom{d-1}{i+1} - \binom{d-2}{i-1} \text{ for } 1 \leq i \leq d-3.$$

The knowledge of this resolution will be a cornerstone for obtaining our results. We are going to prove that the IGC and CMC hold for general sets of any cardinality of points on ACM quasi-minimal surfaces X, up to two sporadic cases (see Theorem 2.2.16). As for the full MRC, we are going to see that it holds for a very wide range of cardinalities of general points on X (see Theorem 2.2.15).

Notice that in terms of the Hilbert scheme $\mathrm{Hilb}^s(X)$ of zero-dimensional subschemes of X, the Minimal Resolution Conjecture for X could be stated saying that there exists a non-empty open subset $U_0^s \subset H_0^s \subset \mathrm{Hilb}^s(X)$, where H_0^s denotes the irreducible component whose general points correspond to a set Z of s *distinct* points on X, such that for any $[Z] \in U_0^s$ we have

$$b_{i+1,r-1}(Z) \cdot b_{i,r}(Z) = 0 \quad \text{for} \quad i = 1, \cdots, n-1.$$

If we do not want to restrict ourselves to set of distinct points, we can wonder how should be the shape of the minimal free resolution of the homogeneous ideal of the 0-dimensional scheme associated to a general point $[Z]$ of any other irreducible component of $\mathrm{Hilb}^s(X)$ and ask if the graded Betti numbers $b_{ij}(Z)$ are as small as possible, i.e. there are no ghost terms in the minimal free resolution of R_Z. Therefore, in **chapter 2** we propose a modified conjecture and say that the *Weak Minimal Resolution Conjecture (WMRC)* holds for s if there is an irreducible component $H^s \subset \mathrm{Hilb}^s(X)$ and a non-empty open subset $U^s \subset H^s \subset \mathrm{Hilb}^s(X)$ such that for any $[Z] \in U^s$ we have

$$b_{i+1,r-1}(Z) \cdot b_{i,r}(Z) = 0 \quad \text{for} \quad i = 1, \cdots, n-1.$$

Regarding the WMRC, we manage to prove that for any integer $d \geq 2$ and for any $s \geq \binom{d+3}{3} - 1$, there exists a $\binom{d+2}{2}$-dimensional family of irreducible generically smooth surfaces $X \subset \mathbb{P}^3$ of degree d satisfying this conjecture (see Theorem 2.3.18).

Of course, in the case of $\mathrm{Hilb}^s(X)$ being irreducible, both conjectures, MRC as proposed by Mustaţă and our modified one should agree. So it turns out to be a crucial question to know when irreducibility of the Hilbert scheme $\mathrm{Hilb}^s(X)$ holds. In general, ever since the existence of the Hilbert scheme $\mathrm{Hilb}^{p(t)}(X)$ parameterizing projective subschemes of a projective variety X with Hilbert polynomial $p(t)$ was shown by Grothendieck in [Gro], the study of the geometrical properties of this moduli space became an area of intense research in Algebraic Geometry. An early result by Hartshorne (cf. [Har66]) affirms that it is always connected. When we specialize to subschemes of constant Hilbert polynomial $p(t) = s$, i.e, when we are dealing with zero-dimensional subschemes of length s, Fogarty proved that, if X is a *smooth* irreducible surface, then the Hilbert scheme $\mathrm{Hilb}^s(X)$ is a smooth irreducible variety of dimension $2s$ (cf. [Fog68]). In larger dimension, Iarrobino in [Iar72] found that irreducibility is no longer true: the Hilbert scheme can be reducible for varieties of dimension ≥ 3. In the short **chapter 3** we focus our attention on singular varieties and ask about the irreducibility of the Hilbert scheme of their 0-dimensional subschemes. The most interesting case, due to Fogarty's result, is that of singular surfaces:

Question. *Is the Hilbert scheme* $\mathrm{Hilb}^s(X)$ *of 0-dimensional schemes of length* s *on a singular surface* X *irreducible?*

We are going to give a negative answer to this question, by constructing singular surfaces whose Hilbert scheme of points is reducible. In fact, our method also works for varieties of larger dimension. We are going to construct generically smooth projective varieties $X \subset \mathbb{P}^N$ of dimension n and degree d with $n > 2$ and $d > 1$ or $n = 2$ and $d > 4$ for which $\mathrm{Hilb}^s(X)$ is reducible for all $s \gg 0$ (see Theorem 3.1.5).

Let us outline now the structure of this thesis and the main results obtained.

Chapter 1 is devoted to recall the notions that will be the subject of the rest of the present work as well as well-known results that will be used throughout it. We also give some examples of the concepts that are involved. We do not claim any originality on this chapter.

We start in section 1.1 introducing the basic notions of minimal free resolution and Betti diagram associated to a graded module M, as well as those of Hilbert function and polynomial. We also introduce the notion of Arithmetically Cohen-Macaulay (ACM) and Arithmetically Gorenstein (AG) scheme.

In section 1.2 we give the rudiments of Liaison Theory which will be a keystone in the proof of the results from chapter 2. Liaison Theory is a very powerful tool to carry over information from a given scheme to a second one which is linked with. We are going to illustrate this feature of Liaison with several important results (as it is Gaeta's theorem). Then we are going to see how the minimal free resolutions of two linked subschemes are related.

Finally, in section 1.3, we focus on moduli spaces. We give an introduction to the Hilbert scheme $\mathrm{Hilb}^{p(t)}(X)$ parameterizing subschemes of a given scheme X with Hilbert polynomial $p(t)$ and to the moduli space $M^s_{X,H}(r; c_1, \ldots, c_{\min(r,n)})$ of μ-stable vector bundles $\mathcal{E}$ on X with fixed rank r and Chern classes c_i.

Chapter 2 provides our contribution to the Minimal Resolution Conjecture, which is basically divided in two parts. First of all, we are going to show that it holds for a wide range of cardinalities of general set of points on a large family of varieties, namely, that of ACM quasi-minimal varieties (up to two sporadic cases). On the other hand, we are also going to deal with the special case of non-reduced zero-dimensional schemes. For these schemes, we are going to state an adapted MRC (namely, the *Weak Minimal Resolution Conjecture (WMRC)*) and prove that it holds in some interesting cases.

In section 2.1, we recall the Minimal Resolution Conjecture (MRC) and give a brief account of the known results around it. In particular, we recall Mustaţă's version of MRC:

Conjecture 2.1.10. Let $X \subset \mathbb{P}^n$ be a projective variety with $d = \dim(X) \geq 1$, $\operatorname{reg}(X) = m$ and with Hilbert polynomial P_X. Let $s \in \mathbb{Z}$ be an integer such that $P_X(r-1) \leq s < P_X(r)$ for some $r \geq m+1$. The *Minimal Resolution Conjecture (MRC for short)* holds for the value s if for every set Z of s general distinct points we have

$$b_{i+1,r-1}(Z)b_{i,r}(Z) = 0 \quad \text{for} \quad i = 1, \cdots, n-1.$$

In section 2.2, we pay attention to ACM quasi-minimal surfaces, i.e., surfaces $X \subseteq \mathbb{P}^d$ of degree d. They include the family of strong del Pezzo surfaces. For this kind of surfaces, we establish first the MRC for two specific cardinalities of points:

Theorem 2.2.13. Let $X \subseteq \mathbb{P}^d$ be an ACM quasi-minimal surface. Assume that X is not the anticanonical model of $F_2 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ or a complete intersection of two quadrics on $\mathbb{P}^4$ with a double line. Let us define:

$$m(r) := \frac{d}{2}r^2 + r\frac{2-d}{2}, \quad n(r) := \frac{d}{2}r^2 + r\frac{d-2}{2}.$$

Then we have:

(1) Let $Z_{n(r)} \subset X$ be a general set of $n(r)$ points, $r \geq 2$. Then the minimal graded free resolution of $I_{Z_{n(r)}|X}$ has the following form:

$$0 \longrightarrow R(-r-d)^{(d-1)r-1} \longrightarrow R(-r-d+1)^{\beta_{d-1,r}} \longrightarrow R(-r-d+2)^{\beta_{d-2,r}} \longrightarrow \dots$$
$$\longrightarrow R(-r-2)^{\beta_{2,r}} \longrightarrow R(-r)^{r+1} \longrightarrow I_{Z_{n(r)}|X} \longrightarrow 0.$$

where

$$\beta_{i,r} = \sum_{l=0}^{1}(-1)^{l+1}\binom{n-l-1}{i-l}\Delta^{l+1}H_X(r+l) + \binom{n}{i}(n(r) - H_X(r-1)).$$

(2) Let $Z_{m(r)} \subset X$ be a general set of $m(r)$ points, $r \geq 2$. Then its minimal graded free resolution has the following form:

$$0 \longrightarrow R(-r-d)^{r-1} \longrightarrow R(-r-d+2)^{\gamma_{d-1,r-1}} \longrightarrow \dots$$
$$\longrightarrow R(-r-1)^{\gamma_{2,r-1}} \longrightarrow R(-r)^{(d-1)r+1} \longrightarrow I_{Z_{m(r)}|X} \longrightarrow 0$$

with

$$\gamma_{i,r-1} = \sum_{l=0}^{1}(-1)^l \binom{n-l-1}{i-l} \Delta^{l+1} P_X(r+l) - \binom{n}{i}(m(r) - P_X(r-1)).$$

In particular, Mustaţă's conjecture works for $n(r)$ and $m(r)$, $r \geq 4$, general distinct points on an ACM quasi-minimal surface $X \subset \mathbb{P}^d$ (except for the two aforementioned cases).

The previous Theorem will allow us to deduce the following results: first we are going to be able to prove that the two weaker conjectures, the Ideal Generation Conjecture and the Cohen-Macaulay type conjecture, hold for any general set of points on ACM quasi-minimal surfaces (except for two sporadic cases):

Theorem 2.2.16. Let $X \subseteq \mathbb{P}^d$ be an ACM quasi-minimal surface. Assume that X is not the anticanonical model of $F_2 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ or a complete intersection of two quadrics on $\mathbb{P}^4$ with a double line. Then for any general set of distinct points Z on X such that $|Z| \geq P_X(3)$ the Cohen-Macaulay type Conjecture and the Ideal Generation Conjecture are true.

Moreover, for general set of distinct points whose cardinalities fall into determinate strips we are able to prove that the whole MRC holds (except for the same two sporadic cases):

Theorem 2.2.15. Let $X \subseteq \mathbb{P}^d$ be an ACM quasi-minimal surface. Assume that X is not the anticanonical model of $F_2 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ or a complete intersection of two quadrics on $\mathbb{P}^4$ with a double line. Let r be an integer such that $r \geq reg(X) + 1 = 4$. Then for any general set of distinct points Z on X such that $P_X(r-1) \leq |Z| \leq m(r)$ or $n(r) \leq |Z| \leq P_X(r)$ the Minimal Resolution Conjecture is true.

For the particular case of integral cubic surfaces, we see that MRC holds for any general set of distinct points.

Theorem 2.2.17. Let $X \subseteq \mathbb{P}^3$ be a integral cubic surface (i.e., an ACM quasi-minimal surface of degree three). Then the Minimal Resolution Conjecture holds for any set of general distinct points on X of cardinality $\geq P_X(3) = 19$.

In section 2.3, we focus our attention on a slightly modified conjecture. Since, in general, $\mathrm{Hilb}^s(X)$ is not irreducible (see [Iar72] for the case of varieties of dimension higher or equal than 3 and chapter 3 for surfaces), we can also search the minimal graded free resolution of the homogeneous ideal of the 0-dimensional scheme associated to a general point of any other irreducible component of the Hilbert scheme $\mathrm{Hilb}^s(X)$ and ask if the graded Betti numbers are as small as possible, i.e. there are no ghost terms in the minimal free resolution. Therefore we state the following Conjecture:

Conjecture 2.3.2. Let $X \subset \mathbb{P}^n$ be a projective variety, let $P_X(t)$ be its Hilbert polynomial and $m = \mathrm{reg}(X)$. Let s be an integer such that $P_X(r-1) \leq s < P_X(r)$ for some $r \geq m+1$. Then, the *Weak Minimal Resolution Conjecture (WMRC)* holds for s if there is an irreducible component $H^s \subset \mathrm{Hilb}^s(X)$ and a non-empty open subset $U^s \subset H^s \subset \mathrm{Hilb}^s(X)$ such that for any $[Z] \in U^s$ we have

$$b_{i+1,r-1}(Z) \cdot b_{i,r}(Z) = 0 \quad \text{for} \quad i = 1, \cdots, n-1.$$

In particular, we are able to prove the following contribution to this Conjecture:

Theorem 2.3.18. Let s be an integer such that $s \geq P_d(d)$, $d \geq 2$. Then there exists a family of dimension $\binom{d+2}{2}$ of irreducible generically smooth surfaces $X \subset \mathbb{P}^3$ of degree d for which WMRC holds, i.e. there exist a non-empty open subset $U^s \subset \mathrm{Hilb}^s(X)$ such that for any $[Z] \in U^s$ we have

$$b_{3,r-1}(I_Z) \cdot b_{2,r}(I_Z) = b_{2,r-1}(I_Z) \cdot b_{1,r}(I_Z) = 0.$$

In **chapter 3** we turn our attention to the reducibility of Hilbert scheme of points. Namely, as it was mentioned, Fogarty proved that, if X is a smooth irreducible surface, then the Hilbert scheme $\mathrm{Hilb}^s(X)$ parameterizing subschemes of length s is a smooth irreducible variety of dimension $2s$. A natural question that arises in this setting is the behavior of the Hilbert scheme of 0-dimensional schemes when the smoothness condition is removed. In this short chapter we are going to construct large families of singular surfaces whose Hilbert scheme of points is reducible. In fact, our method also works for varieties of larger dimension. More concretely, we manage to prove:

Theorem 3.1.5. Let $X = \langle Y, p \rangle \subseteq \mathbb{P}^N$ be an n-dimensional cone with vertex p and base $Y \subseteq \mathbb{P}^{N-1}$. Let us suppose that either $n > 2$ and $\deg X > 1$ or $n = 2$ and $\deg X > 4$. Then there exists $s_0 \in \mathbb{N}$ such that $\mathrm{Hilb}^s(X)$ is reducible for all $s \geq s_0$.

Finally, **chapter 4** is dedicated to the study of ACM vector bundles and in particular to the representation type of some families of varieties. As it has been mentioned, it is an intriguing question to find out the representation type of a given ACM variety since this is a good measure of its complexity. The main goal of this chapter is to provide the first examples of n-dimensional ACM varieties of wild representation type, for arbitrary $n \geq 2$ (cf. Theorems 4.3.13 and 4.4.11). Our source of examples will be Fano blow-ups $X = Bl_Z\mathbb{P}^n$ of $\mathbb{P}^n$ at a finite set of points Z. In the 2-dimensional case, i.e., for del Pezzo surfaces, much more information is obtained. In fact, the vector bundles that we construct share another particular feature: the associated module $\oplus_t \mathrm{H}^0(X, \mathcal{E}(t))$ has the maximal possible number of generators (see Theorem 4.4.11). This property was isolated by Ulrich in [Ulr84, p. 26] for Cohen-Macaulay modules, and since then modules with this property have been called Ulrich modules and correspondingly Ulrich vector bundles in the geometric case. For the case of a general surface $X \subseteq \mathbb{P}^3$ we have been able to prove wildness for $d \leq 9$, relying on a previous result about the existence of rank 2 Ulrich vector bundles on the surface (see [Bea00, Proposition 7.6]). For arbitrary degree d we can at least provide large families of rank 2 and rank 3 ACM vector bundles on a general surface of degree d showing that they are neither of finite nor tame representation type.

This chapter is divided as follows: in section 4.1 we recall the definition and main features of the varieties we are going to be interested in, namely *Fano* blow-up varieties of $\mathbb{P}^n$, $n \geq 2$, and *del Pezzo* surfaces. In section 4.2, we give an account of ACM vector bundles, Ulrich vector bundles, as well it is also discussed the problem of studying the complexity of an ACM variety according the complexity of families of ACM vector bundles that it supports.

In section 4.3, we perform the construction of large families of simple (hence indecomposable) ACM vector bundles on all Fano blow-ups of points in $\mathbb{P}^n$. These families are constructed as the pullback of the kernel of surjective morphisms

$$\mathcal{O}_{\mathbb{P}^n}(1)^b \longrightarrow \mathcal{O}_{\mathbb{P}^n}(2)^a$$

with the property that they are also surjective at the level of global sections. Therefore we are able to prove that Fano blow-ups are varieties of wild representation type. In particular, we prove:

Theorem 4.3.13. Let $X = Bl_Z\mathbb{P}^n$ be a Fano blow-up of points in $\mathbb{P}^n$, $n \geq 3$ and let $r \geq n$.

(i) If n is even, fix $c \in \{0, \dots, n/2 - 1\}$ such that $c \equiv r \bmod n/2$ and set the number $u := \frac{2(r-c)}{n}$. Then there exists a family of rank r simple (hence, indecomposable) ACM vector bundles of dimension $\frac{(n+2)n-4}{4}u^2 - cu - c^2 + 1$.

(ii) If n is odd, fix $c \in \{0, \ldots, n-1\}$ such that $c \equiv r \mod n$ and set $u := \frac{(r-c)}{n}$. Then there exists a family of rank r simple (hence, indecomposable) ACM vector bundles of dimension $((n+2)n-4)u^2 - 2cu - c^2 + 1$.

In particular, Fano blow-ups are varieties of wild representation type.

In section 4.4, we focus our attention on the 2-dimensional case, namely on del Pezzo surfaces, where much more information is obtained. In the first subsection we deal with any del Pezzo surface excluding the case of a quadric surface and we see that the ACM vector bundles that we obtained in the previous section by pullback are simple, Ulrich, and μ-stable with respect to a certain ample divisor H_n:

Theorem 4.4.11. Let $X \subseteq \mathbb{P}^d$ be a del Pezzo surface of degree d. Assume that X is not the smooth quadric embedded in $\mathbb{P}^8$ via the anticanonical divisor $-K_X$. Then for any $r \geq 2$ there exists a family of dimension r^2+1 of simple initialized Ulrich vector bundles of rank r with Chern classes $c_1 = rH$ and $c_2 = \frac{dr^2+(2-d)r}{2}$. Moreover, they are μ-semistable with respect to the polarization $H = 3e_0 - \sum_{i=1}^{9-d} e_i$ and μ-stable with respect to $H_n := (n-3)e_0 + H$ for $n \gg 0$. In particular, del Pezzo surfaces are of wild representation type.

In the intermediate subsection we focus our attention on the quadric surface and we show by an *ad hoc* method that it is a variety of wild representation type:

Theorem 4.4.19. Let $X \subseteq \mathbb{P}^8$ be the smooth quadric surface embedded in $\mathbb{P}^8$ through the very ample anticanonical divisor $H := -K_X$. Then, for any $r \geq 2$, there exists a family of rank r simple (hence indecomposable) Ulrich vector bundles of dimension $r^2 + 1$. In particular, X is a variety of wild representation type.

Finally, in the last subsection, we establish, for a del Pezzo surface X with very ample anticanonical divisor, a version of the well-known Serre correspondence (cf. Theorem 4.4.21). This correspondence will allow us, on one hand, to show, when X is distinct of the quadric surface, that the families of rank r vector bundles constructed in the first subsection could also be obtained from a general set of $m(r) := \frac{d}{2}r^2 + r\frac{2-d}{2}$ distinct points on the surface with minimal free resolution as in Theorem 2.2.13.

Corollary 4.4.22. Let $X \subseteq \mathbb{P}^d$ be a strong del Pezzo surface of degree d, distinct of the quadric surface. Then the rank r initialized Ulrich vector bundles $\mathcal{E}(H)$ given in Theorem 4.4.11 can be recovered as an extension of $\mathcal{I}_{Z,X}(rH)$ by $\mathcal{O}_X^{r-1}$ for general sets Z of $m(r) = 1/2(dr^2 + (2-d)r)$ distinct points on X, $r \geq 2$.

On the other hand, for the quadric surface, we will apply Serre correspondence in the reverse sense to obtain the minimal free resolution of a general set of $m(r)$ distinct points from the Ulrich vector bundles constructed in the previous subsection.

Finally, section 4.5 is devoted to the case of a *general* surface X of arbitrary degree d in $\mathbb{P}^3$. By constructing simple Ulrich bundles of arbitrary even rank as extensions of rank 2 Ulrich bundles, we are able to show that, for $4 \leq d \leq 9$, a general surface $X \subseteq \mathbb{P}^3$ of degree d is of wild representation type:

Theorem 4.5.8. Let $X \subseteq \mathbb{P}^3$ be a general surface of degree $4 \leq d \leq 9$. Then, for any $r = 2s$, $s \geq 2$, there exists a family of rank r simple (hence indecomposable) Ulrich vector bundle of dimension $11(s-1)$. In particular, a general surface $X \subseteq \mathbb{P}^3$ of degree $4 \leq d \leq 9$ is of wild representation type.

In the case of arbitrary degree d, we will be able at least to construct large families of rank 2 and 3 simple ACM vector bundles on a general surface $X \subseteq \mathbb{P}^3$ of degree d, showing that they are not of tame representation type:

Proposition 4.5.10. Let $X \subseteq \mathbb{P}^3$ be a general surface of degree $d \geq 3$. Then there exists a 4-dimensional family of rank 2 initialized μ-stable ACM vector bundles $\mathcal{E}$ with $c_1(\mathcal{E}) = 1$ and $c_2(\mathcal{E}) = d-1$.

Proposition 4.5.11. Let $X \subseteq \mathbb{P}^3$ be a general surface of degree $d \geq 3$. Then there exists an infinite family of rank 3 initialized μ-stable ACM vector bundles $\mathcal{F}$ with $c_1(\mathcal{F}) = 1$ and $c_2(\mathcal{F}) = 2d-3$.

We are going to conclude the chapter giving a general strategy that could be useful to prove that a general surface of arbitrary degree is of wild representation type (see Theorem 4.5.14).

Notation and conventions

Throughout this thesis we are going to work over an algebraically closed field k of characteristic zero; we set $R = k[x_0, \cdots, x_n]$, $\mathfrak{m} = (x_0, \cdots, x_n)$ and $\mathbb{P}^n = \mathrm{Proj}(R)$. All the schemes will be over k. By an *algebraic variety* we mean an integral proper scheme of finite type over k. A *polarized scheme (resp. a polarized variety)* will be a couple $(X, \mathcal{O}_X(1))$, where X is a scheme (resp. an algebraic variety) and $\mathcal{O}_X(1)$ is an ample line bundle on it. When $\mathcal{O}_X(1)$ is very ample, we are going to write $X \subseteq \mathbb{P}^n$ where the embedding is given by $\mathcal{O}_X(1)$.

A scheme $X \subseteq \mathbb{P}^n$ will be *nondegenerate* if it is not contained in any hyperplane. The sheaf of regular functions of X is denoted by $\mathcal{O}_X$. If $Y \subseteq X$ is a closed subscheme, we denote the ideal sheaf of Y in X by $\mathcal{I}_{Y|X}$ and the saturated ideal by $I_{Y|X} = \mathrm{H}^0_*(X, \mathcal{I}_{Y|X}) := \oplus_t \mathrm{H}^0(X, \mathcal{I}_{Y|X} \otimes \mathcal{O}_X(t))$ (or simply I_Y when $X = \mathbb{P}^n$). R_X stands for the homogeneous coordinate ring of X defined as $k[x_0, \ldots, x_n]/I_X$. We denote by $\mathfrak{m}_X := (\overline{x}_0, \cdots, \overline{x}_n)$ the irrelevant maximal ideal of $k[x_0, \ldots, x_n]/I_X$.

For any coherent sheaf $\mathcal{E}$ on $X \subseteq \mathbb{P}^n$ we are going to denote the twisted sheaf $\mathcal{E} \otimes \mathcal{O}_X(l)$ by $\mathcal{E}(l)$. $\mathcal{E}^\vee$ will stand for $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ and $\mathcal{E}nd(\mathcal{E}) := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})$ denotes the sheaf of endomorphisms of $\mathcal{E}$ while $\mathrm{End}(\mathcal{E}) := \mathrm{Hom}(\mathcal{E}, \mathcal{E})$ denotes the group of endomorphisms. Analogously, for a module M over a ring R, $M^\vee$ stands for $\mathrm{Hom}_R(M, R)$. If moreover M has the structure of k-vector space, then $M^{\vee k} := \mathrm{Hom}_k(M, k)$.

As usual, $\mathrm{H}^i(X, \mathcal{E})$ stands for the cohomology groups and $\mathrm{h}^i(X, \mathcal{E})$ for their dimension. We also set $\mathrm{ext}^i(\mathcal{E}, \mathcal{F}) := \dim_k \mathrm{Ext}^i(\mathcal{E}, \mathcal{F})$. We will use the notation $\mathrm{H}^i_*(\mathcal{E})$ for the graded R-module $\bigoplus_{l \in \mathbb{Z}} \mathrm{H}^i(\mathbb{P}^n, \mathcal{E}(l))$ and ω_X will stand for the dualizing sheaf. The i-th Chern class of a coherent sheaf $\mathcal{E}$ on a smooth projective scheme X will be written $c_i(\mathcal{E})$. We write $\mathrm{Pic}(X)$ for the Picard group of X, i.e., the group of line bundles modulo isomorphism. We do not distinguish between a vector bundle and its associated locally free sheaf of sections.

By a *general* homogeneous polynomial of degree d, we mean a polynomial in a suitable Zariski open and dense subset of R_d. Recall that $\dim_k R_d = \binom{n+d}{n}$ and for integers $a, b \in \mathbb{Z}$, we define $\binom{a}{b} := 0$ whenever $a < b$.

Chapter 1

Preliminaries

This preliminary chapter is devoted to introduce the notions that will be the subject of the rest of the present work as well as well-known results that will be used throughout it. We do not claim any originality on this chapter which has been divided as follows: we start in section 1.1 recalling the basic notions of minimal free resolutions and Betti diagrams associated to a graded module M. We also introduce the notion of ACM scheme. In section 1.2 we give the rudiments of the Theory of Liaison which will be a keystone in the proof of the results from chapter 2. Finally, in section 1.3, we present an introduction to the two kind of parameter spaces that we are going to be concerned with throughout this thesis: for a given scheme X, the Hilbert scheme $\mathrm{Hilb}^{p(t)}(X)$ parameterizing subschemes of X with Hilbert polynomial $p(t)$; and the moduli space $M^s_{X,H}(r; c_1, \ldots, c_{\min(r,n)})$ of μ-stable vector bundles $\mathcal{E}$ on X with fixed rank r and Chern classes c_i.

1.1 Minimal free resolutions, Betti numbers and Hilbert functions

Let $R = k[x_0, \ldots, x_n]$ be the homogeneous polynomial ring in $n+1$ variables and let M be a graded R-module. M is said to be of *finite projective dimension* if there exists a graded exact sequence:

$$F_\bullet : 0 \to F_s \xrightarrow{d_s} F_{s-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0 \qquad (1.1.1)$$

where all F_i are free R-modules. The minimum of the length s of such free resolutions is called the *projective dimension* of M and its denoted by $\mathrm{pd}(M)$.

A cornerstone in this context is the following classical result:

Theorem 1.1.1. *Let M be a finitely generated graded R-module and let*

$$0 \to E \xrightarrow{d_{n+1}} F_n \to \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$$

be an exact sequence. Then E is a free R-module. In other words, $\text{pd}(M) \leq n+1$.

Definition 1.1.2. A morphism $\phi : F \longrightarrow M$ of R-modules with F free is said *minimal* if $\phi \otimes id_{R/\mathfrak{m}} : F/\mathfrak{m}F \longrightarrow M/\mathfrak{m}M$ is the zero map in case M is free and an isomorphism in case ϕ surjective. A free resolution of M is minimal if all the morphisms d_i are minimal.

Since from any exact sequence it is possible to extract a minimal one, any finitely generated graded module M has a minimal free resolution of length $\text{pd}(M)$. Moreover, Nakayama's lemma implies that the minimal free resolution of M is unique up to isomorphism of complexes.

Remark 1.1.3. The free resolution $F_\bullet$ of M given in (1.1.1) is minimal if, after choosing basis of F_i, the matrices representing d_i do not have any non-zero scalar.

Definition 1.1.4. Let M be a graded finitely generated R-module and let

$$F_\bullet : 0 \to F_p \xrightarrow{d_p} F_{p-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$$

be its minimal free resolution. The graded *Betti numbers* $b_{ij}(M)$ are defined as

$$F_i = \oplus_{j\in\mathbb{Z}} R(-i-j)^{b_{ij}(M)}, \quad \text{i.e. } b_{ij}(M) = \dim_k \text{Tor}^i(M,k)_{i+j} \tag{1.1.2}$$

and the *Betti diagram* of M has in the $(j,i)-th$ position the Betti number $b_{ij}(M)$. The *Castelnuovo- Mumford* regularity of M is the number:

$$\text{reg}(M) := \max_{i,j}\{j | b_{i,j}(M) \neq 0\},$$

i.e, the index of the last nontrivial row in the Betti diagram of M.

For a finitely generated graded R-module $M = \oplus_t M_t$ the pieces M_t are finite k-vector spaces. The *Hilbert function* H_M of M computes these dimensions:

$$\begin{array}{rccc} H_M : & \mathbb{Z} & \longrightarrow & \mathbb{N} \\ & t & \mapsto & H_M(t) := \dim_k M_t. \end{array}$$

Moreover, we define the *Hilbert series* Ψ_M of M as the generating series of H_M, i.e.,

$$\Psi_M(t) := \sum_i H_M(i)t^i.$$

Since the Hilbert function is additive on graded exact sequences and

$$\dim_k(F_i)_t = \sum_j \dim_k R(-i-j+t) = \sum_j \binom{-i-j+t+n}{n},$$

it is immediate to see that there exists a polynomial $P_M(t) \in \mathbb{Q}[t]$ and $t_0 \in \mathbb{Z}$ such that $P_M(t) = H_M(t)$ for all $t \geq t_0$. $P_M(t)$ is called the *Hilbert polynomial* of M. It can be shown that it has degree $\dim(M) - 1$.

Example 1.1.5. Let us give a toy example produced with the computer program `Macaulay2` ([GS]). Let us consider the polynomial ring $R = k[x, y, z, t]$ in four variables. Let

$$\phi : R(-3)^2 \oplus R(-2)^2 \oplus R(-1)^3 \longrightarrow R^4$$

be the graded morphism represented by the matrix

$$\begin{pmatrix} x & x+y & z+t & x^2 & xy & x^3 & z^3+xt^2 \\ y & x+z & x+y+z & y^2 & xz & y^2t & x^2y+t^3 \\ z & x+t & x+y+t & z^2 & xt & z^2t & yzt \\ t & y+t & y+z+t & t^2 & yt & xyz & 2y^3+xzt \end{pmatrix}.$$

and let us consider the graded R-module $M := \operatorname{Coker}\phi$. Then M has the following free resolution:

$$0 \longrightarrow R(-13) \oplus R(-12)^{14} \longrightarrow R(-11)^{30} \oplus R(-10)^7 \longrightarrow$$

$$R(-10)^{14} \oplus R(-9)^5 \oplus R(-8)^5 \oplus R(-7) \longrightarrow R(-3)^2 \oplus R(-2)^2 \oplus R(-1)^3 \xrightarrow{\phi} R^4 \longrightarrow M \longrightarrow 0.$$

The Betti diagram is

	0	1	2	3	4
0	4	3	–	–	–
1	–	2	–	–	–
2	–	2	–	–	–
3	–	–	–	–	–
4	–	–	–	–	–
5	–	–	1	–	–
6	–	–	5	–	–
7	–	–	5	7	–
8	–	–	14	30	14
9	–	–	–	–	1

from which is straightforward to recover the regularity $\mathrm{reg}(M) = 9$. The Hilbert series can be expressed as

$$\Psi_M(t) = \frac{4 + 9t + 13t^2 + 14t^3 + 12t^4 + 7t^5 - t^6 - 11t^7 - 18t^8 - 17t^9 - t^{10}}{1 - t}.$$

Moreover, $\dim(M) = 1$ and $P_M(t) = 11$.

Now, let $X \subseteq \mathbb{P}^n$ be a subscheme, I_X its saturated homogenous ideal and R_X its coordinate ring. We define the Betti numbers of X as $b_{i,j}(X) := b_{i,j}(R_X)$. Analogously, we define its Hilbert function $H_X := H_{R_X}$ and the Hilbert polynomial $P_X := P_{R_X}$. On the other hand, the regularity of X is defined as the regularity of I_X (i.e, $\mathrm{reg}(X) := \mathrm{reg}(I_X)$) if $X \neq \mathbb{P}^n$ and 1 otherwise. Notice that with these definitions, $\mathrm{reg}(R_X) = \mathrm{reg}(X) - 1$.

Example 1.1.6. Let us supply a concrete example. Let us consider the curve $C \subseteq \mathbb{P}^4$ given by the homogeneous ideal

$$I_C := (x_0x_2 - x_1^2, x_0x_3 - x_1x_2, x_0x_4 - x_3x_1, x_1x_3 - x_2^2, x_1x_4 - x_2x_3).$$

The minimal free resolution of R_C is

$$0 \longrightarrow R(-5) \longrightarrow R(-3)^5 \longrightarrow R(-2)^5 \longrightarrow R \longrightarrow R_C \longrightarrow 0,$$

from which is immediate to obtain the Betti diagram:

	0	1	2	3
0	1	–	–	–
1	–	5	5	–
2	–	–	–	1

Moreover, $\text{reg}(C) = 3$ and the Hilbert polynomial is $P_C(t) = 5t$ which coincides with the Hilbert function $H_C(t)$ for $t \geq 1$. Therefore, C is an elliptic quintic curve in $\mathbb{P}^4$.

Observe that there are natural dependencies among the objects we have just defined. Namely, given a closed scheme $X \subseteq \mathbb{P}^n$, its Betti numbers $b_{i,j}(X)$ determine the Hilbert function H_X of X which in turn determines the Hilbert polynomial P_X. As one might expect, the finer the information the more involved is to find out it. As an example, let us consider a finite set of s points on $\mathbb{P}^n$. Then the Hilbert polynomial of X is trivially equal to s. It is possible to determine the Hilbert function H_X for s general points but very difficult for a particular set of points. Finally, we do not even know the Betti numbers of a set of general s points for all s and n. For a more complete account on these issues we address the reader to chapter 2.

Remark 1.1.7. The regularity of X can be translated in terms of the classical definition given by Mumford, i.e., $\text{reg}(X) \leq m$ if and only if $\text{H}^i(\mathbb{P}^n, \mathcal{I}_X(m-i)) = 0$ for $1 \leq i \leq \dim(X)$.

Remark 1.1.8. It is possible to give a lower bound to the set of integers for which the Hilbert function coincides with the Hilbert polynomial (see [Eis02, Chapter IV,Theorem 4.2]):

$$P_X(t) = H_X(t) \text{ for all } t \geq \text{reg}(X) + \text{pd}(R_X) - n - 1.$$

However, it is an open problem in general to bound, for a nondegenerate projective variety $X \subseteq \mathbb{P}^n$, $\text{reg}(X)$ in terms of the other invariants of X. A famous Conjecture by Eisenbud and Goto claims that

$$\text{reg}(X) \leq \deg X - \text{codim}\, X + 1$$

for a projective variety connected in codimension one. This conjecture has been proven, for instance, in the case of integral curves by Gruson, Lazarsfeld and Peskine (cf. [GLP83, Theorem 1.1]). The analogous result for smooth curves had already been proved by Castelnuovo (cf. [Cas93]). See [Eis02, Chapter 5] for a survey of this results.

Given a closed scheme $X \subseteq \mathbb{P}^n$, recall that by a *regular sequence* of R_X is understood a sequence of elements $F_1, \cdots, F_R$ in $\mathfrak{m}_X$ such that F_i is not a zero divisor of $R_X/(F_1, \cdots, F_{i-1})$ for all $i = 1, \cdots, r$. The maximum of the lengths of regular sequences on R_X is called the *depth* of R_X and it is denoted by depth R_X. It always holds that depth $R_X \leq \dim\, R_X$.

Definition 1.1.9. A closed subscheme $X \subseteq \mathbb{P}^n$ of dimension r is said to be *Arithmetically Cohen-Macaulay (briefly, ACM)* if its homogeneous coordinate ring R_X is a Cohen-Macaulay ring or, equivalently, $\dim R_X = \text{depth } R_X$.

We can provide an equivalent definition of ACM variety. For this, recall that the graded version of the *Auslander-Buchsbaum formula* asserts that for any finitely generated R-module M:

$$\text{pd}(M) = n + 1 - \text{depth}(M).$$

From this formula is immediate to see that a subscheme $X \subseteq \mathbb{P}^n$ is ACM if and only if the projective dimension of R_X is equal to the codimension of X; i.e.

$$\text{pd}(R_X) = \text{codim}\, X. \tag{1.1.3}$$

Hence, if $X \subseteq \mathbb{P}^n$ is a codimension c ACM subscheme, a graded minimal free R-resolution of I_X is of the form:

$$0 \longrightarrow F_c \longrightarrow F_{c-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow I_X \longrightarrow 0$$

where $F_i = \oplus_{j\in\mathbb{Z}} R(-i-j)^{b_{i,j}(X)}$, $i = 1, \ldots, c$.

Example 1.1.10. (i) *Complete intersections.* Let $G_1, \cdots, G_c$, $c \leq n$, be a regular sequence of elements of R. Then the variety $X \subseteq \mathbb{P}^n$ associated to the (saturated) ideal $I = (G_1, \cdots, G_c)$ is called a complete intersection. Since any regular sequence of R can be extended to a maximal one, we see that depth R_X = dim R_X and therefore complete intersections are examples of ACM varieties.

(ii) As an example of ACM curve that it is not a complete intersection, let us consider the *twisted cubic curve* $C \subseteq \mathbb{P}^3$. It is defined as the image of $\mathbb{P}^1$ by the map associated to the complete linear system $|\mathcal{O}_{\mathbb{P}^1}(3)|$. Its minimal free resolution has the following shape:

$$0 \longrightarrow R(-3)^2 \longrightarrow R(-2)^3 \longrightarrow R \longrightarrow R_C \longrightarrow 0.$$

Therefore, since $\text{pd}(R_C) = 2 = \text{codim}\, C$, C is an ACM curve, but it is not a complete intersection since I_X is minimally generated by three quadrics.

(iii) Finally, the smooth rational quartic curve $C \subseteq \mathbb{P}^3$ given as the image of the map

$$\begin{array}{ccc} \mathbb{P}^1 & \longrightarrow & \mathbb{P}^3 \\ (u,t) & \mapsto & (u^4, u^3t, ut^3, t^4), \end{array}$$

is not ACM. In fact, C has minimal free resolution

$$0 \longrightarrow R(-5) \longrightarrow R(-4)^4 \longrightarrow R(-3)^3 \oplus R(-2) \longrightarrow R \longrightarrow R_C \longrightarrow 0$$

from where it is seen that $\mathrm{pd}(R_C) = 3 > \operatorname{codim} C = 2$.

Now let $X \subseteq \mathbb{P}^n$ be an ACM scheme of dimension d. Then we can find a regular sequence $L_1, \dots, L_{d+1} \in R$ of elements of degree one such that $A := R/(I_X + (L_1, \dots, L_{d+1}))$ has dimension zero. A is called an *artinian reduction* of R_X. It is possible to show that we have the following relation between their Hilbert functions:

$$H_A(t) = \Delta^{d+1} H_X(t),$$

for all t. Since A has finite dimension as a k-vector space, the Hilbert function $H_A(t)$ of A has finite support and it is codified in a finite sequence of nonzero integers, $(1, H_A(1), \dots, H_A(e))$. This sequence is called the *h-vector* of X. Of course, H_X can be recover from the h-vector by integrating it $d + 1$ times.

The *socle* of a graded artinian k-algebra $A = R/I$ is defined as the annihilator of the homogeneous maximal ideal $\overline{\mathfrak{m}} = (\overline{x_1}, \cdots, \overline{x_n}) \subseteq A$, namely

$$soc(A) = \{a \in A \mid a\overline{\mathfrak{m}} = 0\}.$$

Definition 1.1.11. We say that an artinian k-algebra A has *socle degrees* $(s_1, \dots, s_t)$ if the minimal generators of its socle (as R-module) have degrees $s_1 \leq \ldots \leq s_t$. Thus, the number of s_j's that equal i is the dimension of the component of the socle of A in degree i. We say that A is *level of type* ρ if the socle $soc(A)$ of A is of dimension ρ and is concentrated in one degree $e = s_t$ (which usually is called the *socle degree*).

On the other hand, if $X \subset \mathbb{P}^n$ is an ACM subscheme then, the rank of the last free R-module in a minimal free R-resolution of I_X is called the *Cohen-Macaulay type* of X. It coincides with the minimal number of generators of K_X. Observe that if $X \subseteq \mathbb{P}^n$ is an ACM scheme and A is the artinian reduction of R_X, then A is level of type ρ if and only if the last free R-module F_c in the minimal free resolution of R_X is of the form $F_c \cong R(-e-c)^\rho$ and hence all minimal generators of K_X have the same degree. With some abuse of terminology we are going to say that an ACM scheme $X \subseteq \mathbb{P}^n$ is *level of type* ρ if its artinian reduction has this property.

Example 1.1.12. Let us give some examples:

(i) The twisted cubic curve $C \subseteq \mathbb{P}^3$ from item (i) of Example 1.1.10 is level of type 2 and socle degree 1.

(ii) The graded artinian ring $A := k[x,y]/(x^2, xy^2, y^3)$ is level of type 2 and socle degree 2 with $soc(A) = \langle xy, y^2 \rangle$.

(iii) On the other hand, the graded artinian ring $B := k[x,y]/(x^2, xy, y^3)$ is not level. In fact, $soc(B) = \langle x, y^2 \rangle$.

1.2 Liaison Theory

Liaison Theory rouse in the nineteenth century as a tool to study curves in the projective space. It has roots in the works of M. Noether, Severi and Macaulay. It turns out that a lot of information is carried over from a curve to its residual and *vice versa*, so the strategy was to pass from a given curve through a sequence of links to a "simpler curve" (basically a complete intersection) and to obtain information about the original curve from its residual simpler curve. Later on, Peskine and Szpiro (cf. [PS74]) set the modern base of Liaison Theory to work for arbitrary varieties. Roughly speaking, liaison is an equivalence relation among subschemes of a fixed dimension in some $\mathbb{P}^n$ and it deals with the study of the properties that are shared by two schemes whose union is well understood. In this section we are just going to recall the basics of this theory that are going to be need in chapter 2 and we refer to the monographies [Mig98], [KMMR+01] and [Mir08] for more details.

One of the properties that is carried over through linkage is the fact of being ACM. In order to prove it, it is necessary to introduce an equivalent characterization of a variety being ACM. Notice that any zero-dimensional scheme $X \subseteq \mathbb{P}^n$ is ACM since depth $R_X = 1$. For schemes of higher dimension, there exists an useful criterion. In order to introduce it, we define the *deficiency modules* of $X \subseteq \mathbb{P}^n$ as $\mathrm{H}^i_*(\mathcal{I}_X)$ for $i = 1, \cdots, d := \dim X$ (when X is a curve, $\mathrm{H}^1_*(\mathcal{I}_X)$ is also called the *Hartshorne-Horrocks-Rao module* of X). These R-modules can also be written in terms of Ext:

$$\mathrm{H}^i_*(\mathcal{I}_X) \cong \mathrm{Ext}^{n-i+1}_R(R_X, R(-n-1))^{\vee k},$$

for $i = 1, \cdots, \dim X$. From this characterization of the deficiency modules it is not difficult to obtain the following result:

Proposition 1.2.1. *If $X \subseteq \mathbb{P}^n$ is a subscheme of dimension $d \geq 1$ then X is ACM if and only if $\mathrm{H}^i_*(\mathcal{I}_X) = 0$ for all $1 \leq i \leq d$.*

Example 1.2.2. From this new point of view it is now very easy to see that the smooth rational quartic curve $C \subseteq \mathbb{P}^3$ from item (iii) of Example 1.1.10 is not ACM. In fact, C is the isomorphic projection from $\mathbb{P}^4$ of the rational normal quartic curve and therefore $\mathrm{H}^1(\mathcal{I}_C(1)) \neq 0$.

The deficiency modules contain a lot of interesting information. For instance, we have the following result (recall that an scheme $X \subseteq \mathbb{P}^n$ is *equidimensional* if all the primary components of the homogenous ideal I_X of X have the same dimension):

Lemma 1.2.3. *A scheme $X \subseteq \mathbb{P}^n$ is locally Cohen-Macaulay and equidimensional if and only if all of its deficiency modules are of finite length.*

Therefore, ACM schemes are locally Cohen-Macaulay and equidimensional.

Remark 1.2.4. Let $X \subseteq \mathbb{P}^n$ be an ACM scheme of codimension c. Then the dual of the minimal free resolution of R_X provides a resolution of a twist of the *canonical module*

$$K_X := \mathrm{Ext}^c_R(R_X, R)(-n-1).$$

of R_X. In fact,

$$0 \longrightarrow F_c \longrightarrow F_{c-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow R \longrightarrow R_X \longrightarrow 0,$$

yields a resolution

$$0 \longrightarrow R \longrightarrow F_1^\vee \longrightarrow \cdots \longrightarrow F_c^\vee \longrightarrow K_X(n+1) \longrightarrow 0.$$

Let us mention here the following important property of ACM schemes that will be used without further mention throughout chapter 2:

Remark 1.2.5. Let $X \subseteq \mathbb{P}^n$ be an ACM scheme of dimension ≥ 1 and let $Y \subseteq X$ be any subscheme. Then the saturated ideal $I_{Y|X} = I_{Y|\mathbb{P}^n}/I_{X|\mathbb{P}^n}$.

Definition 1.2.6. A codimension c subscheme X of $\mathbb{P}^n$ is *arithmetically Gorenstein* (briefly AG) if its homogeneous coordinate ring R_X is a Gorenstein ring or, equivalently, its saturated homogeneous ideal, I_X, has a minimal free graded R-resolution of the following type:

$$0 \longrightarrow R(-t) \longrightarrow F_{c-1} \longrightarrow \ldots \longrightarrow F_1 \longrightarrow I_X \longrightarrow 0.$$

In other words, an AG scheme is an ACM scheme with Cohen-Macaulay type 1.

We have the following equivalent definitions of AG schemes:

Proposition 1.2.7. *Let $X \subseteq \mathbb{P}^n$ be an ACM scheme of codimension c. Then the following conditions are equivalent:*

(i) X is AG.

(ii) $R_X \cong K_X(t)$ for some $t \in \mathbb{Z}$.

(iii) The minimal free resolution of R_X is self-dual, up to twist by $n+1$.

From the above proposition it is easy to see that the h-vector of an AG scheme is symmetric. Moreover, we have the following relations:

Corollary 1.2.8. *Let $X \subseteq \mathbb{P}^n$ be an AG scheme of codimension c with minimal free resolution*

$$0 \longrightarrow R(-t) \longrightarrow F_{c-1} \longrightarrow ...... \longrightarrow F_1 \longrightarrow I_X \longrightarrow 0.$$

Let $\mathcal{O}_X \cong \omega_X(l)$. Assume that the (symmetric) h-vector of X has $e+1$ entries (i.e., its socle degree is e):

$$(1, c, \Delta^{n-c+1}H_X(2), \cdots, \Delta^{n-c+1}H_X(e-2), c, 1).$$

Then:

$$t - c + 1 = \mathrm{reg}(\mathcal{I}_X) = e + 1,$$

and $l = n + 1 - t$.

Example 1.2.9. (i) *Complete intersections.* Let $G_1, \cdots, G_c$, $c \leq n$, be a regular sequence of elements of R of degrees $d_i = \deg G_i$ and consider the complete intersection $I_X = (G_1, \cdots, G_c)$. Its minimal free resolution is given by the *Koszul resolution*:

$$0 \longrightarrow R(-\Sigma_{i=1}^{c} d_i) \cong \bigwedge^{c} F_1 \longrightarrow \bigwedge^{c-1} F_1 \longrightarrow \cdots \bigwedge^{2} F_1 \longrightarrow F_1 \longrightarrow I_X \longrightarrow 0,$$

where $F_1 = \oplus_{i=1}^{c} R(-d_i)$. Therefore complete intersections are examples of AG varieties (note that the socle degree is $e = \Sigma d_i - c$).

(ii) Any AG codimension 2 variety $X \subseteq \mathbb{P}^n$ is a complete intersection. In fact, from the additivity of the rank for exact sequences, we have that in the minimal free resolution of X:

$$0 \longrightarrow R(-t) \longrightarrow F_1 \longrightarrow R \longrightarrow R_X \longrightarrow 0,$$

F_1 should have rank 2 and therefore X is a complete intersection.

(iii) On the other hand, it is not true in higher codimensions that any AG is a complete intersection. For instance, a set Z of $n+2$ points in $\mathbb{P}^n$, $n \geq 3$, in linear general position (i.e., such that any subset of $n+1$ points spans $\mathbb{P}^n$) is AG but it is not a complete intersection. Its minimal graded resolution is (cf. [Hoa93, Theorem 1])

$$0 \longrightarrow R(-n-2) \longrightarrow R(-n)^{\rho_{n-1}} \longrightarrow R(-n+1)^{\rho_{n-2}}$$

$$\longrightarrow \cdots \longrightarrow R(-3)^{\rho_2} \longrightarrow R(-2)^{\rho_1} \longrightarrow I_{Z|\mathbb{P}^n} \longrightarrow 0,$$

where $\rho_i = i\binom{n+1}{i+1} - \binom{n}{i-1}$ for $1 \leq i \leq n-1$. Therefore, Z is AG but it is not a complete intersection since $\rho_1 > n$.

(iv) As an example of ACM curve that it is not AG, let us consider the *twisted cubic curve* $C \subseteq \mathbb{P}^3$. We saw that it has minimal free resolution:

$$0 \longrightarrow R(-3)^2 \longrightarrow R(-2)^3 \longrightarrow R \longrightarrow R_C \longrightarrow 0.$$

Therefore C is a ACM curve of Cohen-Macaulay type 2, and therefore is not AG. Of course, by item (ii), we already knew it since we saw in Example 1.1.10 that the twisted cubic curve is not a complete intersection.

Definition 1.2.10. Two closed subschemes X_1 and X_2 of $\mathbb{P}^n$ are *directly Gorenstein linked* (*directly G-linked* for short) by an AG scheme $G \subseteq \mathbb{P}^n$ if $I_G \subseteq I_{X_1} \cap I_{X_2}$ and $[I_G : I_{X_1}] = I_{X_2}$, $[I_G : I_{X_2}] = I_{X_1}$. We say that X_2 is *residual* to X_1 in G. When G is a complete intersection we talk about a CI-link.

G-liaison (resp. CI-liaison) is the equivalence relation generated by G-links (resp. CI-links):

Definition 1.2.11. Let $X_1, X_2 \subseteq \mathbb{P}^n$ be two equidimensional closed subschemes without embedded components. We say that X_1 and X_2 are in the same *CI-liaison class* (resp. *G-liaison class*) if there exist closed schemes $Y_1, \cdots, Y_r \subseteq \mathbb{P}^n$ for some r such that $X_1 = Y_1$, $X_2 = Y_r$ and Y_i is directly CI-linked (resp. directly G-linked) to Y_{i+1} for $i = 1, \ldots, r-1$.

Remark 1.2.12. When X_1 and X_2 do not share any component, linkage has a clear geometric meaning. Indeed, being directly linked by a scheme G is equivalent to $G = X_1 \cup X_2$ as schemes (i.e., $I_G = I_{X_1} \cap I_{X_2}$).

Remark 1.2.13. Since any complete intersection scheme $X \subseteq \mathbb{P}^n$ is AG (see item (i) of Example 1.2.9), we see that if two subschemes are CI-linked then they are also G-linked. Even more, since in codimension 2 AG schemes and CI schemes coincide (see item (ii) of Example 1.2.9), we get that in codimension 2 CI-liaison and G-liaison agree. This is no longer true in higher codimension: for instance, the rational normal quartic curve $C \subseteq \mathbb{P}^4$ is in the G-liaison class (but it is not in the CI-class) of a complete intersection (see [KMMR+01]).

As was explained at the start of this section, a lot of properties are carried over from an scheme to its residual on a G-link. One of the main tools in order to show this feature is the mapping cone procedure. This procedure can be stated in the setting of free resolutions of finite generated R-modules as well as in the setting of locally free resolutions of coherent sheaves on a projective scheme $X \subseteq \mathbb{P}^n$. Let us recall how it works in the first setting, noticing that an analogous result holds for coherent sheaves (see [Mac63, chapter II, Section 4]):

Lemma 1.2.14 (Mapping cone procedure)**.** *Let*

$$0 \longrightarrow M \xrightarrow{\alpha} N \longrightarrow P \longrightarrow 0$$

be a short exact sequence of finitely generated R-modules and let us consider free resolutions of M,

$$e_\bullet : 0 \longrightarrow G_{n+1} \xrightarrow{e_{n+1}} G_n \longrightarrow \dots \xrightarrow{e_1} G_0 \xrightarrow{e_0} M \longrightarrow 0$$

and N,

$$d_\bullet : 0 \longrightarrow F_{n+1} \xrightarrow{d_{n+1}} F_n \longrightarrow \dots \xrightarrow{d_1} F_0 \xrightarrow{d_0} N \longrightarrow 0.$$

Then the morphism $\alpha : M \to N$ lifts to a morphism between the resolutions $\alpha_\bullet : e_\bullet \longrightarrow d_\bullet$ and a (non necessarily minimal) free resolution for P is

$$0 \longrightarrow G_{n+1} \xrightarrow{c_{n+2}} G_n \oplus F_{n+1} \xrightarrow{c_{n+1}} \dots \xrightarrow{c_3} G_1 \oplus F_2 \xrightarrow{c_2} G_0 \oplus F_1 \xrightarrow{c_1} F_0 \xrightarrow{c_0} P \longrightarrow 0$$

where

$$c_{i+1} = \begin{pmatrix} -e_i & 0 \\ \alpha_i & d_{i+1} \end{pmatrix}, \ 1 \le i \le n.$$

As a first application of the mapping cone procedure (for coherent sheaves) we have:

Lemma 1.2.15. *Let $X_1, X_2 \subseteq \mathbb{P}^n$ be two closed schemes of codimension c directly G-linked by an AG scheme W. Let the sheafified minimal free resolution of I_W be*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t) \xrightarrow{e_c} \mathcal{G}_{c-1} \xrightarrow{e_{c-1}} \dots \mathcal{G}_1 \xrightarrow{e_1} \mathcal{I}_W \longrightarrow 0.$$

Assume that X_1 is locally Cohen Macaulay and that there exists a locally free resolution of $\mathcal{I}_{X_1}$ of the form

$$0 \longrightarrow \mathcal{F}_c \xrightarrow{d_c} \mathcal{F}_{c-1} \xrightarrow{d_{c-1}} \ldots \mathcal{F}_1 \xrightarrow{d_1} \mathcal{I}_{X_1} \longrightarrow 0.$$

Then there is a locally free resolution for $\mathcal{I}_{X_2}$ of the form:

$$0 \longrightarrow \mathcal{F}_1^\vee(-t) \longrightarrow \mathcal{F}_2^\vee(-t) \oplus \mathcal{G}_1^\vee(-t) \longrightarrow \ldots \longrightarrow \mathcal{F}_c^\vee(-t) \oplus \mathcal{G}_{c-1}^\vee(-t) \longrightarrow \mathcal{I}_{X_2} \longrightarrow 0.$$

From the previous Lemma it is possible to deduce the following result:

Theorem 1.2.16. *Let $X_1, X_2 \subseteq \mathbb{P}^n$ be two equidimensional locally Cohen-Macaulay subschemes of the same dimension $d \geq 1$ directly G-linked by an AG subscheme $W \subseteq \mathbb{P}^n$ with a minimal free resolution:*

$$0 \longrightarrow R(-t) \longrightarrow G_{n-d-1} \longrightarrow \ldots G_1 \longrightarrow I_W \longrightarrow 0.$$

Then

$$\mathrm{H}_*^{d-i+1}(\mathcal{I}_{X_2}) \cong (\mathrm{H}_*^i(\mathcal{I}_{X_1}))^\vee(n+1-t)$$

for all $1 \leq i \leq d$.

As a corollary, we see that the property of being ACM is preserved under G-liaison:

Corollary 1.2.17. *Let $X_1, X_2 \subseteq \mathbb{P}^n$ be two equidimensional locally Cohen-Macaulay subschemes of the same dimension $d \geq 1$. Assume that X_1 and X_2 are directly G-linked by an AG subscheme $W \subseteq \mathbb{P}^n$. Then X_1 is ACM if and only if X_2 is ACM.*

Therefore, we see that if a subscheme $X \subseteq \mathbb{P}^n$ is in the G-liaison class of a complete intersection, then X is ACM. In codimension two we have that being ACM is a sufficient condition to be in the liaison class of a complete intersection. This famous result was first proved by Apéry and Gaeta in the case of smooth curves in $\mathbb{P}^3$ and extended to arbitrary codimension two subschemes of the projective space by Peskine and Szpiro:

Theorem 1.2.18. *Let $X \subseteq \mathbb{P}^n$ be a codimension two subscheme. Then X is in the CI-liaison class of a complete intersection if and only if X is ACM.*

The main feature of G-liaison that is going to be exploited in chapter 2 is that through the mapping cone procedure it is possible to pass from the free resolution of an ACM scheme X_1 to a (non necessarily minimal) free resolution of its residual (necessarily ACM, as we have just seen) X_2 on an arithmetically Gorenstein scheme. To see how it works we have the following Lemma which it is analogous to Lemma 1.2.15 but for the case of graded free resolutions instead of the sheafified version. Notice that in the following Lemma we assume that the schemes are ACM:

Lemma 1.2.19. *Let $V_1, V_2 \subseteq \mathbb{P}^n$ be two ACM schemes of codimension c directly G-linked by an AG scheme W. Let the minimal free resolutions of I_{V_1} and I_W be*

$$0 \longrightarrow F_c \xrightarrow{d_c} F_{c-1} \xrightarrow{d_{c-1}} \dots F_1 \xrightarrow{d_1} I_{V_1} \longrightarrow 0$$

and

$$0 \longrightarrow R(-t) \xrightarrow{e_c} G_{c-1} \xrightarrow{e_{c-1}} \dots G_1 \xrightarrow{e_1} I_W \longrightarrow 0$$

respectively. Then the functor $\mathrm{Hom}(-, R(-t))$ *applied to a free resolution of I_{V_1}/I_W gives a (non necessarily minimal) resolution of I_{V_2}:*

$$0 \longrightarrow F_1^\vee(-t) \longrightarrow F_2^\vee(-t)\oplus G_1^\vee(-t) \longrightarrow \dots \longrightarrow F_c^\vee(-t)\oplus G_{c-1}^\vee(-t) \longrightarrow I_{V_2} \longrightarrow 0.$$

Example 1.2.20. (i) On a smooth quadric $Q_2 \subseteq \mathbb{P}^3$, any curve of type (a, a) is a complete intersection (of the quadric and a surface of $\mathbb{P}^3$ of degree a). This remark supplies a lot of examples of linkage in $\mathbb{P}^3$; for instance, any twisted cubic curve is directly CI-linked to any of its secant lines. In general, any two curves of respective types (a, b) and (c, d) on a smooth quadric are directly CI-linked if and only if $a + c = b + d$.

(ii) As an example of G-linkage that it is not CI-linkage, consider four points $Z \subseteq \mathbb{P}^3$ in linear general position and a general point P. Then $X = Z \cup \{P\}$ is an AG scheme and therefore Z and P are directly G-linked.

(iii) As an example of algebraic linkage between schemes with shared components, let $I_X = (x_0x_1, x_0 + x_1) \subseteq k[x_0, x_1, x_2, x_3]$ and $I_C = (x_0, x_1)$. Then $[I_X : I_C] = I_C$ and therefore I_C is self-CI-linked by I_X.

Usually is not easy to find out AG schemes to work with. The following theorem gives a useful way to construct them and it will play an important role in chapter 2. Notice that, since we will want to work with varieties that may be singular, we will have to work in the framework of generalized divisors as introduced in [Har94] and [Har07]. The only general requirements to be fulfilled in order to work in this context are that the schemes we deal with are ACM and satisfy condition G_1. Let us recall this definition here:

Definition 1.2.21. A subscheme $X \subseteq \mathbb{P}^n$ satisfies the condition G_r if for any point $p \in X$ of height $\leq r$ the local ring $\mathcal{O}_{X,p}$ is a Gorenstein ring. Usually this property is quoted as "Gorenstein in codimension $\leq r$", i.e., the nonlocally Gorenstein locus has codimension $\geq r+1$. In particular, G_0 is generically Gorenstein. When X satisfies the condition G_r for $r = \dim(X)$ we just say that X is a (locally) Gorenstein scheme.

Theorem 1.2.22. *(cf. [KMMR+01, Lemma 5.4]) Let $S \subseteq \mathbb{P}^n$ be an ACM scheme satisfying condition G_1. Then any effective divisor in the linear system $|mH_S - K_S|$ is arithmetically Gorenstein.*

1.3 Moduli spaces

Roughly speaking, moduli spaces are schemes whose closed points are in bijection with sets of equivalence classes of certain objects. In this dissertation, given a scheme $X \subseteq \mathbb{P}^n$, there are two sort of objects that we are interested in parameterizing:

(i) Closed subschemes of X with given Hilbert polynomial, and

(ii) μ-stable vector bundles on X with given rank and Chern classes.

The aim of this section will be to recall the main results on this subject.

Let us focus our attention first on Hilbert spaces. So let us fix a projective variety $X \subseteq \mathbb{P}^n$ and a polynomial $p(t) \in \mathbb{Q}[t]$. We define the contravariant functor:

$$\mathcal{H}ilb_{X,p} : (Sch/k) \longrightarrow (Sets),$$

from the category of scheme over k to the category of sets defined by:

$$\mathcal{H}ilb_{X,p}(S) := \{ \text{ flat families } \mathfrak{X} \subseteq X \times S \longrightarrow S \text{ whose fibers over points of S are closed subschemes of } X \text{ with Hilbert polynomial } p(t) \}.$$

Such functor is called the *Hilbert functor*. Grothendieck proved that there exists a unique projective scheme $\mathrm{Hilb}^{p(t)}(X)$ called the *Hilbert scheme* parameterizing a flat family

$$\pi : \mathfrak{W} \subseteq X \times \mathrm{Hilb}^{p(t)}(X) \longrightarrow \mathrm{Hilb}^{p(t)}(X),$$

of closed subschemes of X with Hilbert polynomial $p(t)$ which has the following universal property: for every flat family $\phi : \mathfrak{X} \subseteq X \times S \longrightarrow S$ of closed subschemes of X with Hilbert polynomial $p(t)$ there exists a unique morphism $\psi : S \longrightarrow \mathrm{Hilb}^{p(t)}(X)$ (the classification map for the map ϕ) such that π induces ϕ by base change: $\mathfrak{X} = S \times_{\mathrm{Hilb}^{p(t)}(X)} \mathfrak{W}$.

In categorical language, it means that $(\mathrm{Hilb}^{p(t)}(X), \pi)$ *represents* the moduli functor $\mathcal{H}ilb_{X,p}$, and π is the *universal family*. Once the existence of the Hilbert scheme has been proved it becomes a central question in Algebraic Geometry to find out its properties (irreducibility, rationality, smoothness...). Hartshorne proved that $\mathrm{Hilb}^{p(t)}(X)$ is connected (cf. [Har66, Corollary 5.9]). It is also well known that the Zariski tangent space at any point $[Y] \in \mathrm{Hilb}^{p(t)}(X)$ corresponding to a closed subscheme $Y \subseteq X$ is isomorphic to $\mathrm{H}^0(X, \mathcal{N}_{Y|X})$, where $\mathcal{N}_{Y|X}$ is the normal sheaf of Y in X. Moreover, we have the following bounds for the dimension of the Hilbert scheme at Y:

$$\mathrm{h}^0(X, \mathcal{N}_{Y|X}) \geq \dim_Y \mathrm{Hilb}^{p(t)}(X) \geq \mathrm{h}^0(X, \mathcal{N}_{Y|X}) - \mathrm{h}^1(X, \mathcal{N}_{Y|X}).$$

In particular, if $\mathrm{H}^1(X, \mathcal{N}_{Y|X}) = 0$, then $\mathrm{Hilb}^{p(t)}(X)$ is smooth at Y of dimension $\mathrm{h}^0(X, \mathcal{N}_{Y|X})$.

Example 1.3.1. Let us consider the numerical polynomial $p(t) := \binom{n+t}{n} - \binom{n+t-d}{n}$. Then $\mathrm{Hilb}^{p(t)}(\mathbb{P}^n)$ parameterizes hypersurfaces of $\mathbb{P}^n$ of degree d. It is a classical fact that $\mathrm{Hilb}^{p(t)}(\mathbb{P}^n) \cong \mathbb{P}(k[x_0, \cdots, x_n]_d) = \mathrm{Proj}(k[y_0, \cdots, y_{\binom{n+d}{n}}])$.

Remark 1.3.2. When the polynomial $p(t) = s \in \mathbb{Z}$ is constant, $\mathrm{Hilb}^s(X)$ parameterizes zero-dimensional schemes of length s. These schemes will be the subject of chapter 3 of this thesis.

Let us turn our attention to moduli spaces of μ-stable vector bundles. Let $X \subseteq \mathbb{P}^d$ be a smooth projective n-dimensional variety and let H be an ample divisor on X. Given a coherent sheaf $\mathcal{E}$ on X, the *Euler characteristic* of $\mathcal{E}$ is defined as

$$\chi(\mathcal{E}) := \sum_{i=0}^{n} (-1)^i \, \mathrm{h}^i(\mathcal{E}).$$

It is possible to prove that there exists a polynomial $p_{\mathcal{E}}(t) \in \mathbb{Q}[t]$ such that $p_{\mathcal{E}}(t) = \chi(\mathcal{E}(tH))$ for $t \gg 0$. It is called the *Hilbert polynomial* of $\mathcal{E}$. Following [HL97], we recall that a vector bundle $\mathcal{E}$ on $X \subseteq \mathbb{P}^d$ is *semistable* with respect to H if for every nonzero coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ we have the inequality

$$p_{\mathcal{F}} / \mathrm{rk}(\mathcal{F}) \leq p_{\mathcal{E}} / \mathrm{rk}(\mathcal{E}),$$

where the order is with respect to their asymptotic behavior. If one has the strict inequality for any proper subsheaf the vector bundle is called *stable* with respect to H. There is another notion of stability involving the *slope*, which is defined as

$$\mu_H(\mathcal{E}) := \deg(\mathcal{E})/\operatorname{rk}(\mathcal{E}),$$

with $\deg(\mathcal{E}) := c_1(\mathcal{E})H^{n-1}$. We say that the vector bundle $\mathcal{E}$ is *μ-(semi)stable* with respect to H if for every subsheaf $\mathcal{F}$ of $\mathcal{E}$ with $0 < \operatorname{rk}\mathcal{F} < \operatorname{rk}\mathcal{E}$,

$$\mu_H(\mathcal{F}) < \mu_H(\mathcal{E}) \quad (\text{resp. } \mu_H(\mathcal{F}) \leq \mu_H(\mathcal{E})).$$

Remark 1.3.3. (i) The four notions are related as follows:

$$\mu - stable \Rightarrow stable \Rightarrow semistable \Rightarrow \mu - semistable.$$

(ii) Notice that for a fixed smooth projective variety $X \subseteq \mathbb{P}^d$, stability can depend strongly on the choice of the ample divisor H.

Once fixed a Hilbert polynomial $p(t) \in \mathbb{Q}[t]$, we consider the contravariant moduli functor

$$\mathfrak{M}^{s,H}_{X,p(t)} : (Sch/k) \longrightarrow (Sets),$$

defined for a scheme S as

$\mathfrak{M}^{s,H}_{X,p(t)}(S)$:={ S-flat families $\mathcal{F}$ of vector bundles on $X \times S$ such that for all point $s \in S$, $\mathcal{F}_{|X\times\{s\}}$ is μ-stable with respect to H with Hilbert polynomial $p(t)$ }/$\sim$,

where $\mathcal{F} \sim \mathcal{F}'$ if there exists a line bundle $\mathcal{L}$ on S such that $\mathcal{F} \cong \mathcal{F}' \otimes p^*\mathcal{L}$ where $p : X \times S \longrightarrow S$ is the second projection. In 1977, Maruyama proved (cf. [Mar77, Theorem 5.6]):

Theorem 1.3.4. *The contravariant moduli functor $\mathfrak{M}^{s,H}_{X,p(t)}$ has a coarse moduli scheme $M^{s;H}_{X,p(t)}$ which is separated and of finite type over k, i.e.:*

(i) There exists a natural transformation

$$\Phi : \mathfrak{M}^{s,H}_{X,p(t)}(-) \longrightarrow \operatorname{Hom}(-, M^{s,H}_{X,p(t)})$$

which is bijective on any reduced point x.

(ii) For every scheme N and any natural transformation

$$\Psi : \mathfrak{M}^{s,H}_{X,p(t)}(-) \longrightarrow \mathrm{Hom}(-, N),$$

there exists a unique morphism $\alpha : M^{s,H}_{X,p(t)} \longrightarrow N$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{M}^{s,H}_{X,p(t)}(-) & \xrightarrow{\quad\Phi\quad} & \mathrm{Hom}(-, M^{s,H}_{X,p(t)}) \, . \\ & {\scriptstyle\Psi}\searrow \quad \swarrow{\scriptstyle\alpha_*} & \\ & \mathrm{Hom}(-, N) & \end{array}$$

Moreover, $M^{s,H}_{X,p(t)}$ decomposes as a disjoint union $M^s_{X,H}(r; c_1, \ldots, c_{\min(r,n)})$ of moduli spaces of rank r μ-stable vector bundles with Chern classes $(c_1, \ldots, c_{\min(r,n)})$ up to numerical equivalence.

Remark 1.3.5. It is worthwhile to point out that in general $\mathfrak{M}^{s,H}_{X,p(t)}$ is not representable. In fact, there is not a reason for which

$$\Phi(S) : \mathfrak{M}^{s,H}_{X,p(t)}(S) \longrightarrow \mathrm{Hom}(S, M^{s,H}_{X,p(t)})$$

should be bijective for a general scheme S.

As in the case of Hilbert schemes, one of the main problems in Algebraic Geometry is to determine the local and global structure of $M^s_{X,H}(r; c_1, \ldots, c_{\min(r,n)})$. At least we have the following bounds for the local dimension:

Proposition 1.3.6. *Let X be an n-dimensional smooth irreducible projective variety, let H be an ample divisor and let $\mathcal{E}$ be a μ-stable vector bundle of rank r with Chern classes $c_i \in \mathrm{H}^{2i}(X, \mathbb{Z})$ represented in $M^s_{X,H}(r; c_1, \ldots, c_{\min(r,n)})$ by a point $[\mathcal{E}]$. Then, the Zariski tangent space of $M^s_{X,H}(r; c_1, \ldots, c_{\min(r,n)})$ at $[\mathcal{E}]$ is canonically isomorphic to*

$$T_{[\mathcal{E}]} M^s_{X,H}(r; c_1, \ldots, c_{\min(r,n)}) \cong \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}).$$

Moreover, we have the following bounds:

$$\mathrm{ext}^1(\mathcal{E}, \mathcal{E}) \geq \dim_{[\mathcal{E}]} M^s_{X,H}(r; c_1, \ldots, c_{\min(r,n)}) \geq \mathrm{ext}^1(\mathcal{E}, \mathcal{E}) - \mathrm{ext}^2(\mathcal{E}, \mathcal{E}).$$

In particular, if $\mathrm{Ext}^2(\mathcal{E}, \mathcal{E}) = 0$ then $M^s(r; c_1, \ldots, c_{\min(r,n)})$ is smooth at $[\mathcal{E}]$ and

$$\dim_{[\mathcal{E}]} M^s_{X,H}(r; c_1, \ldots, c_{\min(r,n)}) = \mathrm{ext}^1(\mathcal{E}, \mathcal{E}).$$

One of the characteristics that we are going to exploit in chapter 4 is that μ-stable vector bundles are *simple* and hence *indecomposable*. Let us recall the definitions here:

Definition 1.3.7. Let X be a projective variety and let $\mathcal{E}$ be a vector bundle on X. $\mathcal{E}$ is called *simple* if the only endomorphisms are the homotheties, i.e., $\mathrm{End}(\mathcal{E}) = k$. $\mathcal{E}$ is called *indecomposable* if it can not be written as $\mathcal{E} \cong \mathcal{F} \oplus \mathcal{G}$ where $\mathcal{F}$ and $\mathcal{G}$ non-zero vector bundles.

As it has just been mentioned, these notions are related as follows:

Lemma 1.3.8. *Let $X \subseteq \mathbb{P}^n$ be a projective variety and let $\mathcal{E}$ be a vector bundle on X. Then we have the following implications:*

$$\mathcal{E} \text{ is } \mu - stable \Rightarrow \mathcal{E} \text{ is } \ simple \Rightarrow \mathcal{E} \ \ is \ \ indecomposable.$$

Proof. It is immediate to see that simple vector bundles are indecomposable. On the other hand, the fact that μ-stability implies simplicity is proven in [OSS80, Theorem 1.2.9]. □

Definition 1.3.9. Let $X \subseteq \mathbb{P}^n$ be a projective variety and let $\mathcal{E}$ be a coherent sheaf on X. We are going to say that a sheaf $\mathcal{E}$ on X is *initialized* (with respect to $\mathcal{O}_X(1)$) if

$$\mathrm{H}^0(X, \mathcal{E}(-1)) = 0 \quad \text{but} \quad \mathrm{H}^0(X, \mathcal{E}) \neq 0.$$

We are going to define the *initializing shift* as the integer (if it exists) k_{init} such that $\mathcal{E}_{init} := \mathcal{E}(k_{init})$ is initialized. Notice that if $\mathcal{E}$ is a locally free sheaf, it can always be initialized.

Remark 1.3.10. Given a rank r vector bundle $\mathcal{F}$ on a projective variety X with an ample divisor H, the first two Chern classes of $\mathcal{F}$ are modified by twisting as follows:

- $c_1(\mathcal{F}(lH)) = c_1(\mathcal{F}) + rlH$.
- $c_2(\mathcal{F}(lH)) = c_2(\mathcal{F}) + (r-1)lc_1(\mathcal{F})H + \binom{r}{2}l^2H^2$.

We are going to finish this chapter recalling the Riemann-Roch theorem for surfaces:

Theorem 1.3.11 (Riemann-Roch formula)**.** *Let $\mathcal{E}$ be a rank r vector bundle on the smooth projective surface $X \subseteq \mathbb{P}^d$. Then*

$$\chi(\mathcal{E}) = \frac{c_1(\mathcal{E})(c_1(\mathcal{E}) - K_X)}{2} - c_2(\mathcal{E}) + r\chi(\mathcal{O}_X).$$

Chapter 2

The Minimal Resolution Conjecture

It is a long-standing problem in Algebraic Geometry to determine the Hilbert function of any set Z of distinct points in $\mathbb{P}^n$. It is well-known that

$$H_Z(t) \leq \min\{\binom{n+t}{n}, |Z|\}$$

for any t, and that equality holds if the points are general. A much more subtle question is to find out the exact shape of the minimal free resolution of its coordinate ring R_Z. We know that if Z is a general set of s distinct points in $\mathbb{P}^n$ it has to be of the form (cf. [Lor89, Theorem 2.2]):

$$0 \longrightarrow F_n \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow R_Z \longrightarrow 0$$

with $F_0 = R$ and

$$F_i \cong R(-r-i)^{b_{i,r}} \oplus R(-r-i+1)^{b_{i,r-1}}$$

for $i = 1, \ldots, n$, where r is the unique nonnegative integer such that

$$\binom{r+n-1}{n} \leq s < \binom{r+n}{n}.$$

Moreover we have:

$$b_{i+1,r-1} - b_{i,r} = \binom{r+i-1}{i}\binom{r+n}{n-i} - s\binom{n}{i}.$$

The *Minimal Resolution Conjecture* proposed by Lorenzini in [Lor93, p. 10] says that there exist no *ghost* terms in the minimal free resolution of R_Z, i.e., $b_{i+1,r-1}b_{i,r} = 0$ for all i. Ever since this conjecture was stated, it has attired a lot of attention and positive or negative answers have been obtained in a lot of cases. In section 2.1, we summarize the state of this conjecture. Later on, in [Mus98], Mustaţă proposed a generalized version of this conjecture for general distinct points lying on any projective variety, in particular he conjectured that, as in the case of the projective space, the graded Betti numbers had to be as small as possible.

In this chapter we are interested in tackle Mustaţă's conjecture for points on *ACM quasi-minimal surfaces*. These are ACM surfaces X embedded in the projective space $\mathbb{P}^d$ with degree d. Therefore, they have quasi-minimal degree (since it is well-known that for a nondegenerate variety $X \subseteq \mathbb{P}^n$ it holds $\deg(X) \geq \operatorname{codim}(X) + 1$). This kind of varieties have received a lot of attention recently related to the theory of Δ-genus as developed in [Fuj90], where a satisfactory classification of the quasi-minimal varieties (i.e., varieties of Δ-genus one) was stated. In particular, smooth ACM quasi-minimal surfaces correspond with del Pezzo strong surfaces, which are going to be one of the main topics in chapter 4. We are going to prove that Mustaţă's conjecture holds for a wide range of cardinalities of general distinct points on ACM quasi-minimal surfaces.

The structure of this chapter is as follows: in section 2.1, we recall the Minimal Resolution Conjecture (MRC) and give a brief account of the known results around it.

In section 2.2, we pay attention to ACM quasi-minimal surfaces. We prove that the two weaker conjectures, the Ideal Generation Conjecture and the Cohen-Macaulay type conjecture, hold for any general set of points on ACM quasi-minimal surfaces (except for two sporadic cases, see Theorem 2.2.16). Moreover, for general set of distinct points whose cardinalities fall into determinate strips (see Theorem 2.2.15 for a precise definition) we are able to prove that the whole MRC holds (except for the same two sporadic cases, see Theorem 2.2.15). For the particular case of integral cubic surfaces, we see that MRC holds for any general set of distinct points (see Theorem 2.2.17).

In section 2.3, we focus our attention on a slightly modified conjecture. Since, in general, $\operatorname{Hilb}^s(X)$ is not irreducible (see [Iar72] for the case of varieties of dimension higher or equal than 3 and chapter 3 for surfaces), we can also search the minimal graded free resolution of the homogeneous ideal of the 0-dimensional scheme associated to a general point of any other irreducible component of the

Hilbert scheme $\mathrm{Hilb}^s(X)$ and ask if the graded Betti numbers are as small as possible, i.e. there are no ghost terms in the minimal free resolution. We address this conjecture and, in particular, we prove that for any integer $d \geq 2$ and for any $s \geq \binom{d+3}{3} - 1$, there exists a $\binom{d+2}{2}$-dimensional family of irreducible generically smooth surfaces $X \subset \mathbb{P}^3$ of degree d satisfying it (see Theorem 2.3.18).

Part of the results of this chapter will be published in:

• Miró-Roig, R.M and Pons-Llopis, J., *The Minimal free resolution for points on del Pezzo surfaces*, Algebra and Number Theory (to appear).

• Miró-Roig, R.M and Pons-Llopis, J., *The Minimal free resolution for points on surfaces*, submitted.

2.1 Introduction to the Minimal Resolution Conjecture

The goal of this section will be to introduce the background and known results around the Minimal Resolution Conjecture. Let $X \subseteq \mathbb{P}^n$ be a projective variety of $\dim X \geq 1$, let $Z \subseteq X$ be a set of s distinct points and let $I_{Z|X} := \mathrm{H}^0_*(\mathcal{I}_{Z|X})$. Let $R_Z = R/I_Z$ be its coordinate homogeneous ring. Then R_Z is a Cohen-Macaulay ring of Krull dimension one. Notice that, from the exact sequence

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{I}_{Z|X} \longrightarrow 0,$$

in case X is an ACM variety, we have $I_{Z|X} = I_Z/I_X$ and, in any case, if we write the homogeneous ideal $I_Z/I_X = \oplus_{t\geq 0}(I_Z/I_X)_t$,

$$(I_Z/I_X)_t = \mathrm{H}^0(\mathcal{I}_{Z|X}(t)) \text{ for } t \geq \mathrm{reg}(X).$$

It is well-known that the Hilbert function H_Z of Z is eventually constant and equal to $s = \mathrm{H}^0(\mathcal{O}_Z)$. Let us define the *initial degree*

$$\alpha := \min\{t \geq 0 \mid (I_Z/I_X)_t \neq 0\},$$

i.e., the minimal degree of a generator of I_Z/I_X. Notice that from the definition it follows that $s \geq H_X(\alpha - 1)$.

The next lemma gathers some well-known properties of the Hilbert function H_Z of Z:

Lemma 2.1.1. *(cf. [GM84, Proposition 1.1]) Let $Z \subseteq X \subseteq \mathbb{P}^n$ be a set of s distinct points.*

(i) *H_Z is an increasing function. Moreover, if $H_Z(t) = H_Z(t+1)$ for some t, then $H_Z(t+1) = H_Z(t+2)$.*

(ii) *If we define $e := min\{t \mid H_Z(t) = H_Z(t-1)\}$, then*

$$I_Z/I_X = \langle (I_Z/I_X)_\alpha, \dots, (I_Z/I_X)_e \rangle.$$

(iii) *For $0 \leq t \leq s-1$, we have*

$$t+1 \leq H_Z(t) \leq min\{\binom{n+t}{n}, s\},$$

and for $t \geq s$, $H_Z(t) = s$.

From the previous Lemma we see which are the slowest and fastest possible growth of the Hilbert function of a set of points in $\mathbb{P}^n$. As for the slowest growth, it has a nice geometric interpretation:

Lemma 2.1.2. *(cf. [GM84, Proposition 1.3]) Let $Z \subseteq \mathbb{P}^n$ be a set of s distinct points. Then the following are equivalent:*

(i) *The points of Z lie on a line.*

(ii) *$H_Z(1) = 2$.*

(iii) *$H_Z(t) = t+1$ for $t \leq s-1$ and $H_Z(s) = s$.*

Example 2.1.3. According to the previous Lemma, 23 distinct points lying on a line in $\mathbb{P}^n$ have Hilbert function:

t	0	1	2	3	...	21	22	23	...
$H_Z(t)$	1	2	3	4	...	22	23	23	...

On the other extreme of growth we have:

Definition 2.1.4. We will say that a set of s distinct points $Z \subseteq X$ is in *general position in X* if for all $t \geq 0$ its Hilbert function is given by the formula

$$H_Z(t) = min\{H_X(t), s\}. \tag{2.1.1}$$

Therefore, if $Z \subseteq X$ is a set of s points in general position in X, for the unique integer r such that $H_X(r-1) \leq s < H_X(r)$ it holds that $H_Z(r) = H_Z(r+1)$ and so by Lemma 2.1.1, $I_Z/I_X = \langle (I_Z/I_X)_r, (I_Z/I_X)_{r+1} \rangle$.

Example 2.1.5. Let $X \subseteq \mathbb{P}^3$ be an irreducible quadric surface. Let $Z_1 \subseteq X$ be 7 distinct points in general position and let $Z_2 \subseteq X$ be 7 general distinct points lying on a unique hyperplane section. We have the following Hilbert functions:

t	0	1	2	3	4	...
$H_X(t)$	1	4	9	16	25	...
$H_{Z_1}(t)$	1	4	7	7	7	...
$H_{Z_2}(t)$	1	3	5	7	7	...

Remark 2.1.6. Among those sets of points in general position, the ones with the fastest growth correspond to $s = H_X(r-1)$. In this case $I_Z/I_X = \langle (I_Z/I_X)_r \rangle$.

Example 2.1.7. Let us consider the following two examples of the situation mentioned in the previous remark:

(i) If $Z \subseteq \mathbb{P}^4$ is a set of $\binom{4+3}{4} = 35$ points in general position, its Hilbert function is

t	0	1	2	3	4	...
$H_Z(t)$	1	5	15	35	35	...

and I_Z is generated by 35 quartics.

(ii) Now let us consider the twisted cubic curve $X \subseteq \mathbb{P}^3$. It has Hilbert polynomial $P_X(t) = 3t + 1$ and it is generated by three quadrics. Let $Z \subseteq X$ be a set of $H_X(4) = 13$ distinct points in general position in X. Then the Hilbert functions of X and Z are:

t	0	1	2	3	4	5	...
$H_X(t)$	1	4	7	10	13	16	...
$H_Z(t)$	1	4	7	10	13	13	...

I_Z/I_X is generated by three quintics. Therefore, I_Z is generated by three quadrics and three quintics.

Geramita and Orecchia in the case of points in $\mathbb{P}^n$ and Mustaţă in the more general case of points in a given projective variety showed that a set of general distinct points is in general position:

Proposition 2.1.8. *(cf. [GO81, Theorem 4] and [Mus98, Proposition 1.1]) Given an integer $s \geq 0$ and a variety $X \subseteq \mathbb{P}^n$, the subschemes consisting of s distinct points in X that are in general position form a non-empty open set of* $\mathrm{Hilb}^s(X)$.

In [Mus98], Mustaţă extended previous results about the shape of minimal free resolutions of general set of points $Z \subseteq X$ by Lorenzini (cf. [Lor89]) for the case $X = \mathbb{P}^n$ to arbitrary projective variety $X \subseteq \mathbb{P}^n$. He proved that the first rows of the Betti diagram of a general set of distinct points Z in a projective variety X coincide with the Betti diagram of X and that there are two extra nontrivial rows at the bottom. He also gave lower bounds for the Betti numbers in these last two rows. Let us recall it:

Theorem 2.1.9. *(cf. [Mus98, Propositions 1.5 and 1.6] and [Lor89, Theorem 2.2]). Let $X \subset \mathbb{P}^n$ be a projective variety with $d = \dim(X) \geq 1$, $\operatorname{reg}(X) = m$ and with Hilbert polynomial P_X. Let s be an integer with $P_X(r-1) \leq s < P_X(r)$ for some $r \geq m+1$ and let Z be a set of s general distinct points on X. Then:*

(i) For every i and $j \leq r-2$, $b_{i,j}(Z) = b_{i,j}(X)$.

(ii) $b_{i,j}(Z) = 0$ for $j \geq r+1$ and there exists i such that $b_{i,r-1}(Z) \neq 0$.

(iii) For every $j \geq m$,

$$b_{i,j}(Z) = b_{i-1,j+1}(I_Z/I_X).$$

In other words, if we let

$$0 \to F_n \to F_{n-1} \to \cdots \to F_2 \to F_1 \to R \to R_X \to 0$$

be a minimal free R-resolution of R_X. Then R_Z has a minimal free R-resolution of the following type

$$\begin{gathered} 0 \longrightarrow F_n \oplus R(-r-n+1)^{b_{n,r-1}(Z)} \oplus R(-r-n)^{b_{n,r}(Z)} \longrightarrow \cdots \\ \longrightarrow F_2 \oplus R(-r-1)^{b_{2,r-1}(Z)} \oplus R(-r-2)^{b_{2,r}(Z)} \longrightarrow \\ \longrightarrow F_1 \oplus R(-r)^{b_{1,r-1}(Z)} \oplus R(-r-1)^{b_{1,r}(Z)} \longrightarrow R \longrightarrow R_Z \longrightarrow 0. \end{gathered}$$

Moreover, if we set $Q_{i,r}(s) = b_{i+1,r-1}(Z) - b_{i,r}(Z)$,

$$Q_{i,r}(s) = \sum_{l=0}^{d-1} (-1)^l \binom{n-l-1}{i-l} \Delta^{l+1} P_X(r+l) - \binom{n}{i}(s - P_X(r-1)). \qquad (2.1.2)$$

Therefore, the equation (2.1.2) in Theorem 2.1.9 gives lower bounds for the Betti numbers: $b_{i+1,r-1}(Z) \geq \max\{Q_{i,r}(s), 0\}$ and $b_{i,r}(Z) \geq \max\{-Q_{i,r}(s), 0\}$.

The Minimal Resolution Conjecture asserts that the graded Betti numbers are as small as possible:

Conjecture 2.1.10. *With the notations from the previous Theorem, the* Minimal Resolution Conjecture (MRC for short) *holds for the value s if for every set Z of s general distinct points we have $b_{i+1,r-1}(Z) = max\{Q_{i,r}(s), 0\}$ and $b_{i,r}(Z) = max\{-Q_{i,r}(s), 0\}$. Equivalently,*

$$b_{i+1,r-1}(Z)b_{i,r}(Z) = 0 \quad \textit{for} \quad i = 1, \cdots, n-1.$$

Example 2.1.11. Let $X \subseteq \mathbb{P}^4$ be a smooth complete intersection of two quadric hypersurfaces and let $Z \subseteq X$ be a general set of 28 distinct points. We have $\text{reg}(X) = 4$ and $P_X(3) = 25 < 28 < P_X(4) = 41$. We are going to prove in 2.3.18 that MRC holds in this case. Therefore, the Betti diagram of Z is as follows:

	0	1	2	3	4
0	1	–	–	–	–
1	–	2	–	–	–
2	–	–	1	–	–
3	–	13	32	22	–
4	–	–	–	–	3

Notice that the first three rows coincide with the Betti diagram of X. The minimal graded resolution of R_Z is

$$0 \longrightarrow R(-8)^3 \longrightarrow R(-6)^{22} \longrightarrow R(-4) \oplus R(-5)^{32} \longrightarrow$$
$$R(-2)^2 \oplus R(-4)^{13} \longrightarrow R \longrightarrow R_Z \longrightarrow 0.$$

Lorenzini's original conjecture (cf. [Lor93, p. 10]) dealt with the particular case when $X = \mathbb{P}^n$.

Remark 2.1.12. Let us summarize here what it is known regarding Lorenzini's MRC:

- The MRC is known to hold for any number of points s in $\mathbb{P}^n$ for $n = 2$ (see [Gae51, p. 912]), $n = 3$ ([BG86]) and $n = 4$ ([Wal95, Theorem 1]).
- The MRC is known also to hold for large values of s for any n (see [HS96, p. 468]).
- On the other hand, MRC fails in general for any $n \geq 6$, $n \neq 9$ (see [EPSW02, Theorem 1.1]).

Concerning Mustaţă's conjecture for arbitrary projective varieties, up to now it was known:

- As for the one-dimensional case, MRC holds for large cardinalities of general points on canonical curves $C \subseteq \mathbb{P}^n$ (i.e., curves embedded in $\mathbb{P}^n$ by its canonical divisor) (see [FMP03, Theorems 3.1]).

- Nevertheless, oppositely to the case of the projective space, the MRC fails for sets of points of arbitrarily large length on curves of high degree (see [FMP03, Theorem 2.2]).

- MRC holds for any number of general points on a smooth quadric surface in $\mathbb{P}^3$ (see [GMR96, Theorem 4.3]).

- In [Cas09, Theorem 3.2], Casanellas proved that this conjecture holds for some special cardinalities of sets of general points on a smooth cubic surface.

- In [MP, Theorem 3.1], Migliore and Patnott, independently of our results, have been able to prove it for sets of general distinct points of any cardinality on a cubic surface $X \subseteq \mathbb{P}^3$ with at most isolated double points.

Related to the MRC there exist two weaker conjectures that deal only with a part of the minimal resolution of a general set of points: the *Ideal Generation Conjecture* which says that the minimal number of generators of the ideal of a general set of points will be as small as possible. From Theorem 2.1.9 it is clear that this conjecture can be translated in terms of the Betti numbers:

Conjecture 2.1.13. *Let $X \subset \mathbb{P}^n$ be a projective variety with $d = \dim(X) \geq 1$, $\mathrm{reg}(X) = m$ and with Hilbert polynomial P_X. Let s be an integer with $P_X(r-1) \leq s < P_X(r)$ for some $r \geq m+1$ and let Z be a set of s general distinct points on X. The* Ideal Generation Conjecture *(IGC for short) holds for the value s if $b_{1,r}(Z)b_{2,r-1}(Z) = 0$.*

On the other hand, the *Cohen-Macaulay type Conjecture* affirms that the canonical module $K_Z = \mathrm{Ext}^n(R/I_Z, R(-n-1))$ has as few generators as possible. Since we saw that the dual of the minimal resolution of R_Z provides a (twisted) resolution of K_Z this conjecture can also been translated in terms of Betti numbers:

Conjecture 2.1.14. *Let $X \subset \mathbb{P}^n$ be a projective variety with $d = \dim(X) \geq 1$, $\mathrm{reg}(X) = m$ and with Hilbert polynomial P_X. Let s be an integer with $P_X(r-1) \leq s < P_X(r)$ for some $r \geq m+1$ and let Z be a set of s general distinct points on X. The* Cohen-Macaulay type Conjecture *(CMC for short) holds for the value s if $b_{n-1,r}(Z)b_{n,r-1}(Z) = 0$.*

Regarding these two conjectures, CMC has been proved in full generality in the case of the projective space $X = \mathbb{P}^n$, for any n (see [Tru89, p. 112]). It is also known that the IGC holds for large set of points on curves of degree $d \geq 2g$ (see [FMP03, Theorem 2.2]. In Theorem 2.2.16 we are going to prove that IGC and CMC hold for general set of points on ACM quasi-minimal surfaces (up to two sporadic cases).

