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Classification of Periodic Fatou Components for Rational Maps

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Contents

Introduction

Complex dynamics is concerned with the long term behaviour of the iterates of a given holomorphic function f,

$$f^n := f \circ \overset{n)}{\cdots} \circ f$$

when applied to different initial conditions. In this work we will restrict to rational functions, defined on the extended complex plane $\mathbb{C} \cup \{\infty\}$, the Riemann Sphere. These type of dynamical systems appear in many different contexts, for example as complexifications of real or interval dynamics, or as root finding algorithms like Newton's method in the complex setting.

The dynamics of rational maps on the Riemann sphere is one of the most beautiful and well-known topics in complex dynamics. As holomorphic functions, they can be treated both as power series from the analytic point of view, and as locally conformal mappings from the geometric one. Therefore all powerful tools from complex analysis can be used, gathering the work of many analysts, geometers, topologists and algebraists over the time.

Its origins date back to the manuscripts of Pierre Fatou and Gaston Julia submitted independently to the 1918 Grand Prize in Mathematics promoted by the French Academy of Sciences. Both of them were inspired a few months before by the theory of normal families of the also French mathematician Paul Montel. They realized that sequences of functions in Montel's work could be treated as iterates of a certain map, hence the complex plane could be partitioned into a normality and non-normality sets, nowadays known as Fatou and Julia sets, respectively, with stunning topology and dynamics. Actually the Julia set corresponds to those points with a chaotic behaviour after many iterations, and the Fatou set, as its complementary, can be seen as the set of stable or well-behaved points. Fatou focused his attention on what he called singular domains, since he believed in the existence of components of the normality set that were not in a domain of attraction for a periodic cycle, although it was always wrongly refused by Julia. A few years later Cremer noticed that these hypothetical domains for rational maps should be doubly connected, anticipating Herman rings. In fact he pointed out that the local behaviour near periodic points was not always clear. Later on, many important mathematicians such as Siegel, Arnol'd, Herman or Baker contributed on the topic, but there were many unanswered questions left and gradually the subject lost its interest during the next sixty years.

Fortunately in the early 1980s powerful computer graphics of complex dynamical systems, as well as the fractal geometry and its applications introduced by Mandelbrot, revitalised the study of rational maps. There was a strong interest in the final fate or long-term behaviour of the points on the Fatou and Julia sets, as well as in the properties of their components. At the same time one of the greatest and the most revolutionary contributions in the complex dynamics field was brought to light by Sullivan's No-wandering domains Theorem, a wonderful solution to the old Fatou's conjecture using the recent quasiconformal surgery theory. This result was pleasantly enhanced by the Classification Theorem of periodic domains already developed by Fatou and Cremer, and by some theorems provided by Siegel, Arnol'd and Herman on the existence of rotation domains. This was complemented recently with the Shishikura's sharp inequality for the number of periodic cycles in the Fatou set proved thanks to the relation between critical points and Fatou components, using quasiconformal surgery for his Master's thesis in 1987. All these important results, which were developed throughout the last century, configure a very complete and simple description of the dynamics of rational maps.

The central topic of the current work is the Classification of periodic Fatou components for a rational map into the five possibilities: attractive basins, superattractive basins, parabolic basins, Siegel discs and Herman rings. Several versions of this central theorem have appeared [7, 9, 12] using alternative techniques. Our intention is to give a clear, precise and self-contained proof of this theorem, based on key tools from complex analysis, iteration theory and hyperbolic geometry. For this purpose, these topics are presented in a coherent and reasonable manner. We have made an effort to show the proofs in a way as simple as possible by gathering geometric and analytic ideas in the literature. Important information and detailed exposition of theses subjects as well as further topics in complex dynamics are included in the books of Milnor [9], Beardon [2], Steinmetz [12] or Keen and Lakicv [6], for example.

This work is divided into four chapters:

In chapter 1 we present briefly some definitions and basic results from complex analysis in one dimension that we will be required. The main concepts introduced here are conformal mappings, covering maps and connectivity related to relevant theorems such as the Riemann Mapping Theorem, the Uniformization Theorem or the Riemann-Hurwitz Formula. Some background reading in complex analysis is needed, see for example Ahlfors [1].

In chapter 2 we make an introduction to hyperbolic geometry. We define the hyperbolic metric on the open unit disk and we transfer it to other hyperbolic domains. We study the hyperbolic metric and geodesics on the special case of doubly connected sets. Moreover, we proof the Schwarz-Pick Lemma, a key theorem in these notes.

In chapter 3 we show some basic results and statements about rational functions. We introduce the notion of normality and the Montel's theorem, very important in the Fatou-Julia theory. At this point, we give a formal definition of the Fatou and Julia sets and we prove some relevant properties of them that will be needed.

Finally, in chapter 4 we study the five types of periodic Fatou components by the Classification Theorem for rational functions. This, together with some statements on connectivity, as well as with the No-Wandering Domains Theorem and the Shishikura's inequality, although they are beyond of the current work's scope, allow us to present a complete description of the stable dynamics of rational maps.

Chapter 1

Preliminaries on Complex Analysis

In this chapter we present briefly some definitions and basic results from complex analysis in one dimension that we will need in other sections. We assume the reader has some familiarity with the algebra and geometric representation of complex numbers, and with basic theorems such as the Cauchy formula, the principle of analytic continuation or the maximum modulus principle. As background reading in complex analysis, see for example [1].

1.1 Conformal mappings

We start considering a complex-valued function of one complex variable $f : \Omega \to \mathbb{C}$ in a certain domain $\Omega \subset \mathbb{C}$.

By a domain, we mean an open non-empty and connected subset of $\mathbb{C}$. Recall that a subset of $\mathbb{C}$, or any topological space, is connected if it is not the union of two disjoint open sets. This may be a hard property to verify but luckily, in the complex plane, an open subset is connected if and only if it is pathwise connected, that is any two points in the subset can be joined by a continuous path completely contained in the subset.

In the purpose to extend the tools from calculus to complex analysis, we need the notion holomorphic functions (also called complex differentiable functions), the only ones that can be freely differentiated and integrated in the complex sense.

Definition. A function $f : \Omega \to \mathbb{C}$ is *differentiable* at $z_0 \in \Omega$ if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. In that case, the limit is called the *derivative* of f at z_0 and is denoted by $f'(z_0)$. We say that f is *holomorphic* in Ω if it is differentiable at every point in Ω.

We note that the common differential rules such as the quotient rule or the chain rule are also valid in this context. But the complex differentiability condition is stronger because it implies an approach to the limit from all directions in the complex plane, while in the real case there is only the positive and negative ones.

One important property of holomorphic functions is their analyticity in the complex plane, so that every holomorphic function in a neighbourhood of any point $z_0 \in \Omega$ can be expanded as a convergent power series:

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0),$$

where the coefficients can be given by the Cauchy integral formula, or as the coefficients of the Taylor series since analytic functions are infinitely differentiable in their domains.

Alternatively, from the geometric point of view, a holomorphic function can be treated as a conformal mapping.

Definition. We say that a holomorphic function $f : \Omega \to \mathbb{C}$ is a *conformal mapping* if $f'(z) \neq 0$ for all $z \in \Omega$.

We observe that it is locally injective and preserves oriented-angles between smooth curves. This is due to the behaviour of the function near a point z_0 with non-zero derivative:

$$f(z) - f(z_0) \sim f'(z_0)(z - z_0).$$

Then clearly f scaled the difference $z - z_0$ by a factor $|f'(z_0)|$ and rotate it an angle $\arg(f'(z_0))$.

Conformal mappings theory is extremely rich due to its geometric background. In fact, one powerful example of that is the following classical Schwarz's Lemma, which we are going to use several times through these notes. Its proof is a direct application of the Maximum Modulus Principle.

Theorem 1.1. (Schwarz's Lemma) *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic, and let $f(0) = 0$. Then $|f'(0)| \leq 1$. Moreover, one of the following conditions hold:*

(1) $|f'(0)| < 1$, *and* $|f(z)| < |z|$ *for every* $z \neq 0$.

(2) $|f'(0)| = 1$, *and* $f(z) = e^{i\theta}z$ *for some* $\theta \in \mathbb{R}$.

It essentially says that a holomorphic function from the disc onto the disc that fixes the origin is either a contraction near the origin or a rotation about the origin. In the second case, f is called a conformal automorphism of the unit disc that fixes the origin. Even though such statements are correct in the Euclidean sense, they are invariant under conformal mappings only when we introduce the hyperbolic metric, as we will do in Chapter 2.

Another important result on conformality is the following one that let us transfer the study of the dynamical behaviour of a certain domain to the well-known unit disk $\mathbb{D}$ via a conformal homeomorphism called the Riemann map ϕ.

Theorem 1.2. (Riemann Mapping Theorem) *Let Ω be a domain in $\mathbb{C}$, and let $z_0 \in \Omega$. Then there is a conformal homeomorphism $\phi : \Omega \to \mathbb{D}$ such that $\phi(z_0) = 0$. Moreover, if we impose that $\phi'(z_0) > 0$, then the map ϕ is unique.*

In fact, this is a particular case of the Uniformization Theorem that we will present later,

We would like to recall the definition of some class of maps that we will use in these notes to avoid confusions. An invertible map $f : X \to Y$ between topological spaces is a *homeomorphism* if f and f^{-1} are continuous. In that case, X and Y are called equivalent or homoeomporhic, denoted as $X \cong Y$, such as a donut and a coffee cup. A homeomorphism is called a holomorphic diffeomorphism if f and f^{-1} are holomorphic. Finally a holomorphic diffeomorphism is called conformal or conformal homeomorphism if it is conformal in every point of the domain. In that way, in general we will often use conformal maps such as globally bijective maps by abuse of notation.

Usually we will consider maps defined on domains, but this can be generalized to the notion of Riemann surfaces, i.e. connected complex analytic manifolds of complex dimension one. Roughly speaking, a Riemann surface is a space that is locally equivalent to the complex plane via a proper change of coordinates. Another example is the Riemann sphere that we will introduce in next section.

1.2 Riemann sphere

In complex dynamics, it is useful to consider the ∞ just another point since it has associated a dynamical behaviour such any other point of the plane. With this intention in mind, we consider the extension of the complex plane $\mathbb{C}$ to the *Riemann sphere* by adding the point at infinity, that is

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

In that way, the neighbourhood of ∞ are the complements of the compact subsets of $\mathbb{C}$.

In order to get a natural geometrical idea of this construction, we may use the stereographic projection to map the complex plane $\mathbb{C}$ into the Euclidean sphere

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

The stereographic projection π simply projects each point $z \in \mathbb{C}$ linearly towards (or away from) the north pole, i.e. the point $N = (0, 0, 1)$, such that the ray between z and N intersects the sphere at the point $\pi(z)$, which can be defined in Cartesian coordinates as

$$\pi(z) = \left(\frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right),$$

where $x = Re(z)$ and $y = Im(z)$ are the real and imaginary part of $z \in \mathbb{C}$, respectively.

In fact, points outside the unit disc $\mathbb{D} \subset \mathbb{C}$ map to the northern hemisphere, and points inside $\mathbb{D}$ to the southern hemisphere. We note that the origin $0 \in \mathbb{D}$ is projected to the south pole $S = (0, 0, -1)$. Then we can identify the point at ∞ with the north pole since points far away from the origin are mapped really close to it.

The *chordal distance* on $\hat{\mathbb{C}}$ is given by the Euclidean length of the chord joining $\pi(z)$ to $\pi(w)$, which is defined as:

$$\sigma(z,w) = |\pi(z) - \pi(w)| = \frac{2|z-w|}{(1+|z|^2)^{1/2}(1+|w|^2)^{1/2}}.$$

The chordal metric represents a more useful metric for the points in $\hat{\mathbb{C}}$. In fact, when $w = \infty$, passing to the limit, we have

$$\sigma(z,\infty) = \frac{2}{(1+|z|^2)^{1/2}}.$$

In particular $\sigma(0,\infty) = 2$ as we can observe from the figure. Therefore we can use the chordal metric on $\hat{\mathbb{C}}$ to work with the point ∞ as we do with the Euclidean metric for points on the complex plane.

Now we define an equivalent metric that we will commonly use, for example, to handle convergence properties on $\hat{\mathbb{C}}$:

Definition. The *spherical distance* $d_{\hat{\mathbb{C}}}(z,w)$ is the Euclidean length of the shortest path on $\mathbb{S}^1$ (an arc of a great circle) between two points $\pi(z)$ and $\pi(w)$ of the sphere.

The equivalence is easy to check. If the chord joining these two points in the unit sphere subtends an angle $\theta \in [0, 2\pi)$ at the origin, then

$$d_{\hat{\mathbb{C}}}(z,w) = \theta.$$

From the picture, we deduce the relation $\sigma = 2\sin(\theta/2)$. Therefore the spherical metric and the chordal metric on $\hat{\mathbb{C}}$ are equivalent since

$$\frac{2}{\pi} d_{\hat{\mathbb{C}}}(z,w) \le \sigma(z,w) \le d_{\hat{\mathbb{C}}}(z,w)$$

for all $z, w \in \hat{\mathbb{C}}$.

A really special and useful property of $\hat{\mathbb{C}}$ is that the holomorphic functions on it are just the rationals maps.

Proposition 1.3. *$f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is holomorphic if and only if f is a rational function, that is*

$$f(z) = \frac{p(z)}{q(z)}$$

for all $z \in \hat{\mathbb{C}}$, where $p(z)$ and $q(z)$ are polynomials with complex coefficients.

This can be shown by the Fundamental Theorem of Algebra-

1.3 Möbius transformations

Linear fractional or Möbius transformations are a type of rational maps that will be very useful throughout these notes.

Definition. A *Möbius transformation* is a map of the form

$$f(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

We observe that if $c \neq 0$,

$$f(\infty) = \frac{a}{c}, \qquad f(-\frac{d}{c}) = \infty$$

Otherwise $f(\infty) = \infty$.

We can easily compute the following relations that will be useful later:

$$f'(z) = \frac{ad-bc}{(cz+d)^2}$$

$$Im(f(z)) = \frac{ad-bc}{|cz+d|^2} Im(z)$$

$$Re(f(z)) = \frac{(ac|z|^2+bd)+(ad+bc)Re(z)}{|cz+d|^2}$$

$$f^{-1}(w) = \frac{dw-b}{-cw+a},$$

where $w = f(z)$ is a preimage of z.

The importance of the Möbius transformations in complex analysis is reflected in the following statement:

Proposition 1.4. *The conformal automorphisms of $\hat{\mathbb{C}}$ is equal to the group of Möbius transformations*

$$R(z) = \frac{az+b}{cz+d},$$

where the coefficients are complex numbers with $ad - bc \neq 0$.

There are four special types of conformal automorphism that should be listed:

Proposition 1.5. *Every Möbius transformation is composition of the 4 basic homographies:*

- *Translation:* $T_b(z) = z + b, \quad b \in \mathbb{C}$
- *Rotation:* $R_\theta(z) = e^{i\theta} z, \quad \theta \in \mathbb{R}$
- *Homothecy:* $H_a(z) = az, \quad a \in \mathbb{R}$

- *Inversion:* $I(z) = \frac{1}{z}$

In fact Möbius transformations are determined just by three distinct points:

Proposition 1.6. *Let $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ be three different points. If $w_1, w_2, w_3 \in \hat{\mathbb{C}}$ are three other different points, then there is a unique Möbius transformation such that $f(z_i) = w_i$. In particular, we can suppose $f(z_1) = 0$, $f(z_2) = 1$ and $f(z_3) = \infty$.*

Moreover, an Euclidean circumference of the Riemann sphere is either a straight line in the plane, if it passes through the pole north, i.e. the point ∞, or a circumference in the complex plane. Since three distinct points of the Riemann sphere determine a unique element of the set of all Euclidean circles of $\hat{\mathbb{C}}$, and. Since each generator of the Möbius group preserves this family Ci, it follows that the same hold for all elements of the Möbius group.

Proposition 1.7. *Möbius transformations send circles in $\hat{\mathbb{C}}$ to circles in $\hat{\mathbb{C}}$ (a circle through ∞ is a straight line in $\mathbb{C}$).*

1.4 The Uniformization Theorem

In order to give the central Uniformization Theorem as general as possible, we may give some definitions for Riemann surfaces that, of course, include every domain and the Riemann sphere.

A Riemann surface S is simply connected if any closed curve in it is homotopic to a constant curve, that is it can be continuously deformed to a point. It follows that S is simply connected if and only if its complement is connected, otherwise there will be a hole inside the surface. In that case, the connected component of a point $z \in S$ is the largest connected subset of S that contain z.

Since the connectivity number is a conformal invariant, there is not conformal map from the unit disc onto a multiply connected domain. In that case, we must replace the conformal map by so-called universal cover map in order to get a generalization of the Riemann Mapping Theorem that classifies simply connected domains (or Riemann surfaces).

Definition. Let R and S be Riemann surfaces.

A covering map $\pi : S \to R$ is a *covering map* if every point $p \in R$ has a neighbourhood $U_p \subset R$ such that $\pi^{-1}(U_p) \subset S$ is a disjoint union of open sets, each of which is mapped one-to-one by π onto U_p. S is called a covering space or cover of R.

A covering automorphism or *deck transformation* is a conformal map $\phi : S \to S$ such that $\pi(\phi(p)) = \pi(p)$ for every point $p \in S$.

Therefore the deck transformation allows us to map each connected component of $\pi^{-1}(U)$ conformally to another one that lies on the same Riemann surface S.

Definition. Let R and S be Riemann surfaces. S is *the universal covering space* of R if S is simply connected and $\pi : S \to R$ is a covering map (called the universal covering map).

Then the universal covering is unique up to homeomorphisms, and we denote it as $\tilde{R}$.

Any domain having at least three boundary points is called hyperbolic. In the simply connected case the Uniformization Theorem is nothing other than the Riemann Mapping Theorem.

By using the concept of the universal covering space, we can generalize the Riemann Mapping Theorem to non-simply connected domains. It was proved by Poincaré, Klein and Koebe. A more recent proof is in [*Keen*], for example.

Theorem 1.8. *The universal covering space $\tilde{U}$ of a Riemann surface U is conformally equivalent to either $\hat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{D}$.*

In particular, the universal covering of a hyperbolic surface is $\mathbb{D}$.

A domain or a Riemmann surface U is called hyperbolic if its complementary $\hat{\mathbb{C}} \setminus U$ contains at least three points. By the Uniformization theorem we can study Riemann surfaces by its universal covering space. In fact, this will be key to transfer the hyperbolic metric between different domains as we wiil show in next chapter.

Before that, we present a powerful tool that relate the connectivity of the input and output domains of a proper map, that is $d - to - 1$ map, where $d > 0$ is its topological degree. For example, let $f : D_1 \xrightarrow{d:1} D_2$ is a proper map, that is every point in D_2 has exactly d preimages in D_1. If D_1 and D_2 have connectivity numbers n_1 and n_2, respectively, then $n_2 \leq n_1 \leq dn_2$. In particular, if f is a conformal mapping ($d = 1$ since is locally one-to-one), then $n_1 = n_2$, that is, the connectivity number is a conformal invariant. The Riemann-Hurwitz formula may be regarded as a generalization of this fact.

Theorem 1.9. (Riemann-Hurwitz Formula) *Let f be a proper (holomorphic) map of degree d of some n_1-connected domain D_1 onto some n_2-connected domain D_2. Suppose f has exactly n_c critical points in D_1, including/counting multiplicity. Then*

$$n_1 - 2 = d(n_2 - 2) + n_c.$$

The proof will require to apply the Euler's formula to suitable triