Treball final de grau

GRAU DE MATEMÀTIQUES

Facultat de Matemàtiques
Universitat de Barcelona

THEOREM OF HAYMAN-WU

Míriam Tomàs Jardí

Director: Dr. Xavier Massaneda
Realitzat a: Departament de Matemàtiques i Informàtica

Barcelona, **June 29, 2017**

Abstract

This work consists in the statement and proof of the Hayman-Wu theorem:

Let φ be a conformal mapping from de unit disk $\mathbb{D}$ to a simply connected domain Ω in the complex plane and let L be any line. Then

$$\mathrm{lenght}(\varphi^{-1}(L \cap \Omega)) \leq 4\pi.$$

We will present the elementary proof based on an idea of Knut Øyma, following the sketch in the first chapter of the book by *John B. Garnett and Donald E. Marshall* named *Harmonic Measure* [3].

To state and prove this theorem we study various notions and previous results in the fields of complex analysis and potential theory. Examples of these are: automorphisms of the disk (or of a simply connected domain in general), pseudohiperbolic distance, the Schwarz and Schwarz-Pick lemmas, Riemann's theorem on conformal mapping, harmonic functions, the Dirichlet problem and harmonic measure.

Acknowledgments

I would like to express my gratitude to my director, who has always kindly guided me throughout the project, Xavier Massaneda. So, thank you very much for your dedication and time.

Contents

1 Introduction

The context of this memory is the complex world, just like our life, where everything stops being trivial. Here, the numbers are accompanied by their close friend which is the letter i, although there are real numbers that remain faithfully independent.

I would like to point out that thanks to my passing through the Faculty of Mathematics of the UB, in addition to learning many concepts I did not know, I have got a logical and rational vision of how to face a problem, something which I also apply to real life.

Brief history of the theorem

The Hayman-Wu theorem is stated as follows:

Let φ be a conformal mapping from $\mathbb{D}$ to a simply connected domain Ω and let L be any line. Then

$$\text{lenght}(\varphi^{-1}(L \cap \Omega)) \leq 4\pi.$$

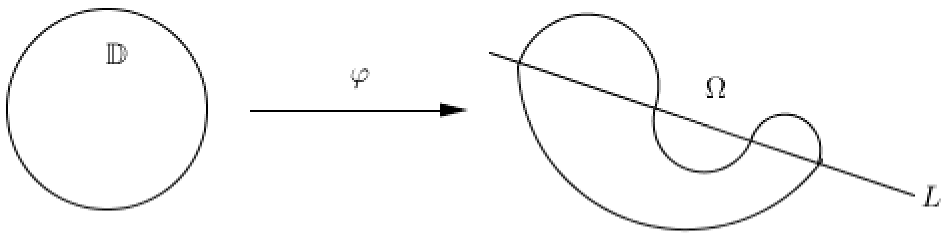

Figure 1: Hayman-Wu theorem.

Many authors have contributed to problems related to the Hayman-Wu theorem. For the sake of brevity we will only mention the results that are directly concerned with the constant C such that $\text{lenght}(\varphi^{-1}(L \cap \Omega)) \leq C$.

Hayman and Wu [4] gave the first proof of the theorem with C being the large constant $C = 2 \cdot 10^{35}$. Garnett, Gehring and Jones [3] obtained a shorter proof, but did not use it to estimate the constant. Fernández, Heinonen and Martio [2] showed that $C \leq 4\pi^2$. Then Knut Øyma [5] showed that $C \leq 4\pi$. Finally, Rohde [8] proved that the best constant is stricly smaller than 4π.

On the other hand again Knut Øyma [6] proved that the best constans is at least π^2.

It is still not known what is the best possible constant.

The proof we present is based on Knut Øyma's elementary proof, which as we mentioned, states that $C \leq 4\pi$.

Structure of the Project

The Hayman-Wu theorem is an easy-to-understand statement, but its proof requires several prior concepts. Therefore, the work is divided into two sections.

The first one includes some preliminaries, in which we will recall basic notions of complex analysis. in particular, we will recall the definition of holomorphic function and its caracterization through the Cauchy Riemann equations. We shall review also how the conformality is the geometric interpretation of these equations, which generically imply preservation of angles.

Then we will see that a special kind of conformal map are the Möbius transformations. These transformations are very important because can naturally extend to a biholomorphism (i.e., a comformal and bijective map) of the Riemann sphere. We will denote by $Aut(\mathbb{D})$ the group of holomorphic automorphisms of $\mathbb{D}$ and we will see that it consists precisely of the Möbius transformations that send the disk to the disk.

We will continue to define the pseudohyperbolic distance. First we will do so for the disk and later, with the help of Riemann's Theorem, we will see how it can be defined on any simply connected domain Ω. This distance is very important in the theory of conformal mapping because of its invariance by automorphisms. Another important feature for this distance is that it is contractive with respect to holomorphic mappings. This will be proved with the help of the Schwarz-Pick lemma, an invariant version of the Schwarz lemma.

After that, we will talk about Riemann's Theorem and its consequences. This is a central theorem, deep in the theory. Riemann's theorem states roughly that any two simply connected domains are conformally equivalent, provided that neither of them is the whole plane. This is an existence theorem; it shows that there is a conformal mapping between two such domains but it does not produce the corresponding conformal mapping.

This theorem will allow us to extend to any simply connected domain Ω many notions previoulsy seen in the disk. For example, we will see how $Aut(\Omega)$, the group of holomorphisms of Ω, can be described through the group $Aut(\mathbb{D})$, or how the solution to the Dirichlet problem in $\mathbb{D}$ is transported to Ω, at least when the boundary of Ω is good enough. As an example of particular interest we will compute the pseudohyperbolic distance in the right half-plane.

We will continue by studying Harmonic functions and the Dirichlet Problem. For the Hayman-Wu theorem it is necessary to use some properties about harmonic functions, which are directly connected to holomorphic functions, such as the Mean-Value property, the Converse to Mean-Value property, the Identity Principle and the Maximum Principle. We will also need to deal with the Dirichlet problem, which consists in finding a harmonic function on a 'nice' domain with prescribed boundary values. We will prove the uniqueness of the solution, and in order to solve the Dirichlet problem in the disk, we will introduce the Poisson kernel transform. This will give directly the solution to the Dirichlet problem in any disk.

Ending this section, we will introduce the Harmonic measure, which can be viewed as the solution to a particular Dirichlet problem. I would like to remark that, as we have indicated before, the solution of the Dirichlet problem in a simply connected domain can be obtained from the solution of the same problem in $\mathbb{D}$ through the map given by Riemann's theorem, at least if the domain is good enough at the boundary. We shall see the explicit form of the harmonic measure in the upper half-plane as a canonical example which will be used for the proof of the Hayman-Wu Theorem.

The final chapter includes the statement and proof of the theorem of Hayman-Wu. This is the core of the project. As we have already mentioned, the Hayman-Wu theorem is a result on conformal mapping. The proof we present uses harmonic measure and the pseudohyperbolic metric seen in the previous chapters. As we have already mentioned, we will give an elementary proof based on an idea of the late Knut Øyma.

2 Preliminaries. Holomorphic functions and conformal mapping

In this chapter we will recall basic notions of complex analysis that will be used through the text. It will also help to fix the notation.

First, we will define holomorphic functions and recall their caracterization through the Cauchy Riemann equations.

Definition 2.1. *A* ***holomorphic function*** *is a complex-valued function of one complex variable that is complex differentiable in a neighborhood of every point in its domain. With other words, given a complex-valued function f of a single complex variable, the derivative of f at a point z_0 in its domain is defined by*

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

This is formally the same as the definition of the derivative for real functions, except that all of the quantities are complex.

Holomorphic functions can be described in terms of the so-called Cauchy Riemann equations.

Proposition 1. *Let $f : \Omega \to \mathbb{C}$, where $\Omega \subseteq \mathbb{C}$ is a domain, and let $z_0 \in \mathbb{C}$ such that $z_0 = x_0 + iy_0$, then the following are equivalent:*

a) f is holomorphic at z_0 with $f'(z_0) = \lambda = \alpha + i\beta$.

b) f is differentiable in $\mathbb{R}^2$ at (x_0, y_0) with $Df(x_0, y_0) = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$.

In case this happens, letting $f = u + iv$, i.e. $u = Re(f)$ and $v = Im(f)$, the ***Cauchy Riemann equations*** *hold:*

$$\begin{cases} u_x = v_y \\ u_y = -v_x. \end{cases} \tag{1}$$

Writing $z = x + iy$ and $f = f(z, \bar{z})$ it is seen that the Cauchy-Riemann equations are equivalent to

$$\frac{df}{d\bar{z}}(z_0) = 0, \tag{2}$$

where

$$\frac{df}{d\bar{z}}(z_0) = \frac{1}{2}\left(\frac{df}{dx} + i\frac{df}{dy}\right).$$

Let's prove this equivalence. A direct computation gives

$$\begin{aligned} \frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \\ &= \frac{1}{2}\left[(u_x + iv_x) + i(u_y + iv_y)\right] \\ &= \frac{1}{2}\left[(u_x - v_y) + i(v_x + u_y)\right] \end{aligned}$$

and therefore the result follows.

Conformality is the geometric interpretation of the Cauchy Riemann equations (1). In general, "conformal" is a term that indicates preservation of angles.

The main object of study in our work is conformal mapping. We recall next the definition of conformality.

Definition 2.2. *Let Ω de a domain in $\mathbb{C}$. A function $f : \Omega \to \mathbb{C}$ is* ***conformal*** *at z_0 if it preserves angles between curves through z_0.*

Let φ_1, φ_2 be differentiable curves in Ω with $\varphi_1(0) = \varphi_2(0) = z_0$ and let $\Gamma_i(t) = f(\varphi_i(t))$, $i = 1, 2$, be the image curves by f. Then f is conformal at z_0 if the (oriented) angle between $\varphi_1'(0)$ and $\varphi_2'(0)$ is the same as the angle between $\Gamma_1'(0)$ and $\Gamma_2'(0)$.

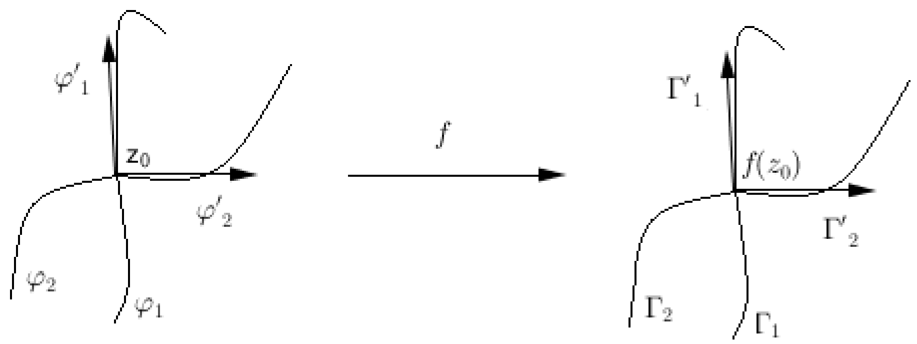

Figure 2: Conformal mapping. Preserves angles between curves.

Conformality has an easy expression in terms of holomorphic functions.