2.2 MRC for points on ACM quasi-minimal surfaces

The goal of this section will be to prove the MRC for points on *ACM quasi-minimal surfaces*. Recall that given a nondegenerate projective variety $X \subseteq \mathbb{P}^d$ it always holds that $\deg(X) \geq \operatorname{codim}(X)+1$. It is a classical result the classification of *minimal varieties*, i.e., varieties for which there is equality in the previous expression (see, for instance, [Dol, Theorem 8.1.1]). We are going to deal with the next case:

Definition 2.2.1. A quasi-minimal variety is a nondegenerate variety $X \subseteq \mathbb{P}^d$ such that $\deg(X) = \operatorname{codim}(X) + 2$.

Example 2.2.2. (i) As examples of quasi-minimal varieties, consider any cubic hypersurface or any complete intersection of two hyperquadrics. They are ACM varieties.

(ii) Any cone over a quasi-minimal variety turns out to be a quasi-minimal variety. Then, for instance, a cone over any nondegenerate curve $C \subseteq \mathbb{P}^{d-1}$ of degree d is a quasi-minimal-surface.

(iii) As for an example of quasi-minimal surface which is not ACM, we have the isomorphic projection to $\mathbb{P}^d$ of the rational normal scroll $Y \subseteq \mathbb{P}^{d+1}$.

Fujita (see [Fuj90]) has a satisfactory classification of such varieties. In this section we are going to restrict our attention to ACM quasi-minimal surfaces. According to [Hoa93, Theorem 1], the minimal free resolution of the coordinate ring of an ACM quasi-minimal surface $X \subseteq \mathbb{P}^d$ has the form:

$$0 \longrightarrow R(-d) \longrightarrow R(-d+2)^{\alpha_{d-3}} \longrightarrow \dots \longrightarrow R(-2)^{\alpha_1} \longrightarrow R \longrightarrow R_X \longrightarrow 0 \tag{2.2.1}$$

where

$$\alpha_i = i\binom{d-1}{i+1} - \binom{d-2}{i-1} \text{ for } 1 \leq i \leq d-3.$$

In particular, the homogenous ideal of an ACM quasi-minimal surface of degree ≥ 4 is generated by quadrics (the case $\deg X = 3$ corresponds to cubic surfaces $X \subseteq \mathbb{P}^3$). Notice that X turns out to be AG and, hence, $\alpha_i = \alpha_{d-2-i}$ for all $i = 1, \dots, d-2$. Moreover, the canonical bundle is $\omega_X \cong \mathcal{O}_X(-1)$. Therefore, when X is a smooth surface, we recover the class of strong del Pezzo surfaces as they will be defined in Definition 4.1.4 of chapter 4.

The main technique used in this section is the theory of Gorenstein liaison. Roughly speaking, knowing that two sets of points are G-linked will allow us to pass from the minimal resolution of the ideal of one of them to the resolution (not necessarily minimal) of the other one through mapping cone procedure as it was explained in section 2 of chapter 2. Then once the MRC is known to hold for a general set of $d+2$ distinct points on an ACM quasi-minimal surface $X \subseteq \mathbb{P}^d$ an induction process will provide us with our main Theorems 2.2.15, 2.2.16 and 2.2.17.

In order to deal with ACM quasi-minimal varieties, the structure of the proof will be as follows. In the next subsection, we establish the MRC for two critical values,

$$m(r) := \frac{d}{2}r^2 + r\frac{2-d}{2}$$

and

$$n(r) := \frac{d}{2}r^2 + r\frac{d-2}{2},$$

for $r \geq 2$, of set of general distinct points on an ACM quasi-minimal surface X (up to two sporadic cases) . Firstly, we will establish the result for $m(2) = d+2$ general distinct points on any ACM quasi-minimal surface $X \subset \mathbb{P}^d$ which gives the initial step for our induction (Lemma 2.2.9). Then, using G-liaison, we prove that if $m(r)$ general distinct points on any ACM quasi-minimal surface satisfy MRC then so do $n(r)$ general distinct points (Proposition 2.2.10). An easy remark gives us that if $n(r)$ general distinct points on X have the expected resolution then $n(r)+1$ general distinct points do as well. Finally, using again G-liaison, we show that if $n(r)+1$ general distinct points on an ACM quasi-minimal surface satisfy MRC then so do $m(r+1)$ (up to four sporadic cases, see Proposition 2.2.12). The final subsection contains the main results of this section: namely that MRC holds on an ACM quasi-minimal surface (up to two of the four aforementioned sporadic cases) for a general set of distinct points whose cardinality falls into the strips $[P_X(r-1), m(r)]$ and $[n(r), P_X(r)]$ for any $r \geq 4 = \text{reg } X + 1$ with $P_X(r)$ the Hilbert polynomial of X (see Theorem 2.2.15). As a corollary, we will get that Mustaţă's conjecture holds for any general set of $t \geq 19$ points on an integral cubic surface in $\mathbb{P}^3$ (Theorem 2.2.17) and the Ideal Generation Conjecture as well as the Cohen-Macaulay type Conjecture holds for any general set of $t \geq 6d+1$ points on an ACM quasi-minimal surface in $\mathbb{P}^d$ (except the two excluded cases, see Theorem 2.2.16).

2.2.1 MRC conjecture for sets of n(r) and m(r) general distinct points on ACM quasi-minimal surfaces

For the rest of the section, $X \subseteq \mathbb{P}^d$ will stand for an ACM quasi-minimal surface as was defined in Definition 2.2.1. The Hilbert polynomial and the regularity of X can be easily computed using the exact sequence (2.2.1) and we have

$$P_X(r) = \frac{d}{2}(r^2+r)+1 \text{ and } \text{reg}(X) = 3. \qquad (2.2.2)$$

Let us consider the following critical values:

$$m(r) := \frac{d}{2}r^2 + r\frac{2-d}{2}, \quad n(r) := \frac{d}{2}r^2 + r\frac{d-2}{2}.$$

Notice that

$$P_X(r-1) < m(r) < n(r) < P_X(r).$$

We also set the following notation.

(i) L is any line on X.

(ii) H denotes a general hyperplane section of X.

(iii) If C is a curve on X, H_C will be a general hyperplane section of C and K_C the canonical divisor on C.

We are going to find out the minimal free resolution for general set of points of the two specific cardinalities $m(r)$ and $n(r)$, for $r \geq 2$. The strategy of the proof is as follows: firstly, we will establish the result for $m(2) = d+2$ general distinct points which gives the starting point for our induction process. Secondly, using G-liaison, we prove that if $m(r)$ general distinct points on any ACM quasi-minimal surface satisfy MRC then so do $n(r)$ general distinct points. Next we observe that if $n(r)$ general distinct points on X have the expected minimal free resolution then $n(r)+1$ general distinct points do as well. And, finally, we show that if $n(r)+1$ general distinct points on an ACM quasi-minimal surface (up to four sporadic cases) satisfy MRC then $m(r+1)$ general distinct points do satisfy the MRC.

First of all, since we are going to heavily rely on Theorem 1.2.22 in order to use G-liaison, we need to ensure that all the curves we are going to work with verify the condition G_1 (see Definition 1.2.21). We need the following result:

Proposition 2.2.3. *([BH93, Prop. 3.1.19]) Let R be a local Gorenstein ring and let $a_1, \dots a_n$ be a R-regular sequence. Then $R/(a_1, \dots a_n)$ is also a Gorenstein ring.*

Proposition 2.2.4. *Let $X \subseteq \mathbb{P}^d$ be an ACM quasi-minimal surface. Assume that X is not one of the following four particular cases: $X \cong \mathbb{P}^2$, $X \cong \mathbb{P}^1 \times \mathbb{P}^1$, X is the anticanonical model of $\mathrm{F}_2 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ or X is a complete intersection of two quadrics on $\mathbb{P}^4$ with a double line. Then there exists a line $L \subseteq X \subseteq \mathbb{P}^d$ such that for any $r \geq 1$ the linear system $|L + rH|$ contains a subsystem $V \subseteq |L + rH|$ of dimension $\geq \mathrm{h}^0(\mathcal{O}_X(rH))$ and an open subset $U \subseteq V$ such that any curve $C \in U$ is locally Gorenstein (i.e., it verifies condition G_1).*

Proof. We are going to divide the proof in several cases, covering the classification of ACM quasi-minimal surfaces as it is presented in [Dol, Section 8.4] (for the normal case) and [Rei94, Theorem 1.1] (for the nonnormal one).

(i) In case $X \subseteq \mathbb{P}^d$ being smooth we are in the case of a strong del Pezzo surface. Any such a surface contains lines L except when $X \cong \mathbb{P}^2$ or $X \cong \mathbb{P}^1 \times \mathbb{P}^1$. For any curve $C \in |L + rH|$, let us consider the exact sequence

$$0 \longrightarrow \mathcal{I}_{C|X} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0. \tag{2.2.3}$$

Then, for any point $p \in C$, $\mathcal{O}_{X,p}$ is regular and therefore $(\mathcal{I}_{C|X})_p$ is generated by a single element. Therefore we can apply Proposition 2.2.3 to the exact sequence

$$0 \longrightarrow (\mathcal{I}_{C|X})_p \longrightarrow \mathcal{O}_{X,p} \longrightarrow \mathcal{O}_{C,p} \longrightarrow 0$$

to conclude that $\mathcal{O}_{C,p}$ is Gorenstein (recall that a regular ring is always Gorenstein).

(ii) When X is singular but normal, [Dol, Section8.4] shows that X contains a line L. Moreover, a general element C of the linear system $|rH|$, does not pass through any of the singular points of X. Let us consider the curve $C' := L \cup C$. For a point $p \in C'$, if $p \in X \setminus \mathrm{Sing}(X)$ the same proof as in the previous case shows that $\mathcal{O}_{C',p}$ is Gorenstein. On the other hand, if $p \in \mathrm{Sing}(X)$, since $p \notin C$, then $\mathcal{O}_{C',p} \cong \mathcal{O}_{L,p}$ is regular and therefore Gorenstein.

(iii) When X is nonnormal, X not isomorphic to a complete intersection of two quadrics on $\mathbb{P}^4$ with a double line, [Rei94, Theorem 1.1] shows that $(\mathrm{Sing}(X))_{red}$ is a line and moreover X is covered by lines. For any such a line $L \neq \mathrm{Sing}(X)$ and for any curve $C \in |rH|$ such that C does not pass through $L \cap \mathrm{Sing}(X)$, we are going to see that $C' := L \cup C$ is a locally Gorenstein curve. So take $p \in C'$. If $p \in X \setminus \mathrm{Sing}(X)$ or $p \in \mathrm{Sing}(X) \cap L$, the same arguments as in item (ii) show that C' is locally Gorenstein. On the other hand, if $p \in \mathrm{Sing}(X) \cap C$, we have $\mathcal{O}_{C',p} \cong \mathcal{O}_{C,p}$. But C was a Cartier divisor and in particular the ideal $(\mathcal{I}_{C,X})_p$ of $\mathcal{O}_{X,p}$ is generated by a single element. Therefore, Proposition 2.2.3 applied to the exact sequence

$$0 \longrightarrow (\mathcal{I}_{C|X})_p \longrightarrow \mathcal{O}_{X,p} \longrightarrow \mathcal{O}_{C,p} \longrightarrow 0$$

shows that $\mathcal{O}_{C',p}$ is Gorenstein (notice that X being AG implies in particular that X is G_l for any l).

□

We will prove the main result of this subsection via a series of Lemmas and Propositions. Since the shape of the minimal free resolution of the homogeneous ideal $I_{X|\mathbb{P}^3}$ of an ACM quasi-minimal surface of degree 3 (i.e, a cubic surface) is slightly different from that of an ACM quasi-minimal surface of degree $d \geq 4$ we need to consider apart the two cases for some arguments. We will give complete proofs in the case of degree $d \geq 4$. The concrete arguments on the case of degree 3 are analogous but much easier to write down and therefore they will be left to the reader.

Lemma 2.2.5. *Let $X \subseteq \mathbb{P}^d$ be any ACM quasi-minimal surface of degree $d \geq 4$ and take $C \in |(r+\epsilon)H|$, $r \geq 2, \epsilon \in \{0,1\}$. Then, any effective divisor G in the linear system $|rH_C|$ is AG (as a subscheme of $\mathbb{P}^d$) and it has a minimal free resolution of the following form:*

$$0 \longrightarrow R(-2r-d-\epsilon) \longrightarrow R(-2r-d+2-\epsilon)^{\alpha_{d-3}} \oplus R(-r-d)^{2-\epsilon} \oplus R(-r-d-1)^{\epsilon} \longrightarrow$$
$$\ldots \longrightarrow M_i \longrightarrow \ldots \longrightarrow R(-2r-\epsilon) \oplus R(-r-2)^{(2-\epsilon)\alpha_1} \oplus R(-r-3)^{\epsilon\alpha_1} \longrightarrow$$
$$\longrightarrow M_1 := R(-r)^{2-\epsilon} \oplus R(-r-1)^{\epsilon} \longrightarrow I_{G|X} \longrightarrow 0,$$

where $M_i = R(-2r-i+1-\epsilon)^{\alpha_{i-2}} \oplus R(-r-i)^{(2-\epsilon)\alpha_{i-1}} \oplus R(-r-i-1)^{\epsilon\alpha_{i-1}}$ *for* $i = 3, \ldots, d-2$ *and* $\alpha_i = i\binom{d-1}{i+1} - \binom{d-2}{i-1}$ *for* $1 \le i \le d-3$.

Proof. A curve $C \in |(r+\epsilon)H|$ has saturated ideal $I_{C|X} = \mathrm{H}^0_*(\mathcal{O}_X(-r-\epsilon))$. From the exact sequence (2.2.1) we have:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^d}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^d}(-d+2)^{\alpha_{d-3}} \longrightarrow \ldots$$
$$\ldots \longrightarrow \mathcal{O}_{\mathbb{P}^d}(-2)^{\alpha_1} \longrightarrow \mathcal{O}_{\mathbb{P}^d} \longrightarrow \mathcal{O}_X \longrightarrow 0 \qquad (2.2.4)$$

with $\alpha_i = i\binom{d-1}{i+1} - \binom{d-2}{i-1}$ for $1 \le i \le d-3$. Twisting the exact sequence (4.5.1) with $\mathcal{O}_{\mathbb{P}^d}(-r-\epsilon)$ and taking global sections we get the minimal graded free resolution of $I_{C|X}$:

$$0 \longrightarrow R(-r-d-\epsilon) \longrightarrow \ldots \longrightarrow R(-r-(i+\epsilon))^{\alpha_{i-1}} \longrightarrow \ldots$$
$$\longrightarrow R(-r-2-\epsilon)^{\alpha_1} \longrightarrow R(-r-\epsilon) \longrightarrow I_{C|X} \longrightarrow 0.$$

Now we can apply the horseshoe lemma to the following exact sequence

$$0 \longrightarrow I_{X|\mathbb{P}^d} \longrightarrow I_{C|\mathbb{P}^d} \longrightarrow I_{C|X} \longrightarrow 0$$

to obtain the minimal free resolution of $I_{C|\mathbb{P}^d}$:

$$0 \longrightarrow R(-r-d-\epsilon) \longrightarrow R(-r-d+2-\epsilon)^{\alpha_{d-3}} \oplus R(-d) \longrightarrow \ldots$$
$$\longrightarrow T_i := R(-r-i-\epsilon)^{\alpha_{i-1}} \oplus R(-(i+1))^{\alpha_i} \longrightarrow \ldots$$
$$\longrightarrow R(-r-\epsilon) \oplus R(-2)^{\alpha_1} \longrightarrow I_{C|\mathbb{P}^d} \longrightarrow 0.$$

This sequence shows that $C \subseteq \mathbb{P}^d$ is an arithmetically Gorenstein variety with canonical module

$$K_C := Ext^{d-1}(R/I_C, R(-d-1)) = R_C(r-1+\epsilon).$$

Therefore $I_{G|C} = H^0_*(\mathcal{O}_C(-r)) = K_C(-2r+1-\epsilon)$. We apply $\mathrm{Hom}(-, R(-d-1))$ to the previous sequence and we get a graded minimal free resolution of K_C:

$$0 \longrightarrow R(-d-1) \longrightarrow R(r-d-1+\epsilon) \oplus R(-d+1)^{\alpha_{d-3}} \longrightarrow \dots$$

$$\longrightarrow T'_i \longrightarrow \dots \longrightarrow R(-1) \oplus R(r-3+\epsilon)^{\alpha_1} \longrightarrow R(r-1+\epsilon) \longrightarrow K_C \longrightarrow 0$$

where $T'_i := T^{\vee}_{d-i}(-d-1) = R(r-i-\epsilon)^{\alpha_{i-1}} \oplus R(-i)^{\alpha_{i-2}}$ for $i = 3, \dots, d-2$. If we twist the previous sequence by $-2r+1-\epsilon$ we get the minimal resolution of $I_{G|C}$:

$$0 \longrightarrow R(-2r-d-\epsilon) \longrightarrow R(-r-d) \oplus R(-2r-d+2-\epsilon)^{\alpha_{d-3}} \longrightarrow \dots \longrightarrow T'_i(-2r+1-\epsilon)$$

$$\longrightarrow \dots \longrightarrow R(-2r-\epsilon) \oplus R(-r-2)^{\alpha_1} \longrightarrow R(-r) \longrightarrow I_{G|C} \longrightarrow 0.$$

Finally, we can apply the horseshoe lemma to the short exact sequence

$$0 \longrightarrow I_{C|X} \longrightarrow I_{G|X} \longrightarrow I_{G|C} \longrightarrow 0$$

to recover the resolution of $I_{G|X}$:

$$0 \longrightarrow R(-2r-d-\epsilon) \longrightarrow R(-2r-d+2-\epsilon)^{\alpha_1} \oplus R(-r-d)^{2-\epsilon} \oplus R(-r-d-1)^{\epsilon}$$

$$\longrightarrow \dots \longrightarrow M_i \longrightarrow \dots \longrightarrow R(-2r-\epsilon) \oplus R(-r-2)^{(2-\epsilon)\alpha_1} \oplus R(-r-3)^{\epsilon\alpha_1}$$

$$\longrightarrow R(-r)^{2-\epsilon} \oplus R(-r-1)^{\epsilon} \longrightarrow I_{G|X} \longrightarrow 0$$

where $M_i = R(-2r-i+1-\epsilon)^{\alpha_{i-2}} \oplus R(-r-i)^{(2-\epsilon)\alpha_{i-1}} \oplus R(-r-i-1)^{\epsilon\alpha_{i-1}}$ for $i = 3, \dots, d-2$. □

Lemma 2.2.6. *Let $X \subseteq \mathbb{P}^3$ be an ACM quasi-minimal surface of degree 3 and take $C \in |(r+\epsilon)H|$, $r \geq 2, \epsilon \in \{0,1\}$. Then, any effective divisor G in the linear system $|rH_C|$ is AG and it has a minimal free resolution of the following form:*

$$0 \longrightarrow R(-2r-3-\epsilon) \longrightarrow R(-2r-\epsilon) \oplus R(-r-3)^{2-\epsilon} \oplus R(-r-4)^{\epsilon} \longrightarrow$$

$$\longrightarrow R(-r)^{2-\epsilon} \oplus R(-r-1)^{\epsilon} \longrightarrow I_{G|X} \longrightarrow 0.$$

Proof. It is completely analogous to Lemma 2.2.5. See also [Cas09, Proposition 3.5]. □

Lemma 2.2.7. *Let $X \subseteq \mathbb{P}^d$ be an ACM quasi-minimal surface distinct to one of the four sporadic cases of Lemma 2.2.4 and let $L \subseteq X$ be a line on it. Take $C \in |L + rH|$ a locally Gorenstein curve, $r \geq 2$, and let G be any effective divisor in the linear system*

$|2rH_C - K_C|$. *Then, G is arithmetically Gorenstein and the minimal free resolution of $I_{G|C}$ has the following form:*

$$0 \longrightarrow R(-2r-d-1) \longrightarrow R(-2r-d+1)^{\alpha_1} \oplus R(-r-d)^{d-1} \longrightarrow \dots$$

$$\longrightarrow R(-2r-i)^{\alpha_{d-i}} \oplus R(-r-i-1)^{\binom{d-1}{d-i}+\alpha_{d-i-1}} \longrightarrow \dots$$

$$\longrightarrow R(-2r-1) \oplus R(-r-3)^{\binom{d-1}{d-2}+\alpha_{d-3}} \longrightarrow R(-r-1) \oplus R(-r-2) \longrightarrow I_{G|C} \longrightarrow 0$$

with $\alpha_i = i\binom{d-1}{i+1} - \binom{d-2}{i-1}$ *for* $1 \leq i \leq d-3$.

Proof. First of all, notice that we are assuming that $C \in |L + rH|$ is locally Gorenstein (i.e., it is G_1) only in order to be able to work with generalized divisors. Let $L \subseteq X$ be any line. Its ideal as a subvariety of $\mathbb{P}^d$ has a resolution:

$$0 \longrightarrow R(-d+1) \longrightarrow \dots \longrightarrow R(-i)^{\binom{d-1}{i}} \longrightarrow \dots \longrightarrow R(-1)^{d-1} \longrightarrow I_{L|\mathbb{P}^d} \longrightarrow 0.$$

Using the mapping cone procedure for the exact sequence $0 \to I_{X|\mathbb{P}^d} \to I_{L|\mathbb{P}^d} \to I_{L|X} \to 0$ we get

$$0 \longrightarrow R(-d) \oplus R(-d+1) \longrightarrow \dots \longrightarrow R(-i)^{\binom{d-1}{i}+\alpha_{i-1}} \longrightarrow$$

$$\longrightarrow \dots \longrightarrow R(-1)^{d-1} \longrightarrow I_{L|X} \longrightarrow 0$$

with $\alpha_i = i\binom{d-1}{i+1} - \binom{d-2}{i-1}$ for $1 \leq i \leq d-3$. Therefore, $C \in |L + rH|$ has the following minimal graded free resolution

$$0 \longrightarrow R(-r-d) \oplus R(-r-d+1) \longrightarrow \dots \longrightarrow R(-r-i)^{\binom{d-1}{i}+\alpha_{i-1}} \longrightarrow \dots$$

$$\longrightarrow R(-r-1)^{d-1} \longrightarrow I_{C|X} \longrightarrow 0. \tag{2.2.5}$$

Now the horseshoe lemma applied to $0 \to I_{X|\mathbb{P}^d} \to I_{C|\mathbb{P}^d} \to I_{C|X} \to 0$ gives us

$$0 \longrightarrow R(-r-d) \oplus R(-r-d+1) \longrightarrow R(-r-d+2)^{\binom{d-1}{d-2}+\alpha_{d-3}} \oplus R(-d) \longrightarrow \dots$$

$$\longrightarrow R(-r-i)^{\binom{d-1}{i}+\alpha_{i-1}} \oplus R(-(i+1))^{\alpha_i} \longrightarrow \dots$$

$$\longrightarrow R(-r-1)^{d-1} \oplus R(-2)^{\alpha_1} \longrightarrow I_{C|\mathbb{P}^d} \longrightarrow 0.$$

Since C is ACM we can apply $\mathrm{Hom}(-, R(-d-1))$ to get a resolution of K_C:

$$0 \longrightarrow R(-d-1) \longrightarrow R(-d+1)^{\alpha_1} \oplus R(r-d)^{d-1} \longrightarrow \dots$$

$$\longrightarrow R(r-i-1)^{\binom{d-1}{d-i}+\alpha_{d-i-1}} \oplus R(-i)^{\alpha_{d-i}} \longrightarrow \dots$$

$$\longrightarrow R(r-3)^{\binom{d-1}{d-2}+\alpha_{d-3}} \oplus R(-1) \longrightarrow R(r-1) \oplus R(r-2) \longrightarrow K_C \longrightarrow 0.$$

Now, since $G \in |2rH_C - K_C|$ we have:

$$0 \longrightarrow R(-2r-d-1) \longrightarrow R(-2r-d+1)^{\alpha_1} \oplus R(-r-d)^{d-1} \longrightarrow \dots$$

$$\longrightarrow R(-r-i-1)^{\binom{d-1}{d-i}+\alpha_{d-i-1}} \oplus R(-2r-i)^{\alpha_{d-i}} \longrightarrow \dots$$

$$\longrightarrow R(-r-3)^{\binom{d-1}{d-2}+\alpha_{d-3}} \oplus R(-2r-1) \longrightarrow R(-r-1) \oplus R(-r-2) \longrightarrow I_{G|C} \longrightarrow 0.$$

□

Lemma 2.2.8. *Let $X \subseteq \mathbb{P}^3$ be an integral cubic surface and let $L \subseteq X$ be a line on it. Take $C \in |L+rH|$ a locally Gorenstein curve, $r \geq 2$, and let G be any effective divisor in the linear system $|2rH_C - K_C|$. Then, G is arithmetically Gorenstein and the minimal free resolution of $I_{G|C}$ has the following form:*

$$0 \longrightarrow R(-2r-4) \longrightarrow R(-2r-1) \oplus R(-r-3)^2 \longrightarrow R(-r-1) \oplus R(-r-2) \longrightarrow I_{G|C} \longrightarrow 0.$$

Proof. It is completely analogous to Lemma 2.2.7. □

Lemma 2.2.9. *A general set Z of $m(2) = d+2$ distinct points on any ACM quasi-minimal surface $X \subset \mathbb{P}^d$ has a minimal free resolution of the following type:*

$$0 \longrightarrow R(-d-2) \longrightarrow R(-d)^{\gamma_{d-1}} \longrightarrow \dots \longrightarrow R(-3)^{\gamma_2} \longrightarrow R(-2)^{2d-1} \longrightarrow I_{Z|X} \longrightarrow 0$$

with

$$\gamma_i = \sum_{l=0}^{1} (-1)^l \binom{d-l-1}{i-l} \Delta^{l+1} H_X(2+l) - \binom{d}{i}(m(2) - H_X(1)).$$

Proof. A general set Z of $d+2$ distinct points on X is in linearly general position (i.e., any subset of Z of $d+1$ points spans $\mathbb{P}^d$). It is well-known that such a Z is AG with minimal free resolution

$$0 \longrightarrow R(-d-2) \longrightarrow R(-d)^{\rho_{d-1}} \longrightarrow R(-d+1)^{\rho_{d-2}} \longrightarrow \cdots$$

$$\longrightarrow R(-3)^{\rho_2} \longrightarrow R(-2)^{\rho_1} \longrightarrow I_{Z|\mathbb{P}^d} \longrightarrow 0.$$

where $\rho_i = i\binom{d+1}{i+1} - \binom{d}{i-1}$ for $1 \leq i \leq d-1$. We now use the mapping cone procedure applied to $0 \to I_X \to I_Z \to I_{Z|X} \to 0$ to obtain a free resolution of the ideal $I_{Z|X}$:

$$0 \longrightarrow R(-d-2) \longrightarrow R(-d)^{\rho_{d-1}+1} \longrightarrow R(-d+1)^{\rho_{d-2}} \longrightarrow R(-d+2)^{\rho_{d-3}-\alpha_{d-3}} \longrightarrow \dots$$
$$\longrightarrow R(-3)^{\rho_2-\alpha_2} \longrightarrow R(-2)^{\rho_1-\alpha_1} \longrightarrow I_{Z|X} \longrightarrow 0$$

with $\alpha_i = i\binom{d-1}{i+1} - \binom{d-2}{i-1}$ for $1 \leq i \leq d-3$. Since there are no ghost terms on the previous exact sequence, it is minimal and the coefficients are forced to be given by the formula from the statement. □

Once we have fixed the starting point of the induction we can deal with the different steps of the procedure.

Fix an integer $r \geq 2$ and let $Z_{m(r)}$ and $Z_{n(r)}$ be general sets of distinct points on X of cardinality $m(r)$ and $n(r)$ respectively. We are going to see that they are directly G-linked by an effective divisor G linearly equivalent to rH_C where C is an AG curve in the linear system $|rH_X|$. Recall that we have:

$$P_X(r-1) < m(r) < n(r) < P_X(r).$$

Let us start with a general set $Z_{m(r)}$ of $m(r)$ distinct points. There are two issues to be checked in order to be sure that the G-liaison is possible. Firstly, we need to check that $\mathrm{h}^0(\mathcal{O}_X(r)) > m(r)$ to guarantee the existence of a curve C in the linear system $|rH_X|$ such that $Z_{m(r)}$ lies on C. On the other hand, we need to verify that $n(r) > p_a(C)$ to be able to apply Riemann-Roch Theorem for (singular) curves which assures that there exists an effective divisor $Z_{n(r)}$ of degree $n(r)$ such that $Z_{m(r)} + Z_{n(r)}$ is linearly equivalent to a divisor rH_C. Notice that, thanks to 1.1.8, $P_X(r) = H_X(r) = \mathrm{h}^0(\mathcal{O}_X(r))$ for any $r \geq 1$. Regarding the first issue, we have $\mathrm{h}^0(\mathcal{O}_X(r)) = P_X(r) > m(r)$ by construction.

Regarding the second issue, let us consider the exact sequence:

$$0 \longrightarrow \mathcal{O}_X(-r) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

Applying the functor of global sections we have

$$0 = H^1(X, \mathcal{O}_X) \longrightarrow H^1(C, \mathcal{O}_C) \longrightarrow H^2(X, \mathcal{O}_X(-r)) \longrightarrow H^2(X, \mathcal{O}_X) = 0$$

and therefore $p_a(C) = \mathrm{h}^1(\mathcal{O}_C) = \mathrm{h}^2(\mathcal{O}_X(-r)) = \mathrm{h}^0(\mathcal{O}_X(r-1))$ where the last equality holds by Serre's duality and taking into account that $\omega_X \cong \mathcal{O}_X(-1)$. Then since $n(r) = dr^2 - m(r) > P_X(r-1) = \mathrm{h}^0(\mathcal{O}_X(r-1)) = p_a(C)$ we are done.

Since this construction can also be performed starting from a general set $Z_{n(r)}$ of $n(r)$ points on the quasi-minimal surface X, we see that a general set of $m(r)$ points is G-linked to a general set of $n(r)$ points and *vice versa*. This allows to state the following Proposition.

Proposition 2.2.10. *Fix $r \geq 2$ and assume that the ideal $I_{Z_{m(r)}|X}$ of $m(r)$ general distinct points on an ACM quasi-minimal surface $X \subseteq \mathbb{P}^d$ has the minimal free resolution*

$$0 \longrightarrow R(-r-d)^{r-1} \longrightarrow R(-r-d+2)^{\gamma_{d-1,r-1}} \longrightarrow \dots$$

$$\longrightarrow R(-r-1)^{\gamma_{2,r-1}} \longrightarrow R(-r)^{(d-1)r+1} \longrightarrow I_{Z_{m(r)}|X} \longrightarrow 0$$

with $\gamma_{i,r-1} = \sum_{l=0}^{1}(-1)^l\binom{d-l-1}{i-l}\Delta^{l+1}P_X(r+l) - \binom{d}{i}(m(r) - P_X(r-1))$. *Then the ideal* $I_{Z_{n(r)}|X}$ *of* $n(r)$ *general distinct points has the minimal free resolution*

$$0 \longrightarrow R(-r-d)^{(d-1)r-1} \longrightarrow R(-r-d+1)^{\beta_{d-1,r}} \longrightarrow \dots$$

$$\longrightarrow R(-r-2)^{\beta_{2,r}} \longrightarrow R(-r)^{r+1} \longrightarrow I_{Z_{n(r)}|X} \longrightarrow 0$$

with $\beta_{i,r} = \sum_{l=0}^{1}(-1)^{l+1}\binom{d-l-1}{i-l}\Delta^{l+1}P_X(r+l) + \binom{d}{i}(n(r) - P_X(r-1))$.

Vice versa, if $n(r)$ general distinct points on an ACM quasi-minimal surface $X \subset \mathbb{P}^d$ have the expected resolution then $m(r)$ general distinct points do as well.

Proof. As mentioned before, we are going to give the complete proof in the case $d \geq 4$. The case $d = 3$ is completely analogous using Lemma 2.2.6 instead of Lemma 2.2.5. So let us suppose that $d \geq 4$. We will check that if $m(r)$ general distinct points have the expected resolution then so do $n(r)$ and we leave to the reader the converse (which is proved analogously). By the above discussion $m(r)$ and $n(r)$ general distinct points on X are G-linked by $G \in |rH_C|$ where C is an AG curve in the linear system $|rH|$. Thanks to Lemma 2.2.5 we know the resolution of $I_{G|X}$ and hence we can apply the mapping cone procedure to the commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & R(-2r-d) & \longrightarrow & R(-r-d)^{r-1} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & R(-2r-d+2)^{\alpha_1} & & & & & & \\
 & & \oplus & \longrightarrow & R(-r-d+2)^{\gamma_{d-1}} & & & & \\
 & & R(-r-d)^2 & & & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \vdots & & \vdots & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & R(-2r-i+1)^{\alpha_{d-i}} & & & & & & \\
 & & \oplus & \longrightarrow & R(-r-i+1)^{\gamma_i} & & & & \\
 & & R(-r-i)^{2\alpha_{i-1}} & & & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \vdots & & \vdots & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & R(-2r) & & & & & & \\
 & & \oplus & \longrightarrow & R(-r-1)^{\gamma_2} & & & & \\
 & & R(-r-2)^{2\alpha_1} & & & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & R(-r)^2 & \longrightarrow & R(-r)^{(d-1)r+1} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & I_{G|X} & \longrightarrow & I_{Z_{m(r)}|X} & \longrightarrow & I_{Z_{m(r)}|G} & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Since $I_{G|X} \subseteq I_{Z_{m(r)}|X}$, we can take as part of the generators of $I_{Z_{m(r)}|X}$ the generators of $I_{G|X}$ and therefore the matrix defining the first horizontal map contains non-zero scalar entries. So the repeated elements can be split off. Therefore we get:

$$
\begin{aligned}
0 \longrightarrow R(-r-d)^{(d-1)r-1} \longrightarrow R(-r-d+1)^{\beta_{d-1,r}} \longrightarrow \dots \\
\longrightarrow R(-2)^{\alpha_1} \oplus R(-r)^{r+1} \longrightarrow I_{Z_{n(r)}|\mathbb{P}^d} \longrightarrow 0.
\end{aligned}
$$

The mapping cone procedure applied now to the exact sequence

$$0 \to I_X \to I_{Z_{n(r)}} \to I_{Z_{n(r)}|X} \to 0$$

gives the desired minimal resolution for $I_{Z_{n(r)}|X}$. □

Lemma 2.2.11. *Let $X \subset \mathbb{P}^d$ be any ACM quasi-minimal surface. Fix $r \geq 2$ and assume that the ideal $I_{Z_{n(r)}|X}$ of a set $Z_{n(r)}$ of $n(r)$ general distinct points on $X \subseteq \mathbb{P}^d$ has the expected minimal free graded resolution then a set of $n(r)+1$ general distinct points does as well.*

Proof. Since $I_{Z_{n(r)}|X}$ has the expected minimal free resolution, we know that the ideal $I_{Z_{n(r)}|X}$ is generated by $r+1$ forms of degree r. Moreover, we know that there are no linear relations among them. We take a general point $p \in X$ and set $Z := Z_{n(r)} \cup \{p\}$. Since $I_{Z|X} \subset I_{Z_{n(r)}|X}$, we can take the r generators of $I_{Z|X}$ in degree r to be a subset of the generators of $I_{Z_{n(r)}|X}$ in degree r; in particular, they do not have linear syzygies. We must add d generators of degree $r+1$ in order to get a minimal system of generators of $I_{Z|X}$. Hence the first module in the minimal free resolution of $I_{Z|X}$ is $R(-r)^r \oplus R(-r-1)^d$ which completely forces the remaining part of the resolution. □

Proposition 2.2.12. *Let $X \subseteq \mathbb{P}^d$ be an ACM quasi-minimal surface distinct to any of the four sporadic cases of Lemma 2.2.4. Fix $r \geq 2$ and assume that the ideal $I_{Z_{p(r)}|X}$ of $p(r) := n(r)+1$ general distinct points on X has the minimal free resolution*

$$0 \longrightarrow R(-r-d)^{(d-1)r} \longrightarrow R(-r-d+1)^{\delta_{d-1,r}} \longrightarrow \dots$$

$$\longrightarrow R(-r-2)^{\delta_{2,r}} \longrightarrow R(-r)^r \oplus R(-r-1)^d \longrightarrow I_{Z_{p(r)}|X} \longrightarrow 0$$

with

$$\delta_{i,r} = \sum_{l=0}^{1}(-1)^{l+1}\binom{d-l-1}{i-l}\Delta^{l+1}H_X(r+l) + \binom{d}{i}(p(r)-H_X(r-1)).$$

Then the ideal $I_{Z_{m(r+1)}|X}$ of $m(r+1)$ general distinct points has the minimal free resolution

$$0 \longrightarrow R(-r-d-1)^r \longrightarrow R(-r-d+1)^{\gamma_{d-1,r}} \longrightarrow \dots$$

$$\longrightarrow R(-r-2)^{\gamma_{2,r}} \longrightarrow R(-r-1)^{(d-1)r+d} \longrightarrow I_{Z_{m(r+1)}|X} \longrightarrow 0$$

with

$$\gamma_{i,r} = \sum_{l=0}^{1}(-1)^{l}\binom{d-l-1}{i-l}\Delta^{l+1}H_X(r+1+l) - \binom{d}{i}(m(r+1)-H_X(r)).$$

Proof. Let $Z_{p(r)}$ be a set of $p(r)$ general distinct points with resolution as in the statement. Let us consider the linear system $|L+rH|$. By Lemma 2.2.4 we can find a locally Gorenstein curve $C \in |L+rH|$ passing through these $p(r)$ points. Notice that $\deg(C) = 1+rd$ and $p_a(C) = d\binom{r}{2} + r$. Since $p_a(C) < m(r+1)$ we can find an effective divisor $Z_{m(r+1)}$ of degree $m(r+1)$ such that $Z_{p(r)}$ and $Z_{m(r+1)}$ are G-linked by a divisor of degree $p(r) + m(r+1) = dr^2 + dr + 2 = deg(2rH_C - K_C)$. This will allowed us to find the resolution of $I_{Z_{m(r+1)}|X}$. First of all, let us find the minimal free resolution of the ideal $I_{Z_{p(r)}|C}$ from the exact sequence

$$0 \to I_{C|X} \to I_{Z_{p(r)}|X} \to I_{Z_{p(r)}|C} \to 0$$

through the mapping cone procedure, with the resolution of $I_{C|X}$ as it was found in (2.2.5). It turns out to be:

$$0 \longrightarrow R(-r-d)^{(d-1)r+1} \longrightarrow R(-r-d+1)^{c_{d-1,r}} \longrightarrow \dots$$

$$\longrightarrow R(-r-2)^{c_{2,r}} \longrightarrow R(-r)^r \oplus R(-r-1) \longrightarrow I_{Z_{p(r)}|C} \longrightarrow 0.$$

Since we have already found out the minimal free resolution of $I_{G|C}$ (see Lemma 2.2.7) we can use the mapping cone procedure applied to the exact sequence $0 \longrightarrow I_{G|C} \longrightarrow I_{Z_{p(r)}|C} \longrightarrow I_{Z_{p(r)}|G} \longrightarrow 0$ to get

$$0 \longrightarrow R(-2r-d-1) \longrightarrow R(-r-d)^{(d-1)r+d} \oplus R(-2r-d+1)^{\alpha_1} \longrightarrow \dots$$

$$\longrightarrow R(-r-i)^{d_{i,r}} \oplus R(-2r-i+1)^{\alpha_{d-i+1}} \longrightarrow \dots$$

$$\longrightarrow R(-r-2)^{d_{2,r}} \longrightarrow R(-r)^r \longrightarrow I_{Z_{p(r)}|G} \longrightarrow 0.$$

($0 \to R(-2r-4) \to R(-r-3)^{2r+3} \oplus R(-2r-1) \to R(-r-2)^{d_{2,r}} \to R(-r)^r \to I_{Z_{p(r)}|G} \to 0$ if $d = 3$).
Finally we obtain the minimal free resolution of $I_{Z_{m(r+1)}|\mathbb{P}^d}$:

$$0 \longrightarrow R(-r-d-1)^r \longrightarrow R(-r-d+1)^{\gamma_{d-1,r}} \longrightarrow R(-r-d+2)^{\gamma_{d-2,r}} \oplus R(-d) \longrightarrow$$

$$\dots \longrightarrow R(-r-i)^{\gamma_{i,r}} \oplus R(-i)^{\alpha_i} \longrightarrow \dots$$

$$\longrightarrow R(-r-1)^{(d-1)r+d} \oplus R(-2)^{\alpha_1} \longrightarrow I_{Z_{m(r+1)}|\mathbb{P}^d} \longrightarrow 0$$

($0 \to R(-r-4)^r \to R(-r-2)^{\gamma_{2,r}} \to R(-r-1)^{2r+3} \oplus R(-3) \to I_{Z_{m(r+1)}|\mathbb{P}^3} \to 0$ if $d = 3$) from which it is straightforward to recover the predicted resolution of $I_{Z_{m(r+1)}|X}$. □

We are now ready to prove the main theorem of this subsection:

Theorem 2.2.13. *Let $X \subseteq \mathbb{P}^d$ be an ACM quasi-minimal surface. Assume that X is not the anticanonical model of* $F_2 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ *or a complete intersection of two quadrics on $\mathbb{P}^4$ with a double line. We have:*

(1) Let $Z_{n(r)} \subset X$ be a general set of $n(r)$ points, $r \geq 2$. Then the minimal graded free resolution of $I_{Z_{n(r)}|X}$ has the following form:

$$0 \longrightarrow R(-r-d)^{(d-1)r-1} \longrightarrow R(-r-d+1)^{\beta_{d-1,r}} \longrightarrow R(-r-d+2)^{\beta_{d-2,r}} \longrightarrow \dots$$

$$\longrightarrow R(-r-2)^{\beta_{2,r}} \longrightarrow R(-r)^{r+1} \longrightarrow I_{Z_{n(r)}|X} \longrightarrow 0.$$

where

$$\beta_{i,r} = \sum_{l=0}^{1}(-1)^{l+1}\binom{n-l-1}{i-l}\Delta^{l+1}H_X(r+l) + \binom{n}{i}(n(r) - H_X(r-1)).$$

(2) Let $Z_{m(r)} \subset X$ be a general set of $m(r)$ points, $r \geq 2$. Then its minimal graded free resolution has the following form:

$$0 \longrightarrow R(-r-d)^{r-1} \longrightarrow R(-r-d+2)^{\gamma_{d-1,r-1}} \longrightarrow \dots$$

$$\longrightarrow R(-r-1)^{\gamma_{2,r-1}} \longrightarrow R(-r)^{(d-1)r+1} \longrightarrow I_{Z_{m(r)}|X} \longrightarrow 0$$

with

$$\gamma_{i,r-1} = \sum_{l=0}^{1}(-1)^{l}\binom{n-l-1}{i-l}\Delta^{l+1}P_X(r+l) - \binom{n}{i}(m(r) - P_X(r-1)).$$

In particular, Mustaţă's conjecture works for $n(r)$ and $m(r)$, $r \geq 4$, general distinct points on an ACM quasi-minimal surface $X \subset \mathbb{P}^d$ (except for the two aforementioned cases).

Proof. Let us first deal with two of the four sporadic cases that had been excluded in the previous Lemma 2.2.12, namely, $X \subseteq \mathbb{P}^9$ being isomorphic to $\mathbb{P}^2$ or $X \subseteq \mathbb{P}^8$ being isomorphic to the smooth quadric. For these two ACM quasi-minimal surfaces, Theorem 4.4.11 and Theorem 4.4.19 of chapter 4 show that there exist Ulrich vector bundles on them with Chern classes $c_1 = rH$ and $c_2 = m(r)$ for any $r \geq 2$. Since we know the minimal free resolution of such vector bundles by Theorem 4.2.16, applying Serre correspondence to these vector bundles (see 4.4.21) we see that $m(r)$, $r \geq 2$, general distinct points have the expected resolution as in the statement. An application of Proposition 2.2.10 gives the analogous result for $n(r)$ distinct points.

For the remaining ACM quasi-minimal surfaces, Lemma 2.2.9 establishes the result for a set of $m(2)$ general distinct points, the starting point of our induction process. Therefore, the result about the resolution of $I_{Z_{n(r)}|X}$ and $I_{Z_{m(r)}|X}$ follows using Lemma 2.2.11, Propositions 2.2.10 and 2.2.12 and applying induction. □

2.2.2 Main Theorems

In this subsection, we are going to prove that the MRC holds for a general set of points Z on any ACM quasi-minimal surface (except for X being the anticanonical model of $\mathrm{F}_2 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ or a complete intersection of two quadrics on $\mathbb{P}^4$ with a double line) when the cardinality of Z falls in the strips of the form $[P_X(r-1), m(r)]$ or $[n(r), P_X(r)]$, $r \geq 4$. We will use the fact that we already know that $n(r)$ and $m(r)$ general distinct points on an ACM quasi-minimal surface satisfy the MRC together with the following lemma which controls how the bottom lines of the Betti diagram of a set of general distinct points on a projective variety change when we add another general point. This lemma will turn out to be a cornerstone in our proof of the MRC for ACM quasi-minimal surfaces:

Lemma 2.2.14. *Let $X \subset \mathbb{P}^n$ be a projective variety with $\dim(X) \geq 2$, $\mathrm{reg}(X) = m$ and with Hilbert polynomial P_X. Let s be an integer with $P_X(r-1) \leq s < P_X(r)$ for some $r \geq m+1$, let Z be a set of s general points on X and let $P \in X \setminus Z$ be a general point. We have:*

(i) $b_{i,r-1}(Z) \geq b_{i,r-1}(Z \cup P)$ for every i.

(ii) $b_{i,r}(Z) \leq b_{i,r}(Z \cup P)$ for every i.

Proof. See [Mus98, Proposition 1.7]. □

We are now ready to state the main result of this section:

Theorem 2.2.15. *Let $X \subseteq \mathbb{P}^d$ be an ACM quasi-minimal surface. Assume that X is not the anticanonical model of $\mathrm{F}_2 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ or a complete intersection of two quadrics on $\mathbb{P}^4$ with a double line. Let r be an integer such that $r \geq reg(X) + 1 = 4$. Then for any general set of distinct points Z on X such that $P_X(r-1) \leq |Z| \leq m(r)$ or $n(r) \leq |Z| \leq P_X(r)$ the Minimal Resolution Conjecture is true.*

Proof. First of all we want to point out that the result was already known in the cases $|Z| = P_X(r-1)$ and $|Z| = P_X(r)$ (see [Mus98, Examples 1 and 2]).

