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Convergence and Divergence of Fourier Series

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Abstract

In this project we study the convergence of Fourier series. Specifically, we first give some positive results about pointwise and uniform convergence, and then we prove two essential negative results: there exists a continuous function whose Fourier series diverges at some point and an integrable function whose Fourier series diverges almost at every point. In the case of divergence, we show that one can use other summability methods in order to represent the function as a trigonometric series.

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Contents

Introduction

In the present work we study Fourier series, specifically their convergence. A Fourier series is the decomposition of a function as a series of sines and cosines, also known as a trigonometric series. This decomposition, in turn, has numerous applications on the resolution of basic and significant problems of mathematics and physics.

Precisely, the origin of Fourier series was due to the lack of any other technique to resolve a problem in physics. Indeed, these arose in the XVIIIth century as a way of trying to solve the differential equation of the vibrating string, which is nothing else but the one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}.$$

The first person to consider that the general solution of this equation could be regarded as a trigonometric series was Daniel Bernoulli in 1753. However, renowned characters such as Euler, Lagrange and d'Alembert disagreed with Bernoulli, claiming that this kind of solution could not be the general one.

Despite having such a controversial birth, Fourier series did not begin to be taken seriously until the beginning of the XIXth century. In 1807, Joseph Fourier, whose name was later given to the series, introduced them together with Fourier integrals to the Paris Academy of Sciences. Specifically, he did so by presenting his resolution to the heat transfer equation

$$\nabla \cdot K \nabla T = c \frac{\partial T}{\partial t}.$$

In spite of his inaccuracy and the opposition of such a notable figure as Lagrange, he managed to spread the word about this technique.

Later, Poisson, Cauchy and Dirichlet were the first to study the convergence of Fourier series, being Dirichlet the one to publish, in 1829, the first accurate result. He limited himself to find sufficient conditions for the convergence rather than try to solve the problem with all generality. His example was followed by personalities like Lipschitz, Jordan and Dini, who either generalized Dirichlet's results or found other criteria for the convergence.

In 1867, Riemann's work on trigonometric series was published. After generalizing Cauchy's integral concept, he found the expression of the coefficients on the Fourier series in order to represent the desired function. Also, he proved some essential results in harmonic analysis such as the Riemman-Lebesgue lemma.

Back then, the majority shared the idea that the Fourier series of a continuous function had to converge. However, this was proven false by du Bois-Reymond [2] in 1873, who showed that there exists a continuous function whose Fourier series diverges at a point. This issue apparently disabled the possibility of recovering a function when only given its Fourier series. Nevertheless, just like Féjer published in 1900, if we modify the way of summing the series, this problem is solved most of the times.

In 1902, Lebesgue introduced his integral and he then generalized the Riemann-Lebesgue lemma. In addition, he defined a summability method similar to that of Féjer and also proved that if a trigonometric series converges to an integrable function, it must be its Fourier series.

The beginning of the XXth century also brought the development of functional analysis, which allowed the resolution of some problems of harmonic analysis and lead to the generalization of its questions. From then on, the pointwise convergence and the p-mean convergence, both in $\mathcal{L}^p$ - the space of functions f such that $|f|^p$ is finite -, began to be considered.

The cases $p = 1$ and $p = \infty$ on p-mean convergence were proven negative, meaning that there exists a function in $\mathcal{L}^p$ such that its Fourier series does not converge in p-mean for these p, whereas the rest of cases were proven positive by Marcel Riesz in 1923, meaning that every function in $\mathcal{L}^p$ has a Fourier series which is convergent in p-mean for $1 < p < \infty$.

In regard to pointwise convergence in $\mathcal{L}^p$, Kolmogorov [8] proved in 1923 that there exists an integrable function, that is from $\mathcal{L}^1$, such that its Fourier series diverges almost everywhere. Therefore, the case $p = 1$ was dismissed. The case $p = 2$ was conjectured to be true by Lusin in 1915 and proven true by Carleson [3] in 1965. The result was later generalized by Hunt [7], who in 1967 showed that there is convergence in $\mathcal{L}^p$ for every $p > 1$.

But this did not settle the question. At that point, the inquiry became: which is the biggest space in which every function has got a convergent Fourier series? Notice that between the space $\mathcal{L}^1$ and the spaces $\mathcal{L}^p \subset \mathcal{L}^1$ for any $p > 1$ we have several spaces on which we can check convergence. Until now, we know that:

- Essentially, the biggest space in which the convergence of the Fourier series of any function has been proven is, as done by Antonov [1] in 1996,

$$\mathcal{L}\log\mathcal{L}\log\log\log\mathcal{L} := \left\{ f;\ \int |f|(\log^+|f|)(\log^+\log^+\log^+|f|) < \infty \right\},$$

  where $\log^+$ denotes the positive part of the logarithm.

- Essentially, the smallest space in which it has been proven that there exists a function whose Fourier series diverges is, as proven by Konyagin [9],[10] in 2000,

$$\mathcal{L}\varphi\mathcal{L} := \left\{ \int |f|\varphi(|f|) < \infty \right\},$$

  where φ is an increasing function such that $\varphi(t) = o\left(\frac{\sqrt{\log t}}{\sqrt{\log\log t}}\right)$ when $t \to \infty$.

As of today, from the spaces chain

$$\mathcal{L}^p \subset \cdots \subset \mathcal{L}\log\mathcal{L}\log\log\log\mathcal{L} \subset \cdots \subset \mathcal{L}\varphi\mathcal{L} \subset \mathcal{L}\varphi \subset \cdots \subset \mathcal{L}^1,$$

we know that the biggest space in which convergence can be granted is somewhere between the Antonov space and the Konyagin one, but such space has not been found yet.

In Chapter 1 of this project, we define Fourier series and prove their basic properties, as well as Bessel's inequality, Theorems 1.10 and 1.12.

In Chapter 2, we start proving the Riemann-Lebesgue lemma, Lemma 2.3, which was originally published by Riemann in 1867 and generalized by Lebesgue 1902. Then, the basic positive results on pointwise and uniform convergence of Fourier series are proven. These give sufficient conditions for the series to converge. The results involving pointwise convergence, Theorems 2.7 and 2.9, are due to Dirichlet and Dini and were published in 1829 and 1880 respectively. In addition to a basic positive result on uniform convergence, a simple negative criterion is given.

For these two chapters, the main references have been Cerdà's book [4] and Duoandikoetxea's notes [6].

In Chapter 3, we prove du Bois-Reymond's and Kolmogorov's results, Theorems 3.1 and 3.4, first published in 1873 and 1923 respectively. For the first one, which grants the existence of a continuous function whose Fourier series diverges at a point, we give both a proof of existence - based on the uniform boundedness principle - and a constructive one. For the second theorem, which claims the existence of an integrable function whose Fourier series diverges almost everywhere, we give the original proof of the author.

The principal references for Chapter 3 have been Duoandikoetxea's notes [6] and book [5], Ul'yanov's paper [13] and Reed's and Simons' book [12], from which I studied the proof of the uniform boundedness principle.

In Chapter 4, the last of this project, we give the Cesàro and the Abel-Poisson summability methods, which can be used as an alternative to the sum of the Fourier series when this one diverges. We study not only the pointwise convergence of the summability criteria but also the p-mean convergence. From the case $p = 2$ we deduce the Plancherel-Parseval identity. Finally, we note the relation between the Abel-Poisson summability and the classic Dirichlet problem, which we solve.

For this chapter, the main sources have been Duoandikoetxea's notes [6] and Körner's book [11].

It should be pointed out that, since I did not take *Anàlisi real i funcional* course, whose content was essential in order to develop this project, I had to independently study it with the help of my thesis director. However, for space reasons this part of my work could not be added to the report. Some of the corresponding results, that were needed throughout the project, are collected in the appendices.

Chapter 1

Fourier Series

In this chapter we give the basic definitions revolving around Fourier series and we prove their basic properties. In the last section we prove Bessel's inequality. The main references for this chapter have been Cerdà's book [4] and Duoandikoetxea's notes [6].

1.1 Trigonometric polynomials and trigonometric series

Definition 1.1. *A trigonometric polynomial of period $T > 0$ is an expression of the form:*

$$\frac{a_0}{2} + \sum_{k=1}^{N} \left[a_k cos\left(\frac{2\pi}{T}kt\right) + b_k sin\left(\frac{2\pi}{T}kt\right)\right],$$

where $t \in \mathbb{R}$ and $a_k, b_k \in \mathbb{C}$. Such a polynomial is said to be of degree N if either of the coefficients a_N or b_N is different from 0.

Remark 1.2. For the sake of simplicity we shall take, from now on, $T = 2\pi$. In this specific case the polynomial above turns into the following way simpler expression:

$$\frac{a_0}{2} + \sum_{k=1}^{N} (a_k coskt + b_k sinkt).$$

The generalization to an arbitrary positive period T is achieved by defining a linear transformation $x = at + b$ which turns $[-\pi, \pi]$ into $[0, T]$.

Remark 1.3. Using the exponential expression of trigonometric functions,

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i},$$

the trigonometric polynomial above can be expressed as a sum of exponential functions:

$$\sum_{k=-N}^{N} c_k e^{ikt},$$

where $c_k \in \mathbb{C}$.

Let us find the relation between the trigonometric coefficients and the exponential ones. Since

$$\sum_{k=-N}^{N} c_k e^{ikt} = \frac{a_0}{2} + \sum_{k=1}^{N} (a_k coskt + b_k sinkt) = \frac{a_0}{2} + \sum_{k=1}^{N} \left(a_k \frac{e^{ikt} + e^{-ikt}}{2} + b_k \frac{e^{ikt} - e^{-ikt}}{2i} \right) =$$

$$\frac{a_0}{2} + \sum_{k=1}^{N} \left(a_k \frac{1}{2} + b_k \frac{1}{2i} \right) e^{ikt} + \left(a_k \frac{1}{2} - b_k \frac{1}{2i} \right) e^{-ikt} = c_0 + \sum_{k=1}^{N} \left(c_k e^{ikt} + c_{-k} e^{-ikt} \right),$$

we can conclude that a solution for this equation is the following one, where $k \geq 1$:

$$c_0 = \frac{a_0}{2} \qquad c_k = \frac{1}{2} a_k + \frac{1}{2i} b_k \qquad c_{-k} = \frac{1}{2} a_k - \frac{1}{2i} b_k. \tag{1.1}$$

In particular, note that $c_{-k} = \overline{c_k}$. These relations can also be expressed as follows, where $k \geq 1$ as well:

$$a_0 = 2c_0 \qquad a_k = c_k + c_{-k} \qquad b_k = i(c_k - c_{-k}).$$

As we will show below, the set composed of the exponential functions with respective arguments ikt turns out to be an orthogonal set, wich implies that the solution above is the only one.

Definition 1.4. *A trigonometric series is a function series of the form:*

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k coskt + b_k sinkt) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}$$

where $a_k, b_k, c_k \in \mathbb{C}$.

Remark 1.5. Usually, $a_k, b_k \in \mathbb{R}$. Otherwise, we separate these coefficients into their real and their imaginary parts.

Observe that the partial sums of a trigonometric series are trigonometric polynomials.

1.2 Orthogonality

The families of functions

$$\{1, \cos t, \cos 2t, \ldots, \sin t, \sin 2t, \ldots\} \qquad \text{and} \qquad \left\{ e^{ikt} \right\}_{k \in \mathbb{Z}}$$

are both orthogonal sets of functions, which means that if φ, ψ are two different functions of the same family, then

$$\int_{-\pi}^{\pi} \varphi(t)\psi(t)dt = 0.$$

The proof is below the level of this project. It suffices to integrate by parts in order to prove it.

1.3 Fourier Series

The main problem of harmonic analysis is to try to express an arbitrary given function f as a trigonometric series.

We will only work with functions that are integrable and 2π-periodic. Otherwise, it is not possible to find suitable trigonometric series for them. From now on, we will use Lebesgue integrability, which is more general than the Riemann one if one does not take into account improper integrals. Therefore, our basic assumption will be that $f \in \mathcal{L}[-\pi, \pi]$. The reason why we only demand f to be defined on $[-\pi, \pi]$ (and only *a.e.*) is its periodicity.

Usually, the functions we will work with will be real. If they are complex, it suffices to split them into their real and imaginary parts so that we can apply them the results that we obtain for strictly real functions.

As stated above, our main goal is to determine whether there exist a_k, $b_k \in \mathbb{R}$ such that the following equality holds for every $t \in [-\pi, \pi]$:

$$f(t) = \frac{a_0}{2} + \sum_{k\geq 1} (a_k coskt + b_k sinkt).$$

An equivalent question is whether there exist $c_k \in \mathbb{C}$ such that, for any $t \in [-\pi, \pi]$:

$$f(t) = \sum_k c_k e^{ikt}.$$

Next, we find the most suitable trigonometric series for the function, which is called its Fourier series. Assume the equalities above hold. This means that the trigonometric series converges pointwisely to the function. Assume also this convergence is uniform. Then, we can integrate the series term-wise. Using this fact and orthogonality properties:

$$\forall\, l \in \mathbb{Z},\ \int_{-\pi}^{\pi} f(t)e^{ilt}dt = \int_{-\pi}^{\pi} \left(\sum_k c_k e^{ikt} \right) e^{ilt}dt = c_{-l} \int_{-\pi}^{\pi} e^{i(-l+l)t}dt = 2\pi c_{-l}.$$

Similarly,

$$\forall\, l \in \mathbb{N},\ \int_{-\pi}^{\pi} f(t)\cos lt = \pi a_l, \qquad \int_{-\pi}^{\pi} f(t)\sin lt = \pi b_l.$$

Definition 1.6. *Let $f \in \mathcal{L}[-\pi, \pi]$. We define its Fourier coefficients as*

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ktdt, \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin ktdt, \qquad c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}dt,$$

which are called the Euler-Fourier formulas. The Fourier series of f is

$$Sf(t) = \frac{a_0}{2} + \sum_{k\geq 1} (a_k coskt + b_k sinkt) = \sum_k c_k e^{ikt},$$

where the coefficients are those defined above. The partial sums of the Fourier series of f,

$$S_N f(t) = \frac{a_0}{2} + \sum_{k=1}^{N} (a_k coskt + b_k sinkt) = \sum_{|k|\leq N} c_k e^{ikt},$$

are called the Fourier sums of f.

Notice that the procedure above will not be, in general, valid unless the uniform convergence is granted. This is why not every function can be expressed as a trigonometric series. Nevertheless, the definitions above are completely general.

Example 1.7. If f is a trigonometric polynomial, we can integrate term-wise and therefore $Sf = f$.

1.4 Properties of Fourier coefficients

Let $f \in \mathcal{L}[-\pi, \pi]$.

Proposition 1.8. *Let us denote $a_k(f), b_k(f), c_k(f)$ the Fourier coefficients of f.*

1. *Fourier coefficients are bounded. Specifically,*

$$|a_k|, |b_k| \leq \frac{1}{\pi}\int_{-\pi}^{\pi} |f(t)|dt, \qquad |c_k| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|dt.$$

2. *Linearity. Let $f, g \in \mathcal{L}[-\pi, \pi]$ and $\lambda, \mu \in \mathbb{R}$. Then, $a_k(\lambda f + \mu g) = \lambda a_k(f) + \mu a_k(g)$, and similarly for b_k and c_k.*

3. *Let $f \in \mathcal{C}^1[-\pi, \pi]$, that is, the derivative of f exists and it is continuous. Then, the Fourier coefficients of f and f' are related according to the following expressions:*

$$a_k(f) = -\frac{b_k(f')}{k}, \qquad b_k(f) = \frac{a_k(f')}{k}, \qquad c_{\pm k}(k) = \mp i\frac{c_{\pm k}(f')}{k}. \tag{1.2}$$

4. *If f has a defined parity, its Fourier coefficients take special forms:*

   (i) f *even* $\Rightarrow a_k = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(t)\cos ktdt$ *and* $b_k = 0$.

   (ii) f *odd* $\Rightarrow a_k = 0$ *and* $b_k = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(t)\cos ktdt$.

Proof.

1. We use the following property of integrals:

$$\left|\int_A f\right| \leq \int_A |f|,$$

   so

$$|a_k| = \left|\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ktdt\right| \leq \frac{1}{\pi}\int_{-\pi}^{\pi} |f(t)|dt,$$

   and similarly for b_k and c_k.

2. These properties are due to the linearity of the integral.

3. Let us integrate by parts:

$$a_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt dt = \left\{ \begin{array}{lll} u = f(t) & \Rightarrow & du = f'(t)dt \\ dv = \cos kt dt & \Rightarrow & v = \frac{1}{k}\sin kt \end{array} \right\} =$$
$$\frac{1}{\pi}\left[\frac{1}{k}f(t)\sin kt\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{k}f'(t)\sin kt dt = -\frac{1}{k}b_k(f'),$$

where the first term vanishes because the argument of the sine is an integer multiple of π. Similarly, we get the analogous result for $b_k(f)$. For the other two equalities, one does not need to integrate again. It suffices to apply the relation between c_k and a_k, b_k, that is equations (1.1), so:

$$c_{\pm k}(f) = \frac{1}{2}a_k \pm \frac{1}{2i}b_k = \frac{1}{2}\cdot\left(-\frac{b_k(f')}{k}\right) \pm \frac{1}{2i}\cdot\left(\frac{a_k(f')}{k}\right) =$$
$$\frac{1}{k}\left[\pm\frac{1}{2i}a_k(f') - \frac{1}{2}b_k(f')\right] = \frac{1}{ik}\left[\pm\frac{1}{2}a_k(f') - \frac{i}{2}b_k(f')\right] =$$
$$-\frac{i}{k}\cdot\left[\pm\frac{1}{2}a_k(f') + \frac{1}{2i}b_k(f')\right] = -\frac{i}{k}\cdot[\pm c_{\pm k}(f')] = \mp\frac{i}{k}c_{\pm k}(f').$$

4. (i) If f is even, then $f \cdot \cos$ is also even and $f \cdot \sin$ is odd.
   (ii) On the other hand, if f is odd, $f \cdot \cos$ is odd too, whereas $f \cdot \sin$ is even. □

In fact, it can be proven that (1.2) holds in more general conditions, specifically if f' is piecewise continuous.

1.5 Bessel's inequality

Along this project, we will study different types of convergence, not only the pointwise or the uniform one. In this section we study a few aspects of the mean square convergence, which is the convergence in norm in the space $(\mathcal{L}^2, ||\cdot||_2)$. Spaces $\mathcal{L}^p$ are defined in Definition A.9.

Let f be a square-integrable function, $f \in \mathcal{L}^2$. Given an arbitrary $N \in \mathbb{N}$, we want to find the trigonometric polynomial of degree N, p_N that best approximates f in mean square convergence, that is, the one that minimizes the integral

$$||f - p_N||_2^2 = \int_{-\pi}^{\pi} |f - p_N|^2.$$

Proposition 1.9. *Let $f \in \mathcal{L}^2$. The trigonometric polynomial of degree N which best approximates f in the norm $||\cdot||_2$ is the Nth Fourier sum of f, $S_N f$.*

Proof. Let

$$p(t) = \frac{c_0}{2} + \sum_{k=1}^{N}(c_k \cos kt + d_k \sin kt)$$

be a trigonometric polynomial of degree N. Let us square the integrand and compute the integrals of the three resulting addends:

$$\int_{-\pi}^{\pi} |f-p|^2 = \int_{-\pi}^{\pi} (f^2 - 2fp + p^2).$$

Using orthogonality properties:

$$\int_{-\pi}^{\pi} p(t)^2 dt = \int_{-\pi}^{\pi} \left[\frac{c_0}{2} + \sum_{k=1}^{N} (c_k \cos kt + d_k \sin kt)\right]^2 dt =$$
$$\int_{-\pi}^{\pi} \left\{\left(\frac{c_0}{2}\right)^2 + \sum_{k=1}^{N} [(c_k \cos kt)^2 + (d_k \sin kt)^2]\right\} dt =$$
$$\frac{c_0^2}{4} \cdot \int_{-\pi}^{\pi} 1 dt + \sum_{k=1}^{N} \left(c_k^2 \cdot \int_{-\pi}^{\pi} \cos^2 kt dt + d_k^2 \cdot \int_{-\pi}^{\pi} \sin^2 kt dt\right) =$$
$$\frac{c_0^2}{4} \cdot 2\pi + \sum_{k=1}^{N} \pi \cdot (c_k^2 + d_k^2) = \pi \cdot \left[\frac{c_0^2}{2} + \sum_{k=1}^{N} (c_k^2 + d_k^2)\right],$$

which is known as Plancherel-Parseval's identity for trigonometric polynomials (in Theorem 4.21 we prove that this equality holds for any square-integrable function). In the second place, using the definition of Fourier coefficients:

$$\int_{-\pi}^{\pi} f(t)p(t) dt = \int_{-\pi}^{\pi} f(t) \cdot \left[\frac{c_0}{2} + \sum_{k=1}^{N} (c_k \cos kt + d_k \sin kt)\right] dt =$$
$$\frac{c_0}{2} \cdot \int_{-\pi}^{\pi} f(t) dt + \sum_{k=1}^{N} \left(c_k \cdot \int_{-\pi}^{\pi} f(t) \cos kt dt + d_k \cdot \int_{-\pi}^{\pi} f(t) \sin kt dt\right) =$$
$$\frac{c_0}{2} \cdot \pi a_0 + \sum_{k=1}^{N} (c_k a_k + d_k b_k)\pi = \pi \cdot \left[\frac{1}{2} a_0 c_0 + \sum_{k=1}^{n} (a_k c_k + b_k d_k)\right].$$

Therefore, using these two expressions and adding and subtracting $\sum(a_k^2 + b_k^2)$ to the result:

$$\int_{-\pi}^{\pi} |f-p|^2 = \int_{-\pi}^{\pi} f^2 + \int_{-\pi}^{\pi} p^2 - 2\int_{-\pi}^{\pi} fp =$$
$$\int_{-\pi}^{\pi} f^2 + \pi \cdot \left[\frac{1}{2} c_0^2 - 2 \cdot \frac{1}{2} a_0 c_0 + \sum_{k=1}^{N} (c_k^2 + d_k^2 - 2a_k c_k - 2b_k d_k)\right] =$$
$$\int_{-\pi}^{\pi} f^2 + \pi \cdot \left[\frac{c_0^2 - 2a_0 c_0 + a_0^2}{2} - \frac{1}{2} a_0^2 + \right.$$
$$\left. + \sum_{k=1}^{N} \left(c_k^2 - 2a_k c_k + a_k^2 - a_k^2 + d_k^2 - 2b_k d_k + b_k^2 - b_k^2\right)\right] =$$
$$\int_{-\pi}^{\pi} f^2 - \pi \cdot \left[\frac{a_0^2}{2} + \sum_{k=1}^{N} (a_k^2 + b_k^2)\right] + \pi \cdot \left\{\frac{(a_0 - c_0)^2}{2} + \sum_{k=1}^{N} \left[(a_k - c_k)^2 + (b_k - d_k)^2\right]\right\}.$$

The first two terms of the last expression do not depend on p, whereas the last one does. Since all the terms in the last term are positive, the only choice to minimize it is $c_k = a_k$ and $d_k = b_k$ for every $0 \leq k \leq N$, which was to be shown. $\square$

From this proof, we also deduce Bessel's inequality (which in Theorem 4.21 will be shown to be, in fact, an equality).

Theorem 1.10 (Bessel's inequality). *Let $f \in \mathcal{L}^2$. Then,*

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} f^2.$$

Notice that the right term in the equality is finite because f is square-integrable by hypothesis.

Proof. Let $p = S_N f$. As we have shown, in this case the last term of the last expression in the proof above vanishes, so

$$0 \leq \int_{-\pi}^{\pi} |f-p|^2 = \int_{-\pi}^{\pi} f^2 - \pi \cdot \left[\frac{a_0^2}{2} + \sum_{k=1}^{N}(a_k^2 + b_k^2)\right]. \qquad \square$$

An essential consequence of this result is that the Fourier coefficients of any square-integrable function tend to 0, otherwise the left term in Bessel's inequality could not be finite.

Corollary 1.11. *Let $f \in \mathcal{L}^2$. Then, $a_k, b_k \to 0$.*

Bessel's inequality also holds for the complex form of the trigonometric series.

Theorem 1.12 (Complex Bessel's inequality). *Let $f \in \mathcal{L}^2$. Then,*

$$\sum_{k=-\infty}^{\infty} |c_k|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2.$$

Notice that now $|\cdot|$ denotes the complex norm instead of the absolute value, for which reason it is not trivial to deduce this form of the theorem from the previous one.

Proof. The integrand takes the following form:

$$\left|f(t) - \sum_{|k|\leq N} c_k e^{ikt}\right|^2 = \left(f(t) - \sum_{|k|\leq N} c_k e^{ikt}\right) \cdot \left(\overline{f}(t) - \sum_{|k|\leq N} \overline{c_k} e^{-ikt}\right) =$$

$$f(t)^2 - \sum_{|k|\leq N}\left[f(t)\overline{c_k}e^{-ikt} + \overline{f}(t)c_k e^{ikt}\right] + \sum_{|k|,|l|\leq N} c_k\overline{c_l}e^{i(k-l)t}.$$

Integrating and using orthogonality properties:

$$0 \leq \int_{-\pi}^{\pi} \left| f(t) - \sum_{|k| \leq N} c_k e^{ikt} \right|^2 =$$

$$\int_{-\pi}^{\pi} f(t)^2 dt - \sum_{|k| \leq N} \left(\overline{c_k} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt + c_k \int_{-\pi}^{\pi} \overline{f}(t) e^{ikt} dt \right) + \sum_{|k|,|l| \leq N} \int_{-\pi}^{\pi} c_k \overline{c_l} e^{i(k-l)t} dt =$$

$$\int_{-\pi}^{\pi} f(t)^2 dt - \sum_{|k| \leq N} (\overline{c_k} c_k \cdot 2\pi + c_k \overline{c_k} \cdot 2\pi) + \sum_{|k|,|l| \leq N} c_k \overline{c_l} \cdot 2\pi \delta(k-l) =$$

$$\int_{-\pi}^{\pi} f(t)^2 dt - 2 \cdot 2\pi \sum_{|k| \leq N} |c_k|^2 + \sum_{|k| \leq N} |c_k|^2 \cdot 2\pi = \int_{-\pi}^{\pi} f(t)^2 dt - 2\pi \sum_{|k| \leq N} |c_k|^2.$$

Since this inequality holds for every N, it also does for the limit $N \to \infty$, which was to be shown. □

Corollary 1.13. *Let $f \in \mathcal{L}^2$. Then, $c_k \to 0$.*

Chapter 2

Pointwise and uniform convergence of Fourier Series: Positive results

In this chapter we study the pointwise and uniform convergence of the Fourier series of a function. More specifically, we shall give some different, independent, sufficient conditions for a function in order for its Fourier series to converge pointwisely or uniformly.

2.1 Dirichlet Kernel

The Dirichlet kernel allows us to write Fourier sums in a very compact, useful integral form. In this section, for which the main source have been Duoandikoetxea's notes [6], we include the definition and properties of this kernel.

Let us write Fourier sums in integral form. For any $N \in \mathbb{N}$:

$$S_N f(x) = \sum_{|k| \leq N} c_k e^{ikx} = \sum_{|k| \leq N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt \right) e^{ikx} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \sum_{|k| \leq N} e^{ik(x-t)} dt.$$

Definition 2.1. *Given $N \in \mathbb{N}$, the Dirichlet Kernel of degree N is defined as the function*

$$D_N(t) := \frac{1}{2} \sum_{|k| \leq N} e^{ikt}.$$

Taking this into account, the integral expression of Fourier sums takes the following form:

$$S_N f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) D_N(x-t) dt. \tag{2.1}$$

If we start from the trigonometric expression of $S_N f$ rather than the exponential one, we get the same expression. The trigonometric form of the Dirichlet kernel is

$$D_N(t) = \frac{1}{2} + \sum_{k=1}^{N} \cos kt. \tag{2.2}$$

Let us see another expression for the Dirichlet Kernel that is way more compact and will be of great utility later. Using the well-known trigonometric identity of the sine of a sum,

$$\sin(x+y) = \cos x \sin y + \sin x \cos y,$$

one can easily prove that, for every $y \in \mathbb{R}$ such that $\sin y \neq 0$:

$$\cos x = \frac{1}{2}\frac{\sin(x+y) - \sin(x-y)}{\sin y}.$$

Therefore, taking $x = kt$ and $y = \dfrac{t}{2}$ (where $1 \leq k \leq N$, and let us assume that t is such that $\sin(t/2) \neq 0$, even though I will further discuss the validity of the resulting expression for any $t \in \mathbb{R}$):

$$\cos kt = \frac{1}{2}\frac{\sin\left(k+\frac{1}{2}\right)t - \sin\left(k-\frac{1}{2}\right)t}{\sin(t/2)},$$

so:

$$D_N(t) = \frac{1}{2} + \sum_{k=1}^{N} \cos kt = \frac{1}{2} + \sum_{k=1}^{N} \frac{1}{2}\frac{\sin\left(k+\frac{1}{2}\right)t - \sin\left(k-\frac{1}{2}\right)t}{\sin(t/2)} =$$

$$\frac{1}{2}\cdot\left\{1 + \frac{1}{\sin(t/2)}\cdot\left[\cancel{\sin\left(\frac{3}{2}t\right)} - \sin\left(\frac{1}{2}t\right) + \cancel{\sin\left(\frac{5}{2}t\right)} - \cancel{\sin\left(\frac{3}{2}t\right)} + \cancel{\cdots} + \right.\right.$$

$$\left.\left. + \sin\left(N+\frac{1}{2}\right)t - \cancel{\sin\left(N-\frac{1}{2}\right)t}\right]\right\} = \frac{1}{2}\cdot\left[\cancel{1} + \frac{\sin\left(N+\frac{1}{2}\right)t - \cancel{\sin(t/2)}}{\sin(t/2)}\right],$$

and therefore

$$D_N(t) = \frac{\sin\left(N+\frac{1}{2}\right)t}{2\sin(t/2)}. \tag{2.3}$$

Notice that the denominator can vanish and hence this expression holds, *a priori*, only if t is not an even multiple of π. However, at these points, the expression on the right of (2.3) shows a removable discontinuity. That is, the left and right limits of the function both exist and they are equal. Furthermore, their value equals the value of the Dirichlet Kernel in that point. Let us show it. Let $x = 2k\pi$ with $k \in \mathbb{Z}$. Then:

$$\lim_{t\to x^{\pm}} \frac{\sin\left(N+\frac{1}{2}\right)t}{2\sin(t/2)} = \lim_{y\to\pi^{\pm}} \frac{\sin\left(N+\frac{1}{2}\right)2ky}{2\sin(2ky/2)} = \lim_{y\to\pi^{\pm}} \frac{\sin k\cdot(2N+1)y}{2\sin ky} =$$

$$\lim_{y\to 0^{\pm}} \frac{\cancel{\pm}\sin k\cdot(2N+1)y}{\cancel{\pm}2\sin ky} = \frac{k\cdot(2N+1)}{2k} = \frac{2N+1}{2},$$

while

$$D_N(x) = \frac{1}{2}\sum_{|l|\leq N} e^{ilx} = \frac{1}{2}\sum_{|l|\leq N} e^{il\cdot 2k\pi} = \frac{1}{2}\sum_{|l|\leq N} 1 = \frac{2N+1}{2}.$$

In conclusion, (2.3) holds for all $t \in \mathbb{R}$ if we convene that, at the points where it is not well-defined, we take its limit at them.

Let us prove some basic still useful properties of the Dirichlet Kernel.

Proposition 2.2.

1. *D_N is 2π-periodic,*
2. *D_N is an even function,*
3. $\frac{1}{\pi}\int_{-\pi}^{\pi} D_N(t)dt = 1.$

Proof. The first two properties are obvious taking into account equation (2.2). To prove the third one, we also use that expression. Note that if we integrate it over $[-\pi, \pi]$, the only addend that will not vanish will be the one corresponding to the constant term $1/2$, since the cosines are being integrated over a (multiple of a) period. So

$$\frac{1}{\pi}\int_{-\pi}^{\pi} D_N(t)dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}dt = \frac{1}{\pi}\cdot 2\pi \cdot \frac{1}{2} = 1. \qquad \square$$

Let us see a couple of alternative integral expressions for the Fourier sums. First of all,

$$S_N f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)dt. \tag{2.4}$$

Indeed:

$$\begin{aligned} S_N f(x) =& \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_N(x-t)dt = \frac{1}{\pi}\int_{-\pi+a}^{\pi+a} f(t)D_N(x-t)dt = \\ & \int_{-\pi}^{\pi} f(s+a)D_N(x-s-a)ds = \frac{1}{\pi}\int_{\pi}^{-\pi} f(x-t)D_N(t)\cdot(-dt) = \\ & \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)dt, \end{aligned}$$

where we have made a couple of changes of variables and used the fact that f and D_N are 2π-periodic, so the integral of $f \cdot D_N$ over $[-\pi, \pi]$ equals the integral over $[-\pi + a, \pi + a]$ for any $a \in \mathbb{R}$.

Another useful expression is:

$$S_N f(x) = \frac{1}{\pi}\int_{0}^{\pi} \left[f(x+t) + f(x-t)\right] D_N(t)dt, \tag{2.5}$$

which can be easily proven using similar arguments.

2.2 Riemann-Lebesgue Lemma

Recall that Bessel's inequality allowed us to claim that Fourier coefficients of any square-integrable function tend to 0 (Corollary 1.11). The Riemann-Lebesgue lemma tells us that this is not only true for square-integrable functions, but in fact it also is for any integrable function. Note that this is a more general result since $\mathcal{L}^2 \subset \mathcal{L}$.

The principal reference for this section has been Cerdà's book [4].

Lemma 2.3 (Riemann-Lebesgue)**.** *Let $f \in \mathcal{L}$ and $\lambda \in \mathbb{R}$. Then,*

$$\lim_{\lambda \to \infty} \int_{-\pi}^{\pi} f(t) \sin \lambda t dt = \lim_{\lambda \to \infty} \int_{-\pi}^{\pi} f(t) \cos \lambda t dt = 0.$$

Remark 2.4. A more general version of the lemma is the following. Let $f \in \mathcal{L}$, $a, b \in \mathbb{R}$ such that $a \leq b$ and $\alpha, \beta \in \mathbb{R}$. Then,

$$\lim_{|\alpha| \to \infty} \int_{a}^{b} f(t) \sin(\alpha t + \beta) dt = 0.$$

The result for the cosine is obtained by taking $\beta = \pi/2$. We only give the proof for Lemma 2.3, being the proof for this last version similar but longer.

One of the most important consequences of this lemma, but not the only one, is the aforementioned:

Corollary 2.5. *If $f \in \mathcal{L}$, then $a_k, b_k \to 0$.*

Proof. (Corollary) It suffices to take $\lambda = k \in \mathbb{N}$ in Lemma 2.3. □

Proof. (Lemma) Let us start proving the lemma for very specific functions. Then, we will generalize until we prove it for any integrable function.

(i) $f = \chi_{(a,b)}$ (characteristic function of an interval).

$$\left| \int_{-\pi}^{\pi} f(t) \sin \lambda t dt \right| = \left| \int_{a}^{b} \sin \lambda t dt \right| = \left| -\frac{1}{\lambda} \cos \lambda t \right|_{a}^{b} = \left| \frac{\cos \lambda a - \cos \lambda b}{\lambda} \right| \leq \frac{2}{|\lambda|} \xrightarrow{\lambda \to \infty} 0,$$

and similarly for the cosine.

(ii) $f = \sum_{k=1}^{N} y_k \chi_{(a_k, b_k)}$ (step function). The corresponding result is easily obtained from (i) by linearity.

(iii) $f = \chi_E$ (characteristic function of a measurable set $E \subset (-\pi, \pi)$). Recall the definition of the Lebesgue measure in Theorem A.2, and that of regular measure, Definition A.3. From Theorem A.4, the Lebesgue measure, which we are using, is regular. By outer regularity, given any $\varepsilon > 0$ there exists an open set $E \subset G \subset (-\pi, \pi)$ such that $|G \setminus E| < \epsilon$, where $|\cdot|$ denotes the Lebesgue measure. Now, let us see that

$$\int_G \sin \lambda t dt \xrightarrow{k} 0.$$

Since $G \subset \mathbb{R}$ is an open set it can be expressed as a countable union of open intervals, $G = \bigcup_k (a_k, b_k)$. Its characteristic function is then $\chi_G = \chi_{\bigcup_k (a_k, b_k)}$, which can be inferiorly approximated by the growing sequence of functions $\{\chi_{G_n}\}_n$, where $G_n := \bigcup_{k=1}^n I_k := \bigcup_{k=1}^n (a_k, b_k)$. Define the set sequence $\{F_m\}_{m\geq 1}$ as follows:

- $F_1 := I_1$,
- $F_k := \text{int}(I_k \setminus I_{k-1})$,

so F_k is a finite, disjoint union of open intervals. Define $G'_n := \bigcup_{k=1}^n F_k$ too, which is also a finite disjoint union of open intervals. We have that:

$$\begin{cases} \chi_G \geq 0 \text{ measurable,} \\ \chi_{G'_{n+1}} \geq \chi_{G'_n} \geq 0 \text{ measurables,} \\ \chi_G \overset{\forall x}{=} \lim_n \chi_{G_n} \overset{a.e.x}{=} \lim_n \chi_{G'_n} \; a.e.\, x, \end{cases}$$

so we can apply MCT↑ and MCT↓ (which can be found in Theorem A.5) to the positive and negative parts of $\chi_G \sin kt$, respectively. Thus, we get:

$$\int_{-\pi}^{\pi} \chi_G \sin \lambda t dt = \lim_n \int_{-\pi}^{\pi} \chi_{G'_n} \sin \lambda t dt, \quad \text{so} \quad \int_G \sin \lambda t dt = \lim_n \int_{-\pi}^{\pi} \chi_{G'_n} \sin \lambda t dt,$$

which, since $\chi_{G'_n}$ are step functions (because $G_{n'}$ are finite unions of intervals), satisfies the theorem by (ii). Consequently,

$$\lim_{\lambda\to\infty} \int_G \sin \lambda t dt = \lim_{\lambda\to\infty} \lim_n \int_{-\pi}^{\pi} \chi_{G'_n} \sin \lambda t dt = \lim_n \lim_{\lambda\to\infty} \int_{-\pi}^{\pi} \chi_{G'_n} \sin \lambda t dt = \lim_n 0 = 0.$$

Finally,

$$\int_G \sin \lambda t dt = \int_{G\setminus E} \sin \lambda t dt + \int_E \sin \lambda t dt \Rightarrow \int_E \sin \lambda t dt = \int_G \sin \lambda t dt - \int_{G\setminus E} \sin \lambda t dt,$$

so by the triangle inequality:

$$\left| \int_E \sin \lambda t dt \right| \leq \left| \int_G \sin \lambda t dt \right| - \left| \int_{G\setminus E} \sin \lambda t dt \right|.$$

The first term on the right is arbitrarily close to 0 when $\lambda \to \infty$. The second term is also arbitrarily close to 0 - regardless of the value of λ - choosing a suitable set G. That is:

$$\forall\, \varepsilon > 0 \; \exists\, 0 < M \in \mathbb{R} \text{ such that } \forall\, \lambda \geq M, \left| \int_E \sin \lambda t dt \right| < \epsilon,$$

which was to be proven.