Theorem 2.1. *Let Ω be a domain in $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be differentiable at $z_0 \in \Omega$. Then f is conformal at z_0 if only if f is holomorphic at z_0 and $f'(z_0) \neq 0$.*

Proof. On the one hand, let $\varphi_i, \Gamma_i, i = 1, 2$, as before. By the chain rule

$$\Gamma_i'(t) = f'(\varphi_i(t))\varphi_i'(t),$$

so for $t = 0$, $\Gamma_i'(0) = f'(z_0)\varphi_i'(0)$. This shows that $\Gamma_i'(0)$ is obtained from $\varphi_i'(0)$ by multiplying always by the same constant $f'(z_0)$. In particular, the angle between $\Gamma_1'(0), \Gamma_2'(0)$ is the same as the angle between $\varphi_1'(0), \varphi_2'(0)$, and we have conformality.

On the other hand, the function f is differentiable (in the real sense) at z_0 because it is conformal, then it is enough to prove that $f = u + iv$ satisfies the Cauchy Riemann equations (1). Without loss of generality, we can assume that $z_0 = 0$. Denote

$$Df(z_0) = Df(0) = \begin{pmatrix} a & c \\ b & d \end{pmatrix},$$

and for each $\theta \in (-\pi, \pi]$ denote the curve

$$\varphi_\theta(t) = e^{i\theta}t = (t\cos\theta, t\sin\theta).$$

Notice that $\varphi_\theta(0) = 0$ and $\varphi'_\theta(0) = e^{i\theta} = (\cos\theta, \sin\theta)$. Then

$$(f \circ \varphi_\theta)'(0) = Df(0)\varphi'_\theta(0) = (a\cos\theta + c\sin\theta, b\cos\theta + d\sin\theta).$$

By hypothesis,

$$\begin{aligned}
\arg(e^{i\theta}) = \theta &= \mathrm{angle}(\varphi'_0(0), \varphi'_\theta(0)) \\
&= \mathrm{angle}((f\circ\varphi_0)'(0), (f\circ\varphi_\theta)'(0)) \\
&= \mathrm{angle}(a+ib, a\cos\theta + c\sin\theta + i(b\cos\theta + d\sin\theta)) = \\
&= \arg\left(\frac{a\cos\theta + c\sin\theta + i(b\cos\theta + d\sin\theta)}{a+ib}\right).
\end{aligned}$$

Equivalently, writing

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}\ , \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$$

and operating:

$$\arg(a+ib) = \arg\left(\frac{1}{2}(a+d+i(b-c)) + \frac{1}{2}e^{-2i\theta}(a-d+i(c+b))\right).$$

Now, this is possible for $\theta \in (-\pi, \pi]$, only if

$$a - d + i(c+b) = 0,$$

which means

$$\begin{cases} a = d \\ c = -d. \end{cases}$$

These are precisely the Cauchy Rimeann equations (1) of $f = u + iv$ at $z_0 = 0$. $\square$

3 Automorphisms of $\mathbb{D}$ and pseudohyperbolic distance

3.1 Möbius transformations and automorphisms of the disk.

A special kind of conformal maps play a special role in many of the results we will see later on.

Definition 3.1. *A **Möbius transformation** is a rational function of the form*

$$f(z) = \frac{az+b}{cz+d}, z \in \mathbb{C}.$$

Here the coefficients a, b, c, d are complex numbers satisfying $ad - bc \neq 0$.

We will denote by $\mathcal{M}$ the group of Möbius transformation.

They are also variously named homographies, homographic transformations, linear fractional transformations, bilinear transformations, or fractional linear transformations.

The Möbius transforms can be seen in the complex plane as the composition of a stereographic projection of the plane on the sphere followed by a rotation or displacement of the sphere to a new location and finally as a stereographic projection, this time from sphere to plane.

This type of transformations are very important because can naturally extend to a biholomorphism (i.e., a comformal and bijective application) of the Riemann sphere.

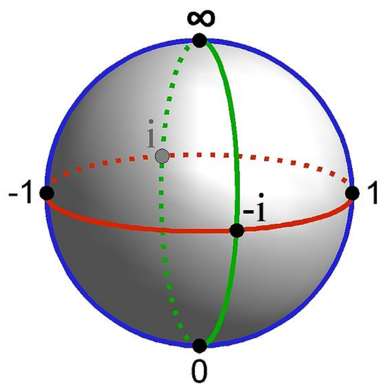

Figure 3: Riemann sphere.

A visual explanation of the Möbius transformations may be seen in `http://www.youtube.com/watch?v=XTrIs_RB_Rc`.

We shall see next the **automorphsims of the disk** $\mathbb{D}$ are a precisely the subgroup of $\mathcal{M}$ that preserves $\mathbb{D}$.

Denote by $Aut(\mathbb{D})$ the group of holomorphic and bijective applications $\varphi : \mathbb{D} \to \mathbb{D}$ with the composition operation.

It can be proved with the help of Schwarz lemma given later in (Theorem 3.1) the following result.

Lemma 1.

$$Aut(\mathbb{D}) := \left\{ e^{i\theta} \frac{z-a}{1-\bar{a}z};\ a \in \mathbb{D},\ \theta \in [0, 2\pi] \right\}.$$

Note that we will use for this group of $Aut(\mathbb{D})$ the following notation:

$$\varphi_{a,\theta}(z) = e^{i\theta} \frac{z-a}{1-\bar{a}z},$$

and also

$$\varphi_a(z) = \varphi_{a,\pi}(z) = \frac{a-z}{1-\bar{a}z}. \tag{3}$$

Proof. On the one hand, let $\varphi_{a,\theta}(z) = e^{i\theta} \frac{z-a}{1-\bar{a}z}$. If $|z| = 1$ then

$$|\varphi(z)| = \left|e^{i\theta}\right| \frac{|z-a|}{|1-\bar{a}z|} = \frac{|z-a|}{|z| \left| \frac{1}{z} - \bar{a} \right|} = \frac{|z-a|}{|\bar{z}-\bar{a}|} = 1.$$

Therefore,

$$\varphi : \{|z| = 1\} \to \{|z| = 1\}.$$

Because φ is a Möbius transformation, φ carries $\mathbb{D}$ either to $\mathbb{D}$ or to the complementary of $\bar{\mathbb{D}}$. Now, we will see that if we apply φ to an arbitrary point of the disk, its image belongs in the disk. We will get the 0 point to see that $\varphi(0) = e^{i\theta}(-a) \in \mathbb{D}$. So we have $\varphi(\mathbb{D}) = \mathbb{D}$. In the end, $\varphi \in Aut(\mathbb{D})$. Moreover, being Möbius transformation, the injectivity is direct.

In order to see the other inclusion, let $\varphi : \mathbb{D} \to \mathbb{D}$ be an arbitrary automorphism of $\mathbb{D}$. Let $a \in \mathbb{D}$ be such that $\varphi(a) = 0$. Such a exists because φ bijective. Consider the automorphism

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z},$$

which has $\varphi_a(0) = a$.

Now, we define

$$g = \varphi \circ \varphi_a.$$

Notice that $g(0) = \varphi(\varphi_a(0)) = \varphi(a) = 0$. Denote

$$g(z) = w, \quad g^{-1}(w) = z.$$

Applying Scharwz Lemma (Theorem 3.1) to g and g^{-1} we have

$$|g(z)| \leq |z|,\ z \in \mathbb{D}$$
$$\left|g^{-1}(w)\right| \leq |w|,\ w \in \mathbb{D}.$$

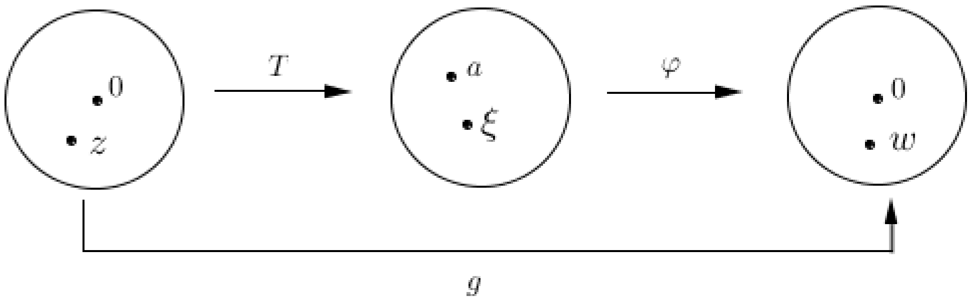

Figure 4: Composition of $g = \varphi \circ T$.

This last inequality in terms of z is

$$|z| \leq |g(z)|\,,\ \ z \in \mathbb{D}$$

so

$$|g(z)| = |z|\,, z \in \mathbb{D}.$$

Therefore there exists $\theta \in [0, 2\pi]$ that

$$g(z) = e^{i\theta} z.$$

Then $\varphi(\varphi_a(z)) = e^{i\theta} z$ and letting $\xi = \varphi_a(z)$, we see finally that

$$\varphi(\xi) = e^{i\theta} {\varphi_a}^{-1}(\xi)$$

is a Möbius transformation. $\square$

Remark 1. *The following easy identity will be helpful in what follows*

$$1 - |\varphi_{a,\theta}(z)|^2 = 1 - \left|\frac{z-a}{1-\bar{a}z}\right|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2}. \tag{4}$$

Proof. Operating

$$\begin{aligned} 1 - \left|\frac{z-a}{1-\bar{a}z}\right|^2 = 1 - \frac{|z-a|^2}{|1-\bar{a}z|^2} &= \frac{|1-\bar{a}z|^2 - |z-a|^2}{|1-\bar{a}z|^2} = \\ &= \frac{1 + |a|^2|z|^2 - 2\,|a|\,|z| - |z|^2 - |a|^2 + 2\,|a|\,|z|}{|1-\bar{a}z|^2} = \\ &= \frac{1 + |a|^2|z|^2 - |a|^2 - |z|^2}{|1-\bar{a}z|^2} \\ &= \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2}. \end{aligned}$$

$\square$

3.2 Pseudohyperbolic distance in $\mathbb{D}$ and the Schwarz-Pick lemma

An important element of the proof of the Hayman-Wu theorem is the pseudohyperbolic metric. We define if first for the unit disk. We shall see in Chapter 4 how it can be defined on any simply connected domain, with the help of Riemann's theorem.

Given $z_1, z_2 \in \mathbb{D}$ define the **pseudohyperbolic distance** as

$$\rho(z_1, z_2) = |\varphi_{z_2}(z_1)| = \left| \frac{z_1 - z_2}{1 - \overline{z_2} z_1} \right|. \tag{5}$$

We get to this definition when looking for a distance which is invariant by automorphisms of $\mathbb{D}$. We first define

$$\rho(z, 0) = |z|,$$

and the rest is determined by the invariance. If you have $z_1, z_2 \in \mathbb{D}$ and take $\varphi_{z_2,0} \in Aut(\mathbb{D})$, then

$$\rho(z_1, z_2) = \rho(\varphi_{z_2,0}(z_1), \varphi_{z_2,0}(z_2)) = \rho(\varphi_{z_2,0}(z_1), 0) = |\varphi_{z_2,0}(z_1)| = \left| \frac{z_1 - z_2}{1 - \bar{z}_2 z_1} \right|.$$

Let us gather the main properties of this distance.

Properties. Let $z_1, z_2 \in \mathbb{D}$. The function $\rho : \mathbb{D} \times \mathbb{D} \to \mathbb{R}_+$ is a distance, i.e. it satisfies:

1) $\rho(z_1, z_2) \geqslant 0$ and $\rho(z_1, z_2) = 0$ if and only if $z_1 = z_2$.

2) Symmetry, $\rho(z_1, z_2) = \rho(z_2, z_1)$.

3) Improved triangle inequality: if also $a \in \mathbb{D}$,

$$\frac{\rho(z_1, a) - \rho(a, z_2)}{1 - \rho(z_1, z_2)\rho(z_2, z_1)} \leq \rho(z_1, z_2) \leq \frac{\rho(z_1, a) + \rho(a, z_2)}{1 + \rho(z_1, z_2)\rho(z_2, z_1)}.$$

Proof. Properties 1) and 2) are immediate from the definition.

In order to see 3) we need first to recall some basic properties of the disks defined by the pseudohyperbolic metric.

For $\alpha \in \mathbb{D}$ and $0 \leq r \leq 1$, the set

$$\Delta(\alpha, r) = \{z \in \mathbb{D} : \rho(z, \alpha) < r\}$$

is known as the pseudohyperbolic disk with center α and radius r. It is a true Euclidian disk, since Möbius transformations preserve circles, but α is not its Euclidian center and r is not its Euclidian radius unless $\alpha = 0$. The Euclidian center and radius of $\Delta(\alpha, r) = D(\beta, R)$ are found to be

$$\beta = \frac{(1 - r^2)\alpha}{1 - r^2|\alpha|^2} \quad \text{and} \quad R = \frac{r(1 - |\alpha|^2)}{1 - r^2|\alpha|^2},$$

respectivily.

Since ρ is Möbius-invariant, we can assume that $\alpha = 0$. Thus, we need to show that

$$\frac{||z_1| - |z_2||}{1 - |z_1|\,|z_2|} \leq |\varphi_{z_2}(z_1)| \leq \frac{||z_1| + |z_2||}{1 + |z_1|\,|z_2|},$$

which is equivalent to

$$1 - \frac{||z_1| + |z_2||^2}{|1 + |z_1|\,|z_2||^2} \leq 1 - |\varphi_{z_2}(z_1)|^2 \leq 1 - \frac{||z_1| - |z_2||^2}{|1 - |z|\,|z_2||^2}.$$

Developing the left and the right hand sides of these inequalities and using (4) we see that this is equivalent to

$$1 - |z|\,|w| \leq |1 - \bar{z}w| \leq 1 + |z|\,|w|,$$

which is straightforward. $\square$

Observation 1. *As we can see in the properties above (see (4)), $\rho(z_1, z_2) \leq 1$, with $\rho(z_1, z_2) = 1$ if and only if either z_1 or z_2 is at the boundary of $\mathbb{D}$.*

The following results show that the pseudohyperbolic distance is contractive, which is an important property in the proof of the Hayman-Wu theorem.

Let $f : \mathbb{D} \to \bar{\mathbb{D}}$ be a holomorphic map. Schwarz lemma states that f is always contractive with respect to pseudohyperbolic distance, i.e. the distance between the images of $f(z_1)$ and $f(z_2)$ is no bigger than the distance between z_1 and z_2, $(z_1, z_2 \in \mathbb{D})$. Its usual form assumes that one of the points is 0.

Theorem 3.1. *Let $f : \mathbb{D} \to \bar{\mathbb{D}}$ be a holomorphic map such that $f(0) = 0$. Then:*

1) $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$,

2) $|f'(0)| \leq 1$.

Moreover, the equality in any of these two statements implies that f is of the form $f(z) = e^{i\theta} z$, $\theta \in [0, 2\pi)$.

Proof. In order to see 1) define

$$g(z) = \begin{cases} \dfrac{f(z)}{z} & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0, \end{cases}$$

which is a holomorphic function on any disk $\overline{D(0, r)}$, $r < 1$.

Applying the maximum modulus principle to g we have, for $z \in D(0, r)$,

$$|g(z)| \leq \max_{|z|=r} |g(z)| = \frac{\max_{|z|=r} |f(z)|}{|r|} \leq \frac{1}{r}.$$

Letting $r \to 1$ we see that $|g(z)| \leq 1$, i.e., $|f(z)| \leq |z|$, as described. Notice also that $g(0) = f'(0)$, which gives 2).

Notice also that if any of the equilities hold then g has a maximum z_0 in the interior of $\mathbb{D}$. Then g is constant, or equivalently $f(z) = cz$ for some $c \in \mathbb{C}$. Since there is z_0 with $|f(z_0)| = |z_0|$ we see that necessarily $|c| = 1$. $\square$

The Schwarz-Pick lemma is the invariant version of the Schwarz lemma.

Theorem 3.2. *Let $f : \mathbb{D} \to \bar{\mathbb{D}}$ be a holomorphic function. Then for all $z_1, z_2 \in \mathbb{D}$,*

$$\rho(f(z_1), f(z_2)) \leq \rho(z_1, z_2),$$

where ρ denotes the pseudohyperbolic distance (5)*.*

Proof. Let

$$\varphi_{z_1}(z) = \frac{z_1 - z}{1 - \bar{z}z_1}$$

defined before at (3), which has $\varphi_{z_1}(z) = 0$ and $\varphi_{z_1}(0) = z_1$.

Similarly consider

$$\varphi_{f(z_1)}(w) = \frac{f(z_1) - w}{1 - \bar{w}f(z_1)}.$$

Then the composition

$$\varphi_{f(z_1)} \circ f \circ \varphi_{z_1} : \mathbb{D} \to \bar{\mathbb{D}}$$

takes 0 to 0. By the Schwarz lemma

$$\left|\varphi_{f(z_1)}(f(\varphi_{z_1}(\xi)))\right| \leq |\xi|\,, \quad \xi \in \mathbb{D}$$

Letting $\varphi_{z_1}(\xi) = z_2$ this is

$$\left|\varphi_{f(z_1)}(f(z_2))\right| \leq \left|{\varphi^{-1}}_{z_1}(z_2)\right|,$$

which is the desired inequality. $\square$

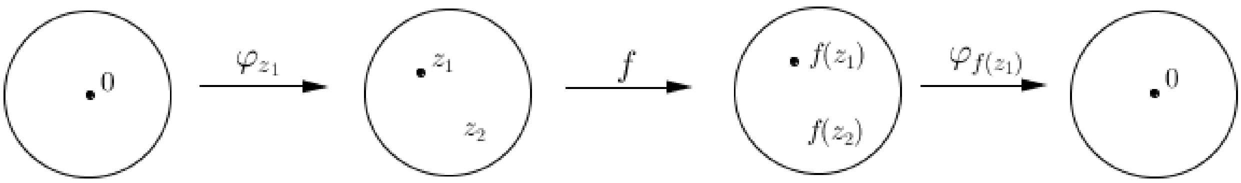

Figure 5: The composition of $\varphi_{f(z_1)} \circ f \circ \varphi_{z_1}$.

4 Riemann's Theorem and its consequences

This is a central theorem, deep in the theory. Riemann's theorem states roughly that any two simply connected domains are conformally equivalent, provided that neither of them is the whole plane. This is an existence theorem, it shows that there is a conformal mapping between two such domains but it does not produce the corresponding conformal mapping. It will not be proved since its proof is outside the content of the work.

Theorem 4.1. *Given a simply connected domain Ω in the plane, $\Omega \neq \mathbb{D}$, and $z_0 \in \Omega$, there exists a unique conformal mapping $f : \Omega \to \mathbb{C}$ normalized with the conditions that $f(z_0) = 0$ and $f'(z_0) > 0$.*

This theorem allows to extend to any simply connected domain Ω many notions previoulsy seen in the disk. For example, given the conformal mapping $f : \Omega \to \mathbb{C}$ we can describe the automorphisms of Ω through the automorphisms of $\mathbb{D}$ given previously. Specifically,

$$Aut(\Omega) = \left\{ f^{-1} \circ \varphi_{\alpha,\theta} \circ f : \Omega \longrightarrow \Omega, \quad \varphi_{\alpha,\theta} \in Aut(\mathbb{D}) \right\}.$$

Similarly, once we know how to solve the Dirichlet problem in the unit disk we can solve it in Ω, as we shall see in Section 5.1

In this chapter we specify the definition of the pseudohyperbolic distance in any simply connected Ω.

After Riemann's theorem and the pseudohyperbolic distance (5) given in the disk, we can define the pseudohyperbolic distance in any simply connected domain Ω. For $z_1, z_2 \in \Omega$ define

$$\rho_\Omega(z_1, z_2) = \rho_{\mathbb{D}}(f(z_1), f(z_2)) = \left| \frac{f(z_1) - f(z_2)}{1 - \bar{f(z_2)} f(z_1)} \right|,$$

where f is a conformal mapping from Ω to $\mathbb{D}$.

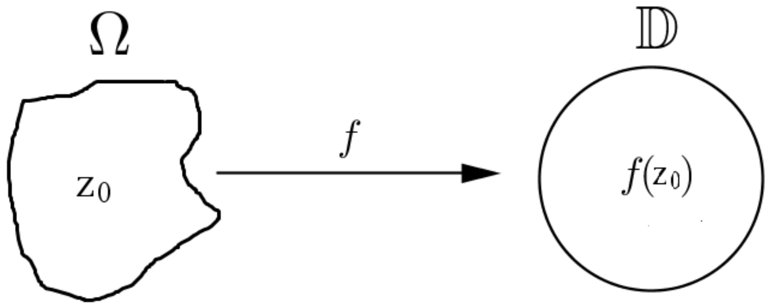

Figure 6: Conformal mapping$f : \Omega \to \mathbb{D}$.

Example 1. *The pseudohyperbolic distance in the right half-plane*

$$\mathbb{T} = \{w \in \mathbb{C} : Re(w) > 0\}$$

is
$$\rho_{\mathbb{T}}(w_1, w_2) = \left|\frac{w_1 - w_2}{\bar{w}_1 + w_2}\right|.$$

In order to see this, consider the following conformal map from the unit disk $\mathbb{D}$ to $\mathbb{T}$:
$$f^{-1}(z) = \frac{1+z}{1-z}, \quad z \in \mathbb{D}.$$
Notice that f^{-1} is a Möbius transformation and that:
$$f^{-1}(0) = 1$$
$$f^{-1}(i) = \infty$$
$$f^{-1}(-1) = 0$$
$$f^{-1}(1) = 1.$$

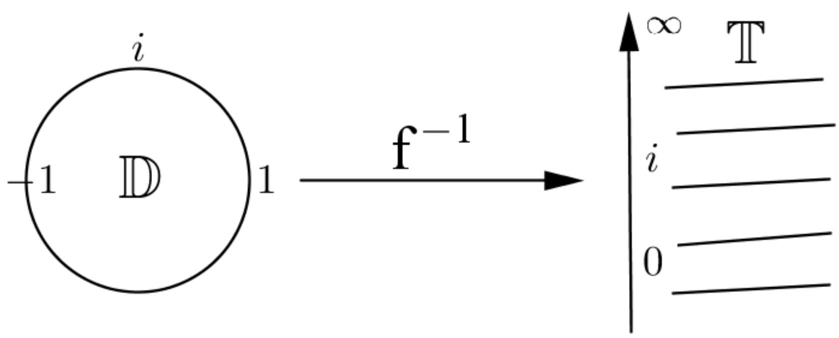

Figure 7: Exemple 1. $f^{-1} : \mathbb{D} \to \mathbb{T}$.

The inverse of f^{-1} is
$$f(w) = \frac{w-1}{w+1}, \ w \in \mathbb{T}, \tag{6}$$
and therefore the pseudohyperbolic distance between two points $w_1, w_2 \in \mathbb{T}$ is
$$\begin{aligned}
\rho_{\mathbb{T}}(w_1, w_2) = \rho_{\mathbb{D}}(f(w_1), f(w_2)) &= \frac{\left|\frac{w_1-1}{w_1+1} - \frac{w_2-1}{w_2+1}\right|}{\left|1 - \frac{\bar{w}_1-1}{\bar{w}_1+1}\frac{w_2-1}{w_2+1}\right|} = \\
&= \frac{|(w_1-1)(w_2+1) - (w_2-1)(w_1+1)|}{|(\bar{w}_1+1)(w_2+1) - (\bar{w}_1-1)(w_2-1)|} = \\
&= \frac{|w_1w_2 - w_2 + w_1 - 1 - w_2w_1 + w_1 - w_2 + 1|}{|\bar{w}_1w_2 + w_2 + \bar{w}_1 + 1 - \bar{w}_1w_2 + w_2 + \bar{w}_1 - 1|} = \\
&= \frac{2\,|w_1 - w_2|}{2\,|\bar{w}_1 + w_2|} = \frac{|w_1 - w_2|}{|\bar{w}_1 + w_2|},
\end{aligned}$$
as stated.

The following corollary will be used in the proof of Hayman-Wu theorem.

Corollary 4.1. *Let Ω_1, Ω_2 be simply connected domains such that $\Omega_1 \subseteq \Omega_2$. Then*

$$\rho_{\Omega_2}(z,w) \leq \rho_{\Omega_1}(z,w), \quad z,w \in \Omega_1. \tag{7}$$

Proof. Let $\varphi_i : \Omega_i \to \mathbb{D}$, $i = 1,2$, be conformal mappings, which exist by Riemann's theorem.

Then $f := \varphi_2 \circ \varphi_1^{-1}$ is a holomorphic map from $\mathbb{D}$ to $\bar{\mathbb{D}}$. By the Schwarz-Pick lemma (Theorem 3.2), since $\varphi_1(z), \varphi_1(w) \in \mathbb{D}$,

$$\rho_{\mathbb{D}}(f(\varphi_1(z)), f(\varphi_1(w))) \leq \rho_{\mathbb{D}}(\varphi_1(z), \varphi_2(w)). \tag{8}$$

But by definition

$$\rho_{\mathbb{D}}(f(\varphi_1(z)), f(\varphi_1(w))) = \rho_{\mathbb{D}}(\varphi_2(z), \varphi_2(w)) = \rho_{\Omega_2}(z,w)$$

and

$$\rho_{\mathbb{D}}(\varphi_1(z), \varphi_1(w)) = \rho_{\Omega_1}(z,w),$$

so we get the desired result. □

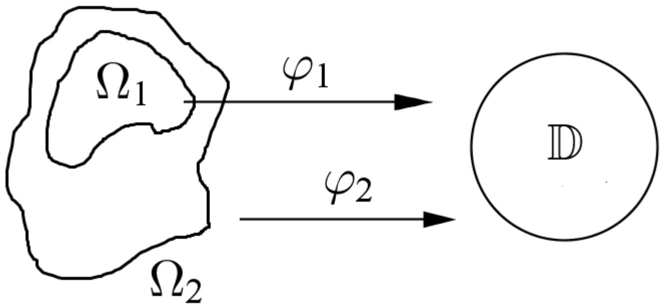

Figure 8: Conformal mappings. $\varphi_i : \Omega_i \to \mathbb{D}$. i=1,2.

5 Harmonic functions and the Dirichlet Problem

5.1 Harmonic functions

For the Hayman-Wu theorem it is necessary to use some properties about harmonic functions, which are directly connected to holomorphic functions. So, before explaining the beautiful applications, we will begin with the formal definitions:

Definition 5.1. *A* ***harmonic function*** *is a twice continuously differentiable function $h : \Omega \to \mathbb{R}$ (where Ω is an open subset of $\mathbb{R}^2$) which satisfies Laplace's equation, i.e.*

$$\Delta h = \frac{\partial^2 h}{\partial x_1^2} + \frac{\partial^2 h}{\partial x_2^2} = 0.$$

Equivalently, with complex notation,

$$\frac{\partial^2 h}{\partial z \partial \bar{z}} = 0. \tag{9}$$

To see this equivalence recall that

$$\frac{\partial h}{\partial z} = \frac{1}{2}\left(\frac{\partial h}{\partial x_1} - i\frac{\partial h}{\partial x_2}\right),$$

$$\frac{\partial h}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial h}{\partial x_1} + i\frac{\partial h}{\partial x_2}\right),$$

and therefore

$$\begin{aligned}
\frac{\partial^2 h}{\partial z \partial \bar{z}} &= \frac{\partial h}{\partial z}\left[\frac{1}{2}\left(\frac{\partial h}{\partial x_1} + i\frac{\partial h}{\partial x_2}\right)\right] = \\
&= \frac{1}{4}\left[\frac{\partial}{\partial x_1}\left(\frac{\partial h}{\partial x_1} + i\frac{\partial h}{\partial x_2}\right) - i\frac{\partial}{\partial x_2}\left(\frac{\partial h}{\partial x_1} + i\frac{\partial h}{\partial x_2}\right)\right] = \\
&= \frac{1}{4}\left(\frac{\partial^2 h}{\partial^2 x_1^2} + i\frac{\partial^2 h}{\partial x_1 \partial x_2} - i\frac{\partial^2 h}{\partial x_1 \partial x_2} - i^2\frac{\partial^2 h}{\partial^2 x_2^2}\right) = \\
&= \frac{1}{4}\left(\frac{\partial^2 h}{\partial^2 x_1^2} + \frac{\partial^2 h}{\partial^2 x_2^2}\right) \\
&= \frac{1}{4}\Delta h.
\end{aligned}$$

We will use the notation $h \in Har(\Omega)$ to indicate that h is an harmonic function in a domain Ω.

Harmonic functions are locally real parts of holomorphic functions.

Theorem 5.1. *Let Ω be a domain in $\mathbb{C}$. Then:*

a) If f is holomorphic on Ω and $h = Re(f)$, then h is harmonic on Ω.

b) If h is harmonic on Ω, and if Ω is simply connected, then $h = Re(f)$ for some f holomorphic on Ω. Moreover, f is unique up to adding a constant.

Proof. a) Using the Cauchy-Riemann equations (1), if $f = h + ik$ is an holomorphic function then $h_x = k_y$ and $h_y = -k_x$. Therefore

$$\Delta h = h_{xx} + h_{yy} = k_{yx} - k_{xy} = 0.$$

b) If $h = Re(f)$ for some function f, say $f = h+ik$ then $f' = h_x + ik_x = h_x - ih_y$. So, if f exists, it is only determined by h up to a constant.

Define $g : \Omega \to \mathbb{C}$,

$$g = h_x - ih_y.$$

Then $g \in C^1(\Omega)$ and is holomorphic on Ω, because it satisfies the Cauchy-Riemann equations:

$$\begin{cases} h_{xx} = -h_{yy} \\ h_{xy} = h_{yx}. \end{cases} \tag{10}$$

Now, fix $z_0 \in \Omega$, and define $f : \Omega \to C$ by

$$f(z) = h(z_0) + \int_{z_0}^{z} f(w)dw$$

the integral being taken over any path in Ω from z_0 to z. As Ω is simply connected, Cauchy's theorem ensures that the integral is independent of the particular path chosen. Then f is holomorphic on Ω and $f' = g = h_x - ih_y$. It only remains to prove the uniqueness.

Writing $j = Re(f)$, we have $j_x - ij_y = f' = h_x - ih_y$, so that $(j-h)_x = 0$ and $(j-h)_y = 0$. It follows that $(j - h)$ is constant on Ω, and puttig $z_0 = z$ shows that the constant is 0. Thus indeed $h = Re(f)$. $\square$

The composition of harmonic functions is not necessarily harmonic. However, the composition of a holomorphic funcion f with an harmonic function h is harmonic.

Property 1. *If $f : \Omega_1 \to \Omega_2$ is a holomorphic map between open subsets Ω_1, Ω_2 of $\mathbb{C}$, and if h is harmonic on Ω_2, then $h \circ f$ is harmonic on Ω_1.*

Proof. Let us see that $\dfrac{\partial^2 (h \circ f)}{\partial z \partial \bar{z}} = 0$. By the chain rule:

$$\frac{\partial^2 (h \circ f)}{\partial z \partial \bar{z}} = \frac{\partial}{\partial z}(\frac{\partial h}{\partial z}\frac{\partial f}{\partial \bar{z}} + \frac{\partial h}{\partial \bar{z}}\frac{\partial f}{\partial \bar{z}}) = \frac{\partial^2 h}{\partial z \partial z}\frac{\partial f}{\partial \bar{z}} + \frac{\partial h}{\partial \bar{z}}\frac{\partial f}{\partial z \partial \bar{z}} = 0.$$

For this, we have used (2) in the second inequality (because f holomorphic) and that h satisfies (9) in the last inequality (because h is harmonic). $\square$

Next we will define some important properties related to harmonic functions. We will denote $h \in Har(\overline{D(z, r)})$ to indicate an harmonic function in a neighbourhood of $\overline{D(z, r)}$.

A property that essentialy characterizes harmonic functions is the following.

Property 2. ***Mean-Value Property:*** *Let $h \in Har(\overline{D(z,r)})$. Then*

$$h(z) = \frac{1}{2\pi} \int_0^{2\pi} h(z + re^{i\theta}) d\theta.$$

Proof. Choose $r' > r$ so that h is harmonic on $D(z, r')$. Applying Theorem 5.1 mentioned before, there exists f holomorphic on $D(z, r')$ such that $h = Re(f)$. By Cauchy's integral formula we have, parametrizing $|a - z| = r$, $a = z + re^{i\theta}$, $\theta \in [0, 2\pi)$,

$$f(z) = \frac{1}{2\pi i} \int_{|a-z|=r} \frac{f(a)}{a - z} da = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta.$$

Then, the result follows upon taking real parts of both sides. □

Property 3. ***Converse to Mean-Value Poperty:*** *Let $h : \Omega \to \mathbb{R}$ be a continuous function on an open subset Ω of $\mathbb{C}$, and suppose that it posseses the local mean-value property, i.e. given $z \in \Omega$, there exixts $p > 0$ such that*

$$h(z) = \frac{1}{2\pi} \int_0^{2\pi} h(z + re^{it}) dt \quad (0 \leq r < p).$$

Then h is harmonic on Ω.

We will see the proof of this property when we introduce the Poisson transform in Section 5.2.

Harmonic functions, like holomorphic functions, have also an idendity principale.

Property 4. ***Identity Principle:*** *Let h and k be harmonic functions on a domain Ω in $\mathbb{C}$. If $h = k$ on a non-empty open subset Ω' of Ω, then $h = k$ throughout Ω.*

Proof. We can suppose, without loss of generality, that $k = 0$. Set $g = h_x - ih_y$. Then the harmonicity of h implies the Cauchy-Riemann equations (1) for g, i.e. g is holomorphic on Ω, and also g=0 on Ω' since h=0 there. By the Identity Principle for holomorphic functions, it follows that g=0 trhoughout Ω, and hence that $h_x = 0$ and $h_y = 0$ on Ω. Therefore h is constant on Ω, and since $h = 0$ on Ω', this constant must be zero. □

Another property we will need is the maximum principle for harmonic functions.

Property 5. ***Maximum Principle:*** *Let h be a holomorphic function in a domain Ω in $\mathbb{C}$. If h extends continuously to $\bar{\Omega}$ and $h \leq 0$ on $\partial\Omega$, then $h \leq 0$ on Ω.*

Proof. a) Suppose that h attains a local maximum at $z \in \Omega$. Then for some $r > 0$ we have $h \leq h(z)$ on $D(z, r)$. By Theorem 5.1, there exists a function f holomorphic on $D(z, r)$ such that $h = Re(f)$ there. Then $\left|e^f\right|$ attains a local maximum at z, so e^f must be constant. Therefore h is constant on $D(z, r)$, and hence on the whole of Ω by the Identity Principle (Property 4).

b) As $\bar{\Omega}$ is compact, h must attain a maximum at some point $z \in \bar{\Omega}$. If $z \in \partial\Omega$, then $h(z) \leq 0$ by assumption, and so $h \leq 0$ on Ω. If $z \in \Omega$, then by part (a) h is constant on Ω, hence $\bar{\Omega}$, and so once again $h \leq 0$ on Ω. □

5.2 The Dirchlet problem and the Poisson kernel

A classical problem in the theory of harmonic functions is the Dirichlet problem, which consists in finding a harmonic function on a 'nice' domain with prescribed boundary values. This is a powerful tool with many applications.

Definition 5.2. *Let Ω be a domain of $\mathbb{C}$, and let be $\phi : \partial\Omega \to \mathbb{R}$ a continuous function.The* ***Dirichlet problem*** *is to find a harmonic function h on Ω that*

$$\lim_{z \to \xi} h(z) = \phi(\xi) \quad \textit{for all} \quad \xi \in \partial\Omega.$$

Theorem 5.2. ***Uniquess:*** *There is at most one solution h to the Dirichlet problem.*

Proof. Suppsose that h and g are two solutions. Then $h - g$ is harmonic on Ω, it extends continuously to $\bar{\Omega}$, and is zero on $\partial\Omega$. Applying the Maximum Principle (Property 5), we conclude that $h - g = 0$ on $\partial\Omega$ and therefore $h - g \leq 0$ on Ω. Similarly $g - h \leq 0$, and therefore $g = h$. □

In order to solve the Dirichlet problem in the disk $\Omega = \mathbb{D}$ we need the following definitions.

Definition 5.3. *a) The Poisson kernel $P : \mathbb{D} \times \partial\mathbb{D} \to \mathbb{R}$ is the function*

$$P(z, \xi) := \frac{1}{2\pi} Re\left(\frac{\xi + z}{\xi - z}\right) = \frac{1}{2\pi} \frac{1 - |z|^2}{|\xi - z|^2}, (|z| < 1, |\xi| = 1).$$

b) If $\phi : \partial\mathbb{D} \to \mathbb{R}$ is a Lebesgue-integrable function, then its Poisson integral $P_{\mathbb{D}}\phi : \mathbb{D} \to \mathbb{R}$ is defined by

$$P_{\mathbb{D}}\phi(z) := \int_0^{2\pi} P(z, e^{i\theta})\phi(e^{i\theta})d\theta \quad (z \in \mathbb{D}, \theta \in [0, 2\pi]).$$

Then, rescaling and translating, a similar expression is obtained for any disk $D(w, p)$:

$$P_{D(w,p)}\phi(z) := \int_0^{2\pi} P\left(\frac{z - w}{p}, e^{i\theta}\right)\phi(w + pe^{i\theta})d\theta \quad (z \in D(w, p), \theta \in [0, 2\pi]).$$

The Poisson kernel transform gives directly the solution to the Dirichlet problem in a disk.

Theorem 5.3. *Let $D(w, p)$ be the disk of center w and radius w. Then*

a) $P_{D(w,p)}\phi$ is harmonic on $D(w, p)$.

b) If ϕ is continuous at $\xi \in \partial D(w, p)$, then

$$\lim_{z \to \xi} P_{D(w,p)}\phi(z) = \phi(\xi).$$

In particular, if ϕ is continuous on the whole of $\partial D(w,p)$, then $h := P_{D(w,p)}\phi$ solves the Dirichlet problem on $D(w,p)$.

Proof. a) Making an affine change of variable if necessary, we can suppose that $w = 0$ and $p = 1$, so that $\mathbb{D} = D(0,1)$. Then since ϕ is a real function,

$$P_{\mathbb{D}}\phi(z) = \frac{1}{2\pi}\int_0^{2\pi} Re\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)\phi(e^{i\theta})d\theta =$$
$$= Re\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\phi(e^{i\theta})d\theta\right).$$

So, it is enough to prove that

$$F(z) = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\phi(e^{i\theta})d\theta$$

is holomorphic in $\mathbb{D}$ (Theorem 6.1.). This can be seen in at least two ways:

i) Checking that $\dfrac{\partial F}{\partial \bar{z}} = 0$ on $\mathbb{D}$. We have

$$\frac{\partial F}{\partial \bar{z}}(z) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial}{\partial \bar{z}}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)\phi(e^{i\theta})d\theta = 0.$$

ii) Applying Morera's theorem. Let γ be a closed curve in $\mathbb{D}$ and let us see that $\int_\gamma F(z)dz = 0$. Observing that $\dfrac{e^{i\theta}+z}{e^{i\theta}-z}$ is holomorphic in $\mathbb{D}$ we have, by Cauchy Theorem,

$$\int_\gamma F(z)dz = \frac{1}{2\pi}\int_0^{2\pi}\left(\int_\gamma \frac{e^{i\theta}+z}{e^{i\theta}-z}dz\right)\phi(e^{i\theta})d\theta = 0.$$

b) It is convenient first to prove the following lemma about the Poisson kernel.

Lemma 5.1. *The Poisson kernel P satisfies:*

i) $P(z,\xi) > 0$, *for* $|z| < 1$ *and* $|\xi| = 1$.

ii) $\displaystyle\int_0^{2\pi} P(z,e^{i\theta})d\theta = 1$ *for* $|z| < 1$.

iii) For $|\xi_0| = 1$ *and fixed* $\delta > 0$,

$$\sup_{|\xi-\xi_0|\geq\delta} P(z,\xi) \to 0 \quad \text{as} \quad z \to \xi_0.$$

Proof. i) This is clear from the definition of Poisson kernel.

ii) Using the Cauchy integral formula and the Residue Theorem, we obtain (as $e^{i\theta} = \xi$)

$$\begin{aligned}\int_0^{2\pi} P(z, e^{i\theta}) d\theta &= Re\left(\frac{1}{i2\pi}\int_{|\xi|=1} \frac{\xi + z}{\xi - z}\frac{d\xi}{\xi}\right) = \\ &= Re\left(\frac{1}{i2\pi}\int_{|\xi|=1} (\frac{2}{\xi - z} - \frac{1}{\xi}) d\xi\right) \\ &= Re(2-1) = 1.\end{aligned}$$

iii) If $|z - \xi_0| < \delta$ then

$$\sup_{|\xi-\xi_0|\geq\delta} P(z,\xi) \leq \frac{1}{2\pi}\frac{1-|z|^2}{(\delta - |\xi_0 - z|)^2}.$$

The result follows easily from this estimate. □

Let us move to the proof of (b). Using (i) and (ii) in Lemma 5.1 we have

$$\begin{aligned}|P_D\phi(z) - \phi(\xi_0)| &= \left|\int_0^{2\pi} P(z, e^{i\theta})(\phi(e^{i\theta}) - \phi(\xi_0)) d\theta\right| \leq \\ &\leq \int_0^{2\pi} P(z, e^{i\theta}) \left|(\phi(e^{i\theta}) - \phi(\xi_0))\right| d\theta,\end{aligned}$$

because $|\xi_0| = 1$. We will separate this integral into two parts.

<u>First part:</u> Points $\xi \in \partial D$ near ξ_0.

Let $\epsilon > 0$. If ϕ is continuous at ξ_0, then there exists $\delta > 0$ such that

$$|\xi - \xi_0| < \delta \Rightarrow |\phi(\xi) - \phi(\xi_0)| < \epsilon.$$

Hence, it follows that

$$\int_{|e^{i\theta}-\xi_0|<\delta} P(z, e^{i\theta}) |\phi(\xi) - \phi(\xi_0)| d\theta \leq \int_0^{2\pi} P(z, e^{i\theta})\epsilon d\theta = \epsilon.$$

<u>Second part:</u> Points $\xi \notin \partial D$ "far" from ξ_0. Assume $\xi = e^{i\theta}$ is such that $|e^{i\theta} - \xi_0| \geq \delta$. By (iii), there exists $\delta^{'} > 0$ such that if $|z - \xi_0| < \delta'$ then

$$\sup_{|\xi-\xi_0|\geq\delta} P(z,\xi) < \epsilon.$$

Then

$$\begin{aligned}\int_{|e^{i\theta}-\xi_0|\geq\delta} P(z, e^{i\theta}) \left|(\phi(e^{i\theta}) - \phi(\xi_0))\right| d\theta &\leq \frac{1}{2\pi}\int_0^{2\pi} \epsilon \left|(\phi(e^{i\theta}) - \phi(\xi_0))\right| d\theta \leq \\ &\leq \epsilon\left[\frac{1}{2\pi}\int_0^{2\pi} \left|\phi(e^{i\theta})\right| d\theta + |(\phi(\xi_0)|\right].\end{aligned}$$

We deduce, putting together both parts, that

$$|P_D\phi(z) - \phi(\xi_0)| \leq \epsilon \left[\frac{1}{2\pi} \int_0^{2\pi} \left|\phi(e^{i\theta})\right| d\theta + |(\phi(\xi_0)| \right].$$

This concludes the proof. $\square$

Now that we have seen the Poisson transform we are in position to prove Property 3 (Converse to Mean-Value Property).

Proof. It is enough to show that h is harmonic on each open disk D with $\bar{D} \subset \Omega$. Fix such a D, and define $k : \bar{D} \to \mathbb{R}$ by

$$k = \begin{cases} h - P_D h, & \text{on } D, \\ 0, & \text{on } \partial D. \end{cases}$$

Then k is continuous on $\bar{D}$ and has the local mean-value property on D. As $\bar{D}$ is compact, k attains a maximum value M at some point of $\bar{D}$. Define

$$A = \{z \in D : k(z) < M\} \text{ and } B = \{z \in D : k(z) = M\}$$

Then A is open, since k is continuous. Also B is open, for if $k(w) = M$, then the local mean-value property forces k to be equal to M on all sufficiently small circles around w. As S and B partition the connected set D, either $A = D$, in which case k attains its maximum on ∂D ans so $M = 0$, or else $B = D$, in which case $k \equiv M$ and again $M = 0$. Thus $k \leq 0$, and a similar argument shows that $k \geq 0$. Hence $h = P_D h$ on D, and since $P_D h$ is harmonic there, so is h. $\square$

Another important property in the proof of the Hayman-Wu theorem is the following.

Theorem 5.4. *Reflection Principle:* *Let $D := D(0, R)$, and write $D^+ = \{z \in D : Im(z) > 0\}$, and $\mathbb{I} = \{z \in D : Im(z) = 0\}$. Suppose that f is a holomorphic function on D^+ such that $Re(f)$ extends continuously to $D^+ \cup \mathbb{I}$ with $Re(f) = 0$ on $\mathbb{I}$. Then f extends holomorphically to the whole of D.*

Note that no assumption is made about continuity of $Im(f)$ on $\mathbb{I}$, this comes from free.

Proof. Define $h : D \to \mathbb{R}$ by

$$h(z) = \begin{cases} Re(f(z)), & z \in D^+, \\ \quad 0, & z \in I, \\ -Re(f(\bar{z})) & \bar{z} \in D^+. \end{cases}$$

Then h is continuous on D and has the local Mean Value Property (Property 2.), so by the Theorem 5.4. it is harmonic on D. Also, there exists a holomorphic function g on D such that $h = Re(g)$. Now $f - g$ is holomorphic on D^+ and takes only imaginary values, so it is constant there by the Cauchy-Riemann equations (1). Adjusting g appropierly, we can make this constant zero. Then g provides the promised holomorphic extension of f to the whole of D. $\square$

6 Harmonic Measure

Given a domain Ω with boundary $\partial\Omega$, given $w \in \Omega$ and given $E \subseteq \partial\Omega$ measurable, the **harmonic measure** of E at w is the solution of the Dirichlet problem for the indicator function in E:

$$\chi_E(t) = \begin{cases} 1, & t \in E \\ 0, & t \in \partial\Omega \setminus E. \end{cases} \tag{11}$$

This harmonic measure is denoted by $\omega(w, E, \Omega)$. As a function of w it is harmonic in Ω (solution to the Dirichlet problem), and as a function of E is a measure on $\partial\Omega$.

Remark 2. *By the maximum principle for harmonic functions we have*

$$0 \le \omega(w, E, \Omega) \le 1, \qquad w \in \Omega.$$

This is so because, if $h(w) = \omega(w, E, \Omega)$,

$$h(w) \le \max_{\partial\Omega} h = 1$$

and

$$(-h)(w) \le \max_{\partial\Omega}(-h) = 0.$$

Remark 3. *As we have indicated before the solution of the Dirichlet problem in a simply connected domain Ω can be obtained from the solution to the same problem in $\mathbb{D}$ through the map given by Riemann's theorem.*

Let $f : \Omega \longrightarrow \mathbb{D}$ be a conformal map and assume f extends continuously to the boundary $\partial\Omega$. Given $\phi \in \mathcal{C}(\partial\Omega)$ consider the function $\psi(\xi) = \phi(f^{-1}(\xi))$, which is continuous in $\partial\mathbb{D}$. Then

$$U(z) := (P_{\mathbb{D}}\psi)(z)$$

is harmonic in $\mathbb{D}$ with boundary values $\psi(\xi)$. Letting $z = f(z)$ and $\xi = f(\eta)$ we see that

$$u(w) := (U \circ f)(w)$$

is harmonic in Ω (by Property 1) and with boundary values $\phi(\eta)$, i.e. u solves the Dirichlet problem in Ω.

In particular

$$\omega(w, E, \Omega) = \omega(f(w), f(E), \mathbb{D}).$$

We will consider next the harmonic measure in the upper half-plane as a canonical example which will be used for the proof of the Hayman-Wu Theorem.

6.1 Harmonic measure in the half-plane

Write $\mathbb{H} = \{w \in \mathbb{C} : Im(w) > 0\}$ for the upper half-plane and $\mathbb{R}$ for the real line. Let $(a, b) = \mathbb{R}$. Consider the following function:

$$\theta = \theta(w) = \arg(\frac{w-b}{w-a}).$$

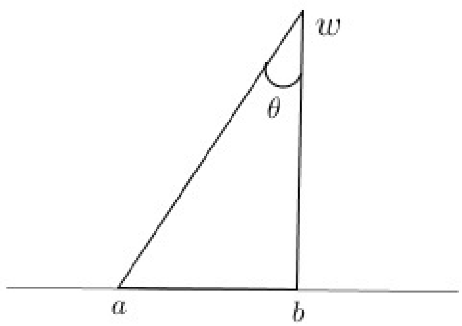

Figure 9: Angle θ on (a, b).

Notice that

$$\theta(w) = \arg\left(\frac{w-b}{w-a}\right) = \operatorname{Re}\left(-i\log\left(\frac{w-b}{w-a}\right)\right),$$

since

$$-i\log\left(\frac{w-b}{w-a}\right) = -i\log\left|\frac{w-b}{w-a}\right| + \arg\left(\frac{w-b}{w-a}\right).$$

Therefore, by Theorem 5.1, $\theta \in Har(\mathbb{H})$.

On the other hand it is clear that

$$\lim_{w\to\xi}\theta(w) = \phi(\xi) = \begin{cases} \pi, & \xi \in (a,b) \\ 0, & \xi \in \mathbb{R}\setminus[a,b]. \end{cases}$$

Therefore

$$\omega(w,(a,b),\mathbb{H}) = \frac{1}{\pi}\theta(w).$$

Let now $E \subset \mathbb{R}$ be slightly more general. Let E be a finite union of open intervals and write $E = \bigcup_{j=1}^{n}(a_j, b_j)$, with $b_{j-1} < a_j < b_j$. Set

$$\theta_j(w) = \arg\left(\frac{w-b_j}{z-w_j}\right).$$

Similarly we see that the **harmonic mesure** of E at $w \in \mathbb{H}$ is now

$$\omega(w,E,\mathbb{H}) = \sum_{j=1}^{n}\frac{\theta(w)_j}{\pi}. \tag{12}$$

As we have seen, this is the unique harmonic function that satisfies the following conditions:

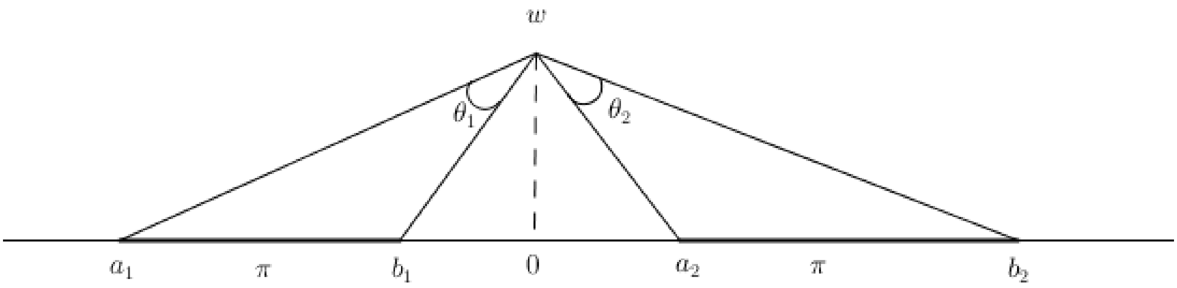

Figure 10: Angle $\theta_i = \pi$ on $(a_i, b_i), i = 1, 2$, and $\theta = 0$ on $\mathbb{R} \diagdown (a_i, b_i)$.

i) $0 < \omega(z, E, \mathbb{H}) < 1$, for $w \in \mathbb{H}$.

ii) $\omega(z, E, \mathbb{H}) \to 1$ as $w \to E$.

iii) $\omega(z, E, \mathbb{H}) \to 0$ as $w \to \mathbb{R} \diagdown \bar{E}$.

Again, this definition can be generalized to any measurable $E \subseteq \mathbb{R}$ by means of the Poisson transform. The Poisson Kernel in $\mathbb{H}$ is obtained by transporting to $\mathbb{H}$ the Poisson Kernel in $\mathbb{D}$. Let $\tau : \mathbb{H} \to \mathbb{D}$ be a conformal mapping, for instance the Möbius transform

$$\tau(w) = i\frac{w-1}{w+1}, \quad w \in \mathbb{H}.$$

Given $w \in \mathbb{H}$ and $t \in \mathbb{R}$ we have

$$P_{\mathbb{H}}(w, t) = P_{\mathbb{D}}(\tau(w), \tau(t)) = \frac{1 - |\tau(w)|^2}{|\tau(t) - \tau(w)|^2}.$$

A computation shows that ($w = x + iy$):

$$P_{\mathbb{H}}(w, t) = \frac{\operatorname{Im}(w)}{(t - \operatorname{Re}(w))^2 + \operatorname{Im}^2(w)} \frac{1}{\pi} = \frac{y}{(t - x)^2 + y^2} \frac{1}{\pi}.$$

Definition 6.1. *If $E \subseteq \mathbb{R}$ is measurable, we define the **harmonic measure** of E at $w \in \mathbb{H}$ and $t \in \mathbb{R}$ to be*

$$\omega(w, E, \mathbb{H}) = \int_E P_{\mathbb{H}}(w, t) dt = \int_E \frac{y}{(t - x)^2 + y^2} \frac{dt}{\pi}.$$

This is the particular solution to the Dirichlet problem for the indicator function

$$\chi_E(t) = \begin{cases} 1, & t \in E \\ 0, & t \notin E. \end{cases}$$

7 The theorem of Hayman-Wu

This chapter is the core of the project. As we have already mentioned, the theorem of Hayman-Wu is a result on conformal mapping. The proof we present uses harmonic measure and the pseudohyperbolic metric seen in the previous chapters. For this, we will give an elementary proof based on an idea of the late Knut Øyma, who proved the estimate

$$\text{lenght}(\varphi^{-1}(L \cap \Omega)) \leq C$$

with $C = 4\pi$.

Many authors have contributed to problems related to the Hayman-Wu theorem. For the sake of brevity we will only mention the results that are directly concerned with the constant C.

Hayman and Wu [4] gave the first proof of the theorem with the large constant $C = 2 \cdot 10^{35}$ instead of 4π. Garnett, Gehring and Jones [3] obtained a shorter proof, but did not use it to estimate the constant. Fernández, Heinonen and Martio [2] showed that $C \leq 4\pi^2$. Later K. Øyma [5] shocked the community by showing in two pages that $C \leq 4\pi$. Finally, Rohde [8] proved that the best constant is stricly smaller than 4π.

On the other hand K. Øyma [6] proved that the best constant is at least π^2.

It is still not known what is the best possible constant.

Theorem 7.1. *Let φ be a conformal mapping from $\mathbb{D}$ to a simply connected domain Ω and let L be any line. Then*

$$\text{lenght}(\varphi^{-1}(L \cap \Omega)) \leq 4\pi.$$

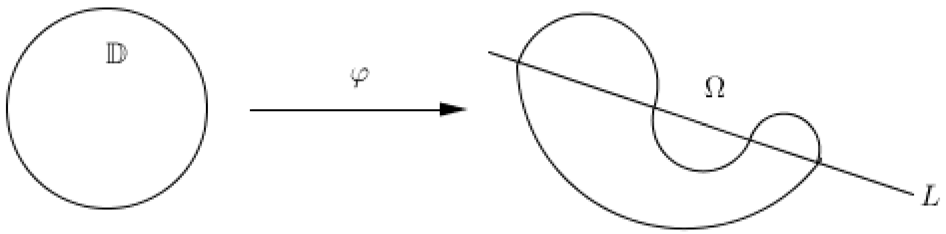

Figure 11: Hayman-Wu theorem.

Proof. First notice that we can assume that φ is analytic in a neighbourhood of $\bar{\mathbb{D}}$. To see this consider $D_r := \{z \in \mathbb{C}; \quad |z| < r\}, r < 1$ and $\Omega_r = \varphi(D_r)$. The function

$$\varphi_r(z) = \varphi(rz)$$

is then a conformal mapping from $\mathbb{D}$ to Ω_r which is analytic in a neighbourhood of $\bar{\mathbb{D}}$ (actually $\varphi_r \in H(D(0, \frac{1}{r}))$) (see Figure 12).

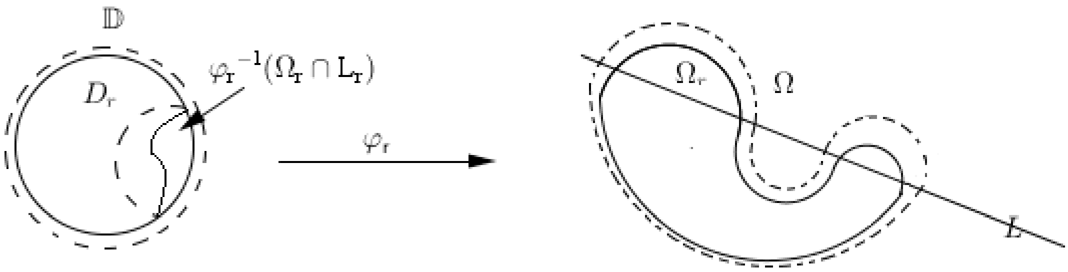

Figure 12: Hayman-Wu theorem. Conformal mapping $\varphi_r : D_r \to \Omega_r$.

If the theorem holds for such φ_r we have

$$\text{lenght}(\varphi_r{}^{-1}(L \cap \Omega_r)) \leq 4\pi.$$

Letting $r \to 1^-$ we get the general result.

$$\text{lenght}(\varphi^{-1}(L \cap \Omega)) \leq 4\pi.$$

Therefore, we can assume that φ is analytic and one-to-one in a neighborhood of $\bar{\mathbb{D}}$. Applying a rotation if necessary, we can assume also that $L = \mathbb{R}$ (see Figure 13).

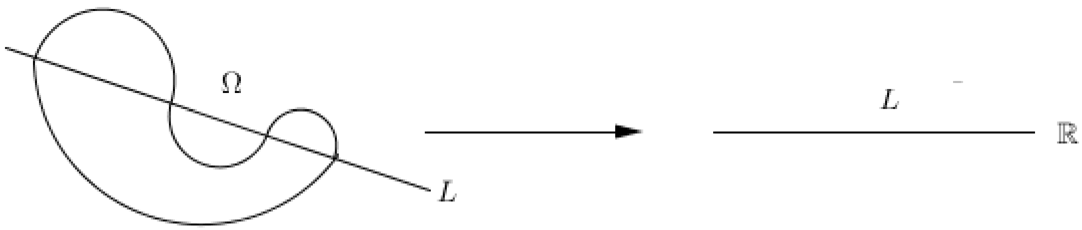

Figure 13: Hayman-Wu theorem. L with rotation happens to be $\mathbb{R}$

Let $L_k, k = 1, 2, ..., N$, denote the components of $\Omega \cap L$ (see Figure 14), and let Ω_k be that component of $\Omega \cap \{\bar{z} : z \in \Omega\}$ such that $L_k \in \Omega_k$.

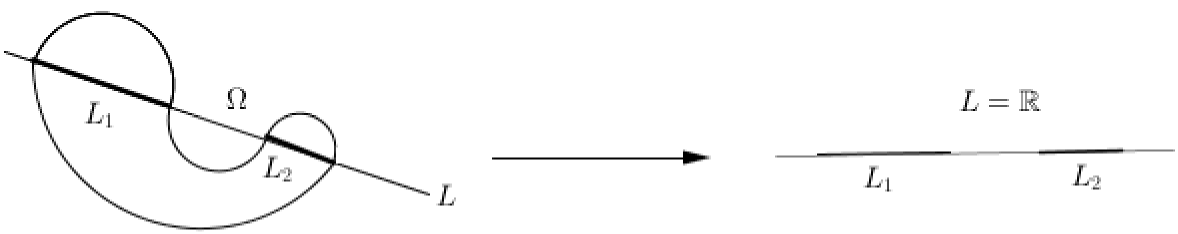

Figure 14: Hayman-Wu theorem. Example with $L_k, k = 1, 2$.

As we can see in Figure 16, Ω_1 is the stripped area and L_1 is a segment that crosses it. Note that Ω_k is a symmetrical domain respect to L_k (Figure 16).

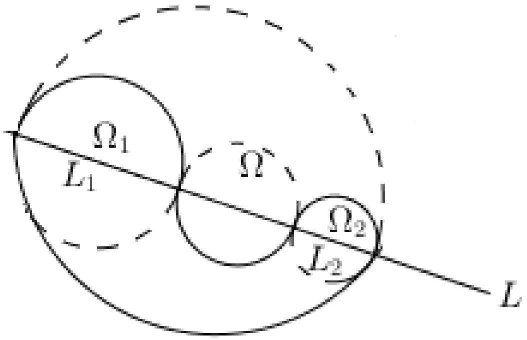

Figure 15: Hayman-Wu theorem. Example with $\Omega_k, k = 1, 2$.

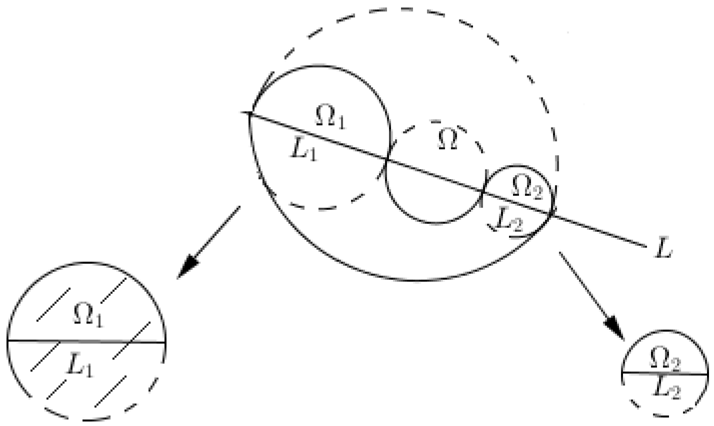

Figure 16: Hayman-Wu theorem.

By Riemann's theorem (Theoremm̃efriemann) there exists a conformal mapping $\psi_k : \Omega_k \to \mathbb{T}$ (see Figure 17) such that $\psi_k(L_k) = \mathbb{R}^+$. Moreover ψ_k extends continuously to $\bar{\Omega}_k$.

Take $\alpha \in \partial\Omega_k \cap \partial\Omega$ and let $\xi \in \partial\mathbb{D} \cap \partial\varphi^{-1}(\Omega_k)$ be such that $\alpha = \varphi(\xi)$. Define also $\beta = {\psi_k}^{-1}(|\psi_k(\alpha)|)$ and $z = \varphi^{-1}(\beta) \in \mathbb{D}$ (see Figure 17).

Notice that the composition

$$\phi = \varphi^{-1} \circ {\psi_k}^{-1}(|\psi_k \circ \varphi|)$$

takes $\varphi^{-1}(\partial\Omega_k \cap \partial\Omega)$ to $\varphi^{-1}(L_k)$. In detail:

- $\psi_k(\varphi(\xi)) = \psi_k(\alpha) \in i\mathbb{R}$,

- $|\psi_k(\varphi(\xi))| \in \mathbb{R}_+$ (denote $x = |\psi_k(\varphi(\xi))|$)

- ${\psi_k}^{-1}(x) \in L_k$,

- $\varphi^{-1}({\psi_k}^{-1}(x)) \in \varphi^{-1}(L_k) \subseteq P$.

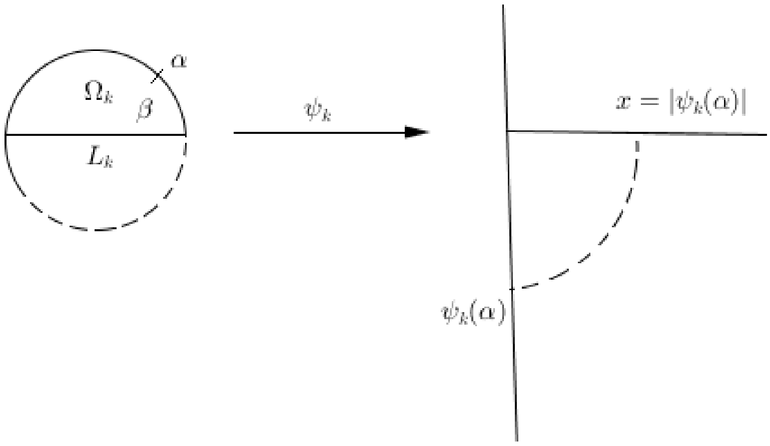

Figure 17: Proof of Hayman-Wu theorem.

Consider the finite sets $P \subseteq \partial\Omega$ and $P' \subseteq \partial\mathbb{D}$ formed by the end points of L_k and $\varphi^{-1}(L_k)$ respectively.

Let

$$E = \varphi^{-1}(\partial\Omega \cap \partial\Omega_k \setminus P)$$

and

$$F = \varphi^{-1}(\bigcup\nolimits_K L_k \setminus P').$$

Then $\phi : E \to F$ is a parametrization of $\varphi^{-1}(\bigcup_K L_k \setminus P')$. Since P' is finite we have

$$\begin{aligned}\text{length}(\varphi^{-1}(\Omega \cap L)) &= \text{length}(\varphi^{-1}(\bigcup\nolimits_k L_k)) \\ &= \text{length}(\varphi^{-1}(\bigcup\nolimits_k L_k \setminus P')) \\ &= \int_E |\nabla\phi(\xi)|\,|d\xi|.\end{aligned}$$

Claim 1. *It is enough to see that* $|\nabla\phi(\xi)| \leq 2$.

This is clear, since $E \subset \partial\mathbb{D}$, and if this is the case then

$$\int_E |\nabla\phi(\xi)|\,|d\xi| \leq 2\,|E| \leq 2 \cdot 2\pi = 4\pi.$$

We will show that $|\nabla\phi(\xi)| \leq 2$ by computing

$$\lim_{\xi \to \xi'} \frac{P_{\mathbb{D}}(\phi(\xi), \phi(\xi'))}{|\xi - \xi'|}.$$

Suppose that $I = (\xi, \xi')$ in open interval contained in $\varphi^{-1}(\partial\Omega_k \cap \partial\Omega)$. Set $\alpha' = \varphi(\xi')$ and $x' = |\psi_k(\alpha')|$.

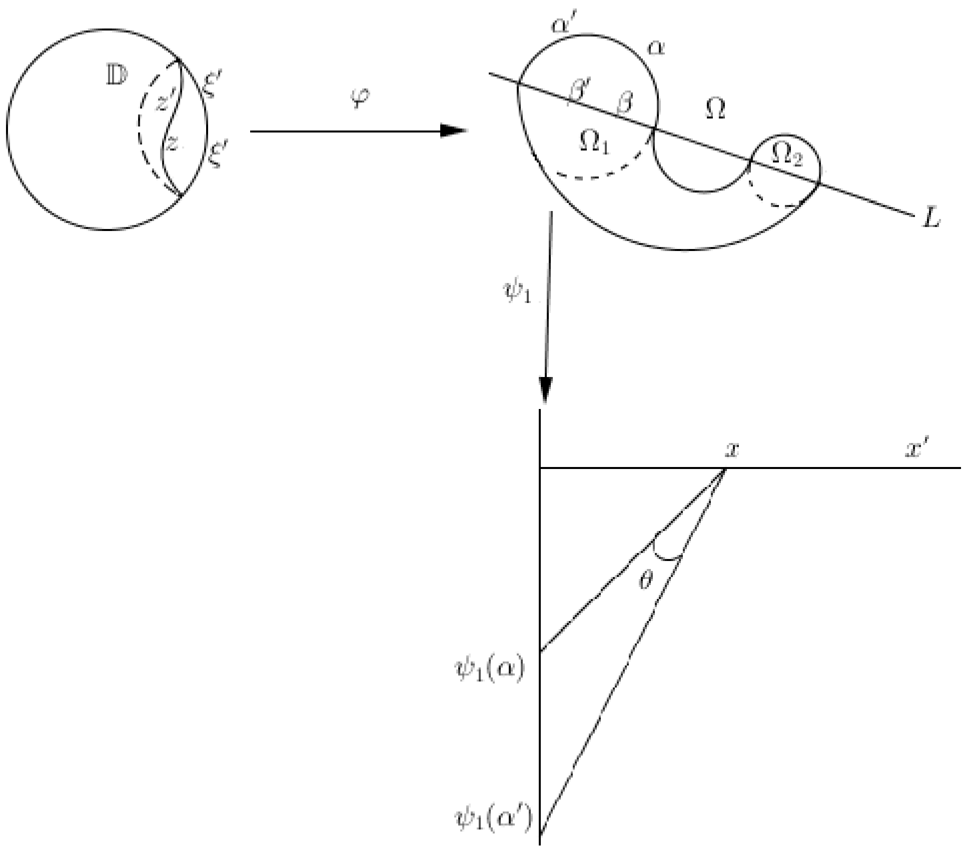

Figure 18: Hayman-Wu theorem. General representation.

By definition and by Corollary 4.1, in the last inequality we have

$$\rho_{\mathbb{D}}(\phi(\xi), \phi(\xi')) = \rho_{\Omega}(\varphi(\phi(\xi)), \varphi(\phi(\xi'))) = \rho_{\Omega}(\beta, \beta') \leq \rho_{\Omega_k}(\beta, \beta').$$

Now as it is seen in Example 1,

$$\rho_{\Omega_k}(\beta, \beta') = \rho_{\mathbb{T}}(\psi_k(\beta), \psi_k(\beta')) = \rho_{\mathbb{T}}(x, x') = \left|\frac{x - x'}{x + x'}\right|.$$

Claim 2. *For x' close to x,*

$$\omega(x, \psi_k(\varphi(I)), \mathbb{T}) \geq \frac{1}{\pi}[1 + (o\,|x - x'|)]\rho_{\mathbb{T}}(x, x')$$

Proof. Here we use what we saw for the harmonic measure in $\mathbb{H}$, which rotated by $-i$ gives the harmonic measure in $\mathbb{T}$. We know that (see (12))

$$\omega(x, \psi_k(\varphi(I)), \mathbb{T}) = \frac{\theta}{\pi},$$

where θ is the angle between the segments $\overrightarrow{x\psi_1(\alpha)}$ and $\overrightarrow{x\psi_1(\alpha')}$ (see Figure 19).

Notice that

$$\tan(\theta + \frac{\pi}{4}) = \frac{x'}{x} = 1 + \delta.$$

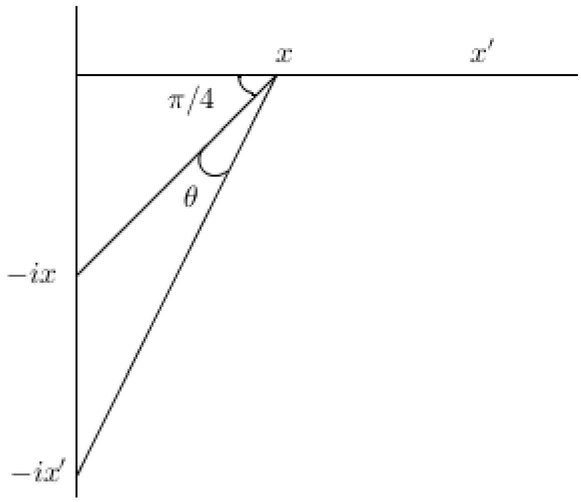

Figure 19: Hayman-Wu theorem.

Since

$$\tan(\theta + \frac{\pi}{4}) = \frac{\tan\theta + 1}{1 - \tan\theta}$$

we get

$$\tan\theta = \frac{\delta}{2+\delta}$$

and therefore

$$\theta = \arctan(\frac{\delta}{2+\delta}) = \frac{\delta}{2} + o(\delta) \qquad \text{as } \delta \to 0.$$

On the other hand,

$$\left|\frac{x - x'}{x + x'}\right| = \frac{\delta}{2+\delta} = \frac{\delta}{2} + o(\delta) \qquad \text{as } \delta \to 0.$$

Hence

$$\omega(x, \psi_k(\varphi(I)), \mathbb{T}) = \frac{\delta}{2\pi} + o(\delta) = [1 + o\,|x - x'|]\frac{1}{\pi}\rho_{\mathbb{T}}(x, x').$$

□

Once this is proved, we have

$$\begin{aligned} \frac{1}{\pi}\rho_{\mathbb{D}}(\phi(\xi), \phi(\xi')) \le \frac{1}{\pi}\rho_{\mathbb{T}}(x, x') &\le (1 + o\,|x - x'|)\ \omega(x, \psi_k(\varphi(I)), \mathbb{T}) \\ &= (1 + o\,|\xi - \xi'|)\ \omega(z, I, \varphi^{-1}(\Omega_k)) \end{aligned}$$

Applying the Maximum Principle for harmonic functions (Property 5.) to the functions

$$h_1(\xi) = \omega(\phi(\xi), I, \varphi^{-1}(\Omega_k)) = h_1, \quad h_2(\xi) = \omega(\phi(\xi), I, \mathbb{D}) = h_2$$

we get $h_1 \le h_2$. This is so because

$$h_1 = 0 \text{ in } \partial\varphi^{-1}(\Omega_k) \setminus I, \quad h_1 = 1 \text{ in } I$$

$$h_2 = 0 \text{ in } \partial\mathbb{D} \setminus I, \quad h_1 = 1 \text{ in } I,$$

and therefore $h_1 - h_2 \le 0$ on $\partial\mathbb{D}$. Letting $z = \phi(\xi)$, we get

$$\omega(z, I, \varphi^{-1}(\Omega_k)) \le \omega(\phi(\xi), I, \mathbb{D}) = \int_I \frac{1 - |\phi(\xi)|^2}{|e^{i\theta} - \phi(\xi)|^2} \frac{d\theta}{2\pi},$$

and therefore

$$\begin{aligned} \frac{1}{\pi} \frac{\rho_{\mathbb{D}}(\phi(\xi), \phi(\xi'))}{|\xi - \xi'|} &\le [1 + o\,|\xi - \xi'|] \frac{1}{|\xi - \xi'|}\ \omega(\phi(\xi), I, \mathbb{D}) = \\ &= [1 + o\,|\xi - \xi'|] \frac{1}{|\xi - \xi'|} \int_I \frac{1 - |\phi(\xi)|^2}{|e^{i\theta} - \phi(\xi)|^2} \frac{d\theta}{2\pi}. \end{aligned}$$

Since

$$\rho_{\mathbb{D}}(\phi(\xi), \phi(\xi')) = \frac{|\phi(\xi) - \phi(\xi')|}{\left|1 - \bar{\phi(\xi')}\phi(\xi)\right|},$$

taking limit as $\xi \to \xi'$ we get

$$\begin{aligned} \lim_{\xi \to \xi'} \frac{1}{\pi} \frac{\rho_{\mathbb{D}}(\phi(\xi), \phi(\xi'))}{|\xi - \xi'|} &= \frac{1}{\pi} \frac{|\nabla\phi(\xi)|}{1 - |\phi(\xi)|^2} \\ &\le \lim_{\xi \to \xi'} \frac{1}{|\xi - \xi'|} \int_{[\xi, \xi']} \frac{1 - |\phi(\xi)|^2}{|e^{i\theta} - \phi(\xi)|^2} \frac{d\theta}{2\pi} \\ &= \frac{1 - |\phi(\xi)|^2}{|\xi - \phi(\xi)|^2} \frac{1}{2\pi} \end{aligned}$$

i.e.

$$|\nabla\phi(\xi)| \le \frac{1}{2} \left| \frac{1 - |\phi(\xi)|^2}{|\xi - \phi(\xi)|} \right|^2 = \frac{1}{2} \left| \frac{1 - |\phi(\xi)|}{|\xi - \phi(\xi)|} \right|^2 (1 + |\phi(\xi)|)^2.$$

Since $\phi(\xi) \in \mathbb{D}$ and $\xi \in \partial\mathbb{D}$ we have $1 + |\phi(\xi)| \le 2$ and $|\xi - \phi(\xi)| \ge 1 - |\phi(\xi)|$. Then, finally

$$|\nabla\phi(\xi)| \le \frac{1}{2} 2^2 = 2,$$

as stated. $\square$

8 Conclusions

The aim of this project has been to state and prove the theorem and Hayman and Wu. We have followed the elementary proof by Knut Øyma's, as sketched in Garnett and Marshall's book Harmonic Measure. Along the way we have introduced several properties of conformal maps, harmonic functions and harmonic measure, which are the elements that appear in the proof. The universal estimate on the length of the preimage of a line has been improved several times since it was first established by Hayman and Wu. It is still not known what is the best possible constant.

I conclude that in mathematics there is usually more than one method to prove a theorem. In this case we have seen several improvements of the original proof, as explained in the introduction. This makes me think that -who knows!- there will be in the future the possibility to find a procedure to improve this result further.

References

[1] J. Bruna, J. Cufí: *Anàlisi complexa*, Manuals de la UAB 49, 2008.

[2] J. Fernández, J. Heinonen, O. Martio: *Quasilines and conformal mappings.* (J.Analyse Math. 52 (1989), pp. 117-132).

[3] J. B. Garnett and D. E. Marshall: *Harmonic Measure*, Cambridge University Press, Cambridge, 2005.

[4] W. Hayman, J.-M. Wu: *Level sets of univalent functions.* (Comm. Math, Helv. 56, pp. 366-403).

[5] K. Øyma: *Harmonic Measure and Conformal Length.* (Proceedings of the American Mathematical Society, Vol. 115, No. 3 (Jul., 1992), pp. 687-689).

[6] K. Øyma: *The Hayman-Wu constant.* (Proceedings of the American Mathematical Society, Vol. 119, No. 1 (Sep., 1993), pp. 337-338).

[7] T. Ransford: *Potential Theory in the Complex Plane*, Cambridge University Press, Cambridge, 1995.

[8] S.Rohde: *On the theorem of Hayman and Wu.* (Proceedings of the American Mathematical Society, Vol. 130, No. 2 (Sep., 2001), pp. 387-394).