Let Z' be a general set of points of cardinality $|Z'| = n(r)$ and add general points to Z' in order to get a set of points Z of cardinality $n(r) \leq |Z| \leq P_X(r)$. By Theorem 2.2.13 we have that

$$b_{i,r-1}(Z') = 0 \text{ for all } i = 2, \ldots, d.$$

Therefore we can apply Lemma 2.2.14 to deduce that

$$b_{i,r-1}(Z) = 0 \text{ for all } i = 2, \ldots, d.$$

Thus, by semicontinuity, MRC holds for a general set of $|Z|$ points. Now if $|Z| \leq m(r)$, we can add general distinct points to Z in order to have a general set Z' including Z and such that $|Z'| = m(r)$. Again from the previous Theorem we have that

$$b_{i,r}(Z') = 0 \text{ for all } i = 1, \ldots, d-1.$$

So we can use again Lemma 2.2.14 to deduce that

$$b_{i,r}(Z) = 0 \text{ for all } i = 1, \ldots, d-1,$$

and therefore MRC holds for Z. □

As a consequence of the aforementioned Theorem 2.2.13 we will prove that the number of generators of the ideal of a general set of points on an ACM quasi-minimal surface is as small as possible and so it is the number of generators of its canonical module as well. In fact, we have:

Theorem 2.2.16. *Let $X \subseteq \mathbb{P}^d$ be an ACM quasi-minimal surface. Assume that X is not the anticanonical model of* $F_2 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ *or a complete intersection of two quadrics on $\mathbb{P}^4$ with a double line. Then for any general set of distinct points Z on X such that $|Z| \geq P_X(3)$ the Cohen-Macaulay type Conjecture and the Ideal Generation Conjecture are true.*

Proof. Let Z be a general set of points on our ACM quasi-minimal surface X. If it is the case that $n(r) \leq |Z| \leq m(r+1)$ the result has been proved on the previous theorem. So we can assume that $m(r) < |Z| < n(r)$ for some r. We know that the MRC holds for a general set $|Z'|$ of $n(r)$ points on X with $Z \subseteq Z'$ and in particular $b_{1,r}(Z') = 0$. Applying Lemma 2.2.14 inductively we see that $b_{1,r}(Z) = 0$. Analogously, since MRC holds for a general set Z'' of $m(r)$ points, $b_{d,r-1}(Z'') = 0$ with $Z'' \subseteq Z$. Applying once again the same Lemma we see that $b_{d,r-1}(Z) = 0$. □

In the particular case of the cubic surface, since the minimal free resolution of its points has length three, we recover one of the main results of [MPLb] (see also [MP] and [Cas09]):

Theorem 2.2.17. *Let $X \subseteq \mathbb{P}^3$ be a integral cubic surface (i.e., an ACM quasi-minimal surface of degree three). Then the Minimal Resolution Conjecture holds for any set of general distinct points on X of cardinality $\geq P_X(3) = 19$.*

Proof. By Theorem 2.2.16 we know that any set Z of general distinct points on X verifies the Cohen-Macaulay type Conjecture and the Ideal Generation Conjecture. But since the codimension is three there is no further term on the resolution left to consider so the general MRC also holds. □

2.3 MRC for points on surfaces $X \subseteq \mathbb{P}^3$

In this last section of chapter 2 we focus our attention on a slightly modified version of the Minimal Resolution Conjecture stated by Mustaţă. In terms of the Hilbert scheme $\mathrm{Hilb}^s(X)$ parameterizing zero-dimensional subschemes Z of X of length s, Mustaţă's conjecture could be stated as follows:

Conjecture 2.3.1. *Let $X \subset \mathbb{P}^n$ be a projective variety, let $P_X(t)$ be its Hilbert polynomial and $m = \mathrm{reg}(X)$. Let s be an integer such that $P_X(r-1) \leq s < P_X(r)$ for some $r \geq m+1$. Let H_0^s be the irreducible component whose general points correspond to a set Z of s distinct points on X. Then, there is a non-empty open subset $U_0^s \subset H_0^s \subset \mathrm{Hilb}^s(X)$ such that for any $[Z] \in U_0^s$ we have*

$$b_{i+1,r-1}(Z) \cdot b_{i,r}(Z) = 0 \quad \textit{for} \quad i = 1, \cdots, n-1.$$

Since, in general, $\mathrm{Hilb}^s(X)$ is not irreducible (see [Iar72] for the case of varieties of dimension ≥ 3 and chapter 3 for the case of surfaces), we can also search the minimal graded free resolution of the homogeneous ideal of the 0-dimensional scheme associated to a general point $[Z]$ of any other irreducible component of $\mathrm{Hilb}^s(X)$ and ask if the graded Betti numbers $b_{ij}(Z)$ are as small as possible, i.e. there are no ghost terms in the minimal free resolution of R_Z. More precisely, we propose the following modification:

Conjecture 2.3.2. *Let $X \subset \mathbb{P}^n$ be a projective variety, let $P_X(t)$ be its Hilbert polynomial and $m = \mathrm{reg}(X)$. Let s be an integer such that $P_X(r-1) \leq s < P_X(r)$ for some $r \geq m+1$. Then, the* Weak Minimal Resolution Conjecture (WMRC) *holds* *for s if there is an irreducible component $H^s \subset \mathrm{Hilb}^s(X)$ and a non-empty open subset $U^s \subset H^s \subset \mathrm{Hilb}^s(X)$ such that for any $[Z] \in U^s$ we have*

$$b_{i+1,r-1}(Z) \cdot b_{i,r}(Z) = 0 \quad \textit{for} \quad i = 1, \cdots, n-1.$$

In the cases where $\mathrm{Hilb}^s(X)$ is irreducible both conjectures agree. However, we can not expect the irreducibility in general. In fact, in [Har66], Hartshorne proved that $\mathrm{Hilb}^s(X)$ is always connected, but in [Iar72], Iarrobino proved that if X is a nonsingular variety of dimension $n > 2$ then there exists an integer s_0 such that $\mathrm{Hilb}^s(X)$ is reducible for all $s \geq s_0$; and in [Fog68], Fogarty proved that if X is a nonsingular surface then $\mathrm{Hilb}^s(X)$ is irreducible. They left open the case of singular surfaces. In the next chapter, we will prove that for any pair of positive integers (d, n) with $n > 2$ and $d > 1$ or $n = 2$ and $d > 4$ there always exists a generically smooth projective variety of dimension n and degree d and an integer s_0 such that $\mathrm{Hilb}^s(X)$ is reducible for all $s \geq s_0$.

In this section, we address Conjecture 2.3.2 for surfaces in $\mathbb{P}^3$. In particular we prove that for any integer $d \geq 2$ and for any $s \geq \binom{d+3}{3} - 1$, there exists a $\binom{d+2}{2}$-dimensional family of irreducible generically smooth surfaces $X \subset \mathbb{P}^3$ of degree d satisfying Conjecture 2.3.2.

The idea of the proof is to tackle independently the *Ideal Generation Conjecture* (IGC for short) and the *Cohen-Macaulay type conjecture* (CMC conjecture for short), as they have been stated in section 2.1 (see Conjectures 2.1.13 and 2.1.14). Since the length of the Minimal Resolution of points in $\mathbb{P}^3$ is three, the truth of both conjectures implies the MRC.

First of all notice that for any $[Z] \in \mathrm{Hilb}^s(X)$ the Hilbert function of R_Z verifies

$$H_Z(t) \leq \min\{H_X(t), s\} \text{ for all } t\,.$$

As in the case of distinct points, we will say that a 0-dimensional scheme $[Z] \in \mathrm{Hilb}^s(X)$ is in *general position* on X if its Hilbert function is given by the formula

$$H_Z(t) = \min\{H_X(t), s\} \text{ for all } t\,. \tag{2.3.1}$$

Remark 2.3.3. It is easy to check that Theorem 2.1.9 works if we replace "Z a set of s general distinct points on X" by "Z a 0-dimensional subscheme of X with Hilbert function given by the formula (2.3.1)". Namely, we have the following result:

Theorem 2.3.4. *(cf. [Mus98, Propositions 1.5 and 1.6]) Let $X \subset \mathbb{P}^n$ be a projective variety with $d = \dim(X) \geq 1$, $\mathrm{reg}(X) = m$ and with Hilbert polynomial P_X. Let s be* *an integer with $P_X(r-1) \leq s < P_X(r)$ for some $r \geq m+1$ and let $Z \subseteq X$ be a zero-dimensional subscheme of length s with Hilbert function given by the formula (2.3.1). Then:*

(i) For every i and $j \leq r-2$, $b_{i,j}(Z) = b_{i,j}(X)$.

(ii) $b_{i,j}(Z) = 0$ for $j \geq r+1$ and there exists i such that $b_{i,r-1}(Z) \neq 0$.

(iii) For every $j \geq m$,

$$b_{i,j}(Z) = b_{i-1,j+1}(I_Z/I_X).$$

A problem that comes up surprisingly often in Algebraic Geometry and which will play an important role on the proof of the IGC and on the computation of a minimal system of generators of the ideal I_Z of a 0-dimensional scheme Z of length s on a surface $X \subset \mathbb{P}^3$, is to determine the Hilbert series of the graded quotient $A = R/I$ where $I = (F_1, \ldots, F_r) \subset R$ is an ideal generated by general forms, that is the formal power series

$$H_A(Z) := \sum_{t \geq 0} H_A(t) Z^t = \sum_{t \geq 0} \dim_k A_t Z^t.$$

If $r \leq n$ then I is a complete intersection and the result is well known. So, assume $r > n$, which in particular means that A is Artinian. Set $d_i := \deg(F_i)$, $1 \leq i \leq r$. In 1985, Fröberg conjectured (see [Fro85])

$$H_{R/I}(Z) = \left| \frac{\prod_{i=1}^{r}(1 - Z^{d_i})}{(1-Z)^n} \right|$$

where

$$\left| \sum_{t \geq 0} a_t Z^t \right| := \sum_{t \geq 0} b_t Z^t$$

with

$$b_t := \begin{cases} a_t & \text{if } a_i \geq 0 \text{ for all } i \leq t \\ 0 & \text{otherwise.} \end{cases}$$

Note that it is easy to see that

$$H_{R/I}(Z) \geq \left| \frac{\prod_{i=1}^{r}(1 - Z^{d_i})}{(1-Z)^n} \right|.$$

Moreover, several contributions to this apparently simple problem have been made and there are at least three possible approaches to this conjecture. First, one could bound the number of variables. The conjecture was proved to be true for $n = 1$ in Fröberg [Fro85] and for $n = 2$ in Anick [Ani86, Corollary 4.14]. Secondly, one could bound the number of generators for the ideal I. The conjecture is easily seen to be true for $r \leq n+1$ and it was proved to be true for $r = n+2$ by Stanley [Sta80]. It is also true if all the generators have the same degree d and $r \geq \frac{1}{n+1}\binom{d+n+1}{n}$ ([Fro85] Example 4, p. 128). Thirdly, one could prove that the conjecture is true for the first terms in the Hilbert series. The first non-trivial statement comes for degree $d+1$ with $d = \min\{d_i\}$ and was proved by Hochster and Laksov in [HL87, Theorem 1].

Another idea that has been around in Commutative Algebra and Algebraic Geometry and which will be crucial for solving the Cohen-Macaulay type Conjecture is the idea of compressed algebra and relatively compressed algebra. The notion of compressed algebra was introduced in 1978 in [EI95]. A graded Artinian algebra R/I is said to be *compressed* if it has maximal Hilbert function among all graded Artinian algebras with fixed socle degrees. In [MMRN05], the notion of compressed algebra was generalized and the following definition was introduced: A graded Artinian algebra R/I is said to be *relatively compressed with respect to* J if it is a quotient of R/J having maximal Hilbert function among all graded Artinian algebras with fixed socle degrees. It is an open problem, also related to Fröberg's conjecture, to find the Hilbert function of a relatively compressed (level) Artinian algebra and beyond finding the Hilbert function, a much more subtle question is to understand all of the syzygies of a relatively compressed (level) Artinian algebra. Fortunately, for us it will be enough to have the Hilbert function of Artinian graded level algebras R/I relatively compressed with respect to $J = (f)$, $f \in R_d$. To state our result we will use the theory of inverse systems as it is introduced in [Iar84] and [IK99] and refer to these sources for the necessary background.

Let us recall some basic facts on *Macaulay-Matlis duality* which will allow us to relate the above mentioned problems. Set

$$R := k[x_0, x_1, \cdots, x_n] \text{ and } \mathcal{R} := k[y_0, y_1, \cdots, y_n].$$

We consider the action of R on $\mathcal{R}$ by partial differentiation

$$\begin{array}{ccc} R_j \times \mathcal{R}_i & \longrightarrow & \mathcal{R}_{i-j} \\ (u, F) & \mapsto & u \cdot F \end{array}$$

making $\mathcal{R}$ into a graded R-module, where for any $u(x_0, x_1, \cdots, x_n) \in R$ and any

$F(y_0, y_1, \cdots, y_n) \in \mathcal{R}$, we define

$$u \cdot F = u(\partial/\partial y_0, \partial/\partial y_1, \cdots, \partial/\partial y_n)F.$$

If $I \subset R$ is a homogeneous ideal, we define the *Macaulay's inverse system* I^{-1} for I as

$$I^{-1} := \{F \in \mathcal{R} \mid u \cdot F = 0 \text{ for all } u \in I\}.$$

I^{-1} is an R-submodule of $\mathcal{R}$ which inherits a grading of $\mathcal{R}$. Conversely, if $M \subset \mathcal{R}$ is a graded submodule, then $Ann(M) := \{u \in R \mid u \cdot F = 0 \text{ for all } F \in M\}$ is a homogeneous ideal in R. The pairing

$$R_i \times \mathcal{R}_i \longrightarrow k \qquad (u, f) \mapsto u \cdot F$$

is exact; it is called the Macaulay-Matlis duality action of R on $\mathcal{R}$. Moreover, for any integer i, we have $h_{R/I}(i) = \dim_k(R/I)_i = \dim_k(I^{-1})_i$. The following Theorem is fundamental.

Theorem 2.3.5. *We have a bijective correspondence*

$$\begin{array}{ccc} \{\textit{ Homogeneous ideals } I \subset R\} & \rightleftharpoons & \{\textit{ Graded } R-\textit{submodules of } \mathcal{R}\} \\ I & \rightarrow & I^{-1} \\ Ann(M) & \leftarrow & M \end{array}.$$

Moreover, I^{-1} is a finitely generated R-module if and only if R/I is an Artinian ring.

Note that a type s Artinian level algebra $A = R/I$ of socle degree d corresponds via the Macaulay-Matlis duality to a unique s-dimensional vector space in $\mathcal{R}_d$ which is nothing but I^{-1}. We have

Proposition 2.3.6. *Let $\mathcal{R} := k[y_0, y_1, \cdots, y_n]$ be the R-module defined above, and consider μ general forms $F_1, \cdots, F_\mu \in \mathcal{R}$ of degree $d_1, \cdots, d_\mu$, respectively. For any integer $c \geq 0$, the subspace of $\mathcal{R}_c$ generated by $R_{d_1-c}F_1, \cdots, R_{d_\mu-c}F_\mu$ has dimension equal to*

$$\min\{\dim_k \mathcal{R}_c, \dim_k \mathcal{R}_{d_1-c} + \cdots + \dim_k \mathcal{R}_{d_\mu-c}\}. \tag{2.3.2}$$

Proof. See [Iar84, Proposition 3.4]. □

The above proposition tells us that generic forms have derivatives as independent as they can be. Hence, they give rise to compressed Artinian algebras.

Example 2.3.7. (1) Take $M = \langle y_0^3y_1 + y_0^2y_1^2 + y_1y_2^3, y_0^3y_2 + y_0^2y_1^2 + y_0^2y_1y_2 + y_1^3y_2\rangle \subset \mathcal{R}$. Then $I = Ann(M) = (x_0^3x_1 - x_2^3x_1, x_0^3x_2 - x_1^3x_2)$ and $A = R/I$ is a Artinian level graded algebra with socle degree 4 and type 2. Its h-vector is 1 3 6 6 2.

(2) Let $G = y_0^2y_1^2y_2^2 \in \mathcal{R}_6$. Then $I = Ann((G)) = (x_0^3, x_1^3, x_2^3)$ and $A = R/I$ is an Artinian complete intersection with h-vector 1 3 6 7 6 3 1.

(3) Take $M = \langle y_0^5+y_1^5, y_0^2y_2^3\rangle \subset \mathcal{R}$. Then $I = Ann(M) = (x_0x_1, x_1x_2, x_0^3x_3, x_3^4, x_0^5 - x_1^5)$ and $A = R/I$ is an Artinian level graded algebra with socle degree 5 and type 2. The h-vector of A is 1 3 4 5 4 2.

Note that only the first example corresponds to a *compressed* Artinian graded. In fact, a *compressed* Artinian level graded algebra A with socle degree s and socle dimension c has Hilbert function (cf. [IK99])

$$H_A(t) = \min\{\dim R_t, c \cdot \dim R_{s-t}\}.$$

For the Hilbert function of a graded level Artinian algebra A that is *relatively compressed* with respect to a complete intersection $J \subset R$, we have

Lemma 2.3.8. *Let $A = R/I$ be a graded level Artinian algebra of socle degree s, type c and relatively compressed with respect to a complete intersection $J \subset R$. Then*

$$H_A(t) \leq \min\{\dim(R/J)_t, c \cdot \dim(R/J)_{s-t}\}.$$

Proof. See [MMRN05, Lemma 2.13]. □

Example 2.3.9. We consider 30 general distinct points $Z \subset \mathbb{P}^3$ on a smooth quadric $Q \subset \mathbb{P}^3$. According to Proposition 2.1.8 the h-vector of X is 1 3 5 7 9 5 and I_Z has a minimal free R-resolution of the following type (see [GMR96, Theorem 4.3]):

$$0 \to R(-8)^5 \to R(-6)^5 \oplus R(-7)^6 \to R(-2) \oplus R(-5)^6 \to R \to R_Z \to 0.$$

Hence, the Artinian reduction of R_Z is an Artinian level algebra with embedding dimension 3, socle degree 5, type 5 and relatively compressed with respect to the ideal generated by a form of degree 2.

Remark 2.3.10. It is easy to check that Lemma 2.2.14 works if we replace "Z a set of s general distinct points on X" by "Z a 0-dimensional subscheme of X with Hilbert function given by the formula (2.3.1)". Namely, we have the following result:

Lemma 2.3.11. *Let $X \subset \mathbb{P}^n$ be a projective variety with $\dim(X) \geq 2$, $\operatorname{reg}(X) = m$ and with Hilbert polynomial P_X. Let s be an integer with $P_X(r-1) \leq s < P_X(r)$ for some $r \geq m+1$, let $Z \subseteq X$ be a zero-dimensional subscheme of length s with Hilbert function given by the formula (2.3.1) on X and let $P \in X \setminus Z$ be a general point. It holds that:*

(i) $b_{i,r-1}(Z) \geq b_{i,r-1}(Z \cup P)$ *for every* i.

(ii) $b_{i,r}(Z) \leq b_{i,r}(Z \cup P)$ *for every* i.

2.3.1 Main Results

In this subsection, we set

$$P_d(t) := \binom{t+3}{3} - \binom{t+3-d}{3},$$

the Hilbert polynomial of a surface $X \subset \mathbb{P}^3$ defined by a homogeneous polynomial f of degree d, $\operatorname{reg}(X) := d$ and $R_X = R/(f)$. We are going to prove that for any $s \geq P_d(d)$ there exists a $\binom{d+2}{2}$-dimensional family of irreducible generically smooth surfaces of degree d in $\mathbb{P}^3$ satisfying the WMRC for s.

Remark 2.3.12. Notice that we bounded ourselves to consider zero dimensional schemes $Z \subseteq X$ of length $s \geq P_d(d)$ in order to fit with the conditions under which the MRC was stated by Mustaţă in [Mus98] and hence also with our WMRC in Conjecture 2.3.2.

Let s be an integer such that $P_d(r-1) \leq s < P_d(r)$ for some $r \geq \operatorname{reg}(X) + 1 = d+1$ and let $Z \subset X$ be a 0-dimensional scheme with Hilbert function given by the formula

$$H_Z(t) = \min\{H_X(t), s\} \quad \text{for all } t\ .$$

First, we are going to address the Ideal Generation Conjecture (IGC) which essentially says that the minimal number of generators of I_Z should be as small as possible, i.e. the conjecture predicts that the morphism

$$(I_Z/(f))_r \otimes (R_X)_1 \longrightarrow (I_Z/(f))_{r+1}$$

should have maximal rank. In particular, when

$$\dim(I_Z/(f))_r \otimes (R_X)_1 \leq \dim(I_Z/(f))_{r+1}$$

then there should be no linear syzygies among the generators, and when

$$\dim(I_Z/(f))_r \otimes (R_X)_1 \geq \dim(I_Z/(f))_{r+1}$$

then $I_Z/(f)$ should have no generators in degree $r+1$.

To be more precise, we define

$$\nu_r(s) := P_d(r) - s$$

and we denote by $g_1, \cdots, g_{\nu_r(s)}$ the $\nu_r(s)$ generators of I_Z of degree r. The IGC says that if

$$3\nu_r(s) \geq P_d(r+1) - P_d(r) = \binom{r+3}{2} - \binom{r+3-d}{2},$$

then I_Z is generated by f and $g_1, \cdots, g_{\nu_r(s)}$; if

$$3\nu_r(s) < P_d(r+1) - P_d(r)$$

and we define

$$\nu_{r+1}(s) = P_d(r+1) - P_d(r) - 3\nu_r(s)$$

then we must add $\nu_{r+1}(s)$ generators of degree $r+1$ to f and $g_1, \cdots, g_{\nu_r}(s)$ in order to get a minimal set of generators of I_Z. Therefore, we have the following formula for the conjectured value of the minimal number of generators of Z:

$$\nu(Z) = max\{1 + P_d(r) - s, 1 + P_d(r+1) - 3\nu_r(s) - s\}.$$

Proposition 2.3.13. *For any $s \geq \binom{d+3}{3} - 1$, there exists a $\binom{d+2}{2}$-dimensional family of irreducible generically smooth surfaces $X \subset \mathbb{P}^3$ of degree d and a non-empty open subset $U^s \subset \mathrm{Hilb}^s(X)$ such that any $[Z] \in U^s$ satisfies the IGC.*

Proof. Set $S = k[x_0, x_1, x_2]$ and $R = k[x_0, x_1, x_2, x_3]$. By semicontinuity, to prove the result, it suffices to explicitly construct, for any irreducible normal surface $X \subseteq \mathbb{P}^3$ defined by a general form $f \in S_d$, a 0-dimensional subscheme $[Z] \in \mathrm{Hilb}^s(X)$ satisfying the IGC. To this end, we choose an integer r such that

$$P_d(r-1) = \binom{r+2}{3} - \binom{r+2-d}{3} \leq s < P_d(r) = \binom{r+3}{3} - \binom{r+3-d}{3}$$

and as before we denote by $\nu_r(s) = P_d(r) - s$. If $s = P_d(r) - 1$, the result is known by [Mus98, Example 2]. So, we assume $\nu_r(s) \geq 2$ and we distinguish two cases:

<u>Case 1:</u> $3\nu_r(s) \leq P_d(r+1) - P_d(r)$. First, we choose $\nu_r(s)$ general homogeneous polynomials $g_1, \cdots, g_{\nu_r(s)} \in S_r$ such that $\dim S_1\langle g_1, \cdots, g_{\nu_r(s)}\rangle = 3\nu_r(s)$. They exist by [HL87, Theorem 1]. Now, we choose a general form $f \in S_d$. By [Ani86, Corollary 4.14], the natural map

$$(S/(g_1, \cdots, g_{\nu_r(s)}))_t \xrightarrow{\times f} (S/(g_1, \cdots, g_{\nu_r(s)}))_{t+d}$$

has maximal rank for all $t \geq 0$ (i.e., it is either injective or surjective). Therefore, the starting values of the Hilbert function of the Artinian ideal

$$J_0 := (f, g_1, \cdots, g_{\nu_r(s)}) \subset S$$

are (note that $\mu(J_0) \geq 3$)

$$H_{S/J_0}(t) = \begin{cases} \binom{t+2}{2} & \text{if } t < d \\ \binom{t+2}{2} - \binom{t+2-d}{2} & \text{if } d \leq t < r \\ \binom{r+2}{2} - \binom{r+2-d}{2} - \nu_r(s) & \text{if } t = r \\ \binom{r+3}{2} - \binom{r+3-d}{2} - 3\nu_r(s) & \text{if } t = r+1. \end{cases}$$

Set $\nu_{r+1}(s) := H_{S/J_0}(r+1)$ and choose $\nu_{r+1}(s)$ forms $h_1, \cdots, h_{\nu_{r+1}(s)} \in S_{r+1}$ such that $\overline{h}_1, \cdots, \overline{h}_{\nu_{r+1}(s)}$ is a k-basis of $(S/J_0)_{r+1}$. Consider the Artinian ideal $J := J_0 + (h_1, \cdots, h_{\nu_{r+1}(s)}) \subset S$. We have

$$H_{S/J}(t) = \begin{cases} H_{S/J_0}(t) & \text{if } t \leq r \\ 0 & \text{if } t \geq r+1. \end{cases} \tag{2.3.3}$$

Therefore, $I := JR \subset R$ defines a 0-dimensional scheme Z of length s which lies on the surface of degree d defined by f and

$$I_Z = I = (f, g_1, \cdots, g_{\nu_r(s)}, h_1, \cdots, h_{\nu_{r+1}(s)})$$

which proves what we want. (Note that $\Delta H_{R/I}(t) = H_{S/J}(t)$ for all t.)

<u>Case 2:</u> $3\nu_r(s) > P_d(r+1) - P_d(r)$. In this case, we choose a general form $f \in S_d$ and $\nu_r(s)$ general forms $g_1, \cdots, g_{\nu_r(s)} \in S_r$. By [Ani86, Corollary 4.14], $J = (f, g_1, \cdots, g_{\nu_r(s)}) \subset S$ is an Artinian ideal with Hilbert function

$$H_{S/J}(t) = \begin{cases} \Delta H_X(t) & \text{if } t < r \\ \Delta H_X(t) - \nu_r(s) & \text{if } t = r \\ 0 & \text{if } t > r. \end{cases}$$

Therefore, $I := JR \subset R$ defines a 0-dimensional scheme Z of length s which lies on the surface X of degree d defined by f and $I_Z = I = (f, g_1, \cdots, g_{\nu_r(s)})$. (Note that $\Delta H_{R/I}(t) = H_{R/J}(t)$ for all t.) □

Remark 2.3.14. It is important to point out that the ideal $J \subset S = k[x_0, x_1, x_2]$ constructed in Proposition 2.3.13, Case 1, has the following minimal graded free resolution:

$$0 \to S(-r-3)^{b_{3,r}} \to S(-r-2)^{b_{2,r}} \to S(-d) \oplus S(-r)^{\nu_r(s)} \oplus S(-r-1)^{\nu_{r+1}(s)} \to J \to 0 \quad (2.3.4)$$

where $b_{3,r} = H_{S/J}(r)$ and $b_{2,r} = b_{3,r} + \nu_r(s) + \nu_{r+1}(s)$. In particular it has the shape predicted by the MRC.

We are now going to address the Cohen-Macaulay type Conjecture, i.e., the expected graded Betti numbers do appear at the end of the resolution.

Proposition 2.3.15. *For any $s \geq \binom{d+3}{3} - 1$, there exists a $\binom{d+2}{2}$-dimensional family of irreducible generically smooth surfaces $X \subset \mathbb{P}^3$ of degree d and a non-empty open subset $U^s \subset \mathrm{Hilb}^s(X)$ such that any $[Z] \in U^s$ satisfies the CMC conjecture.*

Proof. Set $S = k[x_0, x_1, x_2]$, $R = k[x_0, x_1, x_2, x_3]$ and $\mathcal{S} = k[y_0, y_1, y_2]$. Again, by semicontinuity, it suffices to explicitly construct, for any irreducible normal surface $X \subset \mathbb{P}^3$ defined by a general form $f \in S_d$, a 0-dimensional subscheme $[Z] \in \mathrm{Hilb}^s(X)$ satisfying the CMC conjecture. First, we choose an integer r such that

$$P_d(r-1) = \binom{r+2}{3} - \binom{r+2-d}{3} \leq s < P_d(r) = \binom{r+3}{3} - \binom{r+3-d}{3}$$

and we define $\rho_r(s) = s - P_d(r-1)$. We distinguish two cases:

<u>Case 1:</u> $3\rho_r(s) \geq P_d(r-1) - P_d(r-2)$. Set

$$\rho_0 := \min\{\rho_r(s) \mid 3\rho_r(s) \geq P_d(r-1) - P_d(r-2)\}$$

and $s_0 := P_d(r-1) + \rho_0$. We will first construct a 0-dimensional scheme $Z_0 \subset X$ of length s_0 satisfying the CMC conjecture. To this end, we take $f \in S_d$ a general form of degree d defining a smooth plane curve C. The Hilbert function of $S/(f)$ is given by

$$H_{S/(f)}(t) = \begin{cases} \binom{t+2}{2} & \text{if } 0 \leq t \leq d-1 \\ \binom{t+2}{2} - \binom{t+2-d}{2} & \text{if } t \geq d. \end{cases}$$

Set

$$\alpha := H_{S/(f)}(r-1) = P_d(r-1) - P_d(r-2)$$

and write $\alpha = 3\lambda + \mu$ with $0 \leq \mu \leq 2$ (note that $\rho_0 - 1 \leq \lambda \leq \rho_0$). We choose on C a set T of α general points. So, the Hilbert function of T is

$$H_{S/I_T}(t) = \begin{cases} H_{S/(f)}(t) & \text{if } 0 \leq t \leq r-1 \\ \alpha & \text{if } t \geq r-1. \end{cases}$$

If $\mu = 0$, we choose λ disjoint sets $T_1, \cdots, T_\lambda \subset T$ such that $T = \cup_{i=1}^{\lambda} T_i$ and $|T_i| = 3$ for $1 \leq i \leq \lambda$. If $\mu > 0$, we choose $\lambda + 1$ disjoint sets $T_1, \cdots, T_\lambda, T_{\lambda+1} \subset T$ such that $T = \cup_{i=1}^{\lambda+1} T_i$, $|T_i| = 3$ for $1 \leq i \leq \lambda$ and $|T_{\lambda+1}| = \mu$. If the points are general enough the points in T_i, $1 \leq i \leq \lambda$, are not collinear and we have

$$H_{S/I_{T_i}}(t) = \begin{cases} 1 & \text{if } t = 0 \\ 3 & \text{if } t \geq 1 \end{cases} \qquad \text{for } 1 \leq i \leq \lambda;$$

and

$$H_{S/I_{T_{\lambda+1}}}(t) = \begin{cases} 1 & \text{if } t = 0 \\ \mu & \text{if } t \geq 1. \end{cases}$$

Now let $J_i \subset S$, $1 \leq i \leq \lambda + 1$, be general Gorenstein ideals containing I_{T_i}, $1 \leq i \leq \lambda+1$, respectively, whose quotients S/J_i each are Artinian of socle degree r. We construct J_i with S/J_i Artinian of socle degree r (and type 1) by choosing a general enough element $g_i \in (I(T_i)_r)^\perp \cap \mathcal{S}_r$ and defining $J_i := Ann(g_i)$. By [IK99, Lemma 1.17], we have that the Hilbert function of S/J_i is given by

$$H_{S/J_i}(t) := \min(\dim S_t, |T_i|, \dim S_{r-t}).$$

The intersection $J := \cap_{i=1}^{\lambda} J_i$, with $I_C = (f) \subset J \subset S$, (resp. $I_C \subset J := \cap_{i=1}^{\lambda+1} J_i \subset S$) if $\mu = 0$ (resp. $\mu > 0$) gives an Artinian level algebra S/J with Hilbert function

$$H_{S/J}(t) = \begin{cases} \Delta H_X(t) & \text{if } t < r \\ \rho_0 & \text{if } t = r \\ 0 & \text{if } t > r. \end{cases}$$

Therefore, $I := JR \subset R$ defines a 0-dimensional scheme Z_0 of length s_0 which lies on the surface of degree d defined by f and satisfies the CMC conjecture. In fact, $b_{3,r}(Z_0) = \rho_0$ and $b_{3,r-1}(Z_0) = 0$. (Note that $\Delta H_{R/I}(t) = H_{S/J}(t)$ for all t.)

The case $s > s_0$ easily follows from the case s_0. In fact, let K_j, $1 \leq j \leq \rho_r(s) - \rho_0$, be a general Gorenstein ideal of socle degree r and containing I_C (we construct K_j by choosing a general enough element $h_j \in (I(C)_r)^{\perp} \cap \mathcal{S}_r$ and defining $K_j := Ann(h_j)$). Then $I_C \subset K := J \bigcap \cap_{j=1}^{s-s_0} K_j \subset S$ gives an Artinian level algebra S/K with Hilbert function

$$H_{S/K}(t) = \begin{cases} \Delta H_X(t) & \text{if } t < r \\ \rho_r(s) & \text{if } t = r \\ 0 & \text{if } t > r, \end{cases}$$

and $I := KR \subset R$ defines a 0-dimensional scheme Z_s of length s which lies on the surface of degree d defined by f and satisfies the CMC conjecture. In fact, $b_{3,r}(Z_s) = \rho_r(s)$ and $b_{3,r-1}(Z_s) = 0$.

Case 2: $3\rho_r(s) < P_d(r-1) - P_d(r-2)$. In this case, we take $f \in S_d$ a general form of degree d defining a smooth plane curve C and a set T of α general points on C being $\alpha := H_{S/(f)}(r-1) = P_d(r-1) - P_d(r-2)$. So, the Hilbert function of T is

$$H_{S/I(T)}(t) = \begin{cases} H_{S/(f)}(t) & \text{if } 0 \leq t \leq r-1 \\ \alpha & \text{if } t \geq r-1. \end{cases}$$

Define $\rho_{r-1}(s) := \alpha - 3\rho_r(s) > 0$. We choose $\rho_r(s)$ disjoint sets of three no collinear points $T_1, \cdots, T_{\rho_r(s)} \subset T$ and set $\{p_1, \cdots, p_{\rho_{r-1}(s)}\} := T \setminus \cup_{i=1}^{\rho_r(s)} T_i$. We have

$$H_{S/I_{T_i}}(t) = \begin{cases} 1 & \text{if } t = 0 \\ 3 & \text{if } t \geq 1 \end{cases} \qquad \text{for } 1 \leq i \leq \rho_r(s);$$

and

$$H_{S/I_{p_i}}(t) = 1 \text{ for all } t \geq 1 \text{ and } 1 \leq i \leq \rho_{r-1}(s).$$

Now let $J_i \subset S$, for $1 \leq i \leq \rho_r(s)$, be general Gorenstein ideals containing I_{T_i}, $1 \leq i \leq \rho_r(s)$, respectively, whose quotients S/J_i each are Artinian of socle degree r. We construct J_i, as in the previous case, as follows: for any integer $1 \leq i \leq \rho_r(s)$, we choose a general enough element $g_i \in ((I_{T_i})_r)^{\perp} \cap \mathcal{S}_r$ and we define $J_i := Ann(g_i) \subset S$. J_i are general Gorenstein ideals containing I_{T_i} and the quotients S/J_i are Artinian of socle degree r. For any integer $1 \leq j \leq \rho_{r-1}(s)$, we choose a general enough element $h_j \in ((I_{p_j})_{r-1})^{\perp} \cap \mathcal{S}_{r-1}$ and we define $K_j := Ann(h_j) \subset S$. K_j are general Gorenstein ideals containing I_{p_j} and the quotients S/K_j are Artinian of socle degree $r-1$. By [IK99, Lemma 1.17], we have that the Hilbert function of S/J_i is given by $H_{S/J_i}(t) := \min(\dim S_t, 3, \dim S_{r-t})$ for all t and the Hilbert function of S/K_j is given by $H_{S/K_j}(t) := \min(\dim S_t, 1, \dim S_{r-1-t})$ for all t. The intersection

$$J := \bigcap_{i=1}^{\rho_r(s)} J_i \cap \bigcap_{j=1}^{\rho_{r-1}(s)} K_j \subset S$$

gives an Artinian algebra S/J with socle degree $(\overbrace{r-1, \cdots, r-1}^{\rho_{r-1}(s)}, \overbrace{r, \cdots, r}^{\rho_r(s)})$ and Hilbert function

$$H_{S/J}(t) = \begin{cases} \Delta H_X(t) & \text{if } t < r \\ \rho_r(s) & \text{if } t = r \\ 0 & \text{if } t > r. \end{cases}$$

Therefore, $I := JR \subset R$ defines a 0-dimensional scheme Z of length s which lies on the surface of degree d defined by f and Z satisfies the CMC conjecture. In fact, $b_{3,r}(Z) = \rho_r(s)$ and $b_{3,r-1}(Z) = \rho_{r-1}(s)$ which are the expected graded Betti numbers at the end of the minimal resolution of I_Z. □

Recall that (see Definition 1.1.11) a 0-dimensional scheme Z on a surface $X \subset \mathbb{P}^3$ is said to be *level of type* ρ if the last module in its minimal graded free resolution has rank ρ and is concentrated in only one degree.

Remark 2.3.16. (i) It is important to point out that the ideal $J \subset S = k[x_0, x_1, x_2]$ (resp. $K \subset S = k[x_0, x_1, x_2]$) constructed in Proposition 2.3.15, Case 1, is level of type ρ_0 (resp. $\rho_r(s)$) and has the following minimal graded free resolution:

$$0 \longrightarrow S(-r-3)^{\rho_0} \longrightarrow S(-r-1)^{b_{2,r-1}} \oplus S(-r-2)^{b_{2,r}} \longrightarrow \tag{2.3.5}$$
$$S(-d) \oplus S(-r)^{b_{1,r-1}} \oplus S(-r-1)^{b_{1,r}} \longrightarrow J \longrightarrow 0$$

$$\text{(resp.} \quad 0 \longrightarrow S(-r-3)^{\rho_r(s)} \longrightarrow S(-r-1)^{b_{2,r-1}} \oplus S(-r-2)^{b_{2,r}} \longrightarrow \tag{2.3.6}$$
$$S(-d) \oplus S(-r)^{b_{1,r-1}} \oplus S(-r-1)^{b_{1,r}} \longrightarrow K \longrightarrow 0 \text{).}$$

(ii) The ideal $J \subset S = k[x_0, x_1, x_2]$ constructed in Proposition 2.3.15, Case 2, is not level since it has the following minimal graded free resolution:

$$0 \longrightarrow S(-r-3)^{\rho_r(s)} \oplus S(-r-2)^{\rho_{r-1}(s)} \longrightarrow S(-r-1)^{b_{2,r-1}} \longrightarrow \tag{2.3.7}$$
$$S(-d) \oplus S(-r)^{b_{1,r-1}} \oplus S(-r-1)^{b_{1,r}} \longrightarrow J \longrightarrow 0.$$

Remark 2.3.17. Fix s with $P_d(r-1) \leq s < P_d(r)$ and $r \geq d+1$, and set

$$\nu_r(s) := P_d(r) - s.$$

As it was pointed out in Remark 2.3.14 for the Case 1 of Proposition 2.3.13, we already know that the shape of the full minimal free resolution of J and hence of $I = JR$ is the one predicted by Mustaţă's conjecture. On the other hand, notice that in Proposition 2.3.13, Case 2 and in Proposition 2.3.15 we can consider the same general form $f \in k[x_0, x_1, x_2]$ of degree d producing an irreducible generically smooth surface $X \subseteq \mathbb{P}^3$. X is a cone with vertex $p = (0,0,0,1)$ over the plane curve $f(x,y,z) = 0$. Moreover, the 0-dimensional schemes $Z \subset X$ of length s and support p constructed in Proposition 2.3.13, Case 2 (and, hence, satisfying IGC) as well as the 0-dimensional schemes $Z \subset X$ of length s and support p constructed in Proposition 2.3.15 (and, hence, satisfying Cohen-Macaulay type conjecture) are parameterized by a non-empty open subset of the Grassmannian $\mathrm{Grass}(\nu_r(s), (S/(f))_r)$. Since $\mathrm{Grass}(\nu_r(s), (S/(f))_r)$ is an irreducible variety, the open sets of $\mathrm{Hilb}^s(X)$ obtained by semicontinuity from both of the propositions should meet and we are able to conclude the following theorem.

Theorem 2.3.18. *Let s be an integer such that $s \geq P_d(d)$, $d \geq 2$. Then there exists a family of dimension $\binom{d+2}{2}$ of irreducible generically smooth surfaces $X \subset \mathbb{P}^3$ of degree d for which WMRC holds, i.e. there exist a non-empty open subset $U^s \subset \mathrm{Hilb}^s(X)$ such that for any $[Z] \in U^s$ we have*

$$b_{3,r-1}(Z) \cdot b_{2,r}(Z) = b_{2,r-1}(Z) \cdot b_{1,r}(Z) = 0.$$

Proof. First, we choose an integer r such that

$$P_d(r-1) = \binom{r+2}{3} - \binom{r+2-d}{3} \leq s < P_d(r) = \binom{r+3}{3} - \binom{r+3-d}{3}$$

and as before we define

$$\nu_r(s) := P_d(r) - s \qquad \rho_r(s) := s - P_d(r-1).$$

Since the case $3\nu_r(s) < P_d(r+1) - P_d(r)$ and $3\rho_r(s) < P_d(r-1) - P_d(r-2)$ never holds simultaneously we distinguish 3 cases:
<u>Case 1:</u> $3\nu_r(s) \leq P_d(r+1) - P_d(r)$ and $3\rho_r(s) \geq P_d(r-1) - P_d(r-2)$. We define $\nu_{r+1}(s) := \binom{r+3}{2} - \binom{r+3-d}{2} - 3\nu_r(s)$. It follows from Proposition 2.3.13 and Remark 2.3.14 that there exists a family of dimension $\binom{d+2}{2}$ of irreducible generically smooth surfaces $X \subset \mathbb{P}^3$ of degree d and a non-empty open subset $U^s \subset \mathrm{Hilb}^s(X)$

such that for any $[Z] \in U^s$ we have a minimal free graded resolution of the following type:

$$0 \to R(-r-3)^{\rho_r(s)} \to R(-r-2)^{b_{2,r}} \to R(-d) \oplus R(-r)^{\nu_r(s)} \oplus R(-r-1)^{\nu_{r+1}(s)} \to I_Z \to 0$$

where $b_{2,r} = b_{3,r} + \nu_r(s) + \nu_{r+1}(s)$. We clearly have

$$b_{3,r-1}(Z) \cdot b_{2,r}(Z) = b_{2,r-1}(Z) \cdot b_{1,r}(Z) = 0$$

and Z satisfies WMRC.

Case 2: $3\nu_r(s) > P_d(r+1) - P_d(r)$ and $3\rho_r(s) \geq P_d(r-1) - P_d(r-2)$. By Propositions 2.3.13 and 2.3.15 and by Remark 2.3.17 there exists a family of dimension $\binom{d+2}{2}$ of irreducible generically smooth surfaces $X \subset \mathbb{P}^3$ of degree d and a non-empty open subset $U^s \subset \mathrm{Hilb}^s(X)$ such that any $[Z] \in U^s$ satisfies the IGC and the CMC conjecture. Since it satisfies the IGC and $3\nu_r(s) > P_d(r+1) - P_d(r)$, we have $b_{1,r-1}(Z) = \nu_r(s)$ and $b_{1,r}(Z) = 0$. Since it satisfies the CMC conjecture and $3\rho_r(s) \geq P_d(r-1) - P_d(r-2)$, we have $b_{3,r-1}(Z) = 0$ and $b_{3,r}(Z) = \rho_r(s)$. Therefore, I_Z has the following graded minimal free resolution

$$0 \to R(-r-3)^{\rho_r(s)} \to R(-r-2)^{b_{2,r}} \oplus R(-r-1)^{b_{2,r-1}} \to R(-d) \oplus R(-r)^{\nu_r(s)} \to I_Z \to 0$$

where $b_{2,r}, b_{2,r-1}$ are determined by the equations:

$$b_{2,r} + b_{2,r-1} = \nu_r(s) + \rho_r(s) \quad \text{and} \quad d + 3\rho_r(s) = 2b_{2,r} + b_{2,r-1}.$$

Again we have

$$b_{3,r-1}(Z) \cdot b_{2,r}(Z) = b_{2,r-1}(Z) \cdot b_{1,r}(Z) = 0$$

and Z satisfies WMRC.

Case 3: $3\nu_r(s) > P_d(r+1) - P_d(r)$ and $3\rho_r(s) < P_d(r-1) - P_d(r-2)$. Again there exists a family of dimension $\binom{d+2}{2}$ of irreducible generically smooth surfaces $X \subset \mathbb{P}^3$ of degree d and a non-empty open subset $U^s \subset \mathrm{Hilb}^s(X)$ such that any $[Z] \in U^s$ satisfies the IGC and the CMC conjecture. Since it satisfies the IGC and $3\nu_r(s) > P_d(r+1) - P_d(r)$, we have $b_{1,r-1}(Z) = \nu_r(s)$ and $b_{1,r}(Z) = 0$. Since it satisfies the CMC conjecture and $3\rho_r(s) < P_d(r-1) - P_d(r-2)$, we have $b_{2,r}(Z) = 0$, $b_{3,r-1}(Z) = \rho_{r-1}(s)$ and $b_{3,r}(Z) = \rho_r(s)$. Therefore, I_Z has the following graded minimal free resolution

$$0 \to R(-r-3)^{\rho_r(s)} \oplus R(-r-2)^{\rho_{r-1}(s)} \to R(-r-1)^{b_{2,r-1}} \to R(-d) \oplus R(-r)^{\nu_r(s)} \to I_Z \to 0$$

where $b_{2,r-1} = \nu_r(s) + \rho_r(s) + \rho_{r-1}(s)$. Therefore, we have

$$b_{3,r-1}(Z) \cdot b_{2,r}(Z) = b_{2,r-1}(Z) \cdot b_{1,r}(Z) = 0$$

and Z satisfies WMRC. □

Remark 2.3.19. Notice that, thanks to Theorem 2.3.18, for every $s \geq P_d(d)$ we managed to find an open subset V_s of the projective space $Y := \mathbb{P}^{\binom{d+2}{2}-1} = \text{Proj}(S_d)$ in bijection with irreducible normal surfaces $X \subset \mathbb{P}^3$ and for each $[X] \in V_s$ a non-empty open subset $U^s \subset \text{Hilb}^s(X)$ such that any $[Z] \in U^s$ satisfies

$$b_{3,r-1}(Z) \cdot b_{2,r}(Z) = b_{2,r-1}(Z) \cdot b_{1,r}(Z) = 0.$$

Since $\cup_s(Y \setminus V_s)$ does not cover the whole of Y we can conclude that there exists an infinite number of surfaces for which WMRC holds for any $s \geq P_d(d)$.

Let us finish this section with some comments which naturally come up from our work. First, we would like to extend Proposition 2.2.17 (and hence Mustaţă's Conjecture) to any general set Z of s, $s \geq P_X(d)$, distinct points on any smooth surface $X \subset \mathbb{P}^3$ of degree $d \geq 4$ or on any smooth hypersurface $X \subset \mathbb{P}^n$ of degree $d \geq 2$. Second, we would like to extend Theorem 2.3.18 and prove that WMRC holds for infinitely many hypersurfaces (and even more complete intersection varieties) in $\mathbb{P}^n$. Proposition 2.3.15 nicely generalizes and we have

Proposition 2.3.20. *For any $s \geq \binom{d+n}{n} - 1$, there exists a $\binom{d+n-1}{n-1}$-dimensional family of irreducible generically smooth hypersurfaces $X \subset \mathbb{P}^n$ of degree d and a non-empty open subset $U^s \subset \text{Hilb}^s(X)$ such that any $[Z] \in U^s$ satisfies the CMC conjecture.*

Nevertheless, in order to generalize Proposition 2.3.13 from a surface in $\mathbb{P}^3$ to a hypersurface in $\mathbb{P}^n$, we need first to face up Fröberg Conjecture for ideals $I \subset k[x_0, x_1, \cdots, x_n]$, $n \geq 3$, generated by general forms $f_1, \cdots, f_r$ of degree $d_1, \ldots, d_r$, which as far as we know is still an open problem.

Moreover a generalization of Propositions 2.3.13 and 2.3.15 will determine the beginning and the end of the minimal graded free resolution but not the full resolution because the projective dimension of the coordinate ring R_Z increases and we do not have control on the intermediate graded Betti numbers.

Chapter 3

Reducibility of the Hilbert scheme of points for singular surfaces

Given a quasi-projective variety X over a field k and a polynomial $p \in \mathbb{Q}[t]$, the Hilbert scheme $\mathrm{Hilb}^{p(t)}(X)$ parameterizes projective subschemes of X with Hilbert polynomial p. Its existence was shown by Grothendieck in [Gro], where he also showed that if X is projective then $\mathrm{Hilb}^{p(t)}(X)$ turns out also to be a projective scheme. Once its existence is known, a natural problem is the understanding of their geometrical properties: irreducibility, smoothness, dimension, and so on. The first result in this direction was achieved when it was shown that $\mathrm{Hilb}^{p(t)}(X)$ is always connected (cf. [Har66, Corollary 5.9]). In this short chapter, we discuss the irreducibility in the particular interesting case when the Hilbert polynomial is equal to a constant s. In this case, $\mathrm{Hilb}^{s}(X)$ parameterizes 0-dimensional subschemes of X of length s. In [Fog68], Fogarty proved that, if X is a smooth irreducible surface, then the Hilbert scheme $\mathrm{Hilb}^{s}(X)$ is a smooth irreducible variety of dimension $2s$. In larger dimension, Iarrobino in [Iar72] found that irreducibility is no longer true: the Hilbert scheme can be reducible for varieties of dimension ≥ 3. More precisely, Iarrobino proved that if X is a smooth projective variety of dimension $n > 2$, the Hilbert scheme $\mathrm{Hilb}^{s}(X)$ is reducible for all $s > c_0(n)$ and we may take $c_0(3) = 102$, $c_0(4) = 25$, $c_0(5) = 35$ and $c_0(n) = (1+n)(1+\frac{n}{4})$ if $n \geq 6$. They left open the case of singular surfaces. Therefore, a natural question that arises in this setting is the behavior of the Hilbert scheme of 0-dimensional schemes when the smoothness condition is removed: is the Hilbert scheme of 0-dimensional schemes on a singular surface irreducible?

In this short chapter we are going to give a negative answer to this question, by constructing singular surfaces whose Hilbert scheme of points is reducible. In fact, our method also works for varieties of larger dimension. More concretely, we are going to show that for any pair of positive integers (d, n) with $n > 2$ and $d > 1$ or $n = 2$ and $d > 4$ there always exist generically smooth projective varieties $X \subset \mathbb{P}^N$ of dimension n and degree d and an integer s_0 such that $\mathrm{Hilb}^s(X)$ is reducible for all $s \geq s_0$. $X \subset \mathbb{P}^N$ is going to be a projective cone over a projective variety $Y \subset \mathbb{P}^{N-1}$.

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3.1 Reducibility of some punctual Hilbert schemes

We are first going to show that for any n-dimensional cone $X \subset \mathbb{P}^N$ over a projective variety $Y \subset \mathbb{P}^{N-1}$ with vertex the single point $[0, \cdots, 0, 1] \in \mathbb{P}^N$ with $n > 2$ and $\deg X > 1$ or $n = 2$ and $\deg X > 4$ there exists an integer $s_0 \in \mathbb{N}$ such that the Hilbert scheme of points $\mathrm{Hilb}^{s_0}(X)$ parameterizing zero-dimensional subschemes of X of length s_0 is reducible. We will then deduce that the same is true for all $s \geq s_0$.

Notation 3.1.1. Let us fix some notation: $X = \langle Y, p \rangle \subseteq \mathbb{P}^N$ is a projective cone with vertex $p \in \mathbb{P}^N$ and base any $n-1$- dimensional projective variety $Y \subseteq \mathbb{P}^{N-1}$. We can always suppose that $p = [0 : \cdots : 0 : 1]$ and $Y \subseteq \{x_N = 0\}$. If $I_X \subset k[x_0, \ldots, x_N]$ is the homogeneous ideal of X and $I_Y \subset k[x_0, \ldots, x_{N-1}]$ is the homogeneous ideal of Y, then we have $I_X = I_Y k[x_0, \ldots, x_N]$. Analogously, if we denote by $P_X(r)$ (resp. $H_X(r)$) the Hilbert polynomial (resp. the Hilbert function) of X, then the Hilbert polynomial (resp. the Hilbert function) of Y is given by $P_Y(r) = \Delta P_X(r) := P_X(r) - P_X(r-1)$ (resp. $H_Y(r) = \Delta H_X(r) := H_X(r) - H_X(r-1)$).

Remark 3.1.2. There always exists an irreducible component of $\mathrm{Hilb}^s(X)$ parameterizing sets of s distinct points on X. Its dimension is ns. Therefore it will be enough to find an irreducible variety of dimension $> ns$ parameterizing zero-dimensional subschemes of length s on X. The way to obtain it will be through the following construction.

Let us take an integer $r > \text{reg}(X) = \text{reg}(Y)$. In particular, we know that I_X and I_Y are generated by homogeneous forms of degree $< r$. Let t be an integer such that

$$P_X(r) < s := P_X(r) + t < P_X(r+1)$$

and $\mu_{r+1} := P_Y(r+1) - t > n$. Since $\dim(R_Y)_{r+1} = P_Y(r+1) > t$ we can consider the Grassmannian $\text{Grass}(P_Y(r+1) - t, P_Y(r+1))$ which parameterizes vector subspaces $V_{r+1} \subset (R_Y)_{r+1}$ of dimension $P_Y(r+1) - t$. Let $F_1, \cdots, F_{\mu_{r+1}} \in V_{r+1}$ be a basis of V_{r+1} and define the ideal

$$J_0 := I_Y + \langle F_1, \cdots, F_{\mu_{r+1}} \rangle \subset k[x_0, \ldots, x_{N-1}].$$

Set $\mu_{r+2} := \dim(k[x_0, \ldots, x_{N-1}]/J_0)_{r+2}$ (notice that it may be equal to zero). Take $G_1, \cdots, G_{\mu_{r+2}}$ a basis of $(k[x_0, \ldots, x_{N-1}]/J_0)_{r+2}$ and define

$$J := J_0 + \langle G_1, \cdots, G_{\mu_{r+2}} \rangle \subset k[x_0, \ldots, x_{N-1}].$$

By our choices J is an Artinian ideal which contains I_Y and it has Hilbert function

$$H_{R/J}(\ell) = \begin{cases} H_Y(\ell) & \text{if } \ell < r+1 \\ t & \text{if } \ell = r+1 \\ 0 & \text{if } \ell > r. \end{cases}$$

Therefore, $I := Jk[x_0, \ldots, x_N] \subset k[x_0, \ldots, x_N]$ is a saturated ideal which defines a 0-dimensional scheme Z of length s, supported on the vertex p and contained in X.

Since the dimension of $\text{Grass}(P_Y(r+1)-t, P_Y(r+1))$ is $t(P_Y(r+1)-t)$, we have constructed an irreducible family of dimension $t(P_Y(r+1) - t)$ of 0-dimensional schemes of X of length s parameterized by $\text{Grass}(P_Y(r+1) - t, P_Y(r+1))$.

Proposition 3.1.3. *With the above notation, if* $\dim X > 2$ *and* $\deg X > 1$ *or* $\dim X = 2$ *and* $\deg X > 4$*, then there exists* s_0 *such that* $\text{Hilb}^{s_0}(X)$ *is reducible.*

Proof. As was remarked above, since for any integer s there always exists a component of the Hilbert scheme $\text{Hilb}^s(X)$ of dimension ns it will suffice to find an irreducible family parameterizing zero dimensional schemes of length s_0 of dimension larger than ns_0. In order to do that, following the previous notation, we are going to show that for r large enough there exists $t \in (0, P_Y(r+1) - n)$ such that

$$sn = (t + P_X(r))n < t(P_Y(r+1) - t),$$

or, equivalently, that the quadratic polynomial $Q(t) := t^2+(n-P_Y(r+1))t+nP_X(r)$ has a negative integer solution.

The minimum of Q is reached at $t_0 = (P_Y(r+1)-n)/2 \in (0, P_Y(r+1)-n)$. So it will suffice to show that the discriminant

$$\Delta(Q) = n^2 + P_Y(r+1)^2 - 2nP_Y(r+1) - 4nP_X(r)$$

is an increasing function of r. If $\dim X > 2$ then $\deg P_Y(r+1)^2 = 2(n-1)$ is larger than $\deg P_X(r) = n$ and therefore asymptotically $\Delta(Q)$ gets positive.

On the other hand, if $\dim X = 2$, $\Delta(Q)$ can be written as follows:

$$\begin{aligned}
\Delta(Q) &= 4 + P_Y(r+1)^2 - 4P_Y(r+1) - 8P_X(r) \\
&= 4 + (P_X(r+1) - P_X(r))^2 - 4(P_X(r+1) - P_X(r)) - 8P_X(r) \\
&= 4 + P_X(r+1)^2 + P_X(r)^2 - 2P_X(r+1)P_X(r) - 4(P_X(r+1) + P_X(r)) \\
&= \deg X(\deg X - 4)r^2 + (\deg X^2 + 2b\deg X - 4\deg X - 8b)r + \\
&\quad (4 + \tfrac{\deg X^2}{4} + b^2 + b\deg X - 2\deg X - 4b - 8c)
\end{aligned}$$

where we have written $P_X(r) := \frac{\deg X}{2}r^2 + br + c$. Therefore if $\deg X > 4$ asymptotically we get the result. □

Remark 3.1.4. Notice that in the case of $n = 2$ and $\deg X = 4$, the previous result would depend on the Hilbert polynomial of X, since we would have $\Delta(Q) = b^2 - 8c$.

Now we can state the main result of this chapter:

Theorem 3.1.5. *Let $X = \langle Y, p\rangle \subseteq \mathbb{P}^N$ be an n-dimensional cone with vertex p and base $Y \subseteq \mathbb{P}^{N-1}$. Let us suppose that either $n > 2$ and $\deg X > 1$ or $n = 2$ and $\deg X > 4$. Then there exists $s_0 \in \mathbb{N}$ such that $\mathrm{Hilb}^s(X)$ is reducible for all $s \geq s_0$.*

Proof. It is immediate from the previous result once one notices that for any projective variety X of dimension n and for any integer $s \in \mathbb{N}$ the following inequality holds:

$$\mathrm{Hilb}^{s+1}(X) \geq \mathrm{Hilb}^s(X) + n.$$

□

Chapter 4

Ulrich bundles and varieties of wild representation type

A possible way to classify ACM varieties is according to the complexity of the category of ACM sheaves that they support. Much attention has attired this point of view recently and it has even been related to analogous problems in Commutative Algebra and Representation Theory. One of the main achievements in this field has been the classification of the simplest varieties, namely, those that only support a finite number of ACM sheaves (cf. [BGS87] and [EH88]). It turns out that they fall into a very short list (cf. Theorem 4.2.9).

On the other extreme of complexity lie those varieties that have "very large" families of ACM sheaves. In [CHa], a definition, based in analogous definitions coming from Representation Theory, of the meaning of "large" has been proposed. It would correspond to varieties of *wild representation type*, meaning that there exist r-dimensional families of non-isomorphic indecomposable ACM vector bundles for arbitrary large r. Ever since this classification has been proposed, it becomes a problem to find out varieties that fall into this category. In dimension one, where any sheaf is trivially ACM, it has been proved that an ACM curve has wild representation type if and only if its genus is larger or equal than two. For larger dimension, it was proved in [CHb, Theorem 1.1] that smooth cubic surfaces are of wild representation type. In [PLT09, Theorem 5.5], it was shown that del Pezzo surfaces of degree ≤ 6 are of wild representation type. Nevertheless, up to now no example of variety of wild representation type and dimension $n > 2$ was known.

The main goal of this chapter is to provide the first examples of n-dimensional varieties of wild representation type, for arbitrary $n \geq 2$ (cf. Theorems 4.3.13 and 4.4.11) . Our source of examples will be Fano blow-ups $X = Bl_Z\mathbb{P}^n$ of $\mathbb{P}^n$ at a finite set of points Z.

In the 2-dimensional case, i.e., for del Pezzo surfaces, much more information is obtained. In fact, the vector bundles that we construct share another particular feature: the associated module $\oplus_t \mathrm{H}^0(X, \mathcal{F}(t))$ has the maximal possible number of generators (see Theorem 4.4.11). This property was isolated by Ulrich in [Ulr84, p. 26] for Cohen-Macaulay modules, and since then modules with this property have been called Ulrich modules and correspondingly Ulrich vector bundles in the geometric case.

This chapter is divided as follows: in section 4.1 we recall the definition and main features of the varieties we are going to be interested in, namely *Fano* blow-up varieties of $\mathbb{P}^n$, $n \geq 2$, and *del Pezzo* surfaces. In section 4.2, we give an account of ACM vector bundles, Ulrich vector bundles, as well it is also discussed the problem of studying the complexity of an ACM variety according the complexity of families of ACM vector bundles that it supports.

In section 4.3, we perform the construction of large families of simple (hence indecomposable) ACM vector bundles on all Fano blow ups of points in $\mathbb{P}^n$. These families are constructed as the pullback of the kernel of surjective morphisms

$$\mathcal{O}_{\mathbb{P}^n}(1)^b \longrightarrow \mathcal{O}_{\mathbb{P}^n}(2)^a$$

with the property that they are also surjective at the level of global sections. Therefore we are able to prove that Fano blow-ups are varieties of wild representation type (cf. Theorem 4.3.13).

In section 4.4, we focus our attention on the 2-dimensional case, namely on del Pezzo surfaces, where much more information is obtained. In the first subsection we deal with any del Pezzo surface excluding the case of a quadric surface and we see that the ACM vector bundles that we obtained in the previous section by pullback are simple, Ulrich, and μ-stable with respect to a certain ample divisor H_n (cf. Theorem 4.4.11). In the intermediate subsection we focus our attention on the quadric surface and we show by an *ad hoc* method that it is a variety of wild representation type. Finally, in the last subsection, we establish, for a del Pezzo surface X with very ample anticanonical divisor, a version of the well-known Serre correspondence (cf. Theorem 4.4.21). This correspondence will allow us, on one hand, to show, when X is distinct of the quadric surface, that the families of rank r vector bundles constructed in the first subsection could also be obtained from a general set of $m(r) := \frac{d}{2}r^2 + r\frac{2-d}{2}$ distinct points on the surface with minimal free resolution as in Theorem 2.2.13. On the other hand, for the quadric surface, we will apply Serre correspondence in the reverse sense to obtain the minimal free resolution of a general set of $m(r)$ distinct points from the Ulrich vector bundles constructed in the previous subsection.

Finally, section 4.5 is devoted to the case of a *general* surface X of arbitrary degree d in $\mathbb{P}^3$. We manage to show that, for $4 \leq d \leq 9$, a general surface $X \subseteq \mathbb{P}^3$ of degree d is of wild representation type (see Theorem 4.5.8). In the case of arbitrary degree d, we will be able at least to construct large families of rank 2 and 3 simple ACM vector bundles on a general surface $X \subseteq \mathbb{P}^3$ of degree d, showing that they are not of tame representation type (see Propositions 4.5.10 and 4.5.11). We are going to conclude giving a general strategy that could be useful to prove that a general surface of arbitrary degree is of wild representation type (see Theorem 4.5.14).

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• Miró-Roig, R.M and Pons-Llopis, J., *N-dimensional fano varieties of wild representation type*, submitted.

4.1 Fano varieties and del Pezzo surfaces

The first section of this chapter will be devoted to introduce the kind of varieties that will be the subject of our research.

Definition 4.1.1. (cf. [Kol04, Chapter III, Definition 3.1]). A *Fano variety* is defined to be a smooth n-variety X whose anticanonical divisor $-K_X$ is ample. Its degree is defined as K_X^n. If $-K_X$ is very ample, X will be called a *strong Fano variety*.

Remark 4.1.2 (Serre's duality for Fano varieties)**.** Let X be an n-dimensional Fano variety with ample anticanonical divisor $H_X := -K_X$. Given a vector bundle $\mathcal{E}$ on X, Serre's duality states:

$$\mathrm{H}^i(X, \mathcal{E}) \cong \mathrm{H}^{n-i}(X, \mathcal{E}^\vee(-H_X))^\vee,$$

for $i = 0, \cdots, n$.

In the following theorem we summarize the well-known results about the Picard group of the blow-up of $\mathbb{P}^n$ at s points and the intersection product of blow-ups needed in the sequel.

Theorem 4.1.3. *(cf. [Har77, Chapter V, Proposition 4.8]). Let $\{p_1, \ldots, p_s\}$ be a set of s distinct points in $\mathbb{P}^n$ and let $\pi : X \to \mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$ at these points. Let $e_0 \in \mathrm{Pic}(X)$ be the pull-back of a hyperplane in $\mathbb{P}^n$, let e_i be the exceptional divisors (i.e., $\pi(e_i) = p_i$). Then:*

(i) $\mathrm{Pic}(X) \cong \mathbb{Z}^{s+1}$*, generated by $e_0, e_1, \ldots, e_s$.*

(ii) The canonical class is $K_X = -(n+1)e_0 + (n-1)\sum_{i=1}^{s} e_i$.

(iii) If $D \sim ae_0 - \sum_{i=1}^{s} b_i e_i$, then $\chi(\mathcal{O}_X(D)) = \binom{a+n}{n} - \sum_{i=1}^{s} \binom{b_i+n-1}{n}$.

(iv) When $n = 2$, the intersection pairing on the surface X is given by $e_0^2 = 1$, $e_i^2 = -1$, $e_0.e_i = 0$ and $e_i.e_j = 0$ for $i \neq j$.

In the particular two-dimensional case, Fano surfaces are called *del Pezzo surfaces*. Indeed, we have:

Definition 4.1.4. (cf. [Kol04, Chapter III, Definition 3.1]). A *del Pezzo surface* is defined to be a smooth surface X whose anticanonical divisor $-K_X$ is ample. If $-K_X$ is very ample, X will be called a *strong del Pezzo variety*.

The classification of del Pezzo surfaces is classical. Let us recall it here:

Definition 4.1.5. (cf. [Dem80]) A set of s different points $\{p_1, \ldots, p_s\}$ on $\mathbb{P}^2$ with $s \leq 8$ is in *general position* if no three of them lie on a line, no six of them lie on a conic and no eight of them lie on a cubic with a singularity at one of these points.

Theorem 4.1.6. *(cf. [Man86, Chapter IV, Theorems 24.3 and 24.4] and [Dol, Prop. 8.1.9.]). Let X be a del Pezzo surface of degree d. Then $1 \leq d \leq 9$ and*

(i) If $d = 9$, then X is isomorphic to $\mathbb{P}^2$ (and $-K_{\mathbb{P}^2} = 3H_{\mathbb{P}^2}$ gives the usual Veronese embedding in $\mathbb{P}^9$).

(ii) If $d = 8$, then X is isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$ or to a blow-up of $\mathbb{P}^2$ at one point.

(iii) If $7 \geq d \geq 1$, then X is isomorphic to a blow-up of $9-d$ points in general position.

Conversely, any surface described under $(i), (ii), (iii)$ is a del Pezzo surface of the corresponding degree.

Lemma 4.1.7. *(cf. [Kol04, Prop. 3.4]) Let X be the blow-up of $\mathbb{P}^2$ on $0 \leq s \leq 8$ points in general position and let K_X be the canonical divisor. Then:*

(i) If $s \leq 6$, $-K_X$ *is very ample and its global sections yield a closed embedding of* X *in a projective space of dimension*

$$\dim \mathrm{H}^0(X, \mathcal{O}_X(-K_X)) - 1 = K_X^2 = 9 - s.$$

(ii) If $s = 7$, $-K_X$ *is ample and generated by its global sections.*

(iii) if $s = 8$, $-K_X$ *is ample and* $-2K_X$ *is generated by its global sections.*

In the case of dimension $n \geq 3$ we are not allowed to blow up more than one point in $\mathbb{P}^n$ in order to obtain a Fano variety. Indeed, we have:

Theorem 4.1.8. *Let* Z *be a set of* s *distinct points in* $\mathbb{P}^n$, $n \geq 3$, *and let*

$$X := Bl_Z\mathbb{P}^n \longrightarrow \mathbb{P}^n$$

be its blow-up. Then X *is Fano if and only if* $s \leq 1$. *Moreover, in this case* $-K_X$ *is very ample.*

Proof. The fact that the blow-up of $\mathbb{P}^n$ at more than one point is not Fano is an immediate consequence of [Bon02, Theorem 1]. On the other hand, it is obvious that $\mathcal{O}_{\mathbb{P}^n}(n+1)$ is very ample. So let $X = Bl_p\mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$ at one single point p. Its anticanonical divisor is $(n+1)e_0 - (n-1)e_1$, which can be written as $(n-2)(e_0 - e_1) + (3e_0 - e_1)$. Since $e_0 - e_1$ is clearly base-point free and $3e_0 - e_1$ is very ample (which can be proven directly or appealing to the stronger result [Cop02, Theorem 1]) we are done. □

4.2 ACM and Ulrich sheaves

The aim of this section is to provide an account of known results on ACM and Ulrich sheaves (resp. vector bundles) and on the classification of ACM varieties according to the complexity of the families of ACM vector bundles that they support.

Definition 4.2.1. Let $(X, \mathcal{O}_X(1))$ be a polarized scheme. A coherent sheaf $\mathcal{E}$ on X is *Arithmetically Cohen-Macaulay* (ACM for short) if it is locally Cohen-Macaulay (i.e., depth $\mathcal{E}_x = \dim \mathcal{O}_{X,x}$ for every point $x \in X$) and has no intermediate cohomology:

$$\mathrm{H}^i_*(X, \mathcal{E}) := \oplus_{t\in\mathbb{Z}} \mathrm{H}^i(X, \mathcal{E}(t)) = 0 \qquad \text{for all } i = 1, \dots, \dim X - 1.$$

When X is smooth, locally Cohen-Macaulay sheaves are locally free:

Lemma 4.2.2. *Let $(X, \mathcal{O}_X(1))$ be a smooth polarized variety. Then any ACM sheaf $\mathcal{E}$ is locally free.*

Proof. Let $x \in X$. Then by hypothesis, depth $\mathcal{E}_x = \dim \mathcal{O}_{X,x}$. But now, since $\mathcal{O}_{X,x}$ is a regular local ring, the projective dimension of $\mathcal{E}_x$ is finite and we can apply the Auslander-Buchsbaum formula (see [BH93, Theorem 1.3.3])

$$\operatorname{pd} \mathcal{E}_x = \operatorname{depth} \mathcal{O}_{X,x} - \operatorname{depth} \mathcal{E}_x$$

to see that $\operatorname{pd} \mathcal{E}_x = 0$, i.e, $\mathcal{E}_x$ is free. □

From the algebraic point of view there also exist analogous definitions:

Definition 4.2.3. A graded R_X-module E is a *maximal Cohen-Macaulay* module (MCM for short) if $\operatorname{depth} E = \dim E = \dim R_X$.

On an ACM scheme, both definitions are closely related:

Proposition 4.2.4. *(cf. [CH04, Proposition 2.1]) Let $X \subseteq \mathbb{P}^n$ be an ACM scheme. There exists a bijection between ACM sheaves $\mathcal{E}$ on X and MCM R_X-modules E given by the functors $E \to \widetilde{E}$ and $\mathcal{E} \to \mathrm{H}^0_*(X, \mathcal{E})$.*

Proof. First of all, recall that, given a graded finitely generated R_X-module M there exist an exact sequence

$$0 \longrightarrow \mathrm{H}^0_{\mathfrak{m}_X}(M) \longrightarrow M \longrightarrow \mathrm{H}^0_*(X, \widetilde{M}) \longrightarrow \mathrm{H}^1_{\mathfrak{m}_X}(M) \longrightarrow 0,$$

and isomorphisms $\mathrm{H}^i_*(X, \widetilde{M}) \cong \mathrm{H}^{i+1}_{\mathfrak{m}_X}(M)$ for $i \geq 1$, where $\mathfrak{m}_X$ denotes the irrelevant ideal of R_X.

So let $\mathcal{E}$ be an ACM sheaf on X. Then $E := \mathrm{H}^0_*(X, \mathcal{E})$ will be a finitely generated R_X-module that verifies $E \cong \mathrm{H}^0_*(X, \widetilde{E})$ and therefore $\mathrm{H}^i_{\mathfrak{m}_X}(E) = 0$ for $i \leq \dim X = \dim R_X - 1$. This allows us to conclude that E is MCM by the local cohomological criterion of depth.

On the other hand, let E be a MCM R_X-module. Then $\mathcal{E} := \widetilde{E}$ will be a locally Cohen-Macaulay sheaf and again by the previous isomorphisms we will have $\mathrm{H}^i_*(X, \mathcal{E}) = 0$ for $i = 1, \dots, \dim X - 1$. □

This dictionary between modules and sheaves have been deeply exploited to translate results from the algebraic to the geometric side and *vice versa*.

Example 4.2.5. Let us consider the particular case of ACM line bundles. When $\mathrm{Pic}(X) \cong \mathbb{Z}$ there is not too much to say, so as an interesting example let us consider del Pezzo surfaces (cf. [PLT09, Theorems 4.1.5 and 4.2.2]). Let $X \subseteq \mathbb{P}^d$ be a strong del Pezzo surface of degree d embedded through the very ample divisor $-K_X$. Then a line bundle $\mathcal{L}$ on X is initialized and ACM if and only if either $\mathcal{L} \cong \mathcal{O}_X$ or $\mathcal{L} \cong \mathcal{O}_X(D)$ for a rational normal curve $D \subseteq X \subseteq \mathbb{P}^d$ of degree less or equal than d. This allows us to give an explicit list of them.

(i) Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ be a quadric surface and denote by l_1, l_2 the standard basis of $\mathrm{Pic}(X)$. Then there exist exactly (up to twist and isomorphism) 8 initialized ACM line bundles. The initialized ones are given by $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}$ and, in terms of their associated class of divisors,

$$D = l_1 + bl_2 \text{ or } D = bl_1 + l_2 \text{ with } 0 \leq b \leq 3 \,(\deg D = 2 + 2b).$$

(ii) Let X be a blow-up of r general points on $\mathbb{P}^2$, with $0 \leq r \leq 6$. Then, with respect to the very ample $-K_X$, the initialized ACM divisors of X are 0, the exceptional divisors and, up to permutation of the exceptional divisors, the ones listed below:

$\deg D$	D	
$3 - m$	$e_0 - e_1 - \cdots - e_m$	$0 \leq m \leq \min\{2, r\}$
$6 - m$	$2e_0 - e_1 - \cdots - e_m$	$\max\{r-3, 0\} \leq m \leq \min\{5, r\}$
$8 - m$	$3e_0 - 2e_1 - e_2 \cdots - e_m$	$\max\{1, r-1\} \leq m \leq r$
$9 - r$	$4e_0 - 2e_1 - 2e_2 - 2e_3 - e_4 \cdots - e_r$	$r \geq 3$
3	$5e_0 - 2e_1 - 2e_2 - 2e_3 - 2e_4 - 2e_5 - 2e_6$	$r = 6$

A possible way to classify ACM varieties is according to the complexity of the families of ACM *indecomposable* vector bundles that they support. The first result on that direction was Horrocks theorem, that tells us that projective spaces $\mathbb{P}^n$ are, in this sense, as simple as possible:

Theorem 4.2.6. *(cf. [OSS80, Theorem 2.3.1]) A vector bundle $\mathcal{E}$ on $\mathbb{P}^n$, $n \geq 1$, splits into a sum of line bundles if and only if $\mathcal{E}$ is ACM. In other words, the only ACM indecomposable vector bundle (up to isomorphism and twist) on $\mathbb{P}^n$ is $\mathcal{O}_{\mathbb{P}^n}$.*

In the case of a smooth quadric hypersurface $Q^n \subseteq \mathbb{P}^{n+1}$ an analogous result also holds; namely, by [Knö87, Corollary 2.8] and [Buc87, Proposition 3.1], besides $\mathcal{O}_{Q^n}$ there is just one indecomposable ACM vector bundle if n is odd, or two if n is even. The rank of these vector bundles is 2^m where $m = \lfloor \frac{n-1}{2} \rfloor$. They are the so-called *Spinor bundles*.

This result led to the following definition:

Definition 4.2.7. An ACM scheme $X \subseteq \mathbb{P}^n$ is of *finite representation type* if it has, up to twist and isomorphism, only a finite number of indecomposable ACM sheaves.

Remark 4.2.8. Auslander proved that if a graded Cohen-Macaulay R supports only a finite number of MCM modules up to isomorphism and degree shift then R is an isolated singularity (cf. [Yos90, Theorem 4.22]). This means, from the geometric point of view, that an ACM variety with finite number of ACM sheaves is smooth. Therefore, by Lemma 4.2.2, the family of ACM vector bundles would coincide with the family of ACM sheaves.

Varieties of finite representation type have been classified. In fact, they fall into a very short list:

Theorem 4.2.9. *(cf. [BGS87, Theorem C] and [EH88, p. 348]) Let $X \subseteq \mathbb{P}^n$ be an ACM variety of finite representation type. Then X is either three or less reduced points on $\mathbb{P}^2$, a projective space, a smooth quadric hypersurface $X \subset \mathbb{P}^n$, a cubic scroll in $\mathbb{P}^4$, the Veronese surface in $\mathbb{P}^5$ or a rational normal curve.*

In order to study the complexity of ACM varieties in terms of the ACM vector bundles that they support, the following definitions were proposed (cf. [DG01] for the case of curves and [CHa] for the higher dimensional case):

Definition 4.2.10. An ACM scheme $X \subseteq \mathbb{P}^n$ is of *tame representation type* if for each rank r, the indecomposable ACM sheaves of rank r form a finite number of families of dimension at most one. On the other hand, X will be of *wild representation type* if there exist l-dimensional families of non-isomorphic indecomposable ACM sheaves for arbitrary large l.

Remark 4.2.11. We can not expect that the trichotomy finite, tame and wild will be exhaustive in general. For instance, in [CH04, Proposition 6.1], it was proved that a quadric cone supports a countable number of ACM sheaves of rank 2.

Example 4.2.12. Let $C \subseteq \mathbb{P}^n$ be a smooth ACM curve of genus g. In this case, the definition of ACM is trivially satisfied for any vector bundle on C. Then, we have the following:

(i) If $g = 0$, we saw in Theorem 4.2.9 that a rational normal curve is of finite representation type.

(ii) If $g = 1$, the classification of vector bundles on elliptic curves by Atiyah, see [Ati57, Theorems 7 and 10], shows that they are of tame representation type.

(iii) Finally, if $g \geq 2$, by the results proved in [DG01, Theorem 1.6], C turns out to be of wild representation type.

Example 4.2.13. As for the case of hypersurfaces $X \subseteq \mathbb{P}^n$ of degree ≥ 3, $n > 1$, Buchweitz, Greuel and Schreyer proved in [BGS87, Theorem C] that they are not of finite representation type by constructing a surjective map from the set of isomorphism classes of MCM modules over R_X either to $\mathbb{P}^n$ (if $\deg X = 3$) or to a cubic hypersurface in $\mathbb{P}^n$ (if $\deg X > 3$). Nevertheless, it remains open to prove in general that they are of wild representation type. In sections 4.4 and 4.5 we are going to prove that a general surface $X \subseteq \mathbb{P}^3$ of degree $3 \leq d \leq 9$ is of wild representation type.

In the particular case of del Pezzo surfaces, the ACM vector bundles that we construct on them will share a very strong property: they will have the maximum possible number of global sections. The algebraic counterpart was studied by Ulrich in [Ulr84], where he showed that there is a bound on the number of generators of MCM modules depending on the multiplicity and the rank. Ever since, modules (and sheaves) achieving this bound have been called *Ulrich* after him.

Definition 4.2.14. Given a polarized n-dimensional integral scheme $(X, \mathcal{O}_X(1))$, an ACM sheaf $\mathcal{E}$ of positive rank will be called an *Ulrich sheaf* if $\mathrm{h}^0(\mathcal{E}_{init}) = \deg(X)\,\mathrm{rk}(\mathcal{E})$.

When $\mathcal{O}_X(1)$ is very ample, we have the following result that justifies this definition:

Theorem 4.2.15. *Let $X \subseteq \mathbb{P}^n$ be an integral subscheme and $\mathcal{E}$ be an ACM sheaf on X of positive rank. Then the minimal number of generators $m(\mathcal{E})$ of the R_X-module $\mathrm{H}^0_*(\mathcal{E})$ is bounded by*

$$m(\mathcal{E}) \leq \deg(X)\,\mathrm{rk}(\mathcal{E}).$$

Proof. See [CHa, Theorem 3.1]. □

Therefore, since it is obvious that for an initialized sheaf $\mathcal{E}$, $\mathrm{h}^0(\mathcal{E}) \leq m(\mathcal{E})$, the minimal number of generators of Ulrich sheaves is as large as possible. Modules attaining this upper bound were studied by Ulrich in [Ulr84]. A complete account is provided in [ESW03]. In particular, we have:

Theorem 4.2.16. *(cf. [ESW03, Proposition 2.1.]) Let $X \subseteq \mathbb{P}^N$ be an n-dimensional ACM variety and $\mathcal{E}$ be an initialized ACM coherent sheaf on X. The following conditions are equivalent:*

(i) $\mathcal{E}$ *is Ulrich.*

(ii) $\mathcal{E}$ *admits a linear $\mathcal{O}_{\mathbb{P}^N}$-resolution of the form:*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^N}(-N+n)^{a_{N-n}} \longrightarrow \ldots \longrightarrow \mathcal{O}_{\mathbb{P}^N}(-1)^{a_1} \longrightarrow \mathcal{O}_{\mathbb{P}^N}^{a_0} \longrightarrow \mathcal{E} \longrightarrow 0.$$

where $a_0 = \deg(X)\,\mathrm{rk}(\mathcal{E})$ *and* $a_i = \binom{N-n}{i} a_0$ *for all* i.

(iii) $\mathrm{H}^i(\mathcal{E}(-i)) = 0$ *for* $i > 0$ *and* $\mathrm{H}^i(\mathcal{E}(-i-1)) = 0$ *for* $i < n$.

(iv) For some (resp. all) finite linear projections $\pi : X \longrightarrow \mathbb{P}^n$, the sheaf $\pi_*\mathcal{E}$ is the trivial sheaf $\mathcal{O}_{\mathbb{P}^n}^t$ for some t.

In particular, initialized Ulrich sheaves are 0*-regular and therefore they are globally generated.*

Remark 4.2.17. From the previous theorem is immediate to see that an initialized Ulrich sheaf $\mathcal{E}$ has Euler characteristic of the form

$$\chi(\mathcal{E}(t)) = \mathrm{h}^0(\mathcal{E})\binom{t+n}{n}.$$

The existence of Ulrich vector bundles on a given variety is a nontrivial problem in general. Let us give some examples:

Example 4.2.18.
(i) For a del Pezzo surface $X \subseteq \mathbb{P}^d$, among the ACM initialized line bundles listed on Example 4.2.5, the Ulrich ones correspond to those of the form $\mathcal{O}_X(D)$ for D a rational normal curve of maximal degree d.

(ii) In [ESW03, Corollary 6.5] the existence of rank 2 Ulrich vector bundles on arbitrary del Pezzo surfaces was established by Eisenbud, Schreyer and Weyman using elementary transformations.

(iii) On smooth cubic surfaces and threefolds, the existence of Ulrich vector bundles of arbitrary rank has been proved by Casanellas and Hartshorne in [CHb].

(iv) The existence of Ulrich vector bundles on any algebraic curve was settled in [ESW03, Corollary 4.5].

4.3 n-dimensional case

The aim of this section will be to exhibit (as far as we know) the first examples of n-dimensional varieties of wild representation type for arbitrary $n \geq 3$ (for surfaces, some examples were already known. See, for instance, [CHb, Theorem 1.1] and [PLT09, Theorem 5.1.5]). More precisely, in this section we will construct large families of ACM vector bundles on Fano varieties of the form $X = Bl_Z\mathbb{P}^n$ for Z a finite set of s distinct points on $\mathbb{P}^n$. Recall from section 4.1 that in order to X being Fano, we should assume that either $n = 2$ and Z is a set of up to 8 points in general position or $n \geq 3$ and $s = 0, 1$.

In the following section we are going to provide a proof of the fact that strong del Pezzo surfaces are ACM (cf. Theorem 4.4.1). We are going to prove now that the analogous result turns out to be true for Fano blow-ups of $\mathbb{P}^n$, $n \geq 3$. In fact, we have

Proposition 4.3.1. *Let $X = Bl_Z\mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$, $n \geq 3$, at $s \leq 1$ points and let us consider its embedding in $\mathbb{P}^N_k$ through the very ample divisor $-K_X$. Then $X \subseteq \mathbb{P}^N_k$ is an ACM variety.*

Proof. Since it is well-known that Veronese embeddings are ACM, we can suppose that X is the blow-up of $\mathbb{P}^n$ at one single point. First of all, we are going to see that $\mathrm{H}^i_*(X, \mathcal{O}_X) = 0$ for $i = 1, \dots, n-1$. To start with, notice that $\mathrm{H}^i(X, \mathcal{O}_X) = \mathrm{H}^i(\mathbb{P}^n, \pi_*(\mathcal{O}_X)) = \mathrm{H}^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 0$ for $i = 1, \dots, n-1$. On the other hand, by Lemmas 4.3.8 and 4.3.9, we have $\mathrm{H}^i(X, \mathcal{O}_X(t)) = 0$ for $t > 0$ and $i = 1, \dots, n-1$. Finally, the vanishing of $\mathrm{H}^i(X, \mathcal{O}_X(t)) = 0$ for $t < 0$ is obtained by Serre's duality. So it would only remain to prove that $\mathrm{H}^1_*(\mathcal{I}_{X|\mathbb{P}^d}) = 0$, but this is immediate from the fact that $\mathrm{H}^1(X, \mathcal{I}_{X|\mathbb{P}^d}) = 0$ and that $\mathcal{I}_{X|\mathbb{P}^d}$ is 1-regular. □

The notion of regularity with respect to a very ample line bundle is very classical. For our purposes, we will need to work in a slightly wider setting:

Definition 4.3.2. (cf. [Laz04, Definition 1.8.4]) Let X be a projective variety and B an ample line bundle generated by its global sections. A coherent sheaf $\mathcal{F}$ on X is *m-regular with respect to B* if

$$\mathrm{H}^i(X, \mathcal{F} \otimes B^{(m-i)}) = 0 \text{ for } i > 0.$$

Theorem 4.3.3. *(cf. [Laz04, Theorem 1.8.5]) Let X be a projective variety and B an ample line bundle generated by its global sections. Let $\mathcal{F}$ be an m-regular sheaf on X with respect to B. Then for every $k \geq 0$:*

(i) $\mathcal{F} \otimes B^{(m+k)}$ *is generated by its global sections.*

(ii) $\mathcal{F}$ *is* $(m+k)$*-regular with respect to* B*.*

The ACM vector bundles $\mathcal{E}$ on X will be obtained as the kernel of certain surjective morphisms between $\mathcal{O}_X(e_0)^b$ and $\mathcal{O}_X(2e_0)^a$. Following notation from [EH92], let us consider k-vector spaces A and B of respective dimension a and b. Set $V = \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ and let $M = \mathrm{Hom}(B, A \otimes V)$ be the space of $(a \times b)$-matrices of linear forms. It is well-known that there exists a bijection between the elements $m \in M$ and the morphisms $m : B \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow A \otimes \mathcal{O}_{\mathbb{P}^n}(1)$. Taking the tensor with $\mathcal{O}_{\mathbb{P}^n}(1)$ and considering global sections, we have morphisms $\mathrm{H}^0(m(1)) : \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n}(1)^b) \longrightarrow \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n}(2)^a)$. The following result tells us under which conditions the aforementioned morphisms m and $\mathrm{H}^0(m(1))$ are surjective:

Proposition 4.3.4. *([EH92, Proposition 4.1]) For* $a \geq 1$, $b \geq a+n$ *and* $2b \geq (n+2)a$, *the set of elements* $m \in M$ *such that* m *and* $\mathrm{H}^0(m(1))$ *are surjective forms a non-empty open dense subset.*

For a given $r \geq n$, let us fix now the possible ranks a and b, depending on the parity of $n \geq 2$, that we are going to deal with. If n is even, fix $c \in \{0, \dots, n/2-1\}$ such that $c \equiv r \bmod n/2$, set $u := \frac{2(r-c)}{n}$ and also define:

$$a := u \quad \text{and} \quad b := \frac{n+2}{2}u + c. \tag{4.3.1}$$

If n is odd, fix $c \in \{0, \dots, n-1\}$ such that $c \equiv r \bmod n$, set $u := \frac{(r-c)}{n}$ and also define:

$$a := 2u \quad \text{and} \quad b := (n+2)u + c. \tag{4.3.2}$$

Notice that these values verify the conditions of Proposition 4.3.4. So take an element m of the non-empty open and dense subset $U \subseteq M$ provided by Proposition 4.3.4 and consider the exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^b \stackrel{m(1)}{\longrightarrow} \mathcal{O}_{\mathbb{P}^n}(2)^a \longrightarrow 0. \tag{4.3.3}$$

It is immediate to see that $\mathcal{F}$, as a kernel of a surjective morphism of vector bundles, is a vector bundle of rank $r = b - a$. Let us consider now a finite set of s distinct points $Z \subseteq \mathbb{P}^n$ and the blow-up associated to these points

$$X := Bl_Z(\mathbb{P}^n) \stackrel{\pi}{\longrightarrow} \mathbb{P}^n.$$

Pulling-back the exact sequence (4.3.3) we obtain the exact sequence:

$$0 \longrightarrow \pi^*\mathcal{F} \longrightarrow \mathcal{O}_X(e_0)^b \xrightarrow{m(1)} \mathcal{O}_X(2e_0)^a \longrightarrow 0. \tag{4.3.4}$$

The first goal will be to show that $\mathcal{G} := \pi^*\mathcal{F}$ is simple and therefore indecomposable. In order to do that, we are going to argue with the dual exact sequence

$$0 \longrightarrow \mathcal{O}_X(-2e_0)^a \xrightarrow{m(1)^\vee} \mathcal{O}_X(-e_0)^b \longrightarrow \mathcal{G}^\vee \longrightarrow 0. \tag{4.3.5}$$

Notice that the morphism $f := m(1)^\vee : \mathcal{O}_X(-2e_0)^a \longrightarrow \mathcal{O}_X(-e_0)^b$ appearing in the exact sequence (4.3.5) is a general element of the k-vector space

$$M := \operatorname{Hom}(\mathcal{O}_X(-2e_0)^a, \mathcal{O}_X(-e_0)^b) \cong k^{n+1} \otimes k^a \otimes k^b$$

because

$$\operatorname{Hom}(\mathcal{O}_X(-2e_0), \mathcal{O}_X(-e_0)) \cong \mathrm{H}^0(\mathcal{O}_X(e_0)) \cong \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n}(1)) \cong k^{n+1}.$$

In other words, f can be represented by a $b \times a$ matrix A whose entries are elements of $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n}(1))$, i.e., linear forms. Since $\operatorname{Aut}(\mathcal{O}_X(-e_0)^b) \cong GL(b)$ and $\operatorname{Aut}(\mathcal{O}_X(-2e_0)^a) \cong GL(a)$, the group $GL(b) \times GL(a)$ acts naturally on M by

$$\begin{array}{ccc} GL(b) \times GL(a) \times M & \longrightarrow & M \\ (g_1, g_2, A) & \mapsto & g_1^{-1} A g_2. \end{array}$$

Moreover, for all $A \in M$ and $\lambda \in k^*$, $(\lambda\,\mathrm{id}_b, \lambda\,\mathrm{id}_a)$ belongs to the stabilizer of A. Hence $\dim_k Stab(A) \geq 1$. By [Kac80, Theorem 4], we have:

Proposition 4.3.5. *Let $M = k^{n+1} \otimes k^a \otimes k^b$ be endowed with the natural action of $GL(b) \times GL(a)$. If $a^2 + b^2 - (n+1)ab \leq 1$ then, for a general element $A \in M$,*

$$\dim_k Stab(A) = 1.$$

The previous Proposition will allow us to conclude that the general vector bundle given by the exact sequence (4.3.5)

$$0 \longrightarrow \mathcal{O}_X(-2e_0)^a \xrightarrow{f} \mathcal{O}_X(-e_0)^b \xrightarrow{g} \mathcal{E} := \mathcal{G}^\vee \longrightarrow 0 \tag{4.3.6}$$

is simple and, hence, indecomposable. More precisely, we have the following proposition which will be the key result for proving that all Fano blow ups of $\mathbb{P}^n$ at a finite number of points are of wild representation type.

Proposition 4.3.6. *Let $X = Bl_Z\mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$ at a finite set of points, $n \geq 2$ and let $r \geq n$. If n is even, fix $c \in \{0, \dots, n/2 - 1\}$ such that $c \equiv r$ mod $n/2$, set $u := \frac{2(r-c)}{n}$ and define:*

$$a := u \quad \text{and} \quad b := \frac{n+2}{2}u + c. \tag{4.3.7}$$

If n is odd, fix $c \in \{0, \dots, n-1\}$ such that $c \equiv r$ mod n, set $u := \frac{(r-c)}{n}$ and define:

$$a := 2u \quad \text{and} \quad b := (n+2)u + c. \tag{4.3.8}$$

Let $\mathcal{F}$ be the vector bundle obtained as the kernel of a general surjective morphism between $\mathcal{O}_{\mathbb{P}^n}(1)^b$ and $\mathcal{O}_{\mathbb{P}^n}(2)^a$:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^b \stackrel{m(1)}{\longrightarrow} \mathcal{O}_{\mathbb{P}^n}(2)^a \longrightarrow 0. \tag{4.3.9}$$

Then, the vector bundles $\mathcal{E}$ from pulling-back and dualizing $\mathcal{F}$

$$0 \longrightarrow \mathcal{O}_X(-2e_0)^a \stackrel{f}{\longrightarrow} \mathcal{O}_X(-e_0)^b \stackrel{g}{\longrightarrow} \mathcal{E} := (\pi^*\mathcal{F})^\vee \longrightarrow 0 \tag{4.3.10}$$

are simple.

Proof. First of all, notice that the values of a and b appearing in the statement of this Proposition verify the inequality of Proposition 4.3.5. Let A be the element from M associated to the morphism f. We will now check that $\mathcal{E}$ is simple. Otherwise, there exists a non-trivial morphism $\phi : \mathcal{E} \longrightarrow \mathcal{E}$. Then we get a morphism $\overline{\phi} = \phi \circ g : \mathcal{O}_X(-e_0)^b \longrightarrow \mathcal{E}$. Applying $\mathrm{Hom}(\mathcal{O}_X(-e_0)^b, -)$ to the exact sequence (4.3.10) and taking into account that

$$\mathrm{Hom}(\mathcal{O}_X(-e_0)^b, \mathcal{O}_X(-2e_0)^a) = \mathrm{Ext}^1(\mathcal{O}_X(-e_0)^b, \mathcal{O}_X(-2e_0)^a) = 0$$

we get

$$\mathrm{Hom}(\mathcal{O}_X(-e_0)^b, \mathcal{O}_X(-e_0)^b) \cong \mathrm{Hom}(\mathcal{O}_X(-e_0)^b, \mathcal{E}).$$

Hence there is a non-trivial morphism $\widetilde{\phi} \in \mathrm{Hom}(\mathcal{O}_X(-e_0)^b, \mathcal{O}_X(-e_0)^b)$ induced by $\overline{\phi}$ and represented by a matrix $B \neq \mu\,\mathrm{id} \in \mathrm{Mat}_{b\times b}(k)$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_X(-2e_0)^a & \stackrel{f}{\longrightarrow} & \mathcal{O}_X(-e_0)^b & \stackrel{g}{\longrightarrow} & \mathcal{E} & \longrightarrow & 0 \\
 & & \Big\downarrow C & & \Big\downarrow B & \stackrel{\overline{\phi}}{\searrow} & \Big\downarrow \phi & & \\
0 & \longrightarrow & \mathcal{O}_X(-2e_0)^a & \stackrel{f}{\longrightarrow} & \mathcal{O}_X(-e_0)^b & \stackrel{g}{\longrightarrow} & \mathcal{E} & \longrightarrow & 0
\end{array}$$

where $C \in \mathrm{Mat}_{a\times a}(k)$ is the matrix associated to $\widetilde{\phi}_{|\mathcal{O}_X(-2e_0)^a}$. Then the pair $(C, B) \neq (\mu\,\mathrm{id}, \mu\,\mathrm{id})$ verifies $AC = BA$. Let us consider an element $\alpha \in k$ that does not belong to the set of eigenvalues of B and C. Then the pair $(B-\alpha\,\mathrm{id}, C-\alpha\,\mathrm{id}) \in GL(b)\times GL(a)$ belongs to $Stab(f)$ and therefore $\dim_k Stab(f) > 1$ which produces the desired contradiction with Proposition 4.3.5. □

At this point we want to show that the isomorphism class of a vector bundle $\mathcal{E}$ associated to an element $m \in U \subseteq M$ depends only on the orbit of m under the action of $GL(b) \times GL(a)$:

Lemma 4.3.7. *Given two matrices m and m' from the open set $U \subseteq M$ provided by Proposition 4.3.4, the associated vector bundles $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic if and only if m and m' belong to the same orbit under $GL(b) \times GL(a)$.*

Proof. Let us suppose that there exists an isomorphism $\phi : \mathcal{E} \longrightarrow \mathcal{E}'$. This isomorphism lifts to a morphism of resolutions:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_X(-2e_0)^a & \xrightarrow{m} & \mathcal{O}_X(-e_0)^b & \xrightarrow{g} & \mathcal{E} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle C} & & \downarrow{\scriptstyle B} & & \downarrow{\scriptstyle \phi} & & \\
0 & \longrightarrow & \mathcal{O}_X(-2e_0)^a & \xrightarrow{m'} & \mathcal{O}_X(-e_0)^b & \xrightarrow{g'} & \mathcal{E}' & \longrightarrow & 0.
\end{array}$$

with $B \in \mathrm{Mat}_{b\times b}(k)$ and $C \in \mathrm{Mat}_{a\times a}(k)$. Now, in order to show that B belongs to $GL(b)$ (and therefore C belongs to $GL(a)$) we only need to compose this morphism of the resolutions with a morphism of resolutions lifting $\phi^{(-1)}$ and observe that any two morphisms between resolutions are homotopically equivalent and that there is no nonzero morphism between $\mathcal{O}_X(-e_0)$ and $\mathcal{O}_X(-2e_0)$. □

Once the simplicity has been proved, the next goal will be to show that the vector bundles $\mathcal{E}$ given by the exact sequence (4.3.6) are ACM. Since the proof in the surface case is slightly different and moreover one obtains a much stronger result, we postpone the discussion of this case until the next section and for the rest of the current one we only deal with varieties of dimension $n \geq 3$. So, let $Z = \{p_1, \ldots, p_s\} \subseteq \mathbb{P}^n$ be a set of s distinct points on $\mathbb{P}^n$ and let $X = Bl_Z\mathbb{P}^n$ be the blow-up at this set. It is a well-known fact that for any integers $d_1, \cdots, d_s \geq 0$ there is an isomorphism of cohomology groups $\mathrm{H}^i(X, \mathcal{O}_X(ce_0 - \sum_{t=1}^{s} d_t e_t)) \cong \mathrm{H}^i(\mathbb{P}^n, \mathcal{I}_W(c))$ where $\mathcal{I}_W$ is the ideal sheaf of the *fat point* scheme $W = \sum_{t=1}^{s} d_t p_t$ (which is defined locally at p_i by the ideal $\mathcal{I}_{p_i}^{d_i}$ where $\mathcal{I}_{p_i}$ is the maximal ideal of the local ring $\mathcal{O}_{\mathbb{P}^n,p_i}$). This equivalence supplies us with the following standard result:

Lemma 4.3.8. *Let $X = Bl_Z\mathbb{P}^n$ be a blow-up at a set Z of s points in $\mathbb{P}^n$. Assume that one of the following conditions holds:*

(i) $2 \le i \le n-1$, $c \in \mathbb{Z}$ *and* $d_t \ge 0$ *for all* t*;*

(ii) $i = 0$ *and* $c < 0$*; or*

(iii) $i = n$, $c \ge -n$.

Then, we have $\mathrm{H}^i(X, \mathcal{O}_X(ce_0 - \sum_{t=1}^s d_t e_t)) = 0$.

Proof. (i) It is straightforward from the previous remark and the long exact cohomology sequence associated to

$$0 \longrightarrow \mathcal{I}_W(c) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(c) \longrightarrow \mathcal{O}_W(c) \longrightarrow 0.$$

(ii) If $c < 0$, the divisor $ce_0 - \sum_{t=1}^s d_t e_t$ is not effective.
(iii) It follows from (ii) and Serre's duality. □

When we blow-up just one point we get easily the vanishing of some H^1 groups needed later:

Lemma 4.3.9. *Let $X = Bl_Z\mathbb{P}^n$ be a blow-up at $s = 0, 1$ points in $\mathbb{P}^n$. If either $d = 0$ or $c \ge d > 0$ then $\mathrm{H}^1(X, \mathcal{O}_X(ce_0 - de_1)) = 0$.*

Proof. If $d = 0$, then $\mathrm{H}^1(X, \mathcal{O}_X(ce_0)) = \mathrm{H}^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(c)) = 0$. On the other hand, if $c \ge d > 0$, then a single point p of multiplicity d imposes $\binom{d+n-1}{n}$ independent conditions on hypersurfaces of degree c and therefore from the exact sequence

$$0 \longrightarrow \mathcal{I}_{dp}(c) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(c) \longrightarrow \mathcal{O}_{dp}(c) \longrightarrow 0$$

follows immediately the vanishing of $\mathrm{H}^1(X, \mathcal{O}_X(ce_0 - de_1)) = 0$. □

Lemma 4.3.10. *Let $X = Bl_Z\mathbb{P}^n$ be a blow-up at a set Z of $s \le 1$ points in $\mathbb{P}^n$ and let*

$$\mathcal{O}_X(e_0)^b \xrightarrow{f} \mathcal{O}_X(2e_0)^a$$

be a morphism. Suppose that f is surjective on global sections. Then for any $t \ge 1$, the induced morphism $\mathcal{O}_X(e_0 + tH)^b \xrightarrow{f_t} \mathcal{O}_X(2e_0 + tH)^a$ is also surjective on global sections.

Proof. If $s = 0$ the result is obvious. Assume $s = 1$, i.e. $Z = \{p\}$. Let us assume that the morphism

$$\mathrm{H}^0(X, \mathcal{O}_X(e_0)^b) \xrightarrow{\mathrm{H}^0(f)} \mathrm{H}^0(X, \mathcal{O}_X(2e_0)^a)$$

is surjective. Then, after taking the tensor product with $\mathrm{H}^0(\mathcal{O}_X(tH))$, the induced morphism

$$\mathrm{H}^0(f) \otimes id : (\mathrm{H}^0(\mathcal{O}_X(e_0)) \otimes \mathrm{H}^0(\mathcal{O}_X(tH)))^b \cong \mathrm{H}^0(\mathcal{O}_X(e_0)^b) \otimes \mathrm{H}^0(\mathcal{O}_X(tH)) \longrightarrow$$

$$\longrightarrow \mathrm{H}^0(\mathcal{O}_X(2e_0)^a) \otimes \mathrm{H}^0(\mathcal{O}_X(tH)) \cong (\mathrm{H}^0(\mathcal{O}_X(2e_0)) \otimes \mathrm{H}^0(\mathcal{O}_X(tH)))^a$$

is still surjective. For $t \geq 1$, let us consider the following commutative diagram:

$$\begin{array}{ccc} (\mathrm{H}^0(X, \mathcal{O}_X(e_0)) \otimes \mathrm{H}^0(\mathcal{O}_X(tH)))^b & \twoheadrightarrow & (\mathrm{H}^0(X, \mathcal{O}_X(2e_0)) \otimes \mathrm{H}^0(\mathcal{O}_X(tH)))^a \\ \downarrow & & \downarrow \\ (\mathrm{H}^0(X, \mathcal{O}_X(e_0 + tH)))^b & \xrightarrow{\mathrm{H}^0(f_t)} & (\mathrm{H}^0(X, \mathcal{O}_X(2e_0 + tH)))^a. \end{array}$$

So, in order to conclude the result, it will be enough to prove that the right vertical arrow on the previous diagram is surjective. From the above discussion, the previous diagram is equivalent to the following one:

$$\begin{array}{ccc} (\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{I}_W(t(n+1))))^b & \twoheadrightarrow & (\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n}(2)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{I}_W(t(n+1))))^a \\ \downarrow & & \downarrow \\ (\mathrm{H}^0(\mathbb{P}^n, \mathcal{I}_W(t(n+1)+1)))^b & \xrightarrow{\mathrm{H}^0(f_t)} & (\mathrm{H}^0(\mathbb{P}^n, \mathcal{I}_W(t(n+1)+2)))^a. \end{array}$$

where $\mathcal{I}_W$ is the ideal sheaf of the fat point $W = t(n-1)p$. But, since by Lemmas 4.3.8 and 4.3.9, we have $\mathrm{H}^i(\mathcal{I}_W(t(n+1) - 2i)) = 0$ for $1 \leq i \leq n$, $\mathcal{I}_W(t(n+1))$ is 0-regular with respect to $\mathcal{O}_{\mathbb{P}^n}(2)$ and therefore the standard properties of regularity assure us that the right vertical arrow is surjective. □

Now we are ready to prove that the vector bundles that come out from the exact sequence (4.3.10) are ACM in the case of dimension $n \geq 3$.

Proposition 4.3.11. *Let $X = Bl_Z\mathbb{P}^n$ be a blow-up of $\mathbb{P}^n$, $n \geq 3$, on $s = 0, 1$ points. Let $H := -K_X$ be the ample anticanonical divisor. Then the vector bundle $\mathcal{E}$ given by the exact sequence (4.3.10) is ACM with respect to H.*

Proof. Let us consider the following pieces of the long exact cohomology sequence associated to the exact sequence (4.3.10):

$$\dots \longrightarrow \mathrm{H}^i(\mathcal{O}_X(tH-e_0)^b) \longrightarrow \mathrm{H}^i(\mathcal{E}(tH)) \longrightarrow \mathrm{H}^{i+1}(\mathcal{O}_X(tH-2e_0)^a) \longrightarrow \dots$$

for $1 \leq i \leq n-1$, and

$$\dots \longrightarrow \mathrm{H}^n(\mathcal{O}_X(tH-e_0)^b) \longrightarrow \mathrm{H}^n(\mathcal{E}(tH)) \longrightarrow 0.$$

Applying Lemma 4.3.8 we have that $\mathrm{H}^i(\mathcal{E}(tH)) = 0$ for all $i \geq 2$ and $t \geq 0$. On the other hand, for $t \geq 0$, by Lemma 4.3.9 we also get $\mathrm{H}^1(\mathcal{E}(tH)) = 0$ for all $t \geq 0$. To see the remaining vanishings, let us consider the dual exact sequence (4.3.4)

$$0 \longrightarrow \mathcal{E}^\vee \longrightarrow \mathcal{O}_X(e_0)^b \xrightarrow{m(1)} \mathcal{O}_X(2e_0)^a \longrightarrow 0. \tag{4.3.11}$$

Once again let us consider the following pieces of the long exact cohomology sequence

$$\dots \longrightarrow \mathrm{H}^{i-1}(\mathcal{O}_X(tH+2e_0)^a) \longrightarrow \mathrm{H}^i(\mathcal{E}^\vee(tH)) \longrightarrow \mathrm{H}^i(\mathcal{O}_X(tH+e_0)^b) \longrightarrow \dots$$

for $i \geq 2$ and $t \geq 0$. A new application of Lemmas 4.3.8 and 4.3.9 proves that $\mathrm{H}^j(\mathcal{E}(sH)) = \mathrm{H}^{n-j}(\mathcal{E}^\vee((-s-1)H)) = 0$ for $0 \leq j \leq n-2$ and $s \leq -1$. It only remains to show that $\mathrm{H}^{n-1}(\mathcal{E}(tH)) = \mathrm{H}^1(\mathcal{E}^\vee((-t-1)H)) = 0$ for $t < 0$. Notice that by Lemma 4.3.9 $\mathrm{H}^1(e_0 + tH) = 0$ for all $t \geq 0$. Since we were in the case in which $\mathrm{H}^0(m(1))$ were surjective, by Lemma 4.3.10, we have that, in the exact sequence

$$0 \longrightarrow \mathrm{H}^0(\mathcal{E}^\vee(tH)) \longrightarrow \mathrm{H}^0(\mathcal{O}_X(e_0+tH)^b) \xrightarrow{f_t}$$

$$\longrightarrow \mathrm{H}^0(\mathcal{O}_X(2e_0+tH)^a) \longrightarrow \mathrm{H}^1(\mathcal{E}^\vee(tH)) \longrightarrow 0,$$

f_t is surjective for all $t \geq 0$ and therefore we can conclude that $\mathrm{H}^1\ \mathcal{E}^\vee(tH) = 0$ for all $t \geq 0$ which proves what we want. □

Remark 4.3.12. Notice that the vector bundle $\mathcal{E}$ of Proposition 4.3.11 is not Ulrich, since $\mathcal{E}_{init} = \mathcal{E}(H)$ does not have the maximal number of global sections. For instance, when $r = n > 2$ and $s = 0$, following notation from Proposition 4.3.6, we have $a = 2$ and $b = n+2$. Then:

$$\mathrm{h}^0(\mathcal{E}(H)) = (n+2)\binom{2n}{n} - 2\binom{2n-1}{n} = (2n+2)\binom{2n-1}{n} < n(n+1)^n = \mathrm{rk}(\mathcal{E}(H))\deg(X).$$

On the other hand, it will be the case for the vector bundles that we are going to construct on del Pezzo surfaces, as it will be shown in the next section.

We conclude the section gathering the previous results:

Theorem 4.3.13. *Let $X = Bl_Z\mathbb{P}^n$ be a Fano blow-up of points in $\mathbb{P}^n$, $n \geq 3$ and let $r \geq n$.*

(i) If n is even, fix $c \in \{0, \dots, n/2-1\}$ such that $c \equiv r \bmod n/2$ and set $u := \frac{2(r-c)}{n}$. Then there exists a family of rank r simple (hence, indecomposable) ACM vector bundles of dimension $\frac{(n+2)n-4}{4}u^2 - cu - c^2 + 1$.

(ii) If n is odd, fix $c \in \{0, \dots, n-1\}$ such that $c \equiv r \bmod n$ and set $u := \frac{(r-c)}{n}$. Then there exists a family of rank r simple (hence, indecomposable) ACM vector bundles of dimension $((n+2)n-4)u^2 - 2cu - c^2 + 1$.

In particular, Fano blow-ups are varieties of wild representation type.

Proof. Let $X = Bl_Z\mathbb{P}^n$ be a Fano blow-up of $\mathbb{P}^n$, $n \geq 3$. For $r \geq n$, let a and b be natural numbers defined as in (4.3.7) and (4.3.8) (depending on the parity of n) and let A and B k-vector spaces of dimension respectively a and b. We saw in Propositions 4.3.6 and 4.3.11 that there exists a non-empty open and dense subset U of the vector space $M = \mathrm{Hom}(B, A \otimes \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n}(1)))$ such that $m \in U$ provides with a simple ACM vector bundle $\mathcal{E}$ on X of rank r. Since it was proved in Lemma 4.3.7 that these vector bundles are in bijection with the orbits of U under the action of $GL(b) \times GL(a)$ the dimension of the family can be computed as

$$\dim M - \dim \mathrm{Aut}(\mathcal{O}_{\mathbb{P}^n}(1)^b) - \dim \mathrm{Aut}(\mathcal{O}_{\mathbb{P}^n}(2)^a) + 1 = ab(n+1) - a^2 - b^2 + 1,$$

which gives the announced result. Notice that this dimension depends quadratically on u and hence on r. □

Remember that for dimension $n \geq 3$, we are allowed to blow-up a set Z of $s = 0, 1$ points on $\mathbb{P}^n$ in order to get a Fano variety $X = Bl_Z\mathbb{P}^n$. Notice that the meaning of a blow-up of $s = 0$ points is just a change of polarization of $\mathbb{P}^n$, i.e. now we are considering $\mathbb{P}^n$ with the very ample anticanonical divisor $(n+1)H$. Therefore, in this setting the question about the representation type could be stated in the following terms:

Question 4.3.14. *Given a couple of integers (n, r), determine the representation type of the Veronese variety*

$$\nu_{n,r} : \mathbb{P}^n \longrightarrow \mathbb{P}^{\binom{n+r}{n}-1}$$

defined as the image of $\mathbb{P}^n$ by the very ample divisor $rH_{\mathbb{P}^n}$.

From Theorem 4.3.13 it follows that the Veronese varieties $\nu_{n,n+1}(\mathbb{P}^n) \subseteq \mathbb{P}^{\binom{2n+1}{n}-1}$ are of wild representation type for $n \geq 2$. On the other hand, we saw in Theorem 4.2.9 that the only Veronese varieties of finite representation type are given for the couples $(n,r) = (2,2)$ and $(n,r) = (1,r)$ for arbitrary $r \geq 1$. Moreover, it is proved in [ESW03, Corollary 5.7] that any Veronese variety is the support of at least an Ulrich sheaf.

4.4 Del Pezzo surfaces

In this section we focus our attention on 2-dimensional Fano varieties X, i.e., on del Pezzo surfaces. In this case much more information will be obtained. Recall that del Pezzo surfaces are either blow-ups of $\mathbb{P}^2$ on $s \leq 8$ points in general position or the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$. H will stand for the ample anticanonical divisor $-K_X$ on X. The aim is twofold: firstly, we are going to give an alternative construction of the family of ACM vector bundles on X that will come with some extra information about the stability of the vector bundles. Secondly, we are going to show that these ACM vector bundles share a much stronger property: they are Ulrich vector bundles. The quadric case will have to be treated apart and therefore a separated subsection will be devoted to it. In the final subsection, we are going to establish a version of the well-known Serre correspondence, which in particular will allow us to prove that the Ulrich vector bundles that have just been constructed on strong del Pezzo surfaces (except in the quadric case) could be obtained from finite general set of points on X verifying the Minimal Resolution Conjecture.

As in the previous section, let us start showing the well-known fact that strong del Pezzo surfaces are ACM:

Theorem 4.4.1. *(cf. [Dem80, Exposé V, Théorème 1]) Let X be a strong del Pezzo surface of degree d and let us consider its embedding in $\mathbb{P}^d$ through the very ample divisor $-K_X$. Then $X \subseteq \mathbb{P}^d$ is an ACM variety.*

Proof. We are going to prove that $H^1_*(\mathcal{O}_X) = 0$ and $H^1_*(\mathcal{I}_X) = 0$. Let us define $H := -K_X$. Since $H^2 = d$ and H is very ample, by the adjunction formula and by [Har77, Chapter II, Theorem 8.18] we obtain that H is a smooth elliptic curve. In particular, since $K_H \sim 0$, from duality we obtain

$$h^1(\mathcal{O}_H(m)) = h^0(\mathcal{O}_H(-m)) = 0 \text{ for } m > 0.$$

Since X is rational, we can apply Castelnuovo's criterion to conclude that $\mathrm{H}^1(\mathcal{O}_X) = 0$. Next, from the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_H \longrightarrow 0$$

twisting by $m \geq 1$ and taking cohomology

$$\mathrm{H}^1(\mathcal{O}_X(m-1)) \longrightarrow \mathrm{H}^1(\mathcal{O}_X(m)) \longrightarrow \mathrm{H}^1(\mathcal{O}_H(m)) = 0,$$

we obtain that $\mathrm{H}^1(\mathcal{O}_X(m)) = 0$ for any $m \geq 0$. Since

$$\mathrm{H}^1(\mathcal{O}_X(m)) \cong \mathrm{H}^1(\mathcal{O}_X(-m-1)),$$

the vanishing holds for all m.

It remains to prove that $\mathrm{H}^1_*(\mathcal{I}_X) = 0$. Let us consider the exact sequence

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_{\mathbb{P}^d} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Since $\mathrm{H}^2(\mathcal{O}_X(2-2)) \cong \mathrm{H}^0(\mathcal{O}_X(-1)) = 0$, $\mathcal{O}_X$ is 2-regular. Being $\mathcal{O}_{\mathbb{P}^d}$ 3-regular, we have that $\mathcal{I}_X$ is 3-regular and so $\mathrm{H}^1(\mathcal{I}_X(m)) = 0$ for $m \geq 2$. Clearly this also holds for $m \leq 0$. Finally $\mathrm{H}^1(\mathcal{I}_X(1)) = 0$ since X is embedded through the complete linear system $|-K_X|$. □

4.4.1 Construction of Ulrich vector bundles

The aim of this subsection is to recover, for any $r \geq 2$, the r^2+1-dimensional family of rank r ACM vector bundles $\mathcal{E}$ on del Pezzo surfaces X starting from rank r, μ-stable vector bundles $\mathcal{H}$ on $\mathbb{P}^2$ with Chern classes $c_1(\mathcal{H}) = 0$ and $c_2(\mathcal{H}) = r$. The method used in this subsection will not allow us to treat the quadric surface, which will be the subject of the next subsection. Therefore, for the rest of this subsection when we speak of a del Pezzo surface we will be excluding the quadric case.

As was mentioned in Theorem 1.3.4 of chapter 1, given an n-dimensional projective variety X and an ample line bundle H, there exists a coarse moduli space $M^s_{X,H}(r; c_1, \ldots, c_{\min(r,n)})$ parameterizing isomorphism classes of μ-stable rank r vector bundles on X with Chern classes $c_1, \ldots, c_{\min(r,n)}$. However, it is in general a very deep problem to show that $M^s_{X,H}(r; c_1, \ldots, c_{\min(r,n)})$ is non-empty. Nevertheless, for the case of the projective plane, we can prove the existence of rank r, μ-stable vector bundles $\mathcal{H}$ on $\mathbb{P}^2$ with Chern classes $c_1(\mathcal{H}) = 0$ and $c_2(\mathcal{H}) = r$. In order to show that the moduli space $M^s_{\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(1)}(r; 0, r)$ is non-empty we will use the following adapted result from [DL85]. Recall that the *discriminant* of a vector bundle $\mathcal{E}$ on $\mathbb{P}^2$ of rank r and Chern classes c_1, c_2 is defined as:

$$\Delta(r, c_1, c_2) = \frac{1}{r}(c_2 - \frac{(r-1)}{2r}c_1^2).$$

Theorem 4.4.2. *(cf. [DL85, Théorème B]) A sufficient and necessary condition for the existence of a μ-stable vector bundle of rank r and Chern classes $c_1 \in r\mathbb{Z}$ and $c_2 \in \mathbb{Z}$ on $\mathbb{P}^2$ is that $\Delta(r, c_1, c_2) \geq 1$.*

Corollary 4.4.3. *The moduli space $M^s_{\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(1)}(r;0,r)$ is non-empty.*

Proof. Since in our case $\Delta(r;0,r) = 1$, we are done by the previous Theorem. □

Once we have checked the non-emptiness of $M^s_{\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(1)}(r;0,r)$ we can apply [DM03, Proposition 4.3] to assert that a generic element $\mathcal{H}$ from $M_{\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(1)}(r;0,r)$ will have a resolution of the form

$$0 \longrightarrow \bigoplus^{r} \mathcal{O}_{\mathbb{P}^2}(-2) \longrightarrow \bigoplus^{2r} \mathcal{O}_{\mathbb{P}^2}(-1) \longrightarrow \mathcal{H} \longrightarrow 0. \tag{4.4.1}$$

Since this exact sequence is just the dual of the exact sequence constructed in (4.3.3), we see that we are recovering the family of vector bundles we were dealing with in section 3.

The dimension of this family can be easily computed:

Proposition 4.4.4. *The family of μ-stable vector bundles $\mathcal{H}$ of $M^s_{\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(1)}(r;0,r)$ with locally free resolution (4.4.1) has dimension $r^2 + 1$.*

Proof. By [DM03, Theorem 4.4], $M^s_{\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(1)}(r;0,r)$ is a smooth irreducible variety of dimension r^2+1. Since our family forms a non-empty open dense subset of it, we are done. □

Let us take now a set of $0 \leq s \leq 8$ points $Z = \{p_1, \dots, p_s\}$ in general position and let us consider the surface obtained by blowing up these points jointly with the canonical morphism to $\mathbb{P}^2$,

$$\pi : X = Bl_Z(\mathbb{P}^2) \longrightarrow \mathbb{P}^2.$$

We are following the notation introduced in Theorem 4.1.3. Pulling back the vector bundles given by the exact sequence (4.4.1), we obtain the family of vector bundles from (4.3.11)

$$0 \longrightarrow \bigoplus^{r} \mathcal{O}_X(-2e_0) \xrightarrow{f} \bigoplus^{2r} \mathcal{O}_X(-e_0) \longrightarrow \mathcal{E} := \pi^*\mathcal{H} \longrightarrow 0. \tag{4.4.2}$$

If we twist it by the line bundle $\mathcal{O}_X(H)$ we obtain the following family of rank r vector bundles:

$$0 \longrightarrow \bigoplus^{r} \mathcal{O}_X(-2e_0 + H) \longrightarrow \bigoplus^{2r} \mathcal{O}_X(-e_0 + H) \longrightarrow \mathcal{E}(H) \longrightarrow 0. \qquad (4.4.3)$$

Their Chern classes can easily computed with the formulas given in Remark 1.3.10:

$$c_1(\mathcal{E}(H)) = rH \text{ and } c_2(\mathcal{E}(H)) = \frac{H^2r^2 + (2 - H^2)r}{2}.$$

We saw in Proposition 4.3.6 that these vector bundles were simple. Let us, however, provide an alternative proof that gives a stronger result. We will see that they are μ-stable with respect a certain ample divisor and, therefore, simple. We are going to use the following result:

Theorem 4.4.5. *(cf. [Nak93, Theorem 1]) Let X be a surface, let H be an ample line bundle on X and let $\pi : X' \longrightarrow X$ be the blow up of X at l distinct points p_i and denote the exceptional divisors by e_i. Let us define the divisor $H_n := n\pi^*H - \sum_{i=1}^{l} e_i$. Then for $n \gg 0$ there exists an open immersion*

$$\phi : M^s_{X,H}(r, c_1, c_2) \hookrightarrow M^s_{X',H_n}(r, \pi^*c_1, c_2)$$

defined by $\phi(\mathcal{F}) := \pi^*(\mathcal{F})$ on closed points.

Corollary 4.4.6. *The family of vector bundles on the blow up $\pi : X = Bl_Z(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$ defined by the exact sequence*

$$0 \longrightarrow \bigoplus^{r} \mathcal{O}_X(-2e_0 + H) \longrightarrow \bigoplus^{2r} \mathcal{O}_X(-e_0 + H) \longrightarrow \mathcal{E}(H) \longrightarrow 0$$

is μ-stable with respect the ample divisor $ne_0 - \sum e_i$ for $n \gg 0$. In particular, they are simple, i.e., $\mathrm{Hom}(\mathcal{E}(H), \mathcal{E}(H)) = k$.

The last step will be to show, as in the higher dimensional case, that the vector bundles $\mathcal{E}$ are ACM. In fact, much more will be provided in this case: the twisted $\mathcal{E}(H)$ vector bundles are initialized Ulrich vector bundles. For this, we need the following computations.

Remark 4.4.7 (Riemann-Roch for vector bundles on a del Pezzo surface)**.** Let X be a del Pezzo surface. Since X is a rational connected surface we have $\chi(\mathcal{O}_X) = 1$. In particular, Riemann-Roch formula for a vector bundle $\mathcal{E}$ on X of rank r has the form

$$\chi(\mathcal{E}) = \frac{c_1(\mathcal{E})(c_1(\mathcal{E}) - K_X)}{2} + r - c_2(\mathcal{E}).$$

Remark 4.4.8. The Euler characteristic of the involved vector bundles can be computed thanks to the Riemann-Roch formula:

$$\begin{aligned} \chi(\mathcal{O}_X(-2e_0)(lH)) &= \tfrac{9-s}{2}l^2 - \tfrac{3+s}{2}l, \\ \chi(\mathcal{O}_X(-e_0)(lH)) &= \tfrac{9-s}{2}l^2 + \tfrac{3-s}{2}l, \end{aligned} \tag{4.4.4}$$

and

$$\begin{aligned} \chi(\mathcal{E}(lH)) &= 2r\chi(\mathcal{O}_X(-e_0)(lH)) - r\chi(\mathcal{O}_X(-2e_0)(lH)) \\ &= \tfrac{9r-sr}{2}l^2 + \tfrac{9r-sr}{2}l. \end{aligned} \tag{4.4.5}$$

Proposition 4.4.9. *Let X be a del Pezzo surface. The vector bundles $\mathcal{E}(H)$ given by the exact sequence (4.4.3) are initialized simple Ulrich vector bundles. Moreover, in the case of a blow-up of ≤ 7 points, they are globally generated.*

Proof. First of all, notice that, since μ-stability is preserved under duality, $\mathcal{E}^\vee$ is a μ-stable (with respect to H_n) vector bundle with $c_1(\mathcal{E}^\vee) = 0$, and therefore it does not have global sections: $\mathrm{H}^0(\mathcal{E}^\vee) = \mathrm{H}^2(\mathcal{E}(-H)) = 0$. In particular, $\mathrm{H}^2(\mathcal{E}(tH)) = 0$, for all $t \geq -1$. On the other hand, since $\mathrm{H}^2(\mathcal{O}_X(-2e_0)) = \mathrm{H}^0(\mathcal{O}_X(2e_0 - H)) = 0$ and $\mathrm{h}^1(\mathcal{O}_X(-e_0)) = -\chi(\mathcal{O}_X(-e_0)) = 0$ we obtain from the long exact sequence of cohomology associated to (4.4.2) that $\mathrm{H}^1(\mathcal{E}) = 0$. Since $\chi(\mathcal{E}) = 0$, we also conclude that $\mathrm{H}^0(\mathcal{E}) = 0$ and therefore $\mathrm{H}^0(\mathcal{E}(tH)) = 0$ for all $t \leq 0$. Moreover, since we also have that $\chi(\mathcal{E}(-H)) = 0$, we obtain that $\mathrm{H}^1(\mathcal{E}(-H)) = 0$.

Now, it is well-known that $\mathrm{h}^0(\mathcal{O}_X(H)) = H^2 + 1 > 0$ (see for instance [Kol04, Corollary 3.2.5]) and therefore there exists an exact sequence:

$$0 \longrightarrow \mathcal{O}_X(-H) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_H \longrightarrow 0.$$

If we tensor it by $\mathcal{E}$ and we consider the long exact cohomology sequence associated to it we see that

$$0 = \mathrm{H}^0(\mathcal{E}) \longrightarrow \mathrm{H}^0(\mathcal{E}_{|H}) \longrightarrow \mathrm{H}^1(\mathcal{E}(-H)) = 0.$$

This shows that $\mathrm{H}^0(\mathcal{E}_{|H}(-tH)) = 0$ for all $t \geq 0$. Then we can use this last fact jointly with the long exact cohomology sequence associated to

$$0 \longrightarrow \mathcal{E}(-(t+1)H) \longrightarrow \mathcal{E}(-tH) \longrightarrow \mathcal{E}_{|H}(-tH) \longrightarrow 0$$

to show inductively that $\mathrm{H}^1(\mathcal{E}(-tH)) = 0$ for all $t \geq 0$.

In order to complete the proof we need to consider now two different cases:

- *X is the blow-up of $s \leq 7$ points on $\mathbb{P}^2$ in general position*. In this case, by Lemma 4.1.7, H is ample and generated by its global sections. Since we have just seen that $\mathcal{E}(H)$ is 0-regular with respect to H we can conclude by Theorem 4.3.3 that $\mathcal{E}(H)$ is ACM and globally generated. Moreover, $\mathrm{h}^0(\mathcal{E}(H)) = \chi(\mathcal{E}(H)) = (9-s)r = H^2 r$, i.e., $\mathcal{E}(H)$ is an Ulrich vector bundle.

- *X is the blow-up of* 8 *points on $\mathbb{P}^2$ in general position*. In this case, the argument is slightly more involved, since we can use Theorem 4.3.3 only with respect to $2H$, which is ample and globally generated. First of all, since the points are in general position, $\mathrm{H}^0(\mathcal{O}_X(-e_0+H)) = 0$ and from the exact sequence (4.4.3) we get the following exact sequence:

$$0 \longrightarrow \mathrm{H}^0(\mathcal{E}(H)) \longrightarrow \oplus^r\, \mathrm{H}^1(\mathcal{O}_X(-2e_0+H)) \longrightarrow$$
$$\longrightarrow \oplus^{2r}\, \mathrm{H}^1(\mathcal{O}_X(-e_0+H)) \longrightarrow \mathrm{H}^1(\mathcal{E}(H)) \longrightarrow 0.$$

From this sequence and the fact that

$$\mathrm{h}^1(\mathcal{O}_X(-2e_0+H)) = -\chi(\mathcal{O}_X(-2e_0+H)) = 5$$

and

$$\mathrm{h}^1(\mathcal{O}_X(-e_0+H)) = -\chi(\mathcal{O}_X(-e_0+H)) = 2$$

we are forced to conclude that $\mathrm{h}^0(\mathcal{E}(H)) = r$ and $\mathrm{H}^1(\mathcal{E}(H)) = 0$. Now, from what we have gathered up to now, we can affirm that $\mathcal{E}(H)$ is 1-regular with respect to $2H$ and therefore, by Theorem 4.3.3, $\mathrm{H}^1(\mathcal{E}(H+2tH)) = 0$ for all $t \geq 0$. In order to deal with the cancelation of the remaining groups of cohomology, it will be enough to show that $\mathcal{E}(2H)$ is 1-regular with respect to $2H$, i.e., it remains to show that $\mathrm{H}^1(\mathcal{E}(2H)) = 0$. In order to do this consider the exact sequence (the cancelation of $\mathrm{H}^0(\mathcal{O}_X(-e_0+2H))$ is due to the fact that the points are in general position):

$$0 \longrightarrow \mathrm{H}^0(\mathcal{E}(2H)) \longrightarrow \oplus^r\, \mathrm{H}^1(\mathcal{O}_X(-2e_0+2H)) \longrightarrow$$
$$\longrightarrow \oplus^{2r}\, \mathrm{H}^1(\mathcal{O}_X(-e_0+2H)) \longrightarrow \mathrm{H}^1(\mathcal{E}(2H)) \longrightarrow 0.$$

Once again, we control the dimension of these vector spaces:

$$\mathrm{h}^1(\oplus^r \mathcal{O}_X(-2e_0+2H)) = -r\chi(\mathcal{O}_X(-2e_0+2H)) = 9r$$

and

$$\mathrm{h}^1(\oplus^{2r} \mathcal{O}_X(-e_0+2H)) = -2r\chi(\mathcal{O}_X(-e_0+2H)) = 6r.$$

Therefore we are forced to have $h^0(\mathcal{E}(2H)) = 3r$ and $H^1(\mathcal{E}(2H)) = 0$. Notice that in this case $\mathcal{E}(3H)$ is globally generated.

□

Given a del Pezzo surface X, we have just seen that the vector bundles given by the exact sequence (4.4.3) were μ-stable with respect to the ample divisor $H_n := ne_0 - \sum e_i$ for $n \gg 0$. Unfortunately, the proof did not provide an effective value of n. However we are going to prove at least that they are μ-semistable with respect to the anticanonical divisor $H = H_3 = 3e_0 - \sum e_i$. The main tool will be the classification of vector bundles on elliptic curves performed in [Ati57]:

Proposition 4.4.10. *Let X be a del Pezzo surface of degree d. Then a general vector bundle $\mathcal{E}(H)$ given by the exact sequence (4.4.3) is μ-semistable.*

Proof. We follow the structure of the proof given by the case of the cubic surface in [CHa, Proposition 5.2]. We saw in Proposition 4.4.9 that these vector bundles $\mathcal{E}(H)$ were initialized and Ulrich. Moreover, we know that we can take a smooth elliptic curve H as a representative of the anticanonical divisor class (see for instance [Dem80, III,Theorem 1]). From the exact sequence

$$0 \longrightarrow \mathcal{E}(-H) \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_{|H} \longrightarrow 0$$

is deduced that $\mathcal{E}_{|H}(H)$ is also initialized of degree dr and $h^0(\mathcal{E}_{|H}(H)) = dr$. By [Ati57, Theorem 7], $\mathcal{E}_{|H}(H) = \oplus\mathcal{E}_{r_i,d_i}$ with $\mathcal{E}_{r_i,d_i}$ a vector bundle of rank r_i and degree d_i. Since $h^0(\mathcal{E}_{r_i,d_i}(-H)) = 0$, Atiyah's classification forces that $d_i \leq dr_i$. It follows that we have equality and $\mathcal{E}_{|H}(H)$ decomposes as direct sum of μ-semistable vector bundles of the same slope. Thus, $\mathcal{E}_{|H}(H)$ is also μ-semistable of slope d. Therefore it is straightforward to conclude that $\mathcal{E}(H)$ is also μ-semistable. □

Summing up, we get the following result:

Theorem 4.4.11. *Let X be a del Pezzo surface of degree d. Then for any $r \geq 2$ there exists a family of dimension r^2+1 of simple initialized Ulrich vector bundles of rank r with Chern classes $c_1 = rH$ and $c_2 = \frac{dr^2+(2-d)r}{2}$. Moreover, they are μ-semistable with respect to the polarization $H = 3e_0 - \sum_{i=1}^{9-d} e_i$ and μ-stable with respect to $H_n := (n-3)e_0 + H$ for $n \gg 0$. In particular, del Pezzo surfaces are of wild representation type.*

Remark 4.4.12. Notice that the existence of Ulrich vector bundles on a cubic surface $X \subseteq \mathbb{P}^3$ gives a full answer to question (6.3) raised in [BHU87].

In the following lemma we are going to see that, as in the case of surfaces on $\mathbb{P}^3$, explained to us by Mustopa in private communication, it is possible to give bounds for the second Chern class of an Ulrich vector bundle on a strong del Pezzo surface. In fact, following their argument we have:

Lemma 4.4.13. *Let X be a strong del Pezzo surface of degree d and let $\mathcal{E}$ be an initialized Ulrich vector bundle of rank r. Then $c_1(\mathcal{E})H = dr$, $c_2(\mathcal{E}) = \frac{c_1^2+(2-d)r}{2}$ and*

$$\frac{(d-2)r^2+(2-d)r}{2} \leq c_2(\mathcal{E}) \leq \frac{dr^2+(2-d)r}{2}.$$

Proof. We saw in Remark 4.2.17 that an initialized Ulrich vector bundle $\mathcal{E}$ has Hilbert polynomial $dr\binom{t+2}{2}$. Since this polynomial can be computed, thanks to the Riemann-Roch theorem, in terms of its Chern classes, an easy computation gives us the values of $\deg \mathcal{E} = c_1(\mathcal{E})H$ and of $c_2(\mathcal{E})$. Next, recall that $\mathcal{E}$ was a μ-semistable vector bundle (see Proposition 4.4.10) and therefore we can apply Bogomolov inequality (cf. [HL97, Theorem 3.4.1])

$$2rc_2(\mathcal{E}) \geq (r-1)c_1(\mathcal{E})^2,$$

to obtain the lower bound for c_2. Finally, by the Hodge Index Theorem (cf. [Har77, Theorem 1.9,Chapter V]), it holds that $c_1(\mathcal{E})^2H^2 \leq (c_1(\mathcal{E})H)^2$, and an easy computation provides the upper bound. □

Remark 4.4.14. Therefore, by the previous Lemma, the Ulrich vector bundles constructed in Theorem 4.4.11 are extremal with respect to the second Chern class.

4.4.2 The quadric case

This subsection will be devoted to prove the wildness of the unique strong del Pezzo surface that was not treated in the last subsection, namely, the smooth quadric surface $\mathbb{P}^1 \times \mathbb{P}^1 \cong X \subseteq \mathbb{P}^8$ embedded in $\mathbb{P}^8$ through the very ample anticanonical divisor $H_X := -K_X$. We are going to develop an *ad hoc* argument, following the lines of the general case, to construct large families of simple Ulrich vector bundles on X. Let us start recalling the basic facts about the Picard group of X:

Lemma 4.4.15. *Let $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth quadric surface. Then*

(i) $\mathrm{Pic}(X) \cong \mathbb{Z}^2 \cong < l_1, l_2 >$ *with l_i being the pull-back $\pi_i^*\mathcal{O}_{\mathbb{P}^1}(1)$ through the canonical projections $\pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$, $i = 1, 2$.*

(ii) *The canonical divisor of* X *is* $K_X = -2l_1 - 2l_2$.

(iii) *The intersection product is defined as follows:* $l_i^2 = 0$ *for* $i = 1, 2$, *and* $l_1.l_2 = 1$.

We are going to denote a line bundle of the form $\mathcal{O}_X(al_1 + bl_2)$ by $\mathcal{O}_X(a, b)$. As in the case of the rest of del Pezzo surfaces, our rank r Ulrich vector bundles on the quadric X will be constructed as the kernel of certain surjective morphisms between $\mathcal{O}_X(1,0)^r \oplus \mathcal{O}_X(0,1)^r$ and $\mathcal{O}_X(1,1)^r$. We need the following Proposition, which is analogous to Proposition 4.3.4.

Proposition 4.4.16. *Let* X *be a smooth quadric surface. Then, for* $r \geq 2$, *the set of elements* $m \in M := \mathrm{Hom}(\mathcal{O}_X(1,0)^r \oplus \mathcal{O}_X(0,1)^r, \mathcal{O}_X(1,1)^r)$ *such that* m *and the associated morphism of global sections*

$$\mathrm{H}^0(m) : \mathrm{H}^0(\mathcal{O}_X(1,0))^r \oplus \mathrm{H}^0(\mathcal{O}_X(0,1))^r \longrightarrow \mathrm{H}^0(\mathcal{O}_X(1,1))^r$$

are surjective forms a non-empty open dense subset.

Proof. Recall that for any $a, b \geq 0$ we can identify the global sections $\mathrm{H}^0(\mathcal{O}_X(a,b))$ with bihomogeneous polynomials $f(x,y;u,v) \in k[x,y;u,v]$ of bidegree (a,b). Therefore, a morphism as in the statement will be represented by a $r \times 2r$ matrix

$$C = \left(\begin{array}{c|c} A(x,y) & B(u,v) \end{array} \right)$$

where $A(x,y)$ (resp. $B(u,v)$) is a $r \times r$-matrix of linear forms in variables x, y (resp. u, v). By semicontinuity, it will be enough to prove that the set of morphism with the required properties is nonempty. We are going to supply a particular morphism for the cases $r = 2$, $r = 3$ and $r > 4$ separately.

(i) For $r = 2$, we can consider the matrix

$$C_2 = \left(\begin{array}{cc|cc} x & y & u & v \\ 0 & x & v & u \end{array} \right)$$

for which is straightforward to check the required properties.

(ii) For $r = 3$, we can consider the matrix

$$C_3 = \left(\begin{array}{ccc|ccc} x & 0 & y & u & 0 & 0 \\ 0 & x & 0 & u & 0 & v \\ 0 & 0 & x & 0 & v & u \end{array} \right)$$

(iii) Finally, for $r \geq 4$ we can build up the morphism from the rank 2 and 3 examples. Namely, we consider (up to a column permutation) for $r = 2s$ even,

$$C_r = \mathrm{diag}\,(C_2| \overset{s}{\cdots} |C_2)\,,$$

and for $r = 2s + 1$ odd,

$$C_r = \mathrm{diag}\left(C_2| \overset{s-1}{\cdots} |C_2|C_3\right).$$

□

Let us take an element m of the non-empty open and dense subset $U \subseteq M$ provided by Proposition 4.4.16 and consider the exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X(1,0)^r \oplus \mathcal{O}_X(0,1)^r \xrightarrow{m} \mathcal{O}_X(1,1)^r \longrightarrow 0, \tag{4.4.6}$$

from which it is seen that $\mathcal{F}$, as a kernel of a surjective morphism of vector bundles, is a vector bundle of rank r. Now let us consider the vector bundle $\mathcal{E}$ obtained dualizing the exact sequence (4.4.6):

$$0 \longrightarrow \mathcal{O}_X(-1,-1)^r \xrightarrow{m^\vee} \mathcal{O}_X(-1,0)^r \oplus \mathcal{O}_X(0,-1)^r \longrightarrow \mathcal{E} := \mathcal{F}^\vee \longrightarrow 0. \tag{4.4.7}$$

Proposition 4.4.17. *Let $X \subseteq \mathbb{P}^8$ be the smooth quadric surface embedded in $\mathbb{P}^8$ through the very ample anticanonical divisor $H := -K_X$. Then the vector bundle $\mathcal{E}(H)$ from the short exact sequence (4.4.7) twisted by H is an initialized Ulrich vector bundle of rank r.*

Proof. Let us start showing that $\mathcal{E}$ is ACM, namely, that $\mathrm{H}^1_*(\mathcal{E}) = 0$. In order to do this, notice that, since m were surjective at the level of global sections and $\mathrm{h}^0(\mathcal{O}_X(1,0)^r \oplus \mathcal{O}_X(0,1)^r) = 4r = \mathrm{h}^0(\mathcal{O}_X(1,1)^r)$, we have that

$$\mathrm{h}^0(\mathcal{F}) = \mathrm{h}^2(\mathcal{F}^\vee(-H)) = \mathrm{h}^2(\mathcal{E}(-H)) = 0 \ \text{ and } \ \mathrm{h}^1(\mathcal{F}) = 0.$$

On the other hand, from the long exact cohomology sequence associated to the exact sequence (4.4.7), $\mathrm{h}^0(\mathcal{E}) = \mathrm{h}^1(\mathcal{E}) = 0$, from which we see that $\mathcal{E}$ is initialized and 1-regular. Now we can apply Theorem 4.3.3 to obtain $\mathrm{h}^1(\mathcal{E}(tH)) = 0$ for all $t \geq 0$. Finally, $\mathrm{h}^2(\mathcal{F}(-H)) = \mathrm{h}^0(\mathcal{E}) = 0$ shows that $\mathcal{F}$ is also 1-regular and then a new application of Theorem 4.3.3 gives that $\mathrm{h}^1(\mathcal{E}(tH)) = \mathrm{h}^1(\mathcal{F}((-t-1)H)) = 0$ for all $t < 0$, completing the proof.

Once it has been prove that $\mathcal{E}$ is ACM, an easy computation from the exact sequence (4.4.6) gives that $\mathrm{h}^0(\mathcal{E}_{init}) = \mathrm{h}^0(\mathcal{E}(H)) = 8r = \deg(X)\,\mathrm{rk}(\mathcal{E})$ and therefore $\mathcal{E}$ is an Ulrich vector bundle. Notice that, moreover, we can compute its Chern classes: $c_1(\mathcal{E}(H)) = rH$ and $c_2(\mathcal{E}(H)) = 4r^2 - 3r$. □

The last step in the proof of wildness of the quadric surface will be to ensure that the vector bundle $\mathcal{E}(H)$ (or, equivalently $\mathcal{E}$) is indecomposable and to compute the dimension of the family of Ulrich vector bundles constructed via the exact sequence (4.4.7). As in the rest of del Pezzo surfaces, we are going to produce a somehow stronger result, namely, simplicity. We are going to rely again on the Proposition 4.3.5.

Proposition 4.4.18. *Let $X \subseteq \mathbb{P}^8$ be the smooth quadric surface embedded in $\mathbb{P}^8$ through the very ample anticanonical divisor $H := -K_X$. Then the vector bundle $\mathcal{E}$ from the exact sequence (4.4.7):*

$$0 \longrightarrow \mathcal{O}_X(-1,-1)^r \xrightarrow{m^\vee} \mathcal{O}_X(-1,0)^r \oplus \mathcal{O}_X(0,-1)^r \xrightarrow{g} \mathcal{E} \longrightarrow 0 \tag{4.4.8}$$

is simple and, hence, indecomposable.

Proof. Let the morphism $f := m^\vee$ be represented by a $2r \times r$ matrix $A = (A_1|A_2)^t$ where A_1 (resp. A_2) is a matrix of linear forms in variables x, y (resp. u, v). Take a morphism $\phi : \mathcal{E} \longrightarrow \mathcal{E}$. Then we get a morphism

$$\overline{\phi} = \phi \circ g : \mathcal{O}_X(-1,0)^r \oplus \mathcal{O}_X(0,-1)^r \longrightarrow \mathcal{E}.$$

We can apply the functor $\mathrm{Hom}(\mathcal{O}_X(-1,0)^r \oplus \mathcal{O}_X(0,-1)^r, -)$ (as in Proposition 4.3.6 from the last subsection) to the exact sequence (4.4.8) and take into account that

$$\mathrm{Hom}(\mathcal{O}_X(-1,0)\oplus\mathcal{O}_X(0,-1), \mathcal{O}_X(-1,-1)) = \mathrm{Ext}^1(\mathcal{O}_X(-1,0)\oplus\mathcal{O}_X(0,-1), \mathcal{O}_X(-1,-1)) = 0$$

to get

$$\mathrm{End}(\mathcal{O}_X(-1,0)^r \oplus \mathcal{O}_X(0,-1)^r) \cong \mathrm{Hom}(\mathcal{O}_X(-1,0)^r \oplus \mathcal{O}_X(0,-1)^r, \mathcal{E}).$$

Hence there exist matrices B_1, B_2, C from $\mathrm{Mat}_{r\times r}(k)$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X(-1,-1)^r & \longrightarrow & \mathcal{O}_X(-1,0)^r \oplus \mathcal{O}_X(0,-1)^r & \longrightarrow & \mathcal{E} & \longrightarrow & 0 \\ & & \downarrow C & & \downarrow B & \searrow \overline{\phi} & \downarrow \phi & & \\ 0 & \longrightarrow & \mathcal{O}_X(-1,-1)^r & \longrightarrow & \mathcal{O}_X(-1,0)^r \oplus \mathcal{O}_X(0,-1)^r & \longrightarrow & \mathcal{E} & \longrightarrow & 0 \end{array}$$

where

$$B := \left(\begin{array}{c|c} B_1 & 0 \\ \hline 0 & B_2 \end{array}\right).$$

As in Proposition 4.3.6, we can suppose that B and C are invertible and therefore we have the equalities:

$$A_1 = B_1 A_1 C^{-1}$$

and

$$A_2 = B_2 A_2 C^{-1}.$$

Therefore we can apply Proposition 4.3.5 to the two previous matrix equalities (taking into account that, if we write $a = b = r$, the required inequality $a^2 + b^2 - 2ab = 0 \leq 1$ is satisfied) to conclude that $(B_1, C) = (\lambda_1 \operatorname{id}_r, \lambda_1 \operatorname{id}_r)$ and $(B_2, C) = (\lambda_2 \operatorname{id}_r, \lambda_2 \operatorname{id}_r)$. Hence $\lambda := \lambda_1 = \lambda_2$ and $B = \lambda \operatorname{id}_{2r}$ so, *a fortiori*, $\phi = \lambda \operatorname{id}_{2r}$ and we can conclude that $\mathcal{E}$ is simple. □

It is time to harvest the main result of this subsection:

Theorem 4.4.19. *Let $X \subseteq \mathbb{P}^8$ be the smooth quadric surface embedded in $\mathbb{P}^8$ through the very ample anticanonical divisor $H := -K_X$. Then, for any $r \geq 2$, there exists a family of rank r simple (hence indecomposable) Ulrich vector bundles of dimension $r^2 + 1$. In particular, X is a variety of wild representation type.*

Proof. We have seen in Propositions 4.4.17 and 4.4.18 that, for any $r \geq 2$, there exists a bijection between rank r simple Ulrich vector bundles and orbits under $GL(r) \times GL(r) \times GL(r)$ of a non-empty open subset U of

$$M := \operatorname{Hom}(\mathcal{O}_X(1,0)^r \oplus \mathcal{O}_X(0,1)^r, \mathcal{O}_X(1,1)^r).$$

The dimension of this family can be computed as follows:

$$\dim M - 3 \dim GL(r) + 1 = 4r^2 - 3r^2 + 1 = r^2 + 1.$$

□

We have mentioned (see Theorem 4.2.9) that the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1 \cong X \subseteq \mathbb{P}^3$ was a variety of finite representation type with respect to the very ample divisor $l_1 + l_2$. Therefore, the representation type of a variety strongly depends on the chosen polarization. This leads to the following question:

Question 4.4.20. *Given a projective variety $X \subseteq \mathbb{P}^n$, find the minimum N such that X can be embedded in $\mathbb{P}^N$ as a variety of wild representation type.*

For instance, in the case of the quadric $\mathbb{P}^1 \times \mathbb{P}^1$, it follows from Theorems 4.2.9 and 4.4.19 that N should be either 5, 7 or 8.

4.4.3 Serre correspondence

In this last subsection we are going to pay attention to the case of strong del Pezzo surfaces X. In this case, the very ample divisor $-K_X$ provides an embedding $X \subseteq \mathbb{P}^d$, with $d = K_X^2$. For this kind of surfaces we are going to establish a version of the well-known Serre correspondence between coherent sheaves on X and codimension 2 locally complete intersections subschemes Z of X. This correspondence will be used in both directions. In one direction, we are going to show that the (r^2+1)-dimensional family of rank r Ulrich vector bundles given in Theorem 4.4.11 on a strong del Pezzo surface (excluding the quadric) could also be obtained through Serre correspondence from a general set of $m(r) := c_2(\mathcal{E}(H)) = \frac{dr^2+(2-d)r}{2}$ points on X. On the other direction, the existence of Ulrich vector bundles on the quadric $\mathbb{P}^1 \times \mathbb{P}^1 \cong X \subseteq \mathbb{P}^8$, as it is proven in Proposition 4.4.17, allows to prove the Minimal Resolution Conjecture for $m(r)$ general distinct points (see Theorem 2.2.13).

Theorem 4.4.21 (Serre correspondence). *Let $X \subseteq \mathbb{P}^d$ be a strong del Pezzo surface of degree d. Then it holds:*

(i) Given a rank $r \geq 2$ initialized Ulrich vector bundle $\mathcal{E}$ on X with Chern classes $c_1(\mathcal{E}) = rH$ and $c_2(\mathcal{E}) = \frac{dr^2+(2-d)r}{2}$ and a general element of the Grassmannian $\mathrm{Grass}(r-1, \mathrm{H}^0(\mathcal{E}))$ *represented by $r-1$ global sections $s_1, \ldots, s_{r-1}$, there exists a short exact sequence*

$$0 \longrightarrow \mathcal{O}_X^{r-1} \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{Z|X}(rH) \longrightarrow 0, \tag{4.4.9}$$

where $Z \subseteq X$ is a set of $m(r) := c_2(\mathcal{E})$ distinct points. Moreover, $\mathrm{h}^0(\mathcal{I}_{Z|X}((r-1)H)) = 0$ and the scheme Z is level of type $r-1$ and socle degree $e := c_1 + d - 3$. The minimal free resolution of $I_{Z|X}$ is:

$$0 \longrightarrow R(-r-d)^{r-1} \longrightarrow R(-r-d+2)^{\gamma_{d-1,r-1}} \longrightarrow \ldots \tag{4.4.10}$$

$$\longrightarrow R(-r-1)^{\gamma_{2,r-1}} \longrightarrow R(-r)^{(d-1)r+1} \longrightarrow I_{Z|X} \longrightarrow 0$$

with

$$\gamma_{i,r-1} = \sum_{l=0}^{1} (-1)^l \binom{d-l-1}{i-l} \Delta^{l+1} P_X(r+l) - \binom{d}{i} (m(r) - P_X(r-1)).$$

(ii) Reciprocally, given a subset $Z \subseteq X$ of $|Z| = \frac{dr^2+(2-d)r}{2}$, $r \geq 2$, points such that $I_{Z|X}$ has the minimal free resolution (4.4.10), there exists a rank r Ulrich vector bundle $\mathcal{F}$ with Chern classes $c_1(\mathcal{F}) = rH$ and $c_2(\mathcal{F}) = |Z|$ that fits in the short exact sequence (4.4.9).

Proof. (i) As $\mathcal{E}$ is globally generated, $r-1$ general global sections define an exact sequence of the form

$$0 \longrightarrow \mathcal{O}_X^{r-1} \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{Z|X}(D) \longrightarrow 0$$

where $D = c_1(\mathcal{E}) = rH$ is a divisor on X and Z is a zero-dimensional scheme of length

$$c_2(\mathcal{E}) = \frac{dr^2 + (2-d)r}{2}.$$

Moreover, since $\mathcal{E}(H)$ is initialized, $\mathrm{h}^0(\mathcal{I}_{Z|X}(r-1)) = 0$.

(ii) This is a classical argument so we are going to be brief. We follow the lines of the argument given for the cubic case in [CHa, Theorem 4.4]. Let Z be a general set of points of cardinality $m(r)$ with the minimal free resolution of (4.4.10). Let us denote by R_X and R_Z the homogeneous coordinate ring of X and Z. It is well-known that for Arithmetically Cohen-Macaulay varieties, there exists a bijection between ACM vector bundles on X and Maximal Cohen-Macaulay (MCM from now on) graded R_X-modules sending $\mathcal{E}$ to $\mathrm{H}^0_*(\mathcal{E})$ (see Proposition 4.2.4). From the exact sequence

$$0 \longrightarrow I_{Z|X} \longrightarrow R_X \longrightarrow R_Z \longrightarrow 0$$

we get $\mathrm{Ext}^1(I_{Z|X}, R_X(-1)) \cong \mathrm{Ext}^2(R_Z, R_X(-1)) \cong K_Z$ where K_Z denotes the canonical module of R_Z (the last isomorphism is due to the fact that $R_X(-1)$ is the canonical module of X and the codimension of Z in X is 2). Dualizing the exact sequence (4.4.10), we obtain a minimal resolution of K_Z:

$$\ldots \longrightarrow R(r-3)^{\gamma_{d-1,r-1}} \longrightarrow R(r-1)^{r-1} \longrightarrow K_Z \longrightarrow 0.$$

This shows that K_Z is generated in degree $1-r$ by $r-1$ elements. These generators provide an extension

$$0 \longrightarrow R_X^{r-1} \longrightarrow F \longrightarrow I_{Z|X}(r) \longrightarrow 0 \qquad (4.4.11)$$

via the isomorphism $K_Z \cong \mathrm{Ext}^1(I_{Z|X}, R_X(-1))$. F turns out to be a MCM module because $\mathrm{Ext}^1(F, K_X) = 0$ (this last cancelation follows by applying the functor $\mathrm{Hom}_{R_X}(-, K_X)$ to the exact sequence (4.4.11)). If we sheafify the exact sequence (4.4.11) we obtain the sequence

$$0 \longrightarrow \mathcal{O}_X^{r-1} \longrightarrow \widetilde{F} \longrightarrow \mathcal{I}_{Z|X}(r) \longrightarrow 0$$

where $\widetilde{F}$ is an ACM vector bundle on X. From the exact sequence (4.4.10) it can be seen that

$$\mathrm{H}^0(\mathcal{I}_{Z|X}(r-1)) = 0$$

and

$$\mathrm{h}^0(\mathcal{I}_{Z|X}(r)) = (d-1)r + 1$$

and therefore $\widetilde{F}$ will be a Ulrich vector bundle (i.e., $\mathrm{h}^0(\widetilde{F}) = dr$) and initialized. By Theorem 4.2.16, $\widetilde{F}$ will be globally generated. □

Corollary 4.4.22. *Let $X \subseteq \mathbb{P}^d$ be a strong del Pezzo surface of degree d, distinct of the quadric surface. Then the rank r initialized Ulrich vector bundles $\mathcal{E}(H)$ given by the exact sequence (4.4.3) can be recovered as an extension of $\mathcal{I}_{Z,X}(rH)$ by $\mathcal{O}_X^{r-1}$ for general sets Z of $m(r) = 1/2(dr^2 + (2-d)r)$ distinct points on X, $r \geq 2$.*

Proof. We are going to rely on the fact that $m(r) := \frac{dr^2+(2-d)r}{2}$ general points $Z_{m(r)}$ contained in a strong del Pezzo surface $X \subseteq \mathbb{P}^d$ distinct of the quadric satisfies the Minimal Resolution Conjecture. Indeed, by Theorem 2.2.13 from chapter 2, the minimal free resolution of the saturated ideal of $Z_{m(r)}$ in X has the form (4.4.10). Hence, by the previous version of Serre correspondence, we can associate to $Z_{m(r)}$ a rank r initialized Ulrich vector bundle $\mathcal{F}$ with Chern classes $c_1(\mathcal{F}) = rH$ and $c_2(\mathcal{F}) = m(r)$.

It only remains to show that for a general choice of $Z_{m(r)} \subset X$, the associated vector bundle $\mathcal{F} := \widetilde{F}$ just constructed belongs to the family (4.4.3). Since $\mathcal{F}$ is an initialized Ulrich vector bundle of rank r with the expected Chern classes, the problem boils down to a dimension counting. We need to show that the dimension of the family of vector bundles obtained through this construction from a general set $Z_{m(r)}$ is $r^2 + 1$. Since this dimension is given by the formula

$$\dim \mathrm{Hilb}^{m(r)}(X) - \dim \mathrm{Grass}(r-1, \mathrm{H}^0(\mathcal{F})),$$

an easy computation taking into account that

$$\dim \mathrm{Hilb}^{m(r)}(X) = 2m(r),$$

and

$$\dim \mathrm{Grass}(r-1, \mathrm{H}^0(\mathcal{F})) = (r-1)(dr - r + 1),$$

gives the desired result. □

4.5 ACM vector bundles on surfaces $X \subseteq \mathbb{P}^3$

We showed in the previous section that the cubic surface $X \subseteq \mathbb{P}^3$, as a case of del Pezzo surface, is of wild representation type. In this last section we address this question for a *general* surface X of arbitrary degree d in $\mathbb{P}^3$. We manage to show that, for $4 \leq d \leq 9$, a general surface $X \subseteq \mathbb{P}^3$ of degree d is of wild representation type. In the case of arbitrary degree d, we will be able at least to construct large families of rank 2 and 3 simple ACM vector bundles on a general surface $X \subseteq \mathbb{P}^3$ of degree d, showing that they are not of tame representation type (notice that, as it was mentioned in Example 4.2.13, in [BGS87, Theorem C] it was shown that surfaces $X \subseteq \mathbb{P}^3$ of degree > 2 are not of finite representation type). It is worthwhile to remark here that in [HS88] it was proven that there exists no upper bound for the rank of an indecomposable ACM sheaf on a surface $X \subseteq \mathbb{P}^3$ of arbitrary degree.

Let us start with a general surface $X \subseteq \mathbb{P}^3$ of degree $4 \leq d \leq 15$. For such a surface X we are going to construct a positive dimensional family of rank 2 Ulrich vector bundles. Notice that the existence of such a family had already been proved in [Bea00, Proposition 7.6]. However, for sake of completeness we are going to provide such a family through a construction that stresses the relation between Ulrich vector bundles on X and zero-dimensional Gorenstein schemes $Z \subseteq X$ by means of Serre correspondence. We will use the following remark.

Remark 4.5.1. As in the case of strong del Pezzo surfaces (see Lemma 4.4.13) it is possible to show that the degree and the second Chern class of a rank r initialized Ulrich vector bundle $\mathcal{E}$ on a smooth surface $X \subseteq \mathbb{P}^3$ of degree d are given by

$$c_1(\mathcal{E})H = \binom{d}{2} r, \tag{4.5.1}$$

and

$$c_2(\mathcal{E}) = \frac{c_1(\mathcal{E})^2 - \binom{d}{3} r}{2}. \tag{4.5.2}$$

Therefore through Serre correspondence, a rank 2 initialized Ulrich vector bundle $\mathcal{E}$ on a general surface $X \subseteq \mathbb{P}^3$ and a global section $s \in \mathrm{H}^0(\mathcal{E})$ are associated to an AG zero-dimensional scheme $Z \subseteq X$ with a short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{Z|X}(d-1) \longrightarrow 0, \tag{4.5.3}$$

where $|Z| = c_2(\mathcal{E}) = \frac{d(d-1)(2d-1)}{6}$,

$$\mathrm{h}^0(\mathcal{I}_{Z|X}(d-1)) = 2d - 1 \text{ and } \mathrm{h}^0(\mathcal{I}_{Z|X}(d-2)) = 0.$$

The socle degree of Z is $e := 2d-4$ (as it was defined in Definition 1.1.11). Notice that $\mathrm{h}^1(\mathcal{I}_{Z|X}(e-1)) = 1$. Hence Serre duality,

$$\mathrm{H}^1(\mathcal{I}_{Z|X}(e-1))^\vee \cong \mathrm{Ext}^1(\mathcal{I}_{Z|X}(d-1), \mathcal{O}_X),$$

implies that Z will be univocally associated to the pair $(\mathcal{E},\overline{s})$ with $\overline{s} \in \mathbb{P}(\mathrm{H}^0(\mathcal{E}))$. It is straightforward to see that such a zero-dimensional scheme $Z \subseteq X \subseteq \mathbb{P}^3$ has a minimal free resolution of the form

$$0 \longrightarrow R(-2d+1) \longrightarrow R(-d)^{2d-1} \longrightarrow R(-(d-1))^{2d-1} \longrightarrow R \longrightarrow R_Z \longrightarrow 0, \tag{4.5.4}$$

$I(Z)$ is generated by the $2d-1$ pfaffians of a $(2d-1)\times(2d-1)$ skew-symmetric matrix with linear entries and the h-vector of Z is

t	0	1	2	...	$d-3$	$d-2$	$d-1$	...	$2d-4$
$\Delta H_Z(t)$	1	3	6	...	$\binom{d-1}{2}$	$\binom{d}{2}$	$\binom{d-1}{2}$	...	1

(4.5.5)

So our goal will be to show that a general surface $X \subseteq \mathbb{P}^3$ of degree $3 \le d \le 15$ contains a Gorenstein zero-dimensional subscheme $Z \subseteq X$ with minimal free resolution as given in (4.5.4).

Proposition 4.5.2. *Let $X \subseteq \mathbb{P}^3$ be a general surface of degree $3 \le d \le 15$. Then there exists a family of dimension $\alpha := 2d(2d-1) - \binom{d+3}{3}$ of Gorenstein subschemes $Z \subseteq X$ with minimal free resolution as in (4.5.4). In particular, they have length $l := |Z| = \frac{d(d-1)(2d-1)}{6}$.*

Proof. Let us fix the values $3 \le d \le 15$, $l := \frac{d(d-1)(2d-1)}{6}$ and $e := 2d-4$ and let us consider the incidence diagram

$$\begin{array}{ccccc} & \Sigma := \{([Z],[X]) \mid Z \subseteq X\} & \subseteq & \mathcal{G}^h(l,e) \times \mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(d))) & \\ \swarrow^{\phi} & & & & \searrow^{\psi} \\ \mathcal{G}^h(l,e) & & & & \mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(d))) \end{array}$$

where $\mathcal{G}^h(l,e)$ denotes the locally closed subscheme of $\mathrm{Hilb}^l(\mathbb{P}^3)$ parameterizing length l AG subschemes of $\mathbb{P}^3$ with socle degree e and h-vector as given in (4.5.5) and $\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(d)))$ parameterizes surfaces of degree d in $\mathbb{P}^3$. First of all, by an instance of [KMR98, Theorem 2.6], it is possible to compute the dimension of $\mathcal{G}^h(l,e)$:

$$\dim \mathcal{G}^h(l,e) = (2d-1)(2d-3).$$

On the other hand, for a general $[Z] \in \mathcal{G}^h(l, e)$, the dimension of the fibre $\phi^{-1}([Z])$ can also be computed from the minimal free resolution (4.5.4):

$$\dim \phi^{-1}([Z]) = \mathrm{h}^0(\mathcal{I}_Z(d)) - 1 = 4(2d-1) - (2d-1) - 1 = 3(2d-1) - 1.$$

Therefore the dimension of Σ is as follows:

$$\dim \Sigma = (2d-1)(2d-3) + 3(2d-1) - 1 = 2d(2d-1) - 1.$$

Moreover, by [Bea00, Propositions 7.2 and 7.6], ψ is a dominant morphism for $3 \leq d \leq 15$ and hence we can conclude that a generic surface $X \subseteq \mathbb{P}^3$ of degree $3 \leq d \leq 15$ contains a family of AG zero-dimensional schemes Z with minimal free resolution (4.5.4) of dimension

$$\alpha := 2d(2d-1) - 1 - (\binom{d+3}{3} - 1) = 2d(2d-1) - \binom{d+3}{3},$$

which proves what we wanted. □

Proposition 4.5.3. *Let $X \subseteq \mathbb{P}^3$ be a general surface of degree $3 \leq d \leq 15$. Then there exists a family of dimension $\beta := \frac{-d^3+18d^2-35d}{6}$ of initialized rank 2 Ulrich vector bundles with $c_1(\mathcal{E}) = d-1$ and $c_2(\mathcal{E}) = \frac{d(d-1)(2d-1)}{6}$.*

Proof. Since we have just mentioned that there exists a bijection between AG schemes $Z \subseteq X$ with minimal free resolution (4.5.4) and pairs $(\mathcal{E},\overline{s})$, where $\mathcal{E}$ is an initialized Ulrich rank 2 vector bundle with $c_1(\mathcal{E}) = d-1$ and $c_2(\mathcal{E}) = |Z| = \frac{d(d-1)(2d-1)}{6}$, and $\overline{s}_Z \in \mathbb{P}(\mathrm{H}^0(\mathcal{E}))$, Proposition 4.5.2 guarantees the existence of such a family of dimension:

$$\beta = \alpha - (\mathrm{h}^0(\mathcal{E}) - 1) = \alpha - 2d + 1 = \frac{-d^3 + 18d^2 - 35d}{6},$$

as we wanted to prove. □

Once it has been shown the existence of rank 2 Ulrich vector bundles on the general surface X of degree $3 \leq d \leq 15$, the higher rank Ulrich vector bundles will be obtained as their extensions. In order to perform our constructions a keystone will be the following result proven in [PLT09]. Recall that given a projective variety X, and coherent sheaves $\mathcal{F}, \mathcal{G}$ on it, an *extension of $\mathcal{G}$ by $\mathcal{F}$* is a sheaf $\mathcal{E}$ that appears on an exact sequence of the form

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0.$$

Given another extension

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E}' \longrightarrow \mathcal{G} \longrightarrow 0$$

we are going to say that they are *equivalent* if there exists an isomorphism

$$\psi : \mathcal{E} \xrightarrow{\cong} \mathcal{E}'$$

such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \\ & & \| & & \downarrow \psi & & \| & & \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{G} & \longrightarrow & 0. \end{array}$$

A *weak equivalence* of extensions is similarly defined, except that we do not require the morphisms $\mathcal{F} \longrightarrow \mathcal{F}$ and $\mathcal{G} \longrightarrow \mathcal{G}$ to be the identity but only isomorphisms.

It is a well-known result that equivalent classes of extensions of $\mathcal{G}$ by $\mathcal{F}$ correspond bijectively to the elements of $\mathrm{Ext}^1(\mathcal{G}, \mathcal{F})$. If

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0$$

is such an extension, the corresponding element $[\mathcal{E}] \in \mathrm{Ext}^1(\mathcal{G}, \mathcal{F})$ is the image of $\mathrm{id}_{\mathcal{F}}$ under the morphism

$$\mathrm{Hom}(\mathcal{F}, \mathcal{F}) \xrightarrow{\delta} \mathrm{Ext}^1(\mathcal{G}, \mathcal{F})$$

obtained applying $\mathrm{Hom}(-, \mathcal{F})$ to the exact sequence above. We will use the symbol δ for this morphism. The *trivial* extension $\mathcal{F} \oplus \mathcal{G}$ corresponds to $0 \in \mathrm{Ext}^1(\mathcal{G}, \mathcal{F})$. Inside $\mathrm{Ext}^1(\mathcal{G}, \mathcal{F})$ weak equivalence defines an equivalent relation that will be denoted by $\sim_w$.

Proposition 4.5.4. *(cf. [PLT09, Proposition 5.1.3]) Let X be a projective variety over k and $\mathcal{F}_1, \ldots, \mathcal{F}_{r+1}$, with $r \geq 1$, be simple coherent sheaves on X such that*

$$\mathrm{Hom}(\mathcal{F}_i, \mathcal{F}_j) = 0 \textit{ for } i \neq j.$$

Denote also

$$U = \mathrm{Ext}^1(\mathcal{F}_{r+1}, \mathcal{F}_1) - \{0\} \times \cdots \times \mathrm{Ext}^1(\mathcal{F}_{r+1}, \mathcal{F}_r) - \{0\} \subseteq \mathrm{Ext}^1(\mathcal{F}_{r+1}, \bigoplus_{i=1}^{r} \mathcal{F}_i).$$

Then a sheaf $\mathcal{E}$ that comes up from an extension of $\mathcal{F}_{r+1}$ by $\bigoplus_{i=1}^r \mathcal{F}_i$ is simple if and only if $[\mathcal{E}] \in U$ and given two extensions $[\mathcal{E}], [\mathcal{E}'] \in U$ we have that

$$\mathrm{Hom}(\mathcal{E}, \mathcal{E}') \neq 0 \iff [\mathcal{E}] \sim_w [\mathcal{E}'].$$

To be more precise, the simple coherent sheaves $\mathcal{E}$ coming up from an extension of $\mathcal{F}_{r+1}$ by $\bigoplus_{i=1}^r \mathcal{F}_i$

$$0 \longrightarrow \bigoplus_{i=1}^{r} \mathcal{F}_i \longrightarrow \mathcal{E} \longrightarrow \mathcal{F}_{r+1} \longrightarrow 0$$

are parameterized, up to isomorphisms (of coherent sheaves), by

$$(U/\sim_w) \cong \mathbb{P}(\mathrm{Ext}^1(\mathcal{F}_{r+1}, \mathcal{F}_1)) \times \cdots \times \mathbb{P}(\mathrm{Ext}^1(\mathcal{F}_{r+1}, \mathcal{F}_r)).$$

Given a variety $X \subseteq \mathbb{P}^n$, when $\mathrm{Pic}(X) \cong \mathbb{Z} \cong \langle \mathcal{O}_X(1) \rangle$ (as it is the case for our general surface $X \subseteq \mathbb{P}^3$ of degree $d \geq 4$) there exists a useful criterion for a rank 2 or 3 vector bundle on X being μ-stable. To state it, we need to recall that in this situation given a rank r vector bundle $\mathcal{E}$ on X there exists a unique $k_{\mathcal{E}} \in \mathbb{Z}$ such that $c_1(\mathcal{E}(k_{\mathcal{E}})) \in \{0, \ldots, -(r-1)H\}$. We set $\mathcal{E}_{norm} := \mathcal{E}(k_{\mathcal{E}})$. Then we have:

Lemma 4.5.5. *(cf. [OSS80, Lemma 1.2.5 and Remark 1.2.6, Chapter II]) Let $X \subseteq \mathbb{P}^n$ be a variety such that* $\mathrm{Pic}(X) \cong \mathbb{Z} \cong \langle \mathcal{O}_X(1) \rangle$. *Then:*

(i) A rank 2 vector bundle $\mathcal{E}$ on X is μ-stable if and only if $\mathrm{H}^0(\mathcal{E}_{norm}) = 0$.

(ii) A rank 3 vector bundle $\mathcal{F}$ on X is μ-stable if and only if $\mathrm{H}^0(\mathcal{F}_{norm}) = 0$ *and* $\mathrm{H}^0((\mathcal{F}^\vee)_{norm}) = 0$.

Lemma 4.5.6. *Let $X \subseteq \mathbb{P}^3$ be a general surface of degree $4 \leq d \leq 15$. The rank 2 Ulrich vector bundles $\mathcal{E}$ obtained in Proposition 4.5.3 are μ-stable, and therefore, simple.*

Proof. We are going to apply the previous criterion to show μ-stability. Since X is a general surface of degree ≥ 4, by Remark 4.5.1 we have that $c_1(\mathcal{E}) = (d-1)H$ and therefore $\mathcal{E}_{norm} = \mathcal{E}(-\lfloor d/2 \rfloor H)$. Therefore, twisting the exact sequence (4.5.3) by $\mathcal{O}_X(-\lfloor d/2 \rfloor H)$ we have

$$0 \longrightarrow \mathcal{O}_X(-\lfloor d/2 \rfloor H) \longrightarrow \mathcal{E}_{norm} \longrightarrow \mathcal{I}_{Z|X}(\lceil d/2 - 1 \rceil) \longrightarrow 0.$$

Taking global sections and taking into account that $\mathrm{h}^0(\mathcal{O}_X(-\lfloor d/2 \rfloor H)) = 0$ and that $\mathrm{h}^0(\mathcal{I}_{Z|X}(\lceil d/2 - 1 \rceil)) \leq \mathrm{h}^0(\mathcal{I}_{Z|X}(d-2))) = 0$ we obtain that $\mathrm{h}^0(\mathcal{E}_{norm}) = 0$ and therefore, by Lemma 4.5.5, $\mathcal{E}$ is μ-stable. □

Lemma 4.5.7. *Let $X \subseteq \mathbb{P}^3$ be a general surface of degree $4 \leq d \leq 15$. Given two non-isomorphic rank 2 Ulrich vector bundles $\mathcal{E}, \mathcal{E}'$ on X, it holds that* $\mathrm{Hom}(\mathcal{E}', \mathcal{E}) = 0$.

Proof. If we tensor the exact sequence (4.5.3) by $\mathcal{E}'^{\vee}$ we obtain

$$0 \longrightarrow \mathcal{E}'^{\vee} \longrightarrow \mathcal{E} \otimes \mathcal{E}'^{\vee} \longrightarrow \mathcal{E}'^{\vee} \otimes \mathcal{I}_{Z|X}(d-1) \longrightarrow 0.$$

But, by [Har77, Exercise 5.16, chapter II], $\mathcal{E}'^{\vee} \cong \mathcal{E}'(-c_1(\mathcal{E}')) = \mathcal{E}'(-d+1)$. Therefore $\mathrm{H}^0(\mathcal{E}'^{\vee}) = \mathrm{H}^0(\mathcal{E}'(-d+1)) = 0$. On the other hand, since we saw that to a finite set of points $Z \subseteq X$ corresponds uniquely a rank 2 vector bundle $\mathcal{E}$ we have that

$$\mathrm{H}^0(\mathcal{E}'^{\vee} \otimes \mathcal{I}_{Z|X}(d-1)) = \mathrm{H}^0(\mathcal{E}' \otimes \mathcal{I}_{Z|X}) = 0.$$

Therefore $\mathrm{Hom}(\mathcal{E}', \mathcal{E}') = \mathrm{H}^0(\mathcal{E} \otimes \mathcal{E}'^{\vee}) = 0$. □

Theorem 4.5.8. *Let $X \subseteq \mathbb{P}^3$ be a general surface of degree $4 \leq d \leq 9$. Then, for any $r = 2s$, $s \geq 2$, there exists a family of rank r simple (hence indecomposable) Ulrich vector bundle of dimension $11(s-1)$. In particular, a general surface $X \subseteq \mathbb{P}^3$ of degree $4 \leq d \leq 9$ is of wild representation type.*

Proof. Let $r = 2s$ be an integer and let us consider s non-isomorphic rank 2 Ulrich vector bundles $\mathcal{E}_1, \ldots, \mathcal{E}_s$ from the infinite family constructed in Proposition 4.5.3. By Lemmas 4.5.6 and 4.5.7 these vector bundles satisfy the hypothesis of Proposition 4.5.4 and therefore, since any extension of Ulrich vector bundles is an Ulrich vector bundle, there exists a family of rank r simple Ulrich vector bundles $\mathcal{E}$ parameterized by

$$(U/\sim_w) := \mathbb{P}(\mathrm{Ext}^1(\mathcal{E}_s, \mathcal{E}_1)) \times \cdots \times \mathbb{P}(\mathrm{Ext}^1(\mathcal{E}_s, \mathcal{E}_{s-1}))$$

and given as extensions of the form

$$0 \longrightarrow \oplus_{i=1}^{s-1} \mathcal{E}_i \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_s \longrightarrow 0.$$

It only remains to give an explicit lower bound to the dimension of $(U/\sim_w)$. We have

$$\mathrm{ext}^1(\mathcal{E}_i, \mathcal{E}_j) \geq \mathrm{ext}^1(\mathcal{E}_i, \mathcal{E}_j) - \mathrm{ext}^2(\mathcal{E}_i, \mathcal{E}_j) = -\chi(\mathcal{E}_i^{\vee} \otimes \mathcal{E}_j) = -\chi(\mathcal{E}_i^{\vee} \otimes \mathcal{E}_i)$$

where the last equality holds since the Euler characteristic of a vector bundle only depends on its Chern classes. On the other hand, $\chi(\mathcal{E}_i^{\vee} \otimes \mathcal{E}_i)$ can be computed using the Hirzebruch-Riemann-Roch formula as in [HL97, Corollary 4.5.5] to obtain

$$\mathrm{ext}^1(\mathcal{E}_i, \mathcal{E}_j) \geq -\chi(\mathcal{E}_i^{\vee} \otimes \mathcal{E}_j) = -4 - \frac{d-1}{3}(d^2 - 11d + 12) \geq 12$$

for $4 \leq d \leq 9$. Therefore, we obtain a family of simple (hence irreducible) rank $2s$ Ulrich vector bundles of dimension $11(s-1)$. Notice that for $d = 10, \dots, 15$ the value of $-\chi(\mathcal{E}_i^\vee \otimes \mathcal{E}_j) = -4 - \frac{d-1}{3}(d^2 - 11d + 12)$ is negative and hence this method does not allow us to produce higher rank Ulrich vector bundles on these degrees. □

Remark 4.5.9. Notice that, by Remark 4.5.1, a general surface $X \subseteq \mathbb{P}^3$ of even degree can not be the support of an odd rank Ulrich vector bundle. Therefore, in Theorem 4.5.8 we have constructed Ulrich vector bundles on general surfaces of degrees $d = 4, 6, 8$ for all the admissible ranks. It remains open, however, to construct odd rank Ulrich vector bundles on surfaces of odd degree ≥ 5.

To end this section we are going to see that for general surfaces of arbitrary degree, we are able to construct at least rank 2 and rank 3 indecomposable ACM vector bundles. Unfortunately, these vector bundles will not be Ulrich vector bundles.

Proposition 4.5.10. *Let $X \subseteq \mathbb{P}^3$ be a general surface of degree $d \geq 3$. Then there exists a 4-dimensional family of rank 2 initialized μ-stable ACM vector bundles $\mathcal{E}$ with $c_1(\mathcal{E}) = 1$ and $c_2(\mathcal{E}) = d - 1$.*

Proof. Let $X \subseteq \mathbb{P}^3$ be a general surface of degree $d \geq 3$. The construction of the announced vector bundles on X will be performed through Serre correspondence. So let us consider a subset $Z \subseteq X$ of $d-1$ aligned points from the intersection of a general line on $\mathbb{P}^3$ with X. Z is a complete intersection (in $\mathbb{P}^3$) of type $(1, 1, d-1)$ and therefore it has minimal free resolution

$$0 \longrightarrow R(-d-1) \longrightarrow R(-d)^2 \oplus R(-2) \longrightarrow R(-d+1) \oplus R(-1)^2 \longrightarrow R \longrightarrow R_Z \longrightarrow 0.$$

In particular, Z is AG subscheme of X of socle degree $e := d-2$. By Serre duality

$$\mathrm{H}^1(\mathcal{I}_{Z|X}(e-1))^\vee \cong \mathrm{Ext}^1(\mathcal{I}_{Z|X}(1), \mathcal{O}_X),$$

since $\mathrm{h}^1(\mathcal{I}_{Z|X}(e-1) = 1$, we see that there exists an unique coherent sheaf that fits in the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{Z|X}(1) \longrightarrow 0.$$

Notice that $c_1(\mathcal{E}) = 1$ and $c_2(\mathcal{E}) = d-1$. Moreover, since Z was AG, we can assure that $\mathcal{E}$ is an ACM vector bundle. In order to see μ-stability, we just need to realize that $\mathrm{h}^0(\mathcal{E}_{norm}) = \mathrm{h}^0(\mathcal{E}(-1)) = 0$ and apply Lemma 4.5.5. Finally, since we can make this construction from any $d-1$ aligned points on X and since a general line in $\mathbb{P}^3$ cuts out X in d points, we obtained a family of ACM vector bundles of dimension $\dim \mathrm{Grass}(2,4) = 4$. □

Proposition 4.5.11. *Let $X \subseteq \mathbb{P}^3$ be a general surface of degree $d \geq 3$. Then there exists an infinite family of rank 3 initialized μ-stable ACM vector bundles $\mathcal{F}$ with $c_1(\mathcal{F}) = 1$ and $c_2(\mathcal{F}) = 2d-3$.*

Proof. Let $X \subseteq \mathbb{P}^3$ be a general surface of degree $d \geq 3$. In this case, the vector bundles will be obtained from subsets $Z \subseteq X$ of $2d-3$ points. So let $U \subseteq X$ be a finite subset complete intersection of type $(1,2,d)$ and let $V \subseteq U$ be any subset of cardinality 3 which does not lie on a line. Since it is immediate to obtain the minimal free resolution of I_U and I_V, we can apply the mapping cone procedure to

$$0 \longrightarrow I_U \longrightarrow I_V \longrightarrow I_V/I_U \longrightarrow 0$$

to obtain the minimal free resolution of the residual subset $Z := U \setminus V$, G-linked to V by the AG scheme U. It turns out to be:

$$0 \longrightarrow R(-d-1)^2 \longrightarrow R(-d) \oplus R(-3)^4 \longrightarrow R(-d+1) \oplus R(-2) \oplus R(-1) \longrightarrow R \longrightarrow R_Z \longrightarrow 0.$$

In particular, Z is a level zero-dimensional scheme of type 2 and socle degree $e = d-2$. Again, by Serre correspondence (see the proof of 4.4.21), we obtain a unique ACM vector bundle $\mathcal{F}$ that fits in the short exact sequence

$$0 \longrightarrow \mathcal{O}_X^2 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}_{Z|X}(1) \longrightarrow 0. \tag{4.5.6}$$

Notice that $c_1(\mathcal{F}) = 1$ and $c_2(\mathcal{F}) = 2d-3$. In order to prove μ-stability of $\mathcal{F}$ we are going to apply Lemma 4.5.5. Therefore we need to show (since $\mathcal{F}_{norm} = \mathcal{F}(-1)$ and $(\mathcal{F}^\vee)_{norm} = \mathcal{F}^\vee$) that $\mathrm{H}^0(\mathcal{F}(-1)) = 0$ and $\mathrm{H}^0(\mathcal{F}^\vee) = 0$. But the first equality is obvious from the previous exact sequence (4.5.6). Concerning the second equality, notice that we have $\mathcal{F}^\vee \cong \wedge^2\mathcal{F}(-1)$ (see [Har77, Exercise 5.16, chapter II]). Then, if we consider the Eagon-Northcott complex associated to (4.5.6),

$$0 \longrightarrow S^2\mathcal{O}_X \cong \mathcal{O}_X^3 \longrightarrow \mathcal{O}_X^2 \otimes \mathcal{F} \longrightarrow \wedge^2\mathcal{F} \longrightarrow \wedge^2\mathcal{I}_{Z|X}(1) = 0,$$

and we twist it by $\mathcal{O}_X(-1)$ we get

$$0 \longrightarrow \mathcal{O}_X(-1)^3 \longrightarrow \mathcal{F}(-1)^2 \longrightarrow \wedge^2\mathcal{F}(-1) \longrightarrow 0,$$

from which it is immediate to conclude that $\mathrm{H}^0(\mathcal{F}^\vee) = \mathrm{H}(\wedge^2\mathcal{F}(-1)) = 0$. □

Remark 4.5.12. Once the existence of rank 2 and 3 simple ACM vector bundles on a general surface X of arbitrary degree has been proven, we would like to use Proposition 4.5.4 to construct large families of arbitrary rank of simple ACM vector bundles on X. It is easy to check that the rank 2 vector bundles $\mathcal{E}$ (resp. rank 3 vector bundles $\mathcal{F}$) constructed in Proposition 4.5.10 (resp. in Proposition 4.5.11) verify the hypothesis of Proposition 4.5.4 but, unfortunately, we were not able to prove that $\mathrm{ext}^1(\mathcal{E},\mathcal{F}) > 1$ (or $\mathrm{ext}^1(\mathcal{F},\mathcal{E}) > 1$) to conclude.

Remark 4.5.13. A more ambitious goal would be the construction of indecomposable Ulrich vector bundles of arbitrary rank r supported on a general surface $X \subseteq \mathbb{P}^3$ of arbitrary degree d (with the necessary restriction that $(d-1)r$ is even, see Remark 4.5.9). As in the previous cases, a natural strategy to tackle this problem would be through Serre correspondence from a finite set $Z \subseteq X$ of points with some specific properties. We are going to finish the chapter introducing here the precise statement. Notice, however, that in this case, the problem is not related to the Minimal Resolution Conjecture (see Conjecture 2.1.10), since the set of points we are considering are not in general position (except when $d = 3$).

Theorem 4.5.14 (Serre correspondence for Ulrich vector bundles on surfaces)**.** *Let $X \subseteq \mathbb{P}^3$ be a general surface. Then it holds:*
(i) Given a rank r Ulrich vector bundle $\mathcal{E}$ on X and a general element of the Grassmannian $\mathrm{Grass}(r-1, \mathrm{H}^0(\mathcal{E}))$ *represented by $r-1$ global sections $s_1, \dots, s_{r-1}$, there exists a short exact sequence*

$$0 \longrightarrow \mathcal{O}_X^{r-1} \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{Z|X}(c_1(\mathcal{E})) \longrightarrow 0, \tag{4.5.7}$$

where Z is a set of $c_2(\mathcal{E})$ distinct points. Moreover, the scheme Z is level of type $r-1$ and socle degree $e := c_1 + d - 3$. The minimal free resolution of $I_{Z|X}$ is:

$$0 \longrightarrow R(-c_1-d)^{r-1} \longrightarrow R(-c_1-1)^{dr} \longrightarrow R(-c_1)^{(d-1)r+1} \longrightarrow I_{Z|X} \longrightarrow 0. \tag{4.5.8}$$

From the minimal free resolution is immediate to recover the h-vector of Z:

t	0	1	2	...	c_1-2	c_1-1
$\Delta H_Z(t)$	1	3	6	...	$\Delta H_X(c_1-2)$	$\Delta H_X(c_1-1) = (r-1)\Delta H_X(d-2)$

t	c_1	...	$e := c_1 + d - 3$
$\Delta H_Z(t)$	$(r-1)\Delta H_X(d-3)$	...	$r-1$

(ii) Reciprocally, given a subset $Z \subseteq X$ of $|Z| = \frac{c_1(\mathcal{E})^2 - \binom{d}{3} r}{2}$ points such that $I_{Z|X}$ has the minimal free resolution (4.5.8), there exists a rank r Ulrich vector bundle $\mathcal{E}$ with Chern classes $c_1(\mathcal{E}) = (d-1)r/2$ and $c_2 = |Z|$ that fits in the short exact sequence (4.5.7).

Summarizing, the first open case is the existence of a rank 3 Ulrich vector bundle on a general surface X of degree 5 which, according to Theorem 4.5.14, is equivalent to the existence of a level set of 75 distinct points Z on X with h-vector $(1, 3, 6, 10, 15, 20, 12, 6, 2)$.

Resum en català

L'objecte d'aquesta tesi es situa a la cruïlla de tres temes: la construcció de famílies de feixos indescomposables Aritmèticament Cohen-Macaulay a una varietat projectiva donada X, la forma (i.e, els nombres de Betti) de la resolució lliure de conjunts generals de punts a X i la (i)rreductibilitat de l'esquema de Hilbert $\mathrm{Hilb}^s(X)$ que parametritza els subesquemes zero-dimensionals $Z \subseteq X$ de longitud s. Expliquem amb més detall com aquests tres temes estan interrelacionats.

Donada una varietat projectiva $X \subseteq \mathbb{P}^n$ amb anell de coordenades R_X, és normal intentar entendre la complexitat de X en funció de la categoria associada dels fibrats vectorials sobre X. Donat que, en general, aquesta categoria és poc manipulable, es restringeix l'estudi a la categoria de fibrats vectorials (semi)-estables, la qual té un bon comportament i per la qual, en particular, existeix un espai de moduli que els parametritza. Mentre que aquest punt de vista ha sigut exitosament explotat, també és possible fixar-se en una altra propietat d'un fibrat vectorial $\mathcal{E}$: el fet de tenir cohomologia el més simple possible, és a dir, $\mathrm{H}^i(X, \mathcal{E}(l)) = 0$ per a tot $l \in \mathbb{Z}$ i $i = 1, \ldots, \dim(X) - 1$. Els fibrats vectorials amb aquesta propietat s'anomenen fibrat vectorials *Aritmeticament Cohen-Macaulay (ACM)*. Quan X és al seu torn ACM, en termes del R_X-mòdul associat $E = \mathrm{H}^0_*(\mathcal{E}) := \oplus_l \mathrm{H}^0(X, \mathcal{E}(l))$, corresponen als mòduls *Màximament Cohen-Macaulay (MCM)*, és a dir, mòduls que verifiquen $\mathrm{depth}(E) = \dim(R_X)$. Aquesta correspondència ens permet estudiar el problema alternativament des del punt de vista algebraic o geomètric. L'estudi d'aquests fibrats vectorials (o mòduls) té una llarga i interessant història al seu darrere. Un resultat fonamental és degut a Horrocks, el qual afirma que, a l'espai projectiu $\mathbb{P}^n$, qualsevol fibrat vectorial ACM descomposa com a suma directe de fibrats de línia (cf. [Hor64]) o, equivalentment, l'únic fibrat vectorial ACM indescomposable a $\mathbb{P}^n$, mòdul torsió i isomorfisme, és $\mathcal{O}_{\mathbb{P}^n}$. Això correspondria a la idea intuitiva que una varietat "simple"ha de tenir associada una categoria "simple"de fibrats vectorials ACM. Seguint aquestes línies, un resultat fonamental fou la classificació de les varietats ACM de *tipus de representació finit*, és a dir, aquelles varietats que tenen només un nombre finit de fibrats vectorials ACM indescomposables (cf. [BGS87] i [EH88]). Va resultar que formen una llista ben curta: tres o menys punts reduïts a $\mathbb{P}^2$, un espai projectiu, una hipersuperfície quàdrica llisa $X \subset \mathbb{P}^n$, un scroll cúbic a $\mathbb{P}^4$, la superfície de Veronese a $\mathbb{P}^5$ o una corba normal racional.

Per a la resta de varietats ACM, va esdevenir un problema interessant el donar un criteri per a poder dividir-les en una classificació més fina. Un punt de vista prometedor ve donat per la teoria de representació, a on es va demostrar que les àlgebres de dimensió finita i tipus infinit (i.e., aquelles que tenen infinites representacions indescomposables) descomposen en dues classes: o bé són *tame (moderades)*, per les quals les representacions indescomposables d'una dimensió fixada formen un conjunt finit de famílies de dimensió com a màxim u; o bé són *salvatges*, per les quals existeixen famílies de dimensió arbitràriament gran de representacions indescomposables no isomórfiques (cf. [Dro86]). Un resultat anàleg també fou obtingut per la categoria de quivers, per la qual Gabriel va demostrar el següent sorprenent resultat de classificació: un quiver és de tipus de representació finit exactament quan el seu graf no dirigit subjacent és la unió de diagrames de Dynkin de tipus A, D, E (cf. [Gab72]). També l'estudi de la categoria de mòduls Cohen-Macaulay indescomposables sobre anells Cohen-Macaulay ha sigut una branca d'intensiva recerca recentment. Per tot això, motivats per aquests resultats, a [DG01], una tricotomia anàloga (és a dir, tipus de representació finita, moderada i salvatge) va ser proposada per a varietats projectives (veure Definicions 4.2.7 i 4.2.10). En el cas de dimensió u, es va provar que una tal tricotomia és exhaustiva: una corba projectiva llisa és de tipus finit (resp. moderat, salvatge) si i només si té génere 0 (resp. 1, ≥ 2). No obstant, és clar que, per a varietats projectives, aquesta tricotomia no pot ser exhaustiva. A [CH04], es va demostrar que el con quadràtic $X \subseteq \mathbb{P}^3$ té un conjunt infinit però discret de feixos ACM indescomposables. Des d'aquests resultats inicials, ha sigut un problema decidir el tipus de representació d'una varietat ACM donada. Es va demostrar a [CHb] que la superfície cúbica llisa de $\mathbb{P}^3$ és de tipus de representació salvatge. A [PLT09], es va veure que les superfícies del Pezzo de grau ≤ 6 també són de tipus de representació salvatge. Però, de fet, cap exemple de varietat de tipus de representació salvatge de dimensió > 2 era conegut. Per tant es proposa la següent questió:

Questió. *Donada una varietat projectiva $X \subseteq \mathbb{P}^n$, construir famílies de dimensió arbitrària de fibrats vectorials ACM indescomposables per tal de provar que X és de tipus de* *representació salvatge.*

Al **capítol 4**, fem una contribució a aquest problema mostrant que les dues famílies següents de varietats ACM són de tipus de representació salvatge: varietats Fano (i.e., varietats amb divisor anticanònic ample) obtingudes com a explosions de punts de $\mathbb{P}^n$, amb $n \geq 2$; i superfícies generals $X \subseteq \mathbb{P}^3$ de grau $3 \leq d \leq 9$ (veure Teoremes 4.3.13 i 4.5.8). En general, una de les dificultats principals que es troben per provar el fet de ser de tipus salvatge és assegurar la indescomposabilitat dels fibrats vectorials construïts. La estratègia que hem seguit per superar aquesta dificultat ha sigut intentar provar una propietat més forta de un fibrat vectorial que implicaria la indescomposabilitat. De fet, hem aconseguit provar que els fibrats vectorials $\mathcal{E}$ eran o bé simples (i.e., $\mathrm{End}(\mathcal{E}) = k$) o bé, en el millor dels casos, estables.

Entre d'altres característiques d'un fibrat vectorial donat, una especialment rica és el fet d'estar generatat per les seves seccions globals o, al menys, tenir-ne un gran nombre. La contrapartida algebraica havia aconseguit aixecar un gran interés. De fet, Ulrich va provar (cf. [Ulr84]) que per a un anell local (o *local graduat) R hi ha una fita superior pel mínim nombre de generadors d'un Màximament Cohen-Macaulay (MCM) R-mòdul M de rang positiu. Més precisament, si $\mu(M)$ denota el mínim nombre de generadors de M i $e(R)$ denota la multiplicitat de R, llavors es verifica sempre que $\mu(M) \leq e(R)\,\mathrm{rk}(M)$. Els mòduls MCM que assoleixen aquesta fita han sigut anomenats *Ulrich mòduls*. També en aquest cas l'existència d'aquests tipus de R-mòduls ajuda a entendre l'estructura de R. Per exemple, si un anell Cohen-Macaulay R suporta un mòdul Ulrich M verificant $\mathrm{Ext}^i_R(M, R) = 0$ per a $1 \leq i \leq \dim(R)$, llavor R és Gorenstein (cf. [Ulr84]). Per tant, és una questió interessant trobar quins anells Cohen-Macaulay suporten mòduls Ulrich. Una resposta positiva a aquesta questió és obtinguda, per exemple, quan $\dim(R) = 1$, quan R té multiplicitat minima o quan R és una intersecció completa estricte (i.e., R és una intersecció completa local tal que el seu anell graduat associat també és intersecció completa). Aquestes consideracions algebraiques impulsen a definir, per a una varietat projectiva $X \subseteq \mathbb{P}^n$, que un fibrat vectorial $\mathcal{E}$ a X sigui *Ulrich* si és ACM i el seu R_X-mòdul graduat associat $\mathrm{H}^0_*(\mathcal{E})$ és Ulrich. Cal observar que, quan $\mathcal{E}$ és inicialitzat (i.e., $\mathrm{H}^0(X, \mathcal{E}(-1)) = 0$ però $\mathrm{H}^0(X, \mathcal{E}) \neq 0$) llavors l'última condició és equivalent a que $\dim_k \mathrm{H}^0(\mathcal{E}) = \deg(X)\,\mathrm{rk}(\mathcal{E})$. Per a un fibrat vectorial $\mathcal{E}$ inicialitzat, el fet de ser Ulrich tè una interessant interpretació en termes cohomològics (s'ha de verificar que $\mathrm{H}^i(X, \mathcal{E}(-i)) = 0$ per a $i > 0$ i $\mathrm{H}^i(X, \mathcal{E}(-i-1)) = 0$ per a $i < \dim(X)$) i en termes de la seva $\mathcal{O}_{\mathbb{P}^n}$-resolució lliure minimal, ja que aquesta ha de ser linial de longitud $n - \dim(X)$.

Questió. *Donada una varietat projectiva ACM $X \subseteq \mathbb{P}^n$ i un enter $r \in \mathbb{Z}$, construir fibrats vectorials Ulrich de rang r amb suport a X.*

Sobre aquests resultats d'existència, és conegut que una corba arbitrària suporta fibrats vectorials Ulrich de rang u i dos (cf. [ESW03]). En el cas d'una corba plana, existeix una pregona relació entre l'existència d'aquests tipus de fibrats vectorials i la possibilitat d'escriure l'equació de la corba com el determinant (resp. el pfaffià) d'una matriu (resp. una matriu antisimètrica) amb entrades linials (cf. [Bea00]). Pel que respecta la hipersuperfície general $X \subseteq \mathbb{P}^{n+1}$ de grau d, és conegut que per a $n = 2$, X suporta fibrat vectorial Ulrich de rang 2 si i només si $d \leq 15$ i per a $n = 3$, això passa si i només si $d \leq 5$ (cf. [Bea00]). Per a $n \geq 4$ i $d \geq 3$, la hipersuperfície general n-dimensional no suporta un fibrat Ulrich de rang 2 (cf. [CM05]). A les superfícies i sòlids cúbics, l'existència de fibrats vectorials Ulrich de rang arbitrari ha sigut provat per Casanellas i Hartshorne a [CHb]. Al **capítol 4** ens enfrontem a aquests problemes i hi contribuim construint famílies de dimensió gran de fibrats vectorials simples i Ulrich de rang arbitrari sobre qualsevol superfície del Pezzo (veure Teoremes 4.4.11 i 4.4.19). També construim famílies de dimensió gran de fibrats vectorials simples i Ulrich de rang parell arbitrari a una superfície general $X \subseteq \mathbb{P}^3$ de grau $3 \leq d \leq 9$ (veure Teorema 4.5.8).

Una possible aproximació a la construcció de fibrats vectorials ACM i Ulrich sobre una varietat projectiva $X \subseteq \mathbb{P}^n$ ve donada per la coneguda *correspondència de Serre*. Per exemple, en el cas particular d'una superfície X, aquesta correspondència ofereix un diccionari entre fibrats vectorials $\mathcal{E}$ de rang 2 a X amb classes de Chern $c_1(\mathcal{E})$ i $c_2(\mathcal{E})$ i subsquemes zero-dimensionals localment intersecció completa $Z \subseteq X$ de longitud $c_2(\mathcal{E})$ tal que la parella $(\mathcal{O}_X(K_X + c_1(\mathcal{E})), Z)$ tenen la propietat de Cayley-Bacharach (cf. [HL97, Teorema 5.1.1]). Estan relacionats per la successió següent:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{Z|X}(c_1(\mathcal{E})) \longrightarrow 0.$$

A més, és possible traduir altra informació sobre $\mathcal{E}$ en termes de Z i *vice versa*. Per exemple, amb la notació prèvia, el fibrat vectorial $\mathcal{E}$ serà ACM si i només si Z és un esquema aritmèticament Gorenstein. Donat que aquesta propietat pot ser identificada a la resolució lliure minimal de Z, és un problema significatiu esbrinar la forma d'una resolució lliure minimal de l'anell de coordenades R_Z d'un conjunt general de punts Z visquent a una varietat donada X. Per a $X = \mathbb{P}^n$, aquest és un problema clàssic que ha reclamat molta atenció. Sabem que si Z és un conjunt general de punts diferents a $\mathbb{P}^n$ la seva resolució lliure minimal ha de ser de la forma:

$$0 \longrightarrow F_n \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow R_Z \longrightarrow 0$$

amb $F_0 = R := k[x_0, \ldots, x_n]$ i

$$F_i \cong R(-r-i)^{b_{i,r}} \oplus R(-r-i+1)^{b_{i,r-1}}$$

per a $i = 1, \ldots, n$, on r és l'únic enter no negatiu tal que

$$\binom{r+n-1}{n} \leq s < \binom{r+n}{n}.$$

A més es dóna:

$$b_{i+1,r-1} - b_{i,r} = \binom{r+i-1}{i}\binom{r+n}{n-i} - s\binom{n}{i}.$$

La *Conjectura de la Resolució Minimal (MRC)* proposada per Lorenzini (cf. [Lor93]) afirma que no existeixen termes *"ghost"* a la resolució lliure minimal de R_Z, i.e., $b_{i+1,r-1}b_{i,r} = 0$ per a tot i. Una gran quantitat de treball ha estat consagrat a aquesta conjectura. En particular, la MRC es verifica per a qualsevol nombre de punts s de $\mathbb{P}^n$ per a $n = 2$ (veure [Gae51]), $n = 3$ ([BG86]) i $n = 4$ ([Wal95]). MRC és també certa per a valors grans de s per a qualsevol n (veure [HS96]). En canvi, la MRC falla en general per a $n \geq 6$, $n \neq 9$ (veure [EPSW02]).

És també possible fixar-se només en la part inicial i final de la resolució lliure minimal de R_Z i llavors dues conjectures més febles han sigut proposades: la "*Ideal Generation Conjecture (IGC)*", la qual afirma que el nombre mínim de generadors de l'ideal d'un conjunt general de punts serà al més petit possible. En termes dels nombres de Betti, simplement afirma que $b_{1,r}(Z)b_{2,r-1}(Z) = 0$. Per altra banda, la "*Cohen-Macaulay type Conjecture (CMC)*"afirma que el mòdul canònic $K_Z = \mathrm{Ext}^n(R/I_Z, R(-n-1))$ té el menor nombre de generadors. Donat que el dual de la resolució minimal de R_Z dóna una resolució (torçada) de K_Z aquesta conjectura també es pot traduir en funció dels nombres de Betti: $b_{n-1,r}(Z)b_{n,r-1}(Z) = 0$. Respecte aquestes dues conjectures, CMC ha estat provada amb tota generalitat en el cas de l'espai projectiu $X = \mathbb{P}^n$, per a tot n (veure [Tru89, p. 112]). És també sabut que la IGC es verifica per a conjunts grans de punts a corbes de grau $d \geq 2g$ (veure [FMP03]).

Més recentment Mustaţă ha estès els resultats anteriors sobre la forma de la resolució lliure minimal de conjunts generals de punts $Z \subseteq X$ per al cas $X = \mathbb{P}^n$

a una varietat projectiva arbitrària $X \subseteq \mathbb{P}^n$ (cf. [Mus98]). Ha provat que les primeres files del diagrama de Betti d'un conjunt general de punts diferents Z en una varietat projectiva X coincideixen amb les del diagrama de Betti de X i que hi ha dues files afegides al final del diagrama. També ha donat fites inferiors pels nombres de Betti d'aquestes dues últimes files. En altres paraules, si

$$0 \to F_n \to F_{n-1} \to \cdots \to F_2 \to F_1 \to R \to R_X \to 0$$

és una R-resolució lliure minimal de R_X, llavors per a un conjunt general de punts $Z \subseteq X$ tal que $P_X(r-1) \leq |Z| < P_X(r)$ per a algun $r \geq \text{reg}(X)+1$ (on P_X denota el polinomi de Hilbert de X), R_Z té R-resolució lliure minimal de la forma

$$0 \longrightarrow F_n \oplus R(-r-n+1)^{b_{n,r-1}(Z)} \oplus R(-r-n)^{b_{n,r}(Z)} \longrightarrow \cdots$$
$$\longrightarrow F_2 \oplus R(-r-1)^{b_{2,r-1}(Z)} \oplus R(-r-2)^{b_{2,r}(Z)} \longrightarrow$$
$$\longrightarrow F_1 \oplus R(-r)^{b_{1,r-1}(Z)} \oplus R(-r-1)^{b_{1,r}(Z)} \longrightarrow R \longrightarrow R_Z \longrightarrow 0.$$

Com en el cas general, Mustaţă va proposar la Conjectura de la Resolució Minimal en aquest context afirmant que els nombres de Betti graduats seran el més petits possibles: $b_{i+1,r-1}(Z)b_{i,r}(Z) = 0$ per a tot i. Aquesta versió de la conjectura ha estat ja estudiada en alguns casos interessants. Per exemple, la MRC és certa per a tot nombre de punts generals en la quàdrica llisa de $\mathbb{P}^3$ (cf. [GMR96]) i per a algunes cardinalitats especials de conjunts de punts generals a la superfície cúbica llisa. L'estudi de la MRC per a corbes va ser realitzat a [FMP03], on es va demostrar que la conjectura es verifica per a grans cardinalitats de punts generals sobre corbes canòniques $C \subseteq \mathbb{P}^n$ (i.e., corbes submergides a $\mathbb{P}^n$ pel seu divisor canònic). Malgrat tot, contràriament al cas de l'espai projectiu, la MRC és falsa per a conjunts de punts de cardinalitat arbitràriament gran sobre corbes de grau elevat.

Al **capítol 2** ens centrem en les tres conjectures que acabem de mencionar en el cas de conjunts generals de punts a *superfícies ACM quasi-minimals* (no necessàriament llises), les quals estan definides com a varietats ACM no degenerades $X \subseteq \mathbb{P}^d$ tal que $\deg(X) = \text{codim}(X) + 2$. Cal recordar que donada una varietat no degenerada $X \subseteq \mathbb{P}^d$ sempre es verifica que $\deg(X) \geq \text{codim}(X) + 1$. Les *varietats minimals*, i.e., varietats per a les quals es té la igualtat en l'expressió prèvia han sigut classificades clàssicament. El següent cas, el de varietats quasi-minimals, ha sigut el centre d'intensa recerca recentment. Una bona classificació de tals varietats ha estat obtinguda per Fujita (cf. [Fuj90]), relacionada amb la seva teoria de Δ-gènere. En el cas de dimensió dos, la família de superfícies fortes de del Pezzo és un cas particularment significatiu de superfícies ACM quasi-minimals. A [Hoa93], una important contribució es va fer a la comprensió de les varietats quasi-minimals, i en particular a la estructura de les singularitats que poden tenir. Entre altres resultats, la resolució lliure minimal de l'anell de coordenades d'una superfície ACM quasi-minimal $X \subseteq \mathbb{P}^d$ va ser donada:

$$0 \longrightarrow R(-d) \longrightarrow R(-d+2)^{\alpha_{d-3}} \longrightarrow \ldots \longrightarrow R(-2)^{\alpha_1} \longrightarrow R \longrightarrow R_X \longrightarrow 0$$

on

$$\alpha_i = i\binom{d-1}{i+1} - \binom{d-2}{i-1} \text{ per a } 1 \leq i \leq d-3.$$

El coneixement d'aquesta resolució és clau per als resultats que hem obtingut. Provarem que la IGC i la CMC es verifiquen per a conjunts generals de punts de qualsevol cardinalitat a una superfície ACM quasi-minimal X, tret de dos casos esporàdics (veure Teorema 2.2.16). Pel que fa la MRC, provarem que és certa per a un ample ventall de cardinalitats de punts generals a X (veure Teorema 2.2.15).

Cal observar que en termes de l'esquema de Hilbert $\mathrm{Hilb}^s(X)$ de subesquemes zero-dimensionals de X, la Conjectura de la Resolució Minimal per a X pot ser plantejada diguent que existeix un subconjunt obert i no buit $U_0^s \subset H_0^s \subset \mathrm{Hilb}^s(X)$, on H_0^s denota la component irreductible el punt general de la qual correspon a conjunts Z de s punts *distints* a X, tal que per a tot $[Z] \in U_0^s$ es verifica

$$b_{i+1,r-1}(Z) \cdot b_{i,r}(Z) = 0 \quad \text{per a} \quad i = 1, \cdots, n-1.$$

Si no ens volem restringir a conjunt de punts diferents, ens podem preguntar cóm ha de ser la forma de la resolució lliure minimal de l'ideal homogeni de l'esquema 0-dimensional associat a un punt general $[Z]$ de qualsevol altra component irreductible de $\mathrm{Hilb}^s(X)$ i preguntar-nos si els nombres de Betti $b_{ij}(Z)$ són els més petits possibles, i.e., no hi ha termes "ghost"en la resolució lliure minimal de R_Z. Al **capítol 2** proposem una conjectura modificada i diem que la *Conjectura Feble de la Resolució Minimal(WMRC)* es verifica per a s si existeix una component irreductible $H^s \subset \mathrm{Hilb}^s(X)$ i un subconjunt obert i no buit $U^s \subset H^s \subset \mathrm{Hilb}^s(X)$ tal que per a tot $[Z] \in U^s$ es té

$$b_{i+1,r-1}(Z) \cdot b_{i,r}(Z) = 0 \quad \text{per a} \quad i = 1, \cdots, n-1.$$

Respecte la WMRC, provem que per a tot enter $d \geq 2$ i per a tot $s \geq \binom{d+3}{3} - 1$, existeix una família $\binom{d+2}{2}$-dimensional de superfícies irreductibles i genèricament llises $X \subset \mathbb{P}^3$ de grau d satisfent aquesta conjectura (veure Teorema 2.3.18).

Per descomptat, en el cas que $\text{Hilb}^s(X)$ sigui irreductible, ambdues conjectures, la MRC tal com va ser proposada per Mustaţă i la nostra conjectura modificada han de coincidir. Per tant esdevé una questió crucial el saber quan l'esquema de Hilbert $\text{Hilb}^s(X)$ és irreductible. En general, des de que l'existència de l'esquema de Hilbert $\text{Hilb}^{p(t)}(X)$ parametritzant subesquemes projectius d'una varietat projectiva X amb polinomi de Hilbert $p(t)$ va ser demostrada a [Gro] per Grothendieck, l'estudi de les propietats geomètriques d'aquest espai de moduli va esdevenir una àrea d'intensa recerca en Geometria Algebraica. Un resultat primerenc de Hartshorne (cf. [Har66]) afirma que sempre és connex. Quan ens centrem en subesquemes de polinomi de Hilbert constant $p(t) = s$, i.e, quan es tracta de subesquemes zero-dimensionals de longitud s, Fogarty va provar que, si X és una superfície irreductible *llisa*, llavors l'esquema de Hilbert $\text{Hilb}^s(X)$ és una varietat irreductible i llisa de dimensió $2s$ (cf. [Fog68]). En dimensions més grans, Iarrobino a [Iar72] va trobar que la irreductibilitat no té per què donar-se: l'esquema de Hilbert pot ser irreductible per a varietats de dimensió ≥ 3. En el curt **capítol 3** ens centrem en varietats singulars i ens preguntem sobre la irreductibilitat de l'esquema de Hilbert dels seus subsequemes 0-dimensionals. El cas més interessant, degut al resultat de Fogarty, és el de superfícies singulars:

Questió. *És l'esquema de Hilbert* $\text{Hilb}^s(X)$ *de subesquemes 0-dimensionals de longitud* s *en una superfície singular* X *irreductible?*

Donarem una resposta negativa a aquesta questió construint superfícies singulars per les quals els esquemes de Hilbert de punts són reductibles. De fet, el nostre mètode també funciona per a varietats de dimensió més gran. Construirem varietats projectives genèricament llises $X \subset \mathbb{P}^N$ de dimensió n i grau d amb $n > 2$ i $d > 1$ o $n = 2$ i $d > 4$ per a les que $\text{Hilb}^s(X)$ és reductible per a tot $s \gg 0$ (veure Teorema 3.1.5).

Donem ara l'estructura d'aquesta tesi i els principals resultats obtinguts.

El **capítol 1** és dedicat a recordar les nocions que seran l'objecte de la resta del present treball així com resultats ben coneguts que ens seran útils. També donem alguns exemples dels conceptes que hi són involucrats. Aquest capítol no conté cap idea original.

Comencem a la secció 1.1 introduint les nocions bàsiques de resolució lliure minimal i de diagrama de Betti associat a un mòdul graduat M, a més de les de funció i polinomi de Hilbert. També introduim la noció d'esquema Aritmèticament Cohen-Macaulay (ACM) i Aritmèticament Gorenstein (AG).

A la secció 1.2 donem els rudiments de la Teoria de Liaison que es revelarà clau en la prova dels resultats del capítol 2. La Teoria de Liaison és una eina molt potent a l'hora de transportar informació des de un esquema donat a un segon esquema amb el que està lligat. Ilustrarem aquesta propietat de la Liaison amb diversos resultats importants (com és el cas del Teorema de Gaeta). Veurem cóm les resolucions lliures minimals de dos subesquemes lligats estan relacionades.

Finalment, a la secció 1.3, ens fixem en els espais de moduli. Donem una introducció a l'esquema de Hilbert $\mathrm{Hilb}^{p(t)}(X)$ que parametritza subesquemes d'un esquema donat X amb polinomi de Hilbert $p(t)$ i també de l'espai de moduli $M^s_{X,H}(r; c_1, \ldots, c_{\min(r,n)})$ de fibrats vectorials μ-estables $\mathcal{E}$ a X amb rang fixat r i classes de Chern c_i.

Al **capítol 2** s'ofereix la nostra contribució a la Conjectura de la Resolució Minimal, que està bàsicament dividida en dues parts. Primer de tot, demostrarem que es verifica per a un ample ventall de cardinalitats de conjunts generals de punts sobre una gran família de varietats, a saber, la de varietats ACM quasi-minimals (tret de dos casos esporàdics). Per altra banda, treballarem amb el cas especial d'esquemes zero-dimensionals no reduïts. Per aquests esquemes, plantejarem una versió adaptada de la MRC (la *Conjectura Feble de la Resolució Minimal(WMRC)*) i provarem que és satisfeta en alguns casos interessants.

A la secció 2.1, recordem la Conjectura de la Resolució Minimal(MRC) i donem un breu resum dels resultats coneguts entorn d'ella. En particular, recordem la versió de Mustaţă de la MRC:

Conjectura 2.1.10. Sigui $X \subset \mathbb{P}^n$ una varietat projectiva amb $d = \dim(X) \geq 1$, $\mathrm{reg}(X) = m$ i polinomi de Hilbert P_X. Sigui $s \in \mathbb{Z}$ un enter tal que $P_X(r-1) \leq s < P_X(r)$ per a algun $r \geq m+1$. La *Conjectura de la Resolució Minimal(MRC)* es satisfà pel valor s si per a un conjunt Z de s punts generals diferents es té

$$b_{i+1,r-1}(Z)b_{i,r}(Z) = 0 \quad \text{per a} \quad i = 1, \cdots, n-1.$$

A la secció 2.2, ens fixem en les superfícies ACM quasi-minimals, i.e, superfícies $X \subseteq \mathbb{P}^d$ de grau d. Per a aquesta classe de superfícies, establim primer la MRC per a dues cardinalitats específiques de punts:

Teorema 2.2.13. Sigui $X \subseteq \mathbb{P}^d$ una superfície ACM quasi-minimal. Assumim que X no és el model anticanònic de $\mathrm{F}_2 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ o una intersecció de dues quàdriques de $\mathbb{P}^4$ amb una recta doble. Definim:

$$m(r) := \frac{d}{2}r^2 + r\frac{2-d}{2}, \quad n(r) := \frac{d}{2}r^2 + r\frac{d-2}{2}.$$

Llavors es té:

(1) Sigui $Z_{n(r)} \subset X$ un conjunt general de $n(r)$ punts, $r \geq 2$. Llavors la resolució minimal graduada de $I_{Z_{n(r)}|X}$ té la forma:

$$0 \longrightarrow R(-r-d)^{(d-1)r-1} \longrightarrow R(-r-d+1)^{\beta_{d-1,r}} \longrightarrow R(-r-d+2)^{\beta_{d-2,r}} \longrightarrow \dots$$
$$\longrightarrow R(-r-2)^{\beta_{2,r}} \longrightarrow R(-r)^{r+1} \longrightarrow I_{Z_{n(r)}|X} \longrightarrow 0.$$

on

$$\beta_{i,r} = \sum_{l=0}^{1}(-1)^{l+1}\binom{n-l-1}{i-l}\Delta^{l+1}H_X(r+l) + \binom{n}{i}(n(r) - H_X(r-1)).$$

(2) Sigui $Z_{m(r)} \subset X$ un conjunt general de $m(r)$ punts, $r \geq 2$. Llavors la resolució minimal graduada de $I_{Z_{m(r)}|X}$ té la forma:

$$0 \longrightarrow R(-r-d)^{r-1} \longrightarrow R(-r-d+2)^{\gamma_{d-1,r-1}} \longrightarrow \dots$$
$$\longrightarrow R(-r-1)^{\gamma_{2,r-1}} \longrightarrow R(-r)^{(d-1)r+1} \longrightarrow I_{Z_{m(r)}|X} \longrightarrow 0$$

amb

$$\gamma_{i,r-1} = \sum_{l=0}^{1}(-1)^{l}\binom{n-l-1}{i-l}\Delta^{l+1}P_X(r+l) - \binom{n}{i}(m(r) - P_X(r-1)).$$

En particular, la conjectura de Mustaţă es verifica per $n(r)$ i $m(r)$, $r \geq 4$, punts generals diferents sobre una superfície ACM quasi-minimal $X \subset \mathbb{P}^d$ (excepte en els dos casos que hem mencionat).

El Teorema previ ens permetrà deduir els resultats següents: primer provarem que les dues conjectures més febles, la IGC i la CMC són certes per a tot conjunt general de punts a superfícies ACM quasi-minimals (excepte per a dos casos esporàdics):

Teorema 2.2.16. Sigui $X \subseteq \mathbb{P}^d$ una superfície ACM quasi-minimal. Assumim que X no és el model anticanònic de $F_2 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ o una intersecció de dues quàdriques de $\mathbb{P}^4$ amb una recta doble. Llavors per a tot conjunt general de punts diferents Z a X tal que $|Z| \geq P_X(3)$ la CMC i la IGC són certes.

A més, provarem que un conjunt general de punts diferents amb cardinalitat compresa dins de determinades franges verifica la MRC (excepte per als dos mateixos casos esporàdics):

Teorema 2.2.15. Sigui $X \subseteq \mathbb{P}^d$ una superfície ACM quasi-minimal. Assumim que X no és el model anticanònic de $F_2 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ o una intersecció de dues quàdriques de $\mathbb{P}^4$ amb una recta doble. Sigui r un enter tal que $r \geq reg(X) + 1 = 4$. Llavors per a tot conjunt general de punts diferents Z a X tal que $P_X(r-1) \leq |Z| \leq m(r)$ o $n(r) \leq |Z| \leq P_X(r)$ la Conjectura de la Resolució Minimal és verdadera.

Pel cas particular de la superfície cúbica integral, veurem que la MRC es verifica per a tot conjunt general de punts diferents.

Teorema 2.2.17. Sigui $X \subseteq \mathbb{P}^3$ una superfície cúbica integral (i.e., una superfície ACM quasi-minimal de grau tres). Llavors la Conjectura de la Resolució Minimal es verifica per a tot conjunt general de punts diferents a X de cardinalitat $\geq P_X(3) = 19$.

A la secció 2.3, dediquem la nostra atenció a una conjectura lleugerement modificada. Donat que, en general, $\mathrm{Hilb}^s(X)$ no és irreductible (veure [Iar72] pel cas de varietats de dimensió més gran o igual que 3 i el capítol 3 per a superfícies), podem també preguntar-nos per la resolució minimal graduada de l'ideal de l'esquema 0-dimensional associat a un punt general de qualsevol altra component irreductible de l'esquema de Hilbert $\mathrm{Hilb}^s(X)$ i per si els nombres de Betti graduats són el més petits possibles, i.e. si no hi ha termes "ghost"a la resolució lliure minimal. Per tant plantejem la següent conjectura:

Conjectura 2.3.2. Sigui $X \subset \mathbb{P}^n$ una varietat projectiva, sigui $P_X(t)$ el seu polinomi de Hilbert i $m = \mathrm{reg}(X)$. Sigui s un enter tal que $P_X(r-1) \leq s < P_X(r)$ per a algun $r \geq m+1$. Llavors, la *Conjectura Feble de la Resolució Minimal(WMRC)* es satisfà si per a s si existeix una component irreductible $H^s \subset \mathrm{Hilb}^s(X)$ i un subconjunt obert no buit $U^s \subset H^s \subset \mathrm{Hilb}^s(X)$ tal que per a tot $[Z] \in U^s$ es té

$$b_{i+1,r-1}(Z) \cdot b_{i,r}(Z) = 0 \quad \text{per a} \quad i = 1, \cdots, n-1.$$

En particular, podem contribuir de la manera següent a aquesta conjectura:

Teorema 2.3.18. Sigui s un enter tal que $s \geq P_d(d)$, $d \geq 2$. Llavors existeix una família de dimensió $\binom{d+2}{2}$ de superfícies irreductibles genèricament llises $X \subset \mathbb{P}^3$ de grau d per la qual la WMRC es satisfà, i.e., existeix un subconjunt obert no buit $U^s \subset \mathrm{Hilb}^s(X)$ tal que per a tot $[Z] \in U^s$ es té

$$b_{3,r-1}(Z) \cdot b_{2,r}(Z) = b_{2,r-1}(Z) \cdot b_{1,r}(Z) = 0.$$

En el **capítol 3** ens centrem en la reductibilitat de l'esquema de Hilbert de punts. Com ha estat mencionat, Fogarty va provar que, si X és una superfície llisa i irreductible, llavors l'esquema de Hilbert $\mathrm{Hilb}^s(X)$ és una varietat llisa i irreductible de dimensió $2s$. Una pregunta natural que per tant esdevé en aquest context és el comportament de l'esquema de Hilbert quan la condició de ser llisa és eliminada. En aquest capítol construirem famílies de superfícies singulars per a les quals l'esquema de Hilbert de punts és reductible. De fet, el nostre mètode també funciona per a varietats de dimensió superior. Més concretament, tenim:

Teorema 3.1.5. Sigui $X = \langle Y, p\rangle \subseteq \mathbb{P}^N$ un con n-dimensional amb vèrtex p i base $Y \subseteq \mathbb{P}^{N-1}$. Supposem que o bé $n > 2$ i $\deg X > 1$ o bé $n = 2$ i $\deg X > 4$. llavors existeix $s_0 \in \mathbb{N}$ tal que $\mathrm{Hilb}^s(X)$ és reductible per a tot $s \geq s_0$.

Finalment, el **capítol 4** es dedicat a l'estudi de fibrats vectorials ACM i en particular al tipus de representació d'algunes famílies de varietats. Com ha estat mencionat, és una questió interessant trobar el tipus de representació d'una varietat ACM donada ja que és una bona mesura de la seva complexitat. L'objectiu principal d'aquest capítol és donar els primers exemples de varietats n-dimensional ACM de tipus de representació salvatge, per a arbitrari $n \geq 2$ (cf. Teoremas 4.3.13 i 4.4.11). La nostra font d'exemples serà les varietats Fano $X = Bl_Z\mathbb{P}^n$ resultat d'explotar $\mathbb{P}^n$ a un conjunt finit de punts Z. En el cas 2-dimensional , i.e., per a superfícies de del Pezzo, molta més informació serà obtinguda, ja que els fibrats vectorials que construirem comparteixen una característica particular: el mòdul associat $\oplus_t \mathrm{H}^0(X, \mathcal{E}(t))$ té el nombre màxim possible de generadors (veure Teorema 4.4.11). Aquesta propietat va ser aïllada per Ulrich a [Ulr84, p. 26] per a mòduls Cohen-Macaulay, i des de llavors els mòduls amb aquesta propietat s'anomenen mòduls de Ulrich i, respectivament, fibrats vectorials de Ulrich en el cas geomètric. A continuació, pel cas d'una superfície general $X \subseteq \mathbb{P}^3$ hem pogut provar que són de tipus salvatge per a $d \leq 9$, basant-nos en resultats previs sobre l'existència de fibrats vectorials Ulrich de rang 2 en la superfície (veure [Bea00, Proposition 7.6]). Per a grau arbitrari d podem almenys donar famílies grans de fibrat vectorials ACM de rang 2 i 3 a la superfície general de grau d demostrant amb això que no són de tipus finit o moderat.

Aquest capítol es divideix de la manera següent: a la secció 4.1 recordem la definició i característiques principals de les varietats amb les que treballarem, *Fano* blow-ups de $\mathbb{P}^n$, $n \geq 2$, i superfícies de *del Pezzo*. A la secció 4.2, donem un resum dels fibrats vectorials ACM i Ulrich, a més de discutir el problema de estudiar la complexitat d'una varietat ACM en funció de la complexitat de les famílies de fibrats vectorials ACM que suporta.

A la secció 4.3, realitzem la construcció de famílies d'elevada dimensió de fibrats vectorials simples (i per tant indescomposables) i ACM als Fano blow-ups de punts a $\mathbb{P}^n$. Aquestes famílies seran construïdes com el pullback del nucli de morfismes exhaustius

$$\mathcal{O}_{\mathbb{P}^n}(1)^b \longrightarrow \mathcal{O}_{\mathbb{P}^n}(2)^a$$

amb la propietat que també són exhaustius al nivell de seccions globals. Per tant podem demostrar que els Fano blow-ups de punts de $\mathbb{P}^n$ són varietats de tipus de representació salvatge. En particular, provem:

Teorema 4.3.13. Sigui $X = Bl_Z\mathbb{P}^n$ una varietat Fano definida com el blow-up de punts a $\mathbb{P}^n$, $n \geq 3$ i sigui $r \geq n$.

i Si n és parell, fixem $c \in \{0, \dots, n/2 - 1\}$ tal que $c \equiv r \bmod n/2$ i definim $u := \frac{2(r-c)}{n}$. Llavors existeix una família de fibrats vectorials simples i ACM de rang r de dimensió $\frac{(n+2)n-4}{4}u^2 - cu - c^2 + 1$.

ii Si n és senar, fixem $c \in \{0, \dots, n-1\}$ tal que $c \equiv r \bmod n$ i definim $u := \frac{(r-c)}{n}$. Llavors existeix una família de fibrats vectorials simples i ACM de rang r de dimensió $((n+2)n - 4)u^2 - 2cu - c^2 + 1$.

En particular, les varietats Fano obtingudes com a blow-ups de punts de $\mathbb{P}^n$ són de tipus de representació salvatge.

A la secció 4.4, ens centrem en el cas 2-dimensional, és a dir, en el cas de superfícies de del Pezzo, on molta més informació és obtinguda. A la primera subsecció treballem amb qualsevol superfície de del Pezzo excloent el cas de la quàdrica i veiem que els fibrats vectorials ACM obtinguts a la secció anterior per pullback són simples, Ulrich, i μ-estables respecte a cert divisor ample H_n:

Teorema 4.4.11. Sigui $X \subseteq \mathbb{P}^d$ una superfície de del Pezzo de grau d. Assumim que X no és la quàdrica llisa submergida a $\mathbb{P}^8$ mitjançant el divisor anticanònic $-K_X$. Llavors per a tot $r \geq 2$ exiteix una família de dimensió $r^2 + 1$ de fibrats vectorials simples, inicialitzats i Ulrich de rang r amb classes de Chern $c_1 = rH$ i $c_2 = \frac{dr^2+(2-d)r}{2}$. A més, són μ-semiestables respecte de la polarització $H = 3e_0 - \sum_{i=1}^{9-d} e_i$ i μ-estables respecte de $H_n := (n-3)e_0 + H$ per a $n \gg 0$. En particular, les superfícies de del Pezzo són de tipus de representació salvatge.

A la subsecció intermitja ens centrarem en el cas de la quàdrica i demostrarem per un argument *ad hoc* que és una varietat de tipus de representació salvatge:

Teorema 4.4.19. Sigui $X \subseteq \mathbb{P}^8$ una quàdrica llisa submergida a $\mathbb{P}^8$ mitjançant el divisor anticanònic molt ample $H := -K_X$. Llavors, per a qualsevol $r \geq 2$, existeix una família de fibrats vectorials simples (i per tant indescomposables) i Ulrich de rang r de dimensió $r^2 + 1$. En particular, X és una varietat de tipus de representació salvatge.

Finalment, a l'última subsecció, establim, per a una superfície de del Pezzo X amb divisor anticanònic molt ample, una versió de la coneguda correspondència de Serre (cf. Teorema 4.4.21). Aquesta correspondència ens permetrà, per una banda, demostrar, quan X no és la quàdrica, que les famílies de fibrats vectorials de rang r construïdes a la primera subsecció podien ser obtingudes també a partir d'un conjunt general de $m(r) := \frac{d}{2}r^2 + r\frac{2-d}{2}$ punts diferents de la superfície amb resolució lliure minimal com en el Teorema 2.2.13.

Corollary 4.4.22. Sigui $X \subseteq \mathbb{P}^d$ superfície de del Pezzo amb divisor anticanònic molt ample de grau d, diferent de la quàdrica. Llavors els fibrats vectorials $\mathcal{E}(H)$ inicialitzats Ulrich de rang r donats al Teorema 4.4.11 es poden recuperar com a una extensió de $\mathcal{I}_{Z,X}(rH)$ per $\mathcal{O}_X^{r-1}$ per a conjunts generals Z de $m(r) = 1/2(dr^2 + (2-d)r)$ punts diferents de X, $r \geq 2$.

Per altra banda, per a la quàdrica, aplicarem la correspondència de Serre en sentit contrari per obtenir la resolució lliure minimal d'un conjunt de $m(r)$ punts generals a partir dels fibrats vectorials Ulrich construïts a la secció prèvia.

Finalment, la secció 4.5 està dedicada al cas d'una superfície *general* X de grau arbitrari d a $\mathbb{P}^3$. Construint fibrats vectorials simples i Ulrich de rang parell arbitrari com extensions de fibrats Ulrich de rang 2, sarem capaços de demostrar que, per a $4 \leq d \leq 9$, una superfície general $X \subseteq \mathbb{P}^3$ de grau d és de tipus de representació salvatge:

Teorema 4.5.8. Sigui $X \subseteq \mathbb{P}^3$ una superfície general de grau $4 \leq d \leq 9$. Llavors, per a tot $r = 2s$, $s \geq 2$, existeix una família de fibrats vectorials de rang r, simples (i per tant indescomposables) i Ulrich de dimensió $11(s-1)$. En particular, una superfície general $X \subseteq \mathbb{P}^3$ de grau $4 \leq d \leq 9$ és de tipus de representació salvatge.

En el cas de grau arbitrari d, sarem capaços de construir almenys famílies infinites de fibrats vectorials de rang 2 i 3, simples i ACM sobre una superfície general $X \subseteq \mathbb{P}^3$ de grau d, demostrant que no són de tipus de representació moderat:

Proposició 4.5.10. Sigui $X \subseteq \mathbb{P}^3$ una superfície general de grau $d \geq 3$. Llavors existeix una família 4-dimensional de fibrats vectorials $\mathcal{E}$ de rang 2, inicialitzats, μ-estables i ACM amb $c_1(\mathcal{E}) = 1$ i $c_2(\mathcal{E}) = d - 1$.

Proposició 4.5.11. Sigui $X \subseteq \mathbb{P}^3$ una superfície general de grau $d \geq 3$. Llavors existeix una família infinita de fibrats vectorials $\mathcal{F}$ de rang 3, inicialitzats, μ-estables i ACM amb $c_1(\mathcal{F}) = 1$ i $c_2(\mathcal{F}) = 2d - 3$.

Acabarem el capítol donant una estretègia general que podria ser útil per provar que una superfície general de grau arbitrari és de tipus de representació salvatge (veure Teorema 4.5.14).

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