(iv) $f = \sum_{k=1}^{N} \alpha_k \chi_{E_k}$ such that E_k is measurable for every $1 \leq k \leq N$ (simple function). The corresponding result is easily obtained from (iii) by linearity.

(v) $f \in \mathcal{L}$ (general case, integrable function). Now, thanks to Theorem A.1 we know that, given $\varepsilon > 0$, there exists a simple function s such that $\int_{-\pi}^{\pi} |f - s| < \varepsilon/2$, so:

$$\left|\int f(t)\sin\lambda t dt\right| = \left|\int [f(t) - s(t) + s(t)]\sin\lambda t dt\right| =$$
$$\left|\int s(t)\sin\lambda t dt + \int |f(t) - s(t)|\sin\lambda t dt\right| \leq$$
$$\left|\int s(t)\sin\lambda t dt\right| + \left|\int |f(t) - s(t)|\sin\lambda t dt\right|,$$

where the first term can be also bounded by $\varepsilon/2$ according to (iv). Therefore,

$$\lim_{\lambda\to\infty}\left|\int f(t)\sin\lambda t dt\right| = 0.$$

□

2.3 Localization principle

By definition, the Fourier coefficients (and hence the Fourier series) of a function depend on the entire set of values of the function. The localization principle claims that, even so, the value of the Fourier series at a given point only depends on the local values of the function near that point.

The main source for this section - and the remaining ones of this chapter - have been Duoandikoetxea's notes [6].

Theorem 2.6. *(Localization principle)*

(i) Let $f \in \mathcal{L}$ such that there exists $\delta > 0$ such that $f(x) = 0$ for all $x \in (x_0 - \delta, x_0 + \delta)$. Then, $Sf(x_0) = 0$.

(ii) Let $f, g \in \mathcal{L}$ such that there exists $\delta > 0$ such that $f(x) = g(x)$ for all $x \in (x_0 - \delta, x_0 + \delta)$. Then, either $Sf(x_0) = Sg(x_0)$ if either limit exists, or the limits do not exist.

Proof. The proof is purely based on the Riemmann-Lebesgue lemma, Lemma 2.3.

(i) Taking into account (2.3) and (2.4) in the first equality and the hypothesis in the second one, we have:

$$S_N f(x_0) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x_0 - t)\frac{\sin\left(N + \frac{1}{2}\right)t}{2\sin(t/2)}dt = \frac{1}{\pi}\int_{\delta\leq|t|\leq\pi}\frac{f(x_0 - t)}{2\sin(t/2)}\sin\left(N + \frac{1}{2}\right)tdt.$$

The function

$$g(t) := \begin{cases} \dfrac{f(x_0 - t)}{sin(t/2)}, & \text{if } \delta \leq |t| \leq \pi, \\ 0, & \text{if } |t| \leq \delta \end{cases}$$

is integrable. Indeed, it is measurable because

- Both f and sin are measurable and sin does not vanish in $\delta \leq |t| \leq \pi$.
- $\int |g| \leq \frac{1}{|\sin(\delta/2)|} \cdot \int |f| < \infty$ because f is integrable.

Therefore we can apply Lemma 2.3 to g, which taking $\lambda = N$ leads us to:

$$\lim_N \int_{-\pi}^{\pi} g(t) \sin\left(N + \frac{1}{2}\right) t dt = 0,$$

that is, $Sf(x_0) = 0$.

(ii) It is easy to see that $S_N f(x_0) - S_N g(x_0) = S_N(f-g)(x_0)$ (using the integral form 2.4 and linearity of the integral). Taking the limit, $Sf(x_0) - Sg(x_0) = S(f-g)(x_0)$ too. Since $f, g \in \mathcal{L}$, then $f - g$ is also integrable and, by hypothesis, this function satisfies the hypothesis of (i). Therefore, $S(f-g)(x_0) = 0$, and finally $Sf(x_0) = Sg(x_0)$. □

The main positive theorems on pointwise convergence of Fourier series are Dirichlet's and Dini's, which we will state and prove in the following sections.

2.4 Dirichlet's theorem

Theorem 2.7 (Dirichlet). *Let $f \in \mathcal{L}$ and $x_0 \in \mathbb{R}$ such that:*

- *Lateral limits $f(x_0^-), f(x_0^+)$ exist and are real.*
- *Lateral derivatives $f'(x_0^-), f'(x_0^+)$ exist and are real.*

Then,

$$Sf(x_0) = \frac{1}{2}[f(x_0^+) + f(x_0^-)].$$

Remark 2.8. If f is continuous and differentiable at x_0, then $Sf(x_0) = f(x_0)$.

Proof. From (2.5), $Sf(x_0) = \frac{1}{2}[f(x_0^+) + f(x_0^-)]$ is equivalent to

$$\lim_N \frac{1}{\pi} \int_0^{\pi} [f(x+t) + f(x-t)]\, D_N(t) dt = \frac{1}{2}[f(x_0^+) + f(x_0^-)].$$

Taking into account Dirichlet's kernel properties (Proposition 2.2), this is equivalent to

$$\lim_N \int_0^{\pi} [f(x_0+t) + f(x_0-t) - f(x_0^+) - f(x_0^-)] D_N(t) dt = 0.$$

Hence, it suffices to prove the two following equalities:

$$\lim_N \int_0^{\pi} [f(x_0 \pm t) - f(x_0^{\pm})] D_N(t) dt = 0,$$

which written into integral form become:

$$\lim_N \int_0^{\pi} \frac{f(x_0 \pm t) - f(x_0^{\pm})}{2\sin(t/2)} \sin\left(N + \frac{1}{2}\right) t dt = 0.$$

Taking into account the general version of Riemann-Lebesgue's lemma (Remark 2.4), it suffices to see that

$$g_{\pm}(t) := \frac{f(x_0 \pm t) - f(x_0^{\pm})}{2\sin(t/2)} \in \mathcal{L}(0, \pi].$$

First of all, observe that these functions are both measurable, since the numerator is measurable - because f is - and the denominator is measurable (continuous) and does not vanish on $(0, \pi]$. Now we have to see that $\int_0^{\pi} |g_{\pm}| < \infty$.

Since lateral limits $f(x_0^{\pm})$ exist, there exists $\delta > 0$ such that f is continuous on $[x_0 - \delta, x_0) \cup (x_0, x_0 + \delta]$, and therefore both functions $f(x_0 \pm t) - f(x_0^{\pm})$ are continuous on $(0, \delta]$. Functions $g_{\pm}(t)$ are consequently also continuous on $(0, \delta]$, since $\sin(t/2)$ is continuous and does not vanish on this interval. Once δ is defined, we separate the original integral into two integrals and we will later prove that both of them are finite.

$$\int_0^{\pi} |g_{\pm}(t)| dt = \int_0^{\delta} \left| \frac{f(x_0 \pm t) - f(x_0^{\pm})}{2\sin(t/2)} \right| dt + \int_{\delta}^{\pi} \left| \frac{f(x_0 \pm t) - f(x_0^{\pm})}{2\sin(t/2)} \right| dt =: I_1 + I_2.$$

- $I_1 < \infty$. It suffices to see that $g_{\pm}$ are bounded on $(0, \delta]$. In the first place, since $g_{\pm}$ are continuous on this interval, then they are continuous on $[a, \delta]$ for all $0 < a \leq \delta$, and therefore bounded on all these intervals. So it suffices to see that $\lim_{t \to 0^+} g_{\pm}(t) \in \mathbb{R}$, which is indeed true because

  $$\lim_{t\to 0} g_{\pm}(t) = \lim_{t \to 0} \frac{f(x_0 \pm t) - f(x_0^{\pm})}{2\sin(t/2)} = \lim_{t\to 0} \frac{t}{2\sin(t/2)} \cdot \frac{f(x_0 \pm t) - f(x_0^{\pm})}{t} = 1 \cdot f'_{\pm}(x_0)$$

  is real by hypothesis.

- $I_2 < \infty$. Indeed, using the triangle inequality and the fact that $\sin(t/2)$ is increasing between δ and π:

  $$\int_{\delta}^{\pi} \left| \frac{f(x_0 \pm t) - f(x_0^{\pm})}{2\sin(t/2)} \right| dt = \int_{\delta}^{\pi} \left| \frac{f(x_0 \pm t)}{2\sin(t/2)} \right| dt + \int_{\delta}^{\pi} \left| \frac{f(x_0^{\pm})}{2\sin(t/2)} \right| dt =$$

  $$\frac{1}{\sin(\delta/2)} \cdot \left[\int_{\delta}^{\pi} |f(x_0 \pm t)| dt + \int_{\delta}^{\pi} |f(x_0^{\pm})| \, dt \right],$$

  which is finite because $f \in \mathcal{L}$. □

2.5 Dini's criterion

Theorem 2.9 (Dini's criterion)**.** *Let $f \in \mathcal{L}$, $x_0 \in \mathbb{R}$ and $l \in \mathbb{R}$ such that the function*

$$\phi(t) := f(x_0 + t) + f(x_0 - t) - 2l \text{ satisfies } \frac{\phi(t)}{t} \in \mathcal{L}(0, \delta) \text{ for some } \delta > 0.$$

Then, $Sf(x_0) = l$.

Proof.

$$S_N f(x_0) - l = \frac{1}{\pi}\int_0^{\pi}[f(x_0+t)+f(x_0-t)]D_N(t)dt - l =$$

$$\frac{1}{\pi}\int_0^{\pi}[f(x_0+t)+f(x_0-t)-2l]D_N(t)dt = \frac{1}{\pi}\int_0^{\pi}\phi(t)D_N(t)dt =$$

$$\frac{1}{\pi}\int_0^{\pi}\frac{\phi(t)}{2\sin(t/2)}\cdot\sin\left(N+\frac{1}{2}\right)dt.$$

It suffices to see that $\frac{\phi(t)}{2\sin(t/2)} \in \mathcal{L}(0,\pi)$ to apply Lemma 2.3, which would lead us to $Sf(x_0) - l = 0$. In the first place, $\frac{\phi(t)}{2\sin(t/2)}$ is measurable in $(0,\pi)$ because it is the quotient of two measurable functions and the denominator does not vanish. Now, $\int_0^{\pi}\left|\frac{\phi(t)}{2\sin(t/2)}\right|dt < \infty$. To prove it, we separate $(0,\pi) = (0,\delta] \cup (\delta,\pi)$ for any $0 < \delta < \pi$ and see that the integral on both subintervals is finite:

- Since t increases faster than $2\sin(t/2)$ and $\lim_{t\to 0}\frac{t}{2\sin(t/2)} = 1$, then:

$$\int_0^{\delta}\left|\frac{\phi(t)}{2\sin(t/2)}\right|dt = \int_0^{\delta}\left|\frac{\phi(t)}{t}\cdot\frac{t}{2\sin(t/2)}\right|dt \leq \frac{\delta}{2\sin(\delta/2)}\cdot\int_0^{\delta}\left|\frac{\phi(t)}{t}\right|dt,$$

  which is finite by hypothesis.

- On the other hand, since $2\sin(t/2)$ is increasing in (δ,π), then

$$\int_{\delta}^{\pi}\left|\frac{\phi(t)}{2\sin(t/2)}\right|dt \leq \frac{1}{2\sin(\delta/2)}\cdot\int_{\delta}^{\pi}|\phi(t)|dt,$$

  which is finite because $f \in \mathcal{L}$. □

2.6 Uniform convergence

Recall the Weierstrass M-test, Theorem B.1. We can apply it to our problem in the following way. Our function series is the Fourier series of f, the function to analyze:

$$\sum_{n\geq 1} f_n(t) = \frac{a_0}{2} + \sum_{n\geq 1}(a_n\cos nt + b_n\sin nt),$$

and we can bound each term of the series like this:

$$|f_n| = |a_n\cos nt + b_k\sin nt| \leq |a_k| + |b_k| =: M_n,$$

so a sufficient condition for the absolute and uniform convergence of Sf is

$$\sum_{n=1}^{\infty}(|a_n| + |b_n|) < \infty,$$

which can be easily be shown to be equivalent to $\sum_{n=1}^{\infty} |c_n| < \infty$. Therefore we have proven the following simple criterion for uniform convergence.

Proposition 2.10. *Let $f \in \mathcal{L}$ and let a_n, b_n, c_n be its Fourier coefficients. If $\sum_{n=1}^{\infty} (|a_n| + |b_n|) < \infty$, or equivalently $\sum_{n=1}^{\infty} |c_n| < \infty$, then Sf absolutely and uniformly converges to f.*

Another important criterion is the following. Consider the sequence $\{S_N f(x)\}_N$, which is a sequence of continuous functions. Theorem B.2 tells us that, if Sf converges uniformly (to f), then f must be continuous. So we have got a simple negative result on uniform convergence of Fourier series:

Proposition 2.11. *If f is not continuous on $[-\pi, \pi]$, then Sf does not converge uniformly to f.*

Next we enunciate and prove the most basic theorem on uniform convergence of Fourier series. Let us first introduce some notation.

Definition 2.12. *Let f be a function.*

(i) f is said to be piecewise continuous on $[a, b]$ if:

- *f is continuous except, at the most, at a finite set of points, where it does not need to be defined.*
- *At discontinuity points, lateral limits of f do exist and are finite.*

(ii) f is piecewise smooth on $[a, b]$ if f' is piecewise continuous.

Theorem 2.13. *Let f be continuous on $[-\pi, \pi]$ and piecewise smooth on $(-\pi, \pi)$. Then, Sf converges uniformly on $[-\pi, \pi]$.*

Proof. Let us show that the hypothesis of the Weierstrass M-test hold. That is, we will prove that $\sum_k |c_k| < \infty$.

Recall that, from (1.2), the relation between the Fourier coefficients of f and f' is:

$$c_{\pm k}(f) = \pm i \frac{c_{\pm}(f')}{k}.$$

Denoting $c_{\pm k} = c_{\pm k}(f)$ for every $k \geq 1$ and $c'_{\pm k} = c_{\pm k}(f')$, this leads us to $|c_k| = \frac{1}{|k|}|c'_k|$ for every $|k| \geq 1$. So, since

$$0 \leq \left(k \cdot \| c_k| - \frac{1}{k} \right)^2 = k^2 |c_k|^2 + \frac{1}{k^2} - 2k|c_k|\frac{1}{k},$$

then

$$|c_k| \leq \frac{1}{2}\left(\frac{1}{k^2} + k^2 |c_k|^2 \right) = \frac{1}{2}\left(\frac{1}{k^2} + k^2 \frac{1}{k^2} |c'_k|^2 \right) = \frac{1}{2}\left(\frac{1}{k^2} + |c'_k|^2 \right),$$

and it suffices to see that

$$\sum_{|k|\geq 1} \left(\frac{1}{k^2} + |c_k'|^2 \right) < \infty.$$

It is well-known that $\sum_{k\geq 1} \frac{1}{k^2} < \infty$, so we only need to prove that $\sum_{|k|\geq 1} |c_k'|^2 < \infty$. Thus, we have been able to transform an inequality involving Fourier coefficients to an inequality involving the square of Fourier coefficients. The utility of this is that now we can use Bessel's inequality, Theorem 1.12. Since f' is, by hypothesis, piecewise continuous, $(f')^2$ also is, and therefore f' is square-integrable. This assures that Bessel's inequality holds:

$$\sum_k |c_k'|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(t)|^2 dt < \infty.$$

□

Chapter 3

Divergence of Fourier series: Negative results on pointwise convergence

Until now, we have exposed some criteria that set sufficient conditions for Fourier series to converge. How far can we weaken these conditions? Is it enough for a function to be continuous for its Fourier series to converge? Is it enough for it to be integrable? The answer to both questions is negative, as we shall prove in the following sections.

3.1 Divergence of Fourier series of continuous functions

Theorem 3.1. *There exists a function whose Fourier series diverges at a point.*

The first one to prove this fact was Paul du Bois-Reymond [2] in 1873. His counterexample is complex, and simpler constructions of such function were done by Schwarz, Féjer and Lebesgue.

We will give two different proofs of this theorem. The first one is based on the constructive proof of Lebesgue, whereas the other one is a proof of existence based on the uniform boundedness principle.

The main sources for this section have been Duoandikoetxea's notes [6] for the constructive proof and Duoandikoetxea's book [5] and Reed's, Simon's book [12] for the existence one.

Let us first see the constructive proof.

Proof. Consider the following sequences:

- $\{c_n\}_n$ such that $c_n \geq 0$ for every n and $c_n \to 0$,
- $\{\nu_n\}_n$ an increasing sequence of odd positive integers,
- $\{a_n\}_n$ such that $a_n := \nu_0 \cdots \nu_n$,

- $I_n := \left[\dfrac{2\pi}{a_n}, \dfrac{2\pi}{a_{n-1}}\right]$; $n \geq 1$, which is consistent because a_n is an increasing sequence.

Next, let us define the function

$$f(t) := \begin{cases} 0 & \text{if } t = 0, \\ c_n \sin\left(\dfrac{a_n|t|}{2}\right) \dfrac{\sin(|t|/2)}{|t|} & \text{if } |t| \in I_n, \end{cases}$$

Notice that:

- Its domain is $\left[-\dfrac{2\pi}{a_0}, \dfrac{2\pi}{a_0}\right]$.
- It is an even function because $f(t)$ depends only on $|t|$.
- It is well-defined (that is, it is not multivalued at the endpoints of the intervals I_n). Indeed, the value of f at all these points is 0:
  - $f\left(\pm\dfrac{2\pi}{a_n}\right) = c_n \sin\left(\pm\dfrac{a_n 2\pi}{2a_n}\right) \dfrac{sin(t/2)}{t} = 0$, since $\sin\left(\pm\dfrac{a_n 2\pi}{2a_n}\right) = \sin(\pm\pi) = 0$,
  - $f\left(\pm\dfrac{2\pi}{a_{n-1}}\right) = c_n \sin\left(\pm\dfrac{a_n 2\pi}{2a_{n-1}}\right) \dfrac{sin(t/2)}{t} = 0$, since $\sin\left(\pm\dfrac{a_n 2\pi}{2a_{n-1}}\right) = \sin(\pi\nu_n) = 0$ because ν_n is an integer.
- It is continuous on all its domain. Indeed, it is piecewise continuous on all the intervals I_n, it is continuous at the extremes of these intervals as we have just seen, and finally it is continuous at 0. In order to see this, we have to prove that

  $$0 = f(0) \overset{?}{=} \lim_{t\to 0} f(t) = \lim_{\substack{t\to 0 \\ n\to\infty}} c_n \sin\left(\frac{a_n t}{2}\right) \frac{\sin(t/2)}{t}.$$

  Given $t \in \mathbb{R}$, for every $n \in \mathbb{N}$:

  $$0 \leq \left| c_n \sin\left(\frac{a_n t}{2}\right) \frac{\sin(t/2)}{t} \right| = c_n \left| \sin\left(\frac{a_n t}{2}\right) \right| \left| \frac{\sin(t/2)}{t} \right| \leq c_n \left| \frac{\sin(t/2)}{t} \right| \xrightarrow{n} 0$$

  by definition of c_n. Therefore, taking the limit on n, for all $t \in \mathbb{R}$:

  $$0 \leq \lim_n \left| c_n \sin\left(\frac{a_n t}{2}\right) \frac{\sin(t/2)}{t} \right| \leq 0 \Rightarrow \lim_{\substack{t\to 0 \\ n\to\infty}} c_n \sin\left(\frac{a_n t}{2}\right) \frac{\sin(t/2)}{t} = 0,$$

  which was to be proven.

Now, let us extend the definition of f by $f(t) = 0$ for every $\dfrac{2\pi}{a_0} \leq |t| \leq \pi$, making f a continuous function on the whole interval $[-\pi, \pi]$. If $\dfrac{2\pi}{a_0} > \pi$, we don't extend f but restrict it to this interval. Let us now write Fourier sums in their integral form (2.5):

$$S_N f(x) = \frac{1}{\pi} \int_0^{\pi} [f(x+t) + f(x-t)] D_N(t) dt,$$

so using the parity of f:

$$\pi S_N f(0) = \int_0^\pi [f(t)+f(-t)] \frac{\sin\left(N+\frac{1}{2}\right)t}{2\sin(t/2)} dt = \int_0^\pi f(t) \frac{\sin\left(N+\frac{1}{2}\right)t}{2\sin(t/2)} dt =$$

$$\sum_{n\geq 1} \int_{I_n} f(t) \frac{\sin\left(N+\frac{1}{2}\right)t}{\sin(t/2)} dt =$$

$$\sum_{n\geq 1} \int_{I_n} c_n \sin\left(\frac{a_n t}{2}\right) \frac{\sin(t/2)}{t} \frac{\sin\left(N+\frac{1}{2}\right)t}{\sin(t/2)} dt =$$

$$\sum_{n\geq 1} c_n \int_{I_n} \sin\left(\frac{a_n t}{2}\right) \frac{\sin\left(N+\frac{1}{2}\right)t}{t} dt$$

Let us now consider a subsequence of the sequence of Fourier sums. Let $N_k := \dfrac{a_k - 1}{2}$ (which is an integer because a_k is a product of odd integers), then:

$$\pi S_{N_k} f(0) = \int_0^\pi [f(x+t)+f(x-t)] D_N(t) dt = \int_0^{\frac{2\pi}{a_k}} (\ldots) dt + \int_{\frac{2\pi}{a_k}}^\pi (\ldots) dt =$$

$$\int_0^{\frac{2\pi}{a_k}} (\ldots) dt + \sum_{j=1}^k \int_{I_j} (\ldots) dt = \int_0^{\frac{2\pi}{a_k}} f(t) \frac{\sin\left(\frac{a_k t}{2}\right)}{\sin(t/2)} dt + \sum_{j=1}^k c_j \int_{I_j} \frac{\sin\left(\frac{a_j t}{2}\right)\sin\left(\frac{a_k t}{2}\right)}{t} dt.$$

Now, we will bound each of the addends on the previous expression.

- Since $\left|\int g\right| \leq \int |g|$ and $\left|\dfrac{\sin at}{\sin t}\right| \leq \dfrac{\pi a}{2}$, then

$$\left| \int_0^{\frac{2\pi}{a_k}} f(t) \frac{\sin\left(\frac{a_k t}{2}\right)}{\sin(t/2)} dt \right| \leq \frac{\pi a_k}{2} \int_0^{\frac{2\pi}{a_k}} |f(t)| dt.$$

  Now, since $|f(t)|$ is continuous, it is Riemann integrable and thus there exists a primitive $F(t)$ of $|f(t)|$, which allows us to write:

$$\lim_k \frac{\pi a_k}{2} \int_0^{\frac{2\pi}{a_k}} |f(t)| dt = \lim_k \frac{\pi a_k}{2} [F(t)]_0^{\frac{2\pi}{a_k}} = \lim_k \left[F\left(\frac{2\pi}{a_k}\right) - F(0)\right] \frac{\pi a_k}{2},$$

  which, writing $h = \dfrac{2\pi}{a_k}$, equals

$$\lim_{h\to 0} \pi^2 \frac{F(h)-F(0)}{h} = \pi^2 F'(0),$$

that equals $\pi^2 f(0) = 0$ by the Fundamental Theorem of Calculus.

- If $j < k$:

$$\left| c_j \int_{I_j} \frac{\sin\left(\frac{a_j t}{2}\right)\sin\left(\frac{a_k t}{2}\right)}{t} dt \right| \leq c_j \int_{I_j} \frac{1}{t} dt = c_j [\log t]_{\frac{2\pi}{a_j}}^{\frac{2\pi}{a_{j-1}}} = c_j \log \nu_j.$$

- If $j = k$, let us integrate by parts:

$$c_k \int_{I_k} \frac{\sin^2\left(\frac{a_k t}{2}\right)}{t} dt = c_k \left\{ \left[\sin^2\left(\frac{a_k t}{2}\right) \log t \right]_{\frac{2\pi}{a_k}}^{\frac{2\pi}{a_{k-1}}} - \int_{I_k} (\log t) d\left[\sin^2\left(\frac{a_k t}{2}\right)\right] \right\} =$$

$$\left\{ \begin{array}{ccc} u = \sin^2\left(\frac{a_k t}{2}\right) & \Rightarrow & du = d\left[\sin^2\left(\frac{a_k t}{2}\right)\right] \\ dv = \frac{1}{t} dt & \Rightarrow & v = \log t \end{array} \right\} = -c_k \int_{I_k} (\log t) d\left[\sin^2\left(\frac{a_k t}{2}\right)\right] =$$

$$-c_k \int_{I_k} (\log t) d\left[\frac{1}{2}(1 - \cos a_k t)\right] = \frac{1}{2} c_k \int_{I_k} (\log t) d(\cos a_k t),$$

where the first term of the integral by parts vanishes because so does $\sin^2\left(\frac{a_k t}{2}\right)$ at $t = \frac{2\pi}{a_k}, \frac{2\pi}{a_{k-1}}$. Also, we have used the well-known trigonometric identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$. Now, let us integrate again by parts:

$$\frac{1}{2} c_k \int_{I_k} (\log t) d(\cos a_k t) = \left\{ \begin{array}{ccc} u = \log t & \Rightarrow & du = \frac{1}{t} dt \\ dv = d(\cos akt) & \Rightarrow & v = \cos a_k t \end{array} \right\} =$$

$$\frac{1}{2} c_k \left\{ [\log t \cos a_k t]_{\frac{2\pi}{a_k}}^{\frac{2\pi}{a_{k-1}}} - \int_{I_k} \frac{\cos a_k t}{t} dt \right\} = \frac{1}{2} c_k \log \nu_k - \frac{1}{2} c_k \int_{I_k} \frac{\cos(a_k t)}{t} dt,$$

where the last integral is bounded in k, which can be easily seen integrating by parts again.

Once all the addends on the expression of $S_{N_k} f$ are bounded in this way, and using the fact that $\left|\sum_i a_i\right| \geq |a_k| - \sum_{i \neq k} |a_i|$, we can finally write the following bound for the subsequence of the Fourier sums:

$$|\pi S_{N_k} f(0)| \geq \frac{1}{2} c_k \log \nu_k - \left(\sum_{j=1}^{k-1} c_j \log \nu_j + r_k \right),$$

where r_k is a bounded sequence. Now, choosing suitable sequences c_k and ν_k, one can make the sequence on the right tend to $+\infty$. For example, if $c_k = 4^{-k}$ and $\nu_k = 3^{16^k}$

(which satisfy the properties at the beginning of this proof) one gets, after some simple calculation, that

$$|\pi S_{N_k} f(0)| \geq \log(3)\, \frac{1}{6}\, 4^k - r'_k$$

(with r'_k a bounded sequence), which tends to $+\infty$ with k. □

Before giving the existence proof of this theorem, we have to prove the following lemmas.

Lemma 3.2. *There exists a bounded sequence a_n such that $\sum_{j=1}^{n} \frac{1}{j} = \log n + a_n$.*

Proof. Consider the function $\frac{1}{x}\cdot$ We have the following order relation among its integral and its lower and upper sums:

$$\sum_{j=1}^{n} \frac{1}{j+1} \leq \int_1^n \frac{1}{x} dx \leq \sum_{j=1}^{n} \frac{1}{j}\,,$$

so

$$0 \leq \sum_{j=1}^{n} \frac{1}{j} - \log n \leq \sum_{j=1}^{n} \frac{1}{j} - \sum_{j=1}^{n} \frac{1}{j+1} = \sum_{j=1}^{n} \left(\frac{1}{j} - \frac{1}{j+1} \right) = \sum_{j=1}^{n} \frac{1}{j^2+j}\,,$$

and taking the limit superior:

$$0 \leq \overline{\lim_n} \left(\sum_{j=1}^{n} \frac{1}{j} - \log n \right) \leq \lim_n \left(\sum_{j=1}^{n} \frac{1}{j^2+j} \right) = \sum_{j=1}^{\infty} \frac{1}{j^2+j} < \sum_{j=1}^{n} \frac{1}{j^2}\,,$$

which we know to be finite. Consequently,

$$0 \leq \overline{\lim_n} \left(\sum_{j=1}^{n} \frac{1}{j} - \log n \right) < \infty,$$

which means that the sequence $\sum_{j=1}^{n} \frac{1}{j} - \log n$ is bounded, which was to be proven. □

Lemma 3.3. *Lebesgue's numbers $L_N := \frac{1}{\pi} \int_{-\pi}^{\pi} |D_N(t)| dt$ satisfy $L_N = \frac{4}{\pi^2} \log N + R_N$, where R_N is a bounded sequence.*

Proof.

$$L_N = \frac{1}{\pi}\int_{-\pi}^{\pi} |D_N(t)|dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \left| \frac{\sin\left(N+\frac{1}{2}\right)t}{2\sin(t/2)} \right| dt = \frac{2}{\pi}\int_0^{\pi} \frac{\left|\sin\left(N+\frac{1}{2}\right)t\right|}{2\sin(t/2)} dt =$$

$$\frac{2}{\pi}\int_0^{\pi} \left|\sin\left(N+\frac{1}{2}\right)t\right| \left(\frac{1}{2\sin(t/2)} + \frac{1}{t} - \frac{1}{t}\right) dt =$$

$$\frac{2}{\pi}\int_0^{\pi} \frac{\left|\sin\left(N+\frac{1}{2}\right)t\right|}{t} dt + \frac{1}{\pi}\int_0^{\pi} \left|\sin\left(N+\frac{1}{2}\right)t\right| \left[\frac{1}{\sin(t/2)} - \frac{1}{t/2}\right] dt =: I_1 + I_2.$$

I_2 is bounded with respect to N because the integrand is. Indeed:

- $0 \leq \left|\sin\left(N+\frac{1}{2}\right)t\right| \leq 1\,\forall N.$
- Given an arbitrary $0 < \delta < \pi$, the function $\frac{1}{\sin(t/2)} - \frac{1}{t/2}$ is bounded in $[\delta, \pi]$ because it is continuous on this interval. Since δ is arbitrary, in order to see that the function is bounded on $(0, \pi)$ we only need to see that $\lim_{t\to 0}\left(\frac{1}{\sin(t/2)} - \frac{1}{t/2}\right) < \infty$, which is true becase this limit equals 0.

Let us now compute I_1 making a change of variables.

$$\int_0^{\pi} \frac{\left|\sin\left(N+\frac{1}{2}\right)t\right|}{t} dt = \left\{ s := \left(N+\frac{1}{2}\right) t dt \right\} = \int_0^{\left(N+\frac{1}{2}\right)\pi} \frac{|\sin s|}{s} ds =$$
$$\sum_{j=0}^{N-1} \int_{j\pi}^{(j+1)\pi} \frac{|\sin s|}{s} ds + \int_{N\pi}^{\left(N+\frac{1}{2}\right)\pi} \frac{|\sin s|}{s} ds =: I_3 + I_4,$$

and let us bound each one of the resulting integrals:

$$0 \leq I_4 = \int_{N\pi}^{\left(N+\frac{1}{2}\right)\pi} \frac{|\sin s|}{s} ds \leq \int_{N\pi}^{\left(N+\frac{1}{2}\right)\pi} \frac{1}{s} ds = \left[-\frac{1}{s^2}\right]_{N\pi}^{\left(N+\frac{1}{2}\right)\pi} =$$
$$\left(\frac{1}{N\pi}\right)^2 - \left[\frac{1}{\left(N+\frac{1}{2}\right)\pi}\right]^2 =: r_N.$$

Notice that $0 \leq I_4 \leq r_N \leq 1$ and $r_N \to 0$, so $I_4 \xrightarrow{N} 0$. On the other hand,

$$I_3 = \sum_{j=0}^{N-1} \int_{j\pi}^{(j+1)\pi} \frac{|\sin s|}{s} ds = \sum_{j=0}^{N-1} \int_0^{\pi} \frac{|\sin s|}{s + j\pi} ds = \int_0^{\pi} |\sin s| \left(\sum_{j=0}^{N-1} \frac{1}{s+j\pi}\right) ds.$$

Observe that we can superiorly and inferiorly bound the second factor on the integrand as follows:

$$\sum_{j=0}^{N-1} \frac{1}{(j+1)\pi} \leq \sum_{j=0}^{N-1} \frac{1}{s+j\pi} \leq \frac{1}{s} + \sum_{j=1}^{N-1} \frac{1}{j\pi},$$

so there exists a sequence r'_N such that:

- $I_3 = \int_0^{\pi} |\sin s| \left(\sum_{j=0}^{N-1} \frac{1}{(j+1)\pi} \right) ds = \int_0^{\pi} |\sin s| \left(\sum_{j=0}^{N-1} \frac{1}{(j+1)\pi} \right) ds + r'_N$,
- $0 \leq r'_N \leq \int_0^{\pi} |\sin s| \left(\frac{1}{s} - \frac{1}{N\pi} \right) ds = \int_0^{\pi} \frac{|\sin s|}{s} ds - \frac{1}{N\pi} \int_0^{\pi} |\sin s| ds$, which is bounded because:
  - The first term is bounded because the integrand is continuous and satisfies $\lim_{s \to 0} \frac{|\sin s|}{s} = 1$, which is finite.
  - The second term is bound because it is the integral of a finite function over a finite interval.

Therefore,

$$I_3 = \sum_{j=0}^{N-1} \frac{1}{(j+1)\pi} \int_0^{\pi} |\sin s| ds = \frac{2}{\pi} \sum_{j=1}^{N} \frac{1}{j} + r'_N = \frac{2}{\pi} \log N + r'_N,$$

since, according to Lemma 3.2,

$$\sum_{j=1}^{N} \frac{1}{j} = \log N + r''_N$$

with r''_N bounded. Above, we have included r''_N in r'_N, so r'_N remains a bounded sequence. Finally, so:

$$L_N = I_1 + I_2 = I_2 + \frac{2}{\pi} I_3 + \frac{2}{\pi} I_4,$$

and let $R'_N := I_2 + \frac{2}{\pi} I_4$, which is bounded as we have proven. Therefore:

$$L_N = \frac{2}{\pi} I_3 + R'_N = \frac{2}{\pi} \frac{2}{\pi} \log N + \frac{2}{\pi} r'_N + R'_N,$$

so denoting $R_N := \frac{2}{\pi} r'_N + R'_N$, which is bounded, we have

$$L_N = \frac{4}{\pi^2} \log N + R_N,$$

with R_N bounded, which was to be proven. □

Now we can see the second proof of the theorem.

Proof. Let us consider the Banach space $(\mathcal{C}^0[-\pi, \pi], || \cdot ||_\infty)$, the normed space $(\mathbb{R}, |\cdot|)$ and the family of lineal operators $\mathcal{F} = \{T_N\}_N$ defined by

$$\begin{aligned} T_N : \mathcal{C}^0([-\pi, \pi]) &\longrightarrow \mathbb{R} \\ f &\longmapsto S_N f(0) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) D_N(t) dt. \end{aligned}$$

Note that given $N \in \mathbb{N}$, the function D_N has a finite number of zeros, and consequently the function $\text{sgn}(D_N)$ has a finite number of discontinuities. Given $\varepsilon > 0$, this fact allows us to modify $\text{sgn}(D_N)$ on a small neighbourhood of each of the discontinuities in a way that the resulting function f is continuous and satisfies:

- $||f||_\infty = \max\limits_{t \in [-\pi,\pi]} \{\text{sgn}(D_N(t))\} = 1.$
- Modifying $\text{sgn}(D_N)$ conveniently, we can make f to have the same sign as D_N everywhere, so $f \cdot D_N$ is everywhere non negative and:

$$|T_N f| = \left| \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) D_N(t) dt \right| = \frac{1}{\pi} \int_{-\pi}^{\pi} |f(t)||D_N(t)| dt \geq L_N - \varepsilon,$$

  by definition of Lebesgue's numbers.

Since ε was arbitrary, this tells us that $||T_N|| \geq L_N$, which using Lemma 3.3, leads us to $||T_N|| \xrightarrow{N} \infty$. Next we apply the Uniform Boundedness Principle, Theorem A.14. We can do this because all the hypothesis hold:

- $(\mathcal{C}^0([-\pi,\pi]), || \cdot ||_\infty)$ is a Banach space and $(\mathbb{R}, |\cdot|)$ is a normed space.
- Each of the lineal operators $T_N \in \mathcal{F}$ is bounded, because by definition, $|T_N f| \leq L_N ||f||_\infty$.

The contrapositive of Theorem A.14 tells us then that there exists $f \in \mathcal{C}^0[-\pi,\pi]$ such that $|T_N f| = \infty$, that is, $|S_N f(0)| = \infty$, which was to be proven. □

3.2 Almost everywhere divergence of Fourier series of integrable functions

The following theorem is due to Kolmogorov, who published it in 1923. The principal reference for this section has been Ul'yanov's paper [13].

Theorem 3.4. *There exists a function $f \in \mathcal{L}(-\pi,\pi)$ whose Fourier series diverges a.e.*

In this case we give the original proof, which is based on the next lemma. It refers to functions of bounded variation, which are defined in Definition B.4.

Lemma 3.5. *There exists a sequence $\{\varphi_n\}_n \geq 1 \subset \mathcal{L}(-\pi,\pi)$ such that:*

(i) $\varphi_n(t) \geq 0 \, \forall \, t \in [-\pi,\pi]$ *and* $\int_{-\pi}^{\pi} \varphi_n(t) dt = 2 \, \forall n \geq 1.$

(ii) φ_n *is a function of bounded variation* $\forall \, n$*.*

(iii) For every function φ_n *there exists a number* $M_n \in \mathbb{R}$*, a set* $E_n \subset [-\pi,\pi]$ *and an integer* $q_n \in \mathbb{N}$ *such that:*

(a) $M_n \xrightarrow{n} \infty$,

(b) $|E_n| \xrightarrow{n} 2\pi$,

(c) $\forall\, t \in E_n\, \exists p_n = p_n(t)$ *such that* $p_n(t) \leq q_n$ *and* $|S_{p_n(t)}\varphi_n(t)| > M_n \,\forall\, n \geq 1.$

What this lemma states is basically that there exist functions of bounded variation whose $\mathcal{L}^1$-norm is 2 and for which, though, their Fourier sums are arbitrarily large at all the points of a set whose measure is arbitrarily close to 2π.

We first prove the theorem from the lemma and then proceed to prove the lemma.

Proof. (Theorem) Let us inductively define an increasing sequence of natural numbers $\{n_k\}_k$ such that:

(A) $\dfrac{1}{\sqrt{M_{n_k}}} < \dfrac{1}{2^k}\,\forall k \geq 1,$

(B) If $B_n := \sup\limits_{\substack{N\geq 0\\ x\in[0,2\pi]}} |S_N\varphi_n(x)|$, then $\sum\limits_{i=1}^{k-1} B_{n_i} < \dfrac{1}{2}\sqrt{M_{n_k}}\,\forall k \geq 1,$

(C) $2q_{n_i} + 1 \leq \dfrac{\sqrt{M_{n_k}}}{2^k}\,\forall i < k.$

Lemma B.5 assures that, since φ_n are of bounded variation, and $\varphi_n \geq 0$ and $\int_{-\pi}^{\pi} \varphi = 2 \Rightarrow \sup\limits_{t\in[0,2\pi]} |\varphi_n(t)| < \infty$, then the numbers B_n are finite.

Why can we find a sequence that satisfies $(A), (B), (C)$?

(A) Because $M_n \to \infty$,

(B) Because $M_n \to \infty$, n_i are defined for every $i < k$ by induction and for Lemma B.5.

(C) Because $M_n \to \infty$ and n_i are defined for every $i < k$.

Consider the function series $\sum\limits_{i=1}^{\infty} \dfrac{1}{\sqrt{M_{n_i}}}\varphi_{n_i}(t)$ *a.e. t*. From Lemma 3.5.(i) and from (A), we have that

$$\sum_{i=1}^{\infty} \frac{1}{\sqrt{M_{n_i}}} \int_{-\pi}^{\pi} \varphi_{n_i}(t)dt = \sum_{i=1}^{\infty} \frac{2}{\sqrt{M_{n_i}}} < 2 \cdot \sum_{i=1}^{\infty} \frac{1}{2^k} = 2.$$

So, from Beppi-Levo's lemma (Lemma A.6), we have that:

$$\left.\begin{array}{l} 0 \leq \dfrac{1}{\sqrt{M_{n_i}}}\varphi_{n_i} \in \mathcal{L}(-\pi,\pi)\,\forall i \\ \displaystyle\sum_{i=1}^{\infty} \int_{-\pi}^{\pi} \frac{1}{\sqrt{M_{n_i}}}\varphi_{n_i}(t)dt = 2 < \infty \end{array}\right\} \Rightarrow \begin{array}{l} \text{there exists } \Phi(t) \in \mathcal{L}(-\pi,\pi) \\ \text{such that } \displaystyle\sum_{i=1}^{\infty} \frac{1}{\sqrt{M_{n_i}}}\varphi_{n_i}(t) = \Phi(t) \text{ a.e. } t, \end{array}$$

so we can consistently define $\Phi(t) := \sum\limits_{i=1}^{\infty} \dfrac{1}{\sqrt{M_{n_i}}}\varphi_{n_i}(t)$ *a.e. t*, which will be the function whose Fourier series we will prove diverges *a.e.*

Now,

$$c_k(\Phi) = \frac{1}{\pi}\int_{-\pi}^{\pi}\Phi(t)e^{-ikt}dt = \frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{i=1}^{\infty}\frac{1}{\sqrt{M_{n_i}}}\varphi_{n_i}(t)e^{-ikt}dt \overset{MCT}{=}$$
$$\sum_{i=1}^{\infty}\frac{1}{\sqrt{M_{n_i}}}\frac{1}{\pi}\int_{-\pi}^{\pi}\varphi_{n_i}(t)e^{-ikt}dt = \sum_{i=1}^{\infty}\frac{1}{\sqrt{M_{n_i}}}c_k(\varphi_{n_i}),$$

so

$$S_N\Phi(t) = \sum_{i=1}\frac{1}{\sqrt{M_{n_i}}}S_N\varphi_{n_i}(t).$$

Let $k \in \mathbb{N}$ and $t \in E_{n_k}$ and let us consider the Fourier sums corresponding to $N = p_{n_k}(t)$. Then, since $\left|\sum_i a_i\right| \geq |a_k| - \sum_{i\neq k}|a_i|$:

$$\left|S_{p_{n_k}}\Phi(t)\right| = \left|\sum_i \frac{1}{\sqrt{M_{n_i}}}S_{p_{n_k}}\varphi_{n_i}(t)\right| \geq \frac{1}{\sqrt{M_{n_k}}}\left|S_{p_{n_k}}\varphi_{n_k}(t)\right| - \sum_{i\neq k}\frac{1}{\sqrt{M_{n_i}}}\left|S_{p_{n_i}}\varphi_{n_i}(t)\right|,$$

and let us now bound the resulting addends. We separate the series on the right into a finite sum and another series and we bound them.

$$I_1 := \sum_{i=1}^{k-1}\frac{1}{\sqrt{M_{n_i}}}|S_{p_{n_k}}\varphi_{n_i}(t)| \leq \sum_{i=1}^{k-1}\frac{1}{\sqrt{M_{n_i}}}\sup_{\substack{1\leq p<\infty \\ t\in[-\pi,\pi]}}|S_p\varphi_{n_i}(t)| \overset{(B)}{\leq}$$
$$\sum_{i=1}^{k-1}\frac{B_{n_i}}{\sqrt{M_{n_i}}} \overset{(A)}{<} \sum_{i=1}^{k-1}B_{n_i} \overset{(B)}{<} \frac{1}{2}\sqrt{M_{n_k}}.$$

For the following bound, we use the next general bound of Fourier sums:

$$S_Nf(x) = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x-t)D_N(t)dt \Rightarrow |S_Nf(x)| = \left|\frac{1}{\pi}\int_{-\pi}^{\pi}f(x-t)D_N(t)dt\right| \leq$$
$$\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x-t)||D_N(t)|dt \leq \frac{N+\frac{1}{2}}{\pi}\int_{-\pi}^{\pi}|f(x-t)|dt = \frac{2N+1}{2\pi}\int_{-\pi}^{\pi}|f(t)|dt,$$

where we used the fact that $|D_N(t)| = \left|\frac{1}{2}+\sum_{k=1}^{N}\cos kt\right| \leq \frac{1}{2}+N$, and the periodicity of f. Therefore:

$$I_2 := \sum_{i>k}^{\infty}\frac{1}{\sqrt{M_{n_i}}}|S_{p_{n_k}}\varphi_{n_i}(t)| \leq \sum_{i>k}^{\infty}\frac{1}{\sqrt{M_{n_i}}}\left[\frac{1}{2}+p_{n_k}\right]\frac{1}{\pi}\int_{-\pi}^{\pi}\varphi_{n_i}(y)dy \overset{(iii).(c)}{\leq}$$
$$\sum_{i>k}\frac{1}{\sqrt{M_{n_i}}}\left(\frac{1}{2}+q_{n_k}\right)\frac{1}{\pi}\int_{-\pi}^{\pi}\varphi_{n_i}(y)dy = \sum_{i>k}\frac{1}{\sqrt{M_{n_i}}}\frac{2q_{n_k}+1}{2}\frac{1}{\pi}\overbrace{\int_{-\pi}^{\pi}\varphi_{n_i}(y)dy}^{2} =$$
$$\frac{1}{\pi}\sum_{i>k}\frac{2q_{n_k}+1}{\sqrt{M_{n_i}}} \overset{(C)}{\leq} \frac{1}{\pi}\sum_{i>k}\frac{1}{2^i} < \sum_{i>k}\frac{1}{2^i} = \frac{1}{2^k},$$

and

$$\frac{1}{\sqrt{M_{n_k}}}\left|S_{p_{n_k}}\varphi_{n_k}(t)\right| \overset{(iii).(c)}{>} \sqrt{M_{n_k}}.$$

Thus,

$$\left|S_{p_{n_k}}\Phi(t)\right| \geq \frac{1}{\sqrt{M_{n_k}}}\left|S_{p_{n_k}}\varphi_{n_k}(t)\right| - I_1 - I_2 > \sqrt{M_{n_k}} - \frac{1}{2}\sqrt{M_{n_k}} - \frac{1}{2^k} =$$
$$\frac{1}{2}\sqrt{M_{n_k}} - \frac{1}{2^k} \geq \frac{1}{2}\sqrt{M_{n_k}} - \frac{1}{2},$$

which holds for all $t \in E_{n_k}$.

Now let $E := \limsup_k E_{n_k}$, which can also be expressed as $E = \bigcap_{j=1}^{\infty}\bigcup_{k=j}^{\infty} E_{n_k}$, so:

$$2\pi \geq \left|\bigcup_{k=j}^{\infty} E_{n_k}\right| \geq |E_{n_l}|\,\forall\, l \geq j,$$

and taking the limit on l:

$$2\pi \geq \left|\bigcup_{k=j}^{\infty} E_{n_k}\right| \geq \lim_l |E_{n_l}| \overset{(iii).(b)}{=} 2\pi \Rightarrow \left|\bigcup_{k=j}^{\infty} E_{n_k}\right| = 2\pi\,\forall\, k.$$

Then,

$$|E| = \left|\bigcap_{j=1}^{\infty}\bigcup_{k=j}^{\infty} E_{n_k}\right| = 2\pi$$

because it is a numerable intersection of sets with measure 2π, all of them contained in $[-\pi, \pi]$.

Finally, we have that for all $t \in E$ (that is, *a.e.* $t \in [-\pi, \pi]$), since $t \in \bigcap_{j=1}^{\infty}\bigcup_{k=j}^{\infty} E_{n_k}$ then $t \in E_{n_k}$ for an infinite set of different k. Therefore, since

$$\left|S_{p_{n_k}}\Phi(t)\right| \geq \frac{1}{2}\sqrt{M_{n_k}} - \frac{1}{2^k} \geq \frac{1}{2}\sqrt{M_{n_k}} - \frac{1}{2}\,\forall\, t \in E_{n_k},$$

(which holds for every k) and $M_k \to \infty$, then $\limsup_N |S_N\Phi(t)| = \infty$. □

Proof. (Lemma) Let us define the following items:

- $n, n_0 \in \mathbb{N}$ such that $n_0 \leq n$, with n_0 satisfying a certain condition that we will later impose. And, for each n:
- An increasing sequence of odd numbers $\{\lambda_k(n)\}_{1\leq k\leq n}$ such that $\lambda_1 = 1$. We will later impose certain conditions on these sequences.
- A sequence of natural numbers $\{m_k(n)\}_{1\leq k\leq n}$ such that

$$\begin{cases} m_1 := n, \\ 2m_k + 1 = \lambda_k \cdot (2n+1)\,\forall\, 1 \leq k \leq n. \end{cases}$$

- A sequence of real numbers $\{A_k(n)\}_{1\leq k\leq n}$ with $A_k = \dfrac{4\pi}{2n+1}\cdot k$ for every $1 \leq k \leq n$.
- A sequence of intervals $\{\Delta_k(n)\}_{1\leq k\leq n}$ as:

$$\Delta_k := \left[A_k - \frac{1}{m_k^2}, A_k + \frac{1}{m_k^2}\right] \ \forall\, 1 \leq k \leq n.$$

These intervals satisfy:

  - $\{\Delta_k\}$ are all mutually disjoint. Indeed, $\{A_k\}$ is a strictly growing sequence, so it suffices to see that

$$d\left(A_k, A_k \pm \frac{1}{m_k^2}\right) < \frac{1}{2}d(A_k, A_{k\pm 1}),$$

    because in that case the interval Δ_k will be contained in the ball centred at A_k with radius $\frac{1}{2}d(A_k, A_{k\pm1})$, and these balls are all mutually disjoint. Let us see the inequality above holds. It is equivalent to the inequality $\dfrac{1}{m_k^2} < \dfrac{2\pi}{2n+1}$, which holds because:

$$m_k = \frac{\lambda_k(2n+1)-1}{2} \Rightarrow m_k^2 = \frac{[\lambda_k(2n+1)-1]^2}{4} > \frac{[\lambda_k(2n+1)-1]^2}{2\pi} > \frac{2n+1}{2\pi},$$

    where in the last inequality we use that
    * $[\lambda_k(2n+1)-1]^2 \geq (2n)^2$ because $\lambda_k \geq 1$, and
    * $(2n)^2 > 2n+1 \,\forall\, n \geq 1$.
  - $\Delta_k \subset [0, 2\pi]\ \forall k$. Indeed:
    * $\Delta_k \subset [0, +\infty)$ because

$$\frac{1}{m_1^2} < \frac{2\pi}{2n+1} \Rightarrow A_1 - \frac{1}{m_1^2} > \frac{4\pi}{2n+1} - \frac{2\pi}{2n+1} = \frac{2\pi}{2n+1} > 0,$$

      which tells us that Δ_1 (and therefore Δ_k for every $k \geq 1$) lies at the right of 0.
    * $\Delta_k \subset (-\infty, 2\pi]$ because

$$A_n + \frac{1}{m_n^2} < \frac{4\pi}{2n+1}n + \frac{2\pi}{2n+1} = \frac{2\pi}{2n+1}(2n+1) = 2\pi,$$

      which tells us that Δ_n (and therefore Δ_k for every $k \leq n$) lies at the left of 2π.
- A function $\varphi_n : \mathbb{R} \to \mathbb{R}$ such that, on $[0, 2\pi]$:

$$\varphi_n(t) := \begin{cases} \dfrac{m_k^2}{n} & \text{if } t \in \Delta_k,\ 1 \leq k \leq n, \\ 0 & \text{if } t \in [0, 2\pi] \setminus \bigcup_{k=1}^n \Delta_k, \end{cases}$$

and, on $t \in \mathbb{R} \setminus [0, 2\pi]$, we consider the periodic extension of it. Also, we define, for every $1 \leq k \leq n$, the steps $\varphi_n^{(k)} : \mathbb{R} \to \mathbb{R}$ such that, on $[0, 2\pi]$:

$$\varphi_n^{(k)}(t) := \begin{cases} \varphi_n(t) & \text{if } t \in \Delta_k \\ 0 & \text{if } t \in [0, 2\pi] \setminus \Delta_k, \end{cases}$$

and, on $t \in \mathbb{R} \setminus [0, 2\pi]$, we consider the periodic extension of it, so $\varphi_n(t) = \sum_{k=1}^{n} \varphi_n^{(k)}(t)$.

These functions will be the ones satisfying lemma's statements, as we shall prove next.

On the first place, (i) holds because:

- $\varphi_n(t) \geq 0 \, \forall \, t \in [0, 2\pi]$,
- φ_n is 2π-periodic and integrable, and
- $\int_0^{2\pi} \varphi_n(t) dt = \sum_{k=1}^{n} \frac{m_k^2}{n} \frac{2}{m_k^2} = 2 \cdot \sum_{k=1} \frac{1}{n} = 2 \cdot n \cdot \frac{1}{n} = 2.$

Furthermore, φ_n is obviously of bounded variation on $[0, 2\pi]$ because it is a step function. This implies (ii).

Now, let us define, for each n, the sequence of intervals $\{\sigma_k\}_{2 \leq k \leq n}$ as

$$\sigma_k := \left[A_{k-1} + \frac{2}{n^2}, A_k - \frac{2}{n^2} \right].$$

Notice that this definition is consistent because, for every $2 \leq k \leq n$, $A_{k-1} + \frac{2}{n^2} < A_k - \frac{2}{n^2}$. Indeed:

$$\begin{aligned} & A_{k-1} + \frac{2}{n^2} = \frac{4\pi}{2n+1}(k-1) + \frac{2}{n^2} < \frac{4\pi}{2n+1}k - \frac{2}{n^2} = A_k - \frac{2}{n^2} \Leftrightarrow \\ & \frac{4}{n^2} < \frac{4\pi}{2n+1} \Leftrightarrow \frac{2n+1}{\pi} < n^2, \end{aligned}$$

which holds for every $n \geq 1$ because $2n + 1 \leq 3n^2 < \pi n^2$ for such n.

Let us show that these intervals are interspersed among the intervals Δ_k, and therefore σ_k are mutually disjoint. Specifically, σ_k is strictly between Δ_{k-1} and Δ_k, that is, we have the following order relations:

$$A_{k-1} + \frac{1}{m_{k-1}^2} < A_{k-1} + \frac{2}{n^2} < A_k - \frac{2}{n^2} < A_k - \frac{1}{m_k^2}.$$

The second inequality has just been seen, so we only need to prove the first and the last one:

$$\begin{aligned} & \frac{1}{m_{k-1}^2} < \frac{2}{n^2} \;\; \forall 2 \leq k \leq n \Leftrightarrow \frac{1}{m_k^2} < \frac{2}{n^2} \;\; \forall 1 \leq k \leq n-1 \Leftrightarrow \\ & \frac{n^2}{2} < m_k^2 = \frac{[\lambda_k(2n+1) - 1]^2}{2\pi} \Leftrightarrow n^2 < [\lambda_k(2n+1) - 1]^2, \end{aligned}$$

which is true because $\lambda_k \geq 1 \, \forall k \Rightarrow \lambda_k(2n+1) - 1 > n \, \forall \, n \geq 1$. This proves the first inequality, but also the last one because it holds for every k.

Each one of these intervals almost completely fill up the interval $[A_{k-1}, A_k]$ as n grows. That is,

$$|\sigma_k| = |[A_{k-1}, A_k]| - \frac{4}{n^2} = \frac{4\pi}{2n+1} - \frac{4}{n^2} \overset{n}{\sim} \frac{4\pi}{2n+1} = |[A_{k-1}, A_k]|,$$

so their union almost completely fills up the main interval $[0, 2\pi]$, because they are all contained in $[0, 2\pi]$ and:

$$\left|\bigcup_{k=2}^{n} \sigma_k\right| = (n-1)\left(\frac{4\pi}{2n+1} - \frac{4}{n^2}\right) = 4\pi\frac{n-1}{2n+1} - 4\frac{n-1}{n^2} \xrightarrow{n} 2\pi.$$

Let us now impose certain conditions on the sequence $\{m_k\}_k$. Specifically, let us inductively define m_k as follows. Assume we have already defined $\lambda_1, \ldots, \lambda_{k-1}$ (and therefore $m_1, \ldots, m_{k-1}$), and consider the Fourier sums $S_{m_k}\varphi_n^{(i)}(x)$ with $x \in \sigma_k$ and $1 \leq i \leq k-1$:

$$S_{m_k}\varphi_n^{(i)}(x) = \frac{1}{\pi}\int_0^{2\pi} \varphi_n^{(i)}(t) D_{m_k}(t-x)dt = \frac{1}{\pi}\int_{\Delta_i} \frac{m_i^2}{n} \frac{\sin\left[\left(m_k + \frac{1}{2}\right)(t-x)\right]}{2\sin[(t-x)/2]}dt$$
$$\frac{m_i^2}{2\pi n}\int_{\Delta_i} \frac{1}{\sin[(t-x)/2]} \sin\left[\left(m_k + \frac{1}{2}\right)(t-x)\right] dt.$$

Notice that, for any $x \in \sigma_k$ and $t \in \Delta_i$, t and x belong to two disjoint closed intervals. Hence, there exists $\delta > 0$ such that $|t - x| \geq \delta$, and consequently

$$\frac{1}{\sin[(t-x)/2]}$$

is a continuous function of t on the closed interval Δ_i. Therefore, such function is integrable and we can apply the Riemann-Lebesgue lemma (Lemma 2.4) to the integral, leading us to:

$$\lim_{|m_k|\to\infty} \int_{\Delta_i} \frac{1}{\sin[(t-x)/2]} \sin\left[\left(m_k + \frac{1}{2}\right)(t-x)\right] dt = 0,$$

so choosing m_k large enough:

$$\left|S_{m_k}\varphi_n^{(i)}(x)\right| < \frac{1}{n} \ \forall x \in \sigma_k \ \forall i < k.$$

Now we consider the following subsequence of Fourier sums of φ_n, $\{S_{m_k}\varphi_n(x)\}_{m_k}$, whose absolute value we will inferiorly bounded by something arbitrarily bigger in order to prove the last section of the lemma. By definition of φ_n:

$$S_{m_k}\varphi_n(x) = \frac{1}{\pi}\sum_{i=1}^{k-1}\int_{\Delta_i} \varphi_n(t) D_{m_k}(t-x)dt + \frac{1}{\pi}\sum_{i=k}^{n}\int_{\Delta_i} \varphi_n(t) D_{m_k}(t-x)dt =: J_1(x) + J_2(x).$$

Then:

$$|J_1(x)| = \left|\frac{1}{\pi}\sum_{i=1}^{k-1}\int_{\Delta_i} \varphi_n^{(i)}(t) D_{m_k}(t-x)dt\right| \leq \sum_{i=1}^{k-1}\left|\frac{1}{\pi}\int_{\Delta_i} \varphi_n^{(i)}(t) D_{m_k}(t-x)dt\right| =$$
$$\sum_{i=1}^{k-1}\left|S_{m_k}\varphi_n^{(i)}(x)\right| < \sum_{i=1}^{k-1}\frac{1}{n} < 1 \ \forall x \in \sigma_k,$$

where we use the inequality just proved and the fact that $k \leq n$.

On the other hand, let us make some remarks in order to bound the addends of J_2:

(1) $\frac{2m_k+1}{2}(A_i - A_k) = \frac{\lambda_k(2n+1)}{2}\frac{4\pi}{2n+1}(i-k) = (i-k)\lambda_k 2\pi$, which is an even multiple of π, so:

$$\sin\left[\frac{2m_k+1}{2}(A_i - x)\right] = \sin\left[\frac{2m_k+1}{2}(A_i - x) - \frac{2m_k+1}{2}(A_i - A_k)\right] = \sin\left[\frac{2m_k+1}{2}(A_k - x)\right].$$

(2) Dirichlet kernel's derivative satisfies:

$$D_p(t) = \frac{1}{2} + \sum_{\nu=1}^{p} \cos \nu t \Rightarrow D'_p(t) = \sum_{\nu=1}^{p}(-\nu \sin \nu t) \Rightarrow$$
$$|D'_p(t)| \leq \sum_{\nu=1}^{p} |\nu \sin \nu t| \leq \sum_{\nu=1}^{p} |\nu| \leq \sum_{\nu=1}^{p} p = p^2.$$

(3) For all $t \in \Delta_i$, $|A_i - t| \leq \frac{1}{m_i^2}$ by definition of Δ_i, and:

$$\forall\, x \in \sigma_k\ \forall\, i \geq k,\ 0 < A_i - x < A_i - A_{k-1} = \frac{4\pi}{2n+1}(i-k+1) < (i-k+1)\frac{2\pi}{n}.$$

(4)

$$\begin{aligned} S_{m_k}\varphi_n^{(i)}(x) =& \frac{1}{\pi}\int_{\Delta_i} \varphi_n(t) D_{m_k}(t-x)dt = \\ & \frac{1}{\pi}\int_{\Delta_i} \varphi_n(t) D_{m_k}(A_i - x)dt + \frac{1}{\pi}\int_{\Delta_i} \varphi_n(t)[D_{m_k}(t-x) - D_{m_k}(A_i - x)]dt =: \\ & \frac{1}{\pi}\frac{\sin\left[\frac{2m_k+1}{2}(A_i - x)\right]}{2\sin[(A_i-x)/2]}\int_{\Delta_i}\varphi_n(t)dt + K(x) = \\ & \sin\left[\frac{2m_k+1}{2}(A_k - x)\right]\frac{1}{\pi n \sin[(A_i - x)/2]} + K(x). \end{aligned}$$

(5) Now we apply Lagrange's Mean Value theorem (Theorem B.3) to D_{m_k} (which is differentiable) between $t-x$ and $A_i - x$. This tells us that there exists $c \in [t-x, A_i - x]$ such that

$$|D'_{m_k}(c)| = \frac{|D_{m-k}(t-x) - D_{m_k}(A_i - x)|}{|A_i - t|}.$$

Then, from (2), $|D_{m-k}(t-x) - D_{m_k}(A_i - x)| \leq m_k^2 |A_i - t|$, which, from (3), is less than or equal 1 for all $t \in \Delta_i$. Therefore:

$$\begin{aligned} |K(x)| =& \left|\frac{1}{\pi}\int_{\Delta_i}\varphi_n(t)[D_{m_k}(t-x) + D_{m_k}(A_i - x)]dt\right| \leq \\ & \frac{1}{\pi}\int_{\Delta_i}\varphi_n(t)|D_{m_k}(t-x) + D_{m_k}(A_i - x)|dt \leq \frac{1}{\pi}\frac{m_k^2}{n}\frac{1}{m_k^2}\cdot 1 < \frac{1}{n}. \end{aligned}$$

So, from (4):

$$S_{m_k}\varphi_n^{(i)}(x) = \sin\left[\left(m_k+\frac{1}{2}\right)(A_k-x)\right]\frac{1}{\pi n\sin[(a_i-x)/2]}+\frac{\tau}{n},$$

with $|\tau|<1$. Hence:

$$\begin{aligned}J_2(x) =&\frac{1}{\pi}\sum_{i=k}^{n}\int_{\Delta_i}\varphi_n(t)D_{m_k}(t-x)dt = \sum_{i=k}^{n} S_{m_k}\varphi_n^{(i)}(x) = \\ &\sin\left[\left(m_k+\frac{1}{2}\right)(A_k-x)\right]\sum_{i=k}^{n}\frac{1}{\pi n\sin[(A_i-x)/2]}+\tau.\end{aligned}$$

Now from (3) we have that for all $x\in\sigma_k$:

- $0<A_i-x<\dfrac{2\pi}{n}(i-k+1)\le\pi$, so
- $\dfrac{A_i-x}{2}\in(0,\pi)\Rightarrow \sin\left(\dfrac{A_i-x}{2}\right)<\left(\dfrac{A_i-x}{2}\right)$,

from which

$$\begin{aligned}\sum_{i=k}^{n}\frac{1}{\pi n\sin[(A_i-x)/2]} > \sum_{i=k}^{n}\frac{1}{\pi n[(A_i-x)/2]} > \sum_{i=k}^{n}\frac{1}{\pi n\frac{\pi}{n}(i-k+1)} = \\ \frac{1}{\pi^2}\sum_{i=k}^{n}\frac{1}{i-k+1} = \frac{1}{\pi^2}\sum_{j=q}^{n-k+1}\frac{1}{j}.\end{aligned}$$

Thus, since $|a+b|\ge|a|-|b|$:

$$\begin{aligned}|J_2(x)| = &\left|\sin\left[\left(m_k+\frac{1}{2}\right)(A_k-x)\right]\sum_{i=k}^{n}\frac{1}{\pi n\sin[(A_i-x)/2]}+\tau\right| > \\ &\frac{1}{\pi^2}\left(\sum_{j=1}^{n-k+1}\frac{1}{j}\right)\left|\sin\left[\left(m_k+\frac{1}{2}\right)(A_k-x)\right]\right|-1\ge \\ &\frac{1}{\pi^2}\left|\sin\left[\left(m_k+\frac{1}{2}\right)(A_k-x)\right]\right|\log(n-k+2)-1,\end{aligned}$$

as we saw in Lemma 3.2.

Using again that $|a+b|\ge|a|-|b|$, we have that for all $x\in\sigma_k$ and every $2\le k\le n$:

$$|S_{m_k}\varphi_n(x)|\ge|J_2(x)|-|J_1(x)|\ge\left|\sin\left[\left(m_k+\frac{1}{2}\right)(A_k-x)\right]\right|\log(n-k+2)-2. \quad (3.1)$$

We can finally find the sequences M_n, q_n, E_n. Take, in the inequality above, $k\le n-\sqrt{n}$ (which indeed satisfies $2\le k\le n$ for every $n\ge 4$):

$$|S_{m_k}\varphi_n(x)|\ge\frac{1}{\pi^2}|\sin(\ldots)|\log(n-n+\sqrt{n}+2)-2\ge\frac{1}{2\pi^2}|\sin(\ldots)|\log n-2.$$

Let us define then:

- $M_n = \sqrt{\log n} - 2$,
- $q_n = m_n$,
- $E_n = \bigcup_{2 \le k \le n-\sqrt{n}} E_n^{(k)}$, with $E_n^{(k)} := \left\{ x \in \sigma_k \, ; \, \left| \sin \left[\left(m_k + \frac{1}{2} \right) (A_k - x) \right] \right| \ge \frac{2\pi^2}{\sqrt{\log n}} \right\}$.

Choosing n_0 and m_2 large enough,

$$\left| \left\{ x \in [A_{k-1}, A_k] \, ; \, \left| \sin \left[\left(m_k + \frac{1}{2} \right) (A_k - x) \right] \right| < \frac{2\pi^2}{\sqrt{\log n}} \right\} \right| = \mathcal{O} \left(\frac{1}{n\sqrt{\log n}} \right),$$

and

$$|E_n^{(k)}| = |\sigma_k| - \mathcal{O} \left(\frac{1}{n\sqrt{\log n}} \right)$$

when $n \to \infty$. Therefore:

$$|E_n| = \sum_{2 \le k \le n-\sqrt{n}} |\sigma_k| - \mathcal{O} \left(\frac{1}{\sqrt{\log n}} \right) = \left(\frac{4\pi}{2n+1} - \frac{4}{n^2} \right) (n - \sqrt{n}) - \mathcal{O} \left(\frac{1}{\sqrt{\log n}} \right) =$$
$$2\pi - o(1),$$

that is, $|E_n| \xrightarrow{n} 2\pi$.

From (3.1) we have finally that

$$\forall x \in E_n^{(k)} \; \forall 2 \le k \le n - \sqrt{n}, |S_{m_k} \varphi_n(x)| \ge \sqrt{\log n} - 2 = M_n,$$

with $M_n > 0$ a sequence such that $M_n \xrightarrow{n} \infty$. Thus, (iii).(a),(b) and (c) are satisfied for every $n \ge n_0$. For $1 \le n \le n_0$ we can take $\varphi_n(t) = \varphi_{n_0}(t)$, so we have completed the proof. □

Kolmogorov also proved that there exists an integrable function whose Fourier series diverges everywhere, but we will not prove it here.

Chapter 4

Summability of Fourier series

Consider that we only know, from an unknown 2π-periodic integrable function, its Fourier coefficients. How can we recover the function from them?

We know that, under certain conditions (for instance the hypothesis on Dirichlet's theorem, Theorem 2.7) it suffices to sum the Fourier series of f. Nevertheless, as seen in Chapter 3, in general we cannot recover f from its Fourier series, not even almost everywhere.

In this chapter we show that we can sometimes use other series in order to recover f. These series converge to f whenever Sf does, but they can even converge to f when the Fourier series does not.

For the following sections, the main reference have been Duoandikoetxea's notes [6].

4.1 Cesàro summability

In this case we study the limit of the arithmetic mean of the Fourier sums. This summability method is inspired by the following basic result for numerical sequences and series.

Proposition 4.1 (Cesàro). *Let $\{a_n\}_n$ be a convergent sequence and let $l := \lim_n a_n$. Then, the sequence*

$$\left\{ b_n := \frac{a_1 + \cdots + a_n}{n} \right\}_n$$

also converges to l.

Féjer was the first one to apply this summability method to Fourier series. Specifically, he considered the Cesàro partial sums of the sequence of Fourier sums of f. We will denote these partial sums as $\sigma_N f$:

$$\sigma_N f(x) := \frac{1}{N+1} \sum_{j=0}^{N} S_j f(x). \tag{4.1}$$

Also, we denote $\sigma f(x) = \lim_N \sigma_N f(x)$ the Cesàro sum of $f(x)$.

Remark 4.2. The index N in σ_N denotes the maximum degree of the Fourier sums that we add. This is why we have N on the left and $N+1$ on the denominator on the right, because we compute the mean of the first $N+1$ Fourier sums, from 0 to N.

Let us now rewrite (4.1). Expressing each $S_j f$ in its integral form:

$$\sigma_n f(x) = \frac{1}{N+1}\sum_{j=0}^{n}\frac{1}{\pi}\int_{-\pi}^{\pi} D_j(t)f(x-t)dt = \frac{1}{N+1}\frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)\sum_{j=0}^{N} D_j(t)dt.$$

Now, let us rewrite the integrand in a more compact form:

$$\sum_{j=0}^{N} D_j(t) = \sum_{j=0}^{N}\frac{\sin\left(j+\frac{1}{2}\right)t}{2\sin(t/2)} = \frac{1}{[2\sin(t/2)]^2}\sum_{j=0}^{N} 2\sin(t/2)\sin\left(j+\frac{1}{2}\right)t. \tag{4.2}$$

This can be rewritten in a simpler form using the following trigonometric expression:

$$\sin x\sin y = \cos x\cos y - \cos(x+y),$$

which leads us to

$$2\sin x\sin y = \cos(x-y) - \cos(x+y),$$

and taking $x = t/2$ and $y = \left(j+\frac{1}{2}\right)t$:

$$2\sin(t/2)\sin\left(j+\frac{1}{2}\right) = \cos jt - \cos(j+1)t.$$

Replacing this in (4.2):

$$\sum_{j=0}^{N} D_j(t) = \frac{1}{[2\sin(t/2)]^2}[1-\cos(N+1)t],$$

which using the trigonometric identity

$$2\sin^2 x = 1 - \cos 2x$$

can be finally expressed as

$$\sum_{j=0}^{N} D_j(t) = \frac{2\sin^2[(N+1)(t/2)]}{4\sin^2(t/2)} = \frac{1}{2}\left(\frac{\sin[(N+1)(t/2)]}{\sin(t/2)}\right)^2.$$

Definition 4.3. *We define the Féjer kernel as*

$$F_N(t) := \frac{\sum_{j=0}^{N} D_j(t)}{N+1} = \frac{1}{2(N+1)}\left(\frac{\sin[(N+1)(t/2)]}{\sin(t/2)}\right)^2.$$

Hence, the integral form of the Cesàro sums is:

$$\sigma_N f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} F_N(t)f(x-t)dt.$$

Taking into account the different integral expressions of the Fourier sums in terms of the Dirichlet kernel, (2.1) and (2.5), one can easily find the following integral expressions for Cesàro sums in terms of the Féjer kernel:

$$\sigma_N f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)F_N(x-t)dt,$$

$$\sigma_N f(x) = \frac{1}{\pi}\int_{0}^{\pi} F_N(t)[f(x+t)+f(x-t)]dt.$$

Let us show some basic properties of the Féjer kernel.

Proposition 4.4.

1. *F_N is an even, non negative and 2π-periodic function,*
2. $\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} F_N(t)dt = 1\ \forall\, N,$
3. $\forall\, \delta > 0, F_N(t) \overset{N}{\rightrightarrows} 0$ *on* $[-\pi,-\delta]\cup[\delta,\pi]$.

Proof.

1. To see that F_N is an even, 2π-periodic function it suffices to look at the expression $F_N(t) := \dfrac{\sum_{j=0}^{N} D_j(t)}{N+1}$, which is a sum of even, 2π-periodic functions. To prove that it is not negative, we only have to look at $\dfrac{1}{2(N+1)}\left(\dfrac{\sin[(N+1)(t/2)]}{\sin(t/2)}\right)^2$, which is the square of a real function.
2. From Dirichlet kernel properties, Proposition 2.2:
$$\frac{1}{\pi}\int_{-\pi}^{\pi} D_N(t)dt = 1 \Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} F_N(t)dt = \frac{1}{\pi}\frac{\sum_{j=0}^{N} D_N(t)}{N+1}dt = \frac{N+1}{N+1} = 1.$$
3. Given $\delta > 0$, we have that for all $\delta \le |t| \le \pi$, since $\sin^2(t/2)$ is an increasing function in these intervals, $\sin^2(t/2) \ge \sin^2(\delta/2)$. So, for all $\delta \le |t| \le \pi$:
$$F_N(t) = \frac{1}{2(N+1)}\frac{\sin^2[(N+1)(t/2)]}{\sin^2(t/2)} \le \frac{1}{2(N+1)\sin^2(t/2)} \le \frac{1}{2(N+1)\sin^2(\delta/2)}.$$
Notice that this bound tends to 0 with N independently of t, that is, uniformly on the intervals $[-\pi,-\delta]\cup[\delta,\pi]$, which was to be proven.

□

Next we show the main result of this section, Féjer's theorem, which assures that continuity is a sufficient condition for Cesàro sums to converge to the function. Notice that, taking into account du Bois-Reymond example, this was not the case for the sum of the Fourier series.

Theorem 4.5 (Féjer)**.** *Assume $f \in \mathcal{L}$ has got lateral limits at x. Then,*

$$\sigma f(x) = \frac{1}{2}[f(x^+) + f(x^-)].$$

Remark 4.6. If f is continuous at x, then $\sigma f(x) = f(x)$.

Proof. From Féjer's kernel properties (Proposition 4.4), we can write:

$$\sigma_N f(x) - \frac{1}{2}\int_0^{\pi} F_N(t)[f(x-t) + f(x+t) - f(x^-) - f(x^+)]dt.$$

Therefore, for every N:

$$\left|\sigma_N f(x) - \frac{1}{2}[f(x^+) + f(x^-)]\right| \leq$$
$$\frac{1}{\pi}\left\{\left|\int_0^{\pi} F_N(t)[f(x-t) - f(x^-)]dt\right| + \left|\int_0^{\pi} F_N(t)[f(x+t) - f(x^+)]dt\right|\right\},$$

and we can bound each addend in the following way:

$$\left|\int_0^{\pi} F_N(t)[f(x-t) - f(x^-)]dt\right| \leq \int_0^{\pi} |F_N(t)[f(x-t) - f(x^-)]dt =$$
$$\int_0^{\delta} F_N(t)|f(x-t) - f(x^-)|dt + \int_{\delta}^{\pi} F_N(t)|f(x-t) - f(x^-)|dt,$$

where we have used the fact that, given $\varepsilon > 0$ we can choose $\delta > 0$ such that

$$\sup_{0\leq t\leq\delta} |f(x-t) - f(x^-)| < \frac{1}{\pi}\cdot\frac{\varepsilon}{2},$$

because $f(x^-) := \lim_{t\to 0^+} f(x-t)$. Therefore, we can bound the first term like:

$$\int_0^{\delta} F_N(t)|f(x-t) - f(x^-)|dt \leq \int_0^{\delta} F_N(t)dt \cdot \sup_{0\leq t\leq\delta} |f(x-t) - f(x^-)| <$$
$$\int_{-\pi}^{\pi} F_N(t)dt\frac{1}{\pi}\cdot\frac{\varepsilon}{2} = \frac{\varepsilon}{2}.$$

The second term can be bounded like:

$$\int_{\delta}^{\pi} F_N(t)|f(x-t) - f(x^-)|dt \leq \sup_{\delta<t\leq\pi} F_N(t)\cdot\int_{\delta}^{\pi} |f(x-t) - f(x^-)|dt,$$

and

$$\int_{\delta}^{\pi} |f(x-t) + f(x^-)|dt \leq \int_{\delta}^{\pi} |f(x-t)|dt + \int_{\delta}^{\pi} |f(x^-)|dt \leq$$
$$\int_{-\pi}^{\pi} |f(x-t)|dt + \int_0^{\pi} |f(x^-)|dt = \int_{-\pi}^{\pi} |f(t)|dt + \pi|f(x^-)|,$$

where the last equality holds by the periodicity of f. Notice that the result is finite because f is integrable. Also, from the last property of Fejér's kernel on Proposition 4.4, for N large enough

$$\sup_{\delta<t\leq\pi} F_N(t) < \frac{1}{\int|f| + \pi|f(x^-)|}\cdot\frac{\varepsilon}{2},$$

so

$$\int_{\delta}^{\pi} F_N(t)|f(x-t) - f(x^-)|dt < \frac{\varepsilon}{2}.$$

Therefore,

$$\forall \varepsilon > 0\, \exists N_0 \in \mathbb{N} \text{ such that } \forall\, N \geq N_0, \left| \int_0^{\pi} F_N(t)[f(x-t) - f(x^-)]dt \right| < \varepsilon,$$

so

$$\int_0^{\pi} F_N(t)[f(x-t) - f(x^-)]dt \xrightarrow{N} 0.$$

Similarly, $\int_0^{\pi} F_N(t)[f(x+t) - f(x^+)]dt \xrightarrow{N} 0$, so finally

$$\sigma_N f(x) - \frac{1}{2}[f(x^+) + f(x^-)] \xrightarrow{N} 0,$$

which was to be proven. □

4.2 Abel-Poisson summability

Now we study another kind of series, specifically a power series that is obtained with the Fourier coefficients of the function. This kind of summability is based on the following known result for numerical series.

Proposition 4.7 (Abel). *Let* $\sum_{n=1}^{\infty} a_n = s$ *be a convergent numerical series. Then:*

- *The function* $S(r) := \sum_{n=1}^{\infty} r^n a_n$ *is well defined (convergent)* $\forall\, 0 \leq r \leq 1$, *and*
- $\lim_{r \to 1^-} S(r) = s$.

We now apply this summability method to Fourier series. Consider the following Abel power series, where the coefficients are the Fourier coefficients of f:

$$S_r f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} r^k (a_k \cos kx + b_k \sin kx), \tag{4.3}$$

of which we have to study its limit $S_{1^-} f(x) := \lim_{r \to 1^-} S_r f(x)$, what we call the Abel sum of $f(x)$.

Note that the exponential form of (4.3) is not $\sum_{k=-\infty}^{\infty} r^k c_k e^{ikx}$, but:

$$S_r f(x) = \sum_{k=-\infty}^{\infty} r^{|k|} c_k e^{ikx}. \tag{4.4}$$

Indeed,

$$\forall\, k \geq 0,\ c_{\pm k} = \frac{1}{2}a_k \pm \frac{1}{2i}b_k \Rightarrow \frac{1}{2}(r^k a_k) \pm \frac{1}{2i}(r^k b_k) = (r^k c_{\pm k}),$$

that is,

$$r^k \left(\frac{1}{2}a_k \pm \frac{1}{2i}b_k\right) = r^k c_{\pm k},$$

and the sign of k on the power r^k is always positive.

The Abel power series (4.4) are well defined, that is, convergent, for all $0 \leq r < 1$. Let us show, in fact, that for any given $0 \leq r < 1$, $S_r f(x)$ is uniformly convergent with respect to x. It suffices to see that, for any $0 \leq r < 1$, defining $f_n(x) = r^{|n|} c_n e^{inx}$, then

$$\sum_n \sup_x |f_n(x)| < \infty\ \forall\, x \in [-\pi, \pi].$$

Indeed, in that case, using the Weierstrass M-test (Theorem B.1), $S_r f(x) = \sum_n f_n(x)$ converges uniformly with respect to x. We have that:

$$\sum_n \sup_x |f_n(x)| = \sum_n \sup_x |r^{|n|} c_n e^{inx}| = \sum_n r^{|n|} |c_n| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)| dt \cdot \sum_n r^{|n|},$$

which is finite because $f \in \mathcal{L}$ and $r < 1 \Rightarrow \sum_n r^{|n|} < \infty$, which was to be shown.

Now let us find the integral form of the Abel power series:

$$S_r f(x) = \sum_k (r^{|k|} c_k e^{ikx}) = \sum_k \left[r^{|k|} e^{ikx} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt \right] = \frac{1}{2\pi} \sum_k \int_{-\pi}^{\pi} r^{|k|} f(t) e^{ik(x-t)} dt =$$
$$\frac{1}{\pi} \int_{-\pi}^{\pi} \left[f(t) \sum_k \left(\frac{1}{2} r^{|k|} e^{ik(x-t)} \right) \right] dt.$$

Definition 4.8. *We define the Poisson kernel as*

$$P_r(t) := \sum_{k=-\infty}^{\infty} \frac{1}{2} r^{|k|} e^{ikt}\ \ \forall\, 0 \leq r < 1.$$

Therefore,

$$S_r f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) P_r(x-t) dt.$$

Let us see some alternative expressions for the Poisson kernel. In the first place, its trigonometric expression can be obtained with Euler's formula $e^{ix} = \cos x + i \sin x$:

$$P_r(t) = \sum_{k=-\infty}^{\infty} \frac{1}{2} r^{|k|} e^{ikt} = \sum_{k=-\infty}^{\infty} \frac{1}{2} r^{|k|} (\cos kt + i \sin kt) = \sum_{k=-\infty}^{\infty} \frac{1}{2} r^{|k|} \cos kt = \frac{1}{2} + \sum_{k \geq 1} r^k \cos kt,$$

where we have used sine's and cosine's parities. Thus,

$$P_r(t) = \frac{1}{2} + \sum_{k=1}^{\infty} r^k \cos kt.$$

Like Dirichlet's and Féjer's kernels, the Poisson one can also be expressed as a quotient of trigonometric functions. Let $S := P_r(t) - \dfrac{1}{2} = \sum_{k\geq 1} r^k \cos kt$. Recall the trigonometric identity

$$\cos(x+y) = \cos x \cos y - \sin x \sin y,$$

which leads us to

$$2\cos x \cos y = \cos(x+y) + \cos(x-y),$$

so:

$$2S\cos t = \sum_{k\geq 1} r^k 2\cos t \cos kt = \sum_{k\geq 1} r^k [\cos(k+1)t + \cos(k-1)t] =$$
$$\frac{1}{r}\sum_{k\geq 2} r^k \cos kt + r\sum_{k\geq 0} r^k \cos kt = \frac{1}{r}(S - r\cos t) + r(S+1) = \left(r + \frac{1}{r}\right) S + r - \cos t,$$

and isolating S:

$$S = \frac{r\cos t - r^2}{1 - 2r\cos t + r^2}.$$

Finally, $P_r(t) = S + 1/2$, so:

$$P_r(t) = \frac{1-r^2}{2(1-2r\cos t + r^2)}.$$

Let us show the basic properties of the Poisson kernel.

Proposition 4.9.

1. *P_r is an even, non negative and 2π-periodic function.*
2. $\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} P_r(t)dt = 1.$
3. $\forall\, \delta > 0,\ P_r(t) \overset{r\to 1^-}{\rightrightarrows} 0$ *on* $[-\pi, -\delta] \cup [\delta, \pi]$.

Proof.

1. From its exponential form we see that it is non negative. From its trigonometric form we see that it is even and 2π-periodic.
2. The Poisson kernel is a particular case of an Abel power series, which we proved to be uniformly convergent. Therefore, we can swap the integral and the series:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} P_r(t)dt = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2} + \sum_{k\geq 1} r^k \cos kt\right) dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2} dt = 1.$$

3. Given $\delta > 0$, we have that for every $\delta \leq |t| \leq \pi$:

$$2(1-2r\cos t + r^2) = 2(1 - 2r + r^2 - 2r\cos t + 2r) = 2[(1-r)^2 + 2r(1-\cos t)] \geq$$
$$2[(1-r)^2 + 2r(1-\cos\delta)] \Rightarrow P_r(t) \leq \frac{1-r^2}{2[(1-r)^2 + 2r(1-\cos\delta)]}.$$

This bound tends to 0 when $r \to 1^-$ independently of t, that is, uniformly. Indeed:

$$\frac{1-r^2}{2[(1-r)^2+2r(1-\cos\delta)]} \overset{r\to 1^-}{\sim} \frac{1-r^2}{4r(1-\cos\delta)} \xrightarrow{r\to 1^-} 0. \qquad \square$$

It is easy to see that a couple of alternative integral expressions for the Abel power series in terms of the Poisson kernel are:

$$S_r f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)P_r(t)dt,$$

$$S_r f(x) = \frac{1}{\pi}\int_{0}^{\pi} P_r(t)[f(x-t)+f(x+t)]dt.$$

Let us now show the main result about Abel summability of Fourier series.

Theorem 4.10. *Let $f \in \mathcal{L}$ bounded. If f has got lateral limits at x, then*

$$S_{1^-}f(x) = \frac{1}{2}[f(x^+)+f(x^-)].$$

Remark 4.11. If f is continuous at x, then $S_{1^-}f(x) = f(x)$.

The proof is completely similar to the Féjer's theorem one, Theorem 4.5, the key argument being the uniform convergence of the kernel on any closed subinterval of $(0,\pi]$.

4.3 Uniform summability

We already claimed that, if Sf converges uniformly, then f must be continuous. The same happens if, instead of studying the convergence of the Fourier series, we study the summability, either the Abel or the Cesàro one. Indeed, since $\sigma_N f$ and $S_r f$ are continuous for every N and for all r respectively, if they converge uniformly then f must be continuous.

The interesting fact about summability is that the continuity of the limit (in case of uniform convergence) is not only necessary but also sufficient, which thanks to du Bois-Reymond, Theorem 3.1, we know to be in general false for the convergence of the Fourier series.

Theorem 4.12. *Let $f \in \mathcal{C}^0$. Then, $\sigma_N f \overset{N}{\rightrightarrows} f$ and $S_r f \overset{r}{\rightrightarrows} f$ on $[-\pi,\pi]$.*

Proof. We only give the proof for σ_N. The one for the Abel power series is similar.

$$\begin{aligned} |\sigma_N f(x) - f(x)| &= \left|\frac{1}{\pi}\int_0^{\pi} F_N(t)[f(x+t)+f(x-t)]dt - f(x)\right| = \\ &\left|\frac{1}{\pi}\int_0^{\pi} F_N(t)[f(x+t)-f(x)+f(x-t)-f(x)]dt\right| \le \\ &\frac{1}{\pi}\int_0^{\pi} F_N(t)|f(x+t)-f(x)|dt + \frac{1}{\pi}\int_0^{\pi} F_N(t)|f(x-t)-f(x)|dt, \end{aligned}$$

and let us bound both terms in the same, following way:

$$\int_0^{\pi} F_N(t)|f(x \pm t) - f(x)|dt = \int_0^{\delta} (\dots)dt + \int_{\delta}^{\pi} (\dots)dt,$$

and:

- $\displaystyle\int_0^{\delta} F_N(t)|f(x \pm t) - f(x)|dt \leq \overbrace{\int_0^{\pi} F_N(t)dt}^{\pi/2} \cdot \sup_{0\leq t\leq \delta} |f(x \pm t) - f(x)|.$
- Since f is continuous, let $M := \max\limits_{\delta \leq t \leq \pi} |f(t)|$. Then:

$$\int_{\delta}^{\pi} F_N(t)|f(x \pm t) - f(x)|dt \leq \int_{\delta}^{\pi} F_N(t)(|f(x \pm t)| + |f(x)|)dt \leq 2M \int_{\delta}^{\pi} F_N(t)dt,$$

so:

$$\int_0^{\pi} F_N(t)|f(x \pm t) - f(x)|dt \leq \frac{1}{\pi} 2 \left[\frac{\pi}{2} \sup_{0\leq t\leq \delta} |f(x \pm t) - f(x)| + 2M \int_{\delta}^{\pi} F_N(t)dt \right] = \sup_{0\leq t\leq \delta} |f(x \pm t) - f(x)| + \frac{4M}{\pi} \int_{\delta}^{\pi} F_N(t)dt.$$

We want to see uniform convergence, so we need to prove that

$$\forall \varepsilon > 0 \ \exists \delta > 0, \ \exists N_0 \text{ such that } |\sigma_N f(x) - f(x)| < \varepsilon \ \forall x, \forall N \geq N_0 \text{ with } \delta \neq \delta(x).$$

On the one hand, recall that for all $\delta > 0$, $F_N(t) \overset{N}{\rightrightarrows} 0$ on $[-\pi, -\delta] \cup [\delta, \pi]$, so:

$$\frac{4M}{\pi} \int_{\delta}^{\pi} F_N(t)dt \leq \frac{4M}{\pi} \frac{\pi}{2} \sup_{\delta \leq t \leq \pi} F_N(t) = 2M \sup_{\delta \leq t \leq \pi} F_N(t) \overset{N}{\rightrightarrows} 0,$$

so for N_0 great enough, this term is, for any $N \geq N_0$, bounded by $\varepsilon/2$.

On the other hand, by continuity of f on $[-\pi, \pi]$, then f is uniformly continuous on $[-\pi, \pi]$. Hence, by definition of uniform continuity, given $\varepsilon > 0$ there exists $\delta > 0$ such that $\sup\limits_{0\leq t\leq \delta} |f(x - t) - f(x)| < \varepsilon/2$.

Therefore, for this $\delta > 0$ and for every $N \geq N_0$, $|\sigma_N f(x) - f(x)| < \varepsilon$, which was to be proven. □

Corollary 4.13. *Let $f \in \mathcal{C}^0$. Then there exists a sequence $\{p_n\}_n$ of trigonometric polynomials that converges uniformly to f.*

Proof. From the theorem, since f is continuous then $\sigma_N f \overset{N}{\rightrightarrows} f$, where $\{\sigma_N\}_N$ is a sequence of trigonometric polynomials. □

Corollary 4.14. *Two continuous functions with the same Fourier series are equal.*

Proof. If $Sf = \sum_n f_n$ and $Sg = \sum_n g_n$ are the same, then $f_n = g_n \ \forall n$. Therefore, the respective partial sums are the same, that is, $S_j f = S_j g$ for every j, so $\sigma_N f = \sigma_N g$ for every N. Since f and g are continuous, from the previous corollary $\sigma_N f \overset{N}{\rightrightarrows} f$ and $\sigma_N g \overset{N}{\rightrightarrows} g$, but since the limit is unique, then $f = g$. □

4.4 Convergence in $\mathcal{L}^p$

Until now we only considered pointwise and/or uniform convergence of Fourier series or Cesàro and Abel sums. But there exist another kinds of convergence, one of the most important being the convergence in $\mathcal{L}^p$.

Definition 4.15. *The sequence of functions $\{g_N\}_N$ is said to converge in p-mean - or, simply, in $\mathcal{L}^p$ -, for $1 \le p \le \infty$, to a function f on $[-\pi,\pi]$ if*

$$\lim_N \int_{-\pi}^{\pi} |g_N(t) - g(t)|^p dt = 0,$$

that is, if $\lim_N ||g_N - g||_p^p = 0$.

If $p = 1$, the convergence is said to be in mean, and if $p = 2$, it is said to be in (root) mean square.

Let us see the main results of this section, which are referred to the converge in $\mathcal{L}^p$ of the Cesàro and Abel series. We consider separately the case $p = 1$ and the cases $p > 1$.

Theorem 4.16. *Let $f \in \mathcal{L}$. Then, $\lim_N \int_{-\pi}^{\pi} |\sigma_n f - f| = 0$ and $\lim_{r\to 1^-} \int_{-\pi}^{\pi} |S_r f - f| = 0$.*

Proof. Once again, we only give the proof for the Cesàro sums, since the one for the Abel series is similar. For any $\delta > 0$,

$$\int_{-\pi}^{\pi} |\sigma_N f(x) - f(x)| dx = \int_{-\pi}^{\pi} \left| \frac{1}{\pi} F_N(t)[f(x-t) - f(x)]dt \right| dx \le$$
$$\int_{-\pi}^{\pi}\int_{-\delta}^{\delta} F_N(t)|f(x-t) - f(x)|dtdx + \int_{-\pi}^{\pi}\int_{\delta\le|t|\le\pi} F_N(t)|f(x-t) - f(x)|dtdx \le$$
$$\int_{-\pi}^{\pi}\int_{-\delta}^{\delta} F_N(t)|f(x-t) - f(x)|dtdx + 2||f||_1\pi \sup_{\delta\le|t|\le\pi} F_N(t),$$

where the last inequality is due to:

$$\int_{-\pi}^{\pi}\int_{\delta\le|t|\le\pi} F_N(t)|f(x-t) - f(x)|dtdx = \int_{\delta\le|t|\le\pi}\int_{-\pi}^{\pi} F_N(t)|f(x-t) - f(x)|dtdx \le$$
$$\int_{\delta\le|t|\le\pi} F_N(t)\int_{-\pi}^{\pi}(|f(x-t)| + |f(x)|)dxdt = \int_{\delta\le|t|\le\pi} F_N(t)\int_{-\pi}^{\pi} 2|f(x)|dxdt =$$
$$2||f||_1 \int_{\delta\le|t|\le\pi} F_N(t)dt \le 2||f||_1\pi \sup_{\delta\le|t|\le\pi} F_N(t).$$

Now, from the last property of the Féjer kernel on Proposition 4.4, we know that for any $\delta > 0$, $2||f||_1\pi \sup_{\delta\le|t|\le\pi} F_N(t) \xrightarrow{N} 0$.

We know from Theorem A.15 that

$$\lim_{t\to 0}\int_{-\pi}^{\pi} |f(x-t) - f(x)|dx = 0,$$

so, for any $\varepsilon > 0$:

$$\int_{-\pi}^{\pi}\int_{-\delta}^{\delta} F_N(t)|f(x-t)-f(x)|dtdx = \int_{\delta}^{\delta} F_N(t)\int_{-\pi}^{\pi}|f(x-t)-f(x)|dxdt \leq$$
$$\frac{\varepsilon}{2\pi}\int_{-\delta}^{\delta} F_N(t)dt \leq \frac{\varepsilon}{2\pi}\pi = \frac{\varepsilon}{2}. \qquad \square$$

Theorem 4.17. *Let $f \in \mathcal{L}^p$ for $1 < p < \infty$. Then, $\lim_N \int_{-\pi}^{\pi} |\sigma_N f - f|^p = 0$ and $\lim_{r\to 1^-}\int_{-\pi}^{\pi}|S_r f - f|^p = 0$.*

Proof. This proof is similar to the previous one.

$$|\sigma_N f(x) - f(x)|^p \leq \left(\frac{1}{\pi}\int_{-\pi}^{\pi} F_N(t)|f(x-t)-f(x)|dt\right)^p.$$

Now, let us apply Hölder's inequality (Theorem A.11) to this integral. Specifically:

$$\begin{cases} g(t) := F_N(t)^{1/p}|f(x-t)-f(x)| \\ h(t) := F_N(t)^{1/q} \text{ such that } \dfrac{1}{q} = 1 - \dfrac{1}{p} \end{cases} \implies ||gh||_1 \leq ||g||_p||h||_q,$$

which translates into:

$$\int_{-\pi}^{\pi} F_N(t)|f(x-t)-f(x)|dt \leq \left(\int_{-\pi}^{\pi} F_N(t)|f(x-t)-f(x)|^p dt\right)^{\frac{1}{p}} \cdot \left(\int_{-\pi}^{\pi} F_N(t)dt\right)^{\frac{1}{q}} \Rightarrow$$
$$\left(\frac{1}{\pi}\int_{-\pi}^{\pi} F_N(t)|f(x-t)-f(x)|dt\right)^p \leq \frac{1}{\pi^p}\int_{-\pi}^{\pi} F_N(t)|f(x-t)-f(x)|^p dt \left(\int_{-\pi}^{\pi} F_N(t)dt\right)^{\frac{p}{q}}$$
$$= \frac{\pi^{p/q}}{\pi^p}\int_{-\pi}^{\pi} F_N(t)|f(x-t)-f(x)|^p dt \underset{q\geq 1}{\leq} \frac{1}{\pi}F_N(t)|f(x-t)-f(x)|^p dt,$$

so

$$|\sigma_N f(x) - f(x)|^p \leq \frac{1}{\pi}F_N(t)|f(x-t)-f(x)|^p dt,$$

and

$$\int_{-\pi}^{\pi}|\sigma_N f(x) - f(x)|^p dx \leq \int_{-\pi}^{\pi}\int_{-\delta}^{\delta} F_N(t)|f(x-t)-f(x)|^p dtdx +$$
$$+ \int_{-\pi}^{\pi}\int_{\delta\leq|t|\leq\pi} F_N(t)|f(x-t)-f(x)|^p dtdx.$$

Let us bound both addends:

$$\int_{-\pi}^{\pi}\int_{\delta\leq|t|\leq\pi} F_N(t)|f(x-t)-f(x)|^p dtdx = \int_{\delta\leq|t|\leq\pi} F_N(t)\int_{-\pi}^{\pi}|f(x-t)-f(x)|^p dxdt,$$

which defining $g_t(x) = f(x-t)$, equals

$$\int_{\delta\leq|t|\leq\pi} F_N(t)||g_t(x)-f(x)||_p^p dt.$$

Since $||\cdot||_p$ is a norm (see Theorem A.10), then the triangle inequality tells us that:

$$\int_{\delta\le|t|\le\pi} F_N(t)||g_t(x)-f(x)||_p^p dt \le \int_{\delta\le|t|\le\pi} F_N(t)(||g_t(x)||_p+||f(x)||_p)^p dt,$$

which by definition of g_t and periodicity of f equals

$$\int_{\delta\le|t|\le\pi} F_N(t)(2||f||_p)^p dt = 2^p||f||_p^p\int_{\delta\le|t|\le\pi} F_N(t)dt \le 2^p||f||_p^p\pi \sup_{\delta\le|t|\le\pi} F_N(t) \xrightarrow{N} 0,$$

because $F_N \overset{N}{\rightrightarrows} 0$ on $\delta\le|t|\le\pi$. Now, for the second addend, we bound it similarly to the case $p=1$. We use Theorem A.15, that is,

$$\lim_{h\to 0}\int_X |f(x+h)-f(x)|^p dx = 0 \ \ \forall f\in\mathcal{L}^p.$$

Then,

$$\int_{-\pi}^{\pi}\int_{-\delta}^{\delta} F_N(t)|f(x-t)-f(x)|^p dtdx = \int_{-\delta}^{\delta} F_N(t)\int_{-\pi}^{\pi}|f(x-t)-f(x)|^p dxdt \le$$
$$\frac{\varepsilon}{2\pi}\int_{-\delta}^{\delta}F_N(t)dt \le \frac{\varepsilon}{2\pi}\pi = \frac{\varepsilon}{2}.$$

□

We have seen that two continuous functions with the same Fourier series are in fact the same function. For integrable functions, we have the following result.

Corollary 4.18. *If $f,g\in\mathcal{L}$ have the same Fourier series, then* $\displaystyle\int_{-\pi}^{\pi}|f-g| = 0.$

Remark 4.19. Recall that if $\int_X f = 0$ then $f=0$ *a.e.* X. Therefore, the corollary tells us that if two integrable functions have the same Fourier series, then they are equal almost everywhere.

Proof. Let $h=f-g$. If $Sf=Sg$, then $Sh=S(f-g)=Sf-Sg=0$, so all the Fourier coefficients of h are 0. Thus, $\sigma_N h = 0$ for all N and, from the theorem, $\int|h|=0$, which was to be proven. □

4.5 Mean square convergence: Plancherel-Parseval's identity

We apply the results of the previous section to the case $p=2$.

Corollary 4.20. *Let $f\in\mathcal{L}^2$. Then,* $\displaystyle\lim_N\int_{-\pi}^{\pi}|S_Nf-f|^2 = 0.$

Proof. Recall Theorem 1.9, which tells us that, among all the trigonometric polynomials of degree N, the one that best approximates f in square mean is $S_N f$. So the trigonometric polynomial $\sigma_N f$ of degree N satisfies

$$\int_{-\pi}^{\pi} |S_N f - f|^2 \leq \int_{-\pi}^{\pi} |\sigma_N f - f|^2 \ \forall N.$$

Therefore, from the previous section:

$$0 \leq \lim_N \int_{-\pi}^{\pi} |S_N f - f|^2 \leq \int_{-\pi}^{\pi} |\sigma_N f - f|^2 = 0. \qquad \square$$

Now we show that Bessel's inequality is, in fact, an equality, which is called Plancherel-Parseval's identity.

Corollary 4.21 (Plancherel-Parseval's identity)**.** *Let $f \in \mathcal{L}^2$ bounded. Then,*

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f^2 = \frac{a_0^2}{2} + \sum_{k \geq 1} (a_k^2 + b_k^2).$$

In exponential form, the identity is

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f^2 = \sum_{k=-\infty}^{\infty} |c_k|^2,$$

which can be easily proven from the relation between a_k, b_k and c_k, that is equations (1.1).

Proof. In the proof of Theorem 1.9 we saw that

$$\int_{-\pi}^{\pi} |S_N f - f|^2 = \int_{-\pi}^{\pi} f^2 - \pi \left[\frac{a_0^2}{2} + \sum_{k=1}^{N} (a_k^2 + b_k^2) \right],$$

and taking the limit $N \to \infty$ we get the identity. $\square$

Corollary 4.22. *The trigonometric basis is a complete basis of $\mathcal{L}^2(-\pi, \pi)$.*

4.6 Dirichlet problem

In this section we solve the classic Dirichlet problem. The main reference for this section has been Körner's book [11].

The general Dirichlet problem consists on trying to solve Laplace's equation, $\Delta u = 0$, on an open region $\Omega \subset \mathbb{R}^n$, with the requirement that, for a given function f, then $f = u$ on $\partial\Omega$. The problem can be generalized for any differential equation on any differentiable manifold. Nevertheless, we will study the original problem:

Let $f : \partial D \to \mathbb{R}$ be a continuous funtion. We are searching for a function $u : \overline{D} \to \mathbb{R}$ such that:

(i) u is harmonic, that is, $u \in \mathcal{C}^2(D)$ and satisfies Laplace's equation $\Delta u = 0$ on D.

(ii) u is continuous on $\overline{D}$.

(iii) $u = f$ on ∇D.

It is convenient to work in polar coordinates.

Theorem 4.23. *Let $f \in \mathcal{C}^0[-\pi, \pi]$. Then, the function*

$$\begin{array}{rcl} u : D & \to & \mathbb{R} \\ (r, \theta) & \mapsto & S_r f(\theta) \end{array}$$

is the unique function such that

(i) $\Delta u = 0$ *on* D*,*

(ii) $\lim_{\substack{r \to 1^- \\ \theta \to \theta_0}} u(r, \theta) = f(\theta_0) \ \forall \, \theta_0$.

Therefore, the function

$$\begin{array}{rcl} u : \overline{D} & \to & \mathbb{R} \\ (r, \theta) & \mapsto & \left\{ \begin{array}{l} S_r f(\theta) \ \forall \, 0 \leq r < 1, \\ f(\theta) \ \text{ if } r = 1, \end{array} \right. \end{array}$$

is the only solution of Dirichlet's problem.

Proof. For (i), recall the complex expression of Abel power series (4.4)

$$S_r f(\theta) = \sum_{k=-\infty}^{\infty} r^{|k|} c_k e^{ik\theta},$$

which, writing $z = re^{i\theta}$ and recalling that $c_{-k} = \overline{c_k}$ becomes

$$\begin{aligned} S_r f(\theta) = & c_0 + \sum_{k \geq 1} c_k r^k e^{ik\theta} + \sum_{k \geq 1} c_{-k} r^k e^{-ik\theta} = c_0 + \sum_{k \geq 1} c_k (re^{i\theta})^k + \sum_{k \geq 1} \overline{c_k} (\overline{re^{i\theta}})^k = \\ & \left(\frac{1}{2} c_0 + \sum_{k \geq 1} c_k z^k \right) + \left(\overline{\frac{1}{2} c_0 + \sum_{k \geq 1} c_k z^k} \right) = 2Re \left(\frac{1}{2} c_0 + \sum_{k \geq 1} c_k z^k \right), \end{aligned}$$

which is the real part of an analytic complex function (because it can be written as a well-defined, - uniformly - convergent series), and therefore, by Cauchy-Riemann's equations, is harmonic. For (ii), one only needs to apply Abel-Poisson summability, Theorem 4.10. □

Conclusions

Since I first knew about Fourier series and integrals and their applications, I have always found them quite appealing. The fact that I never studied harmonic analysis accurately made me contemplate choosing it as my undergraduate thesis subject, what I finally did. Honestly, I did not know what I would be facing, but once the project is over, I'm glad to say that my joy for this branch is greater than then. One of the reasons that made me stay curious about it is the fact that the problem which I study in this project, the convergence of Fourier series, is still open. My interest has been reinforced by my director's attitude towards my lacks and difficulties; instead of ignoring them, she gave me the tools to overcome them by myself. For instance, when she found out that I had not taken *Anàlisi Real i Funcional* course, which I needed to develop this project, she encouraged me to study it by myself. Even though I could not include this part of my work in the report, I also enjoyed it.

The task of bibliographic collection has been a double-edged sword. On the one hand, it has been quite frustrating not to instantly find every result that I needed to check. One time, this matter stopped me from advancing for some days. Furthermore, the lack of a main source to look up forces you to compare the different versions of the authors about a similar subject. On the other hand, this is an essential ability in the professional world, either at research or at teaching, and I have improved my search skills thanks to this project.

To sum up, my experience has been positive. I found what I currently consider my favourite branch of mathematics, to which I am looking forward to dedicate more than this project.

Appendices

Appendix A

Real and Functional Analysis

In this appendix I include the definitions and the studied results that could not be added to the main body of the project but are required at some point.

A.1 Measure theory and the integral with respect of a measure

Theorem A.1. *Let $f : X \to \mathbb{R}^+$ a measurable function. Then, there exists an increasing sequence $\{s_n\}_n$ of simple, non negative functions such that $s_n(x) \to f(x)$ for all $x \in X$.*

Theorem A.2. *Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing function. The exterior measure associated to the elemental function $\beta(a, b] := F(b) - F(a)$ is a metric outer measure.*

- *If, in addition, F is right-continuous, the corresponding measure is called the Lebesgue-Stieltjes measure, and it satisfies*

$$\mu(a, b] = F(b) - F(a) \text{ and } \mu(\{a\}) = F(a) - F(a^-).$$

- *If $F(x) = x$, the measure is called the Lebesgue measure.*

Definition A.3. *A measure is said to be:*

- *Outer regular if, for any measurable set E,*

$$\mu(E) = \inf\{\mu(U); E \subset U, U \text{ is open }\}.$$

- *Inner regular measure if, for any measurable set E,*

$$\mu(E) = \sup\{\mu(K); K \subset E, K \text{ is compact }\}.$$

- *Regular if it is inner and outer regular.*

Theorem A.4. *Every Lebesgue-Stieltjes measure is regular.*

Theorem A.5 (Monotone Convergence Theorem). *Let f be a positive measurable function and $\{f_n\}_n$ a sequence of measurable functions such that $f_n \leq f_{n+1}$ for every n. If $f(x) = \lim_n f_n(x)$ a.e. x, then*

$$\int f d\mu = \lim_n \int f_n d\mu.$$

This is the increasing version of the theorem. The analogous decreasing version also holds.

Lemma A.6 (Beppo-Levi). *Let $\{f_n : E \to \mathbb{R}\}_n$ be a sequence of non negative, integrable functions. If*

$$\sum_{n=1}^{\infty} \int_E f_n(t)dt < +\infty,$$

then there exists an integrable, everywhere finite function f such that

$$f(t) = \sum_{n=1}^{\infty} f_n(t) \text{ a.e. } t.$$

A.2 Banach Spaces

Definition A.7. *A normed space is a vector space with a norm.*

Definition A.8. *A Banach space is a normed, complete space.*

Definition A.9. *Given a measure μ, we define the Lebesgue spaces $\mathcal{L}^p(\mu)$ for each $1 \leq p \leq \infty$ as follows:*

- *For all $1 \leq p < \infty$, $\mathcal{L}^p(\mu) := \{f \text{ measurable} \,;\, ||f||_p < \infty\}$, where*

$$||f||_p := \left(\int_X |f(x)|^p d\mu(x) \right)^{1/p}.$$

- *$\mathcal{L}^\infty := \{f \text{ measurable} \,;\, ||f||_\infty < \infty\}$, where*

$$||f||_\infty := \inf \{M > 0 \,;\, |f(x)| \leq M \text{ a.e. } x\}$$

is the infinity norm, also-known as the supreme norm or the Chebyshev norm. According to this definition, one can write $\mathcal{L}^\infty(\mu) = \{f \text{ measurable} \,;\, f \text{ bounded a.e. } x\}$.

Theorem A.10. *For all $1 \leq p \leq \infty$, $(\mathcal{L}^p, || \cdot ||_p)$ is a Banach space.*

Theorem A.11 (Hölder's inequality). *Let $f \in \mathcal{L}^p$ and $g \in \mathcal{L}^q$ such that $\frac{1}{p} + \frac{1}{q} = 1$ (if $p = 1$, $q = \infty$ and conversely). Then, $fg \in \mathcal{L}^1$ and:*

$$\int_X |f(x)g(x)|d\mu(x) \leq \left(\int_X |f(x)|^p d\mu(x) \right)^{1/p} \cdot \left(\int_X |g(x)|^q d\mu(x) \right)^{1/q}.$$

Definition A.12. *Let E, F be vector spaces over a field K. A linear operator is a K-linear function from E to F.*

Definition A.13. *Let $T : E \to F$ be a linear operator. Its norm is defined as*

$$||T|| := \sup_{||x||_E \leq 1} ||Tx||_F,$$

if such supreme exists. In this case, T is said to be bounded.

Theorem A.14 (Uniform boundedness principle). *Let E be a Banach space and F a normed space. Consider the family of linear and bounded operators $\mathcal{F} = \{T : E \to F\}$. If, for every $x \in E$, the set $\{||Tx||_F \mid T \in \mathcal{F}\}$ is bounded, then the set $\{||T|| \mid T \in \mathcal{F}\}$ is also bounded.*

Lemma A.15. *Let $f \in \mathcal{L}^p$ for some $1 \leq p < \infty$. Then,*

$$\lim_{h \to 0} \int |f(x+h) - f(x)|^p dx = 0.$$

Appendix B

Other basic results and definitions

Here we include well-known results or results which are below the level of this project but that are needed at some point.

Theorem B.1 (Weierstrass M-test)**.** *Let* $\sum_n f_n$ *be a function series and* $\sum_n M_n$ *a convergent numerical series such that* $M_n \geq 0\,\forall\, n$*. If* $|f_n(x)| \leq M_n$ *for all* $x \in D$ *and every* n*, then the function series converges absolutely and uniformly on* D*.*

Theorem B.2. *Let* $\{f_n\}_n$ *be a sequence of continuous functions. If there exists* f *such that* $f_n \rightrightarrows f$*, then* f *is continuous.*

Theorem B.3. *Let* $f : (a, b) \to \mathbb{R}$ *be a differentiable function. Then, there exists* $c \in (a, b)$ *such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Definition B.4. *Let* $f : [a, b] \to \mathbb{R}$*. For every partition* $\tau \in \mathcal{P}$*, let*

$$v(f, \tau) := \sum_{k=1}^{n} |f(\tau_k) - f(\tau_{k-1})|.$$

The total variation of f *on* $[a, b]$ *is*

$$V_a^b(f) := \sup_{\tau \in \mathcal{P}[a,b]} v(f, \tau).$$

We say f *is of bounded variation on* $[a, b]$ *if* $V_a^b(f) < \infty$*.*

Lemma B.5. *Let* $f \in \mathcal{L}$ *be a* 2π*-periodic function of bounded variation on* $[0, 2\pi]$*. Then,*

$$\forall\, x \in [0, 2\pi],\ \forall N \geq 0,\ |S_N f(x)| \leq \sup_{y \in [0, 2\pi]} |f(y)| + V_0^{2\pi}(f).$$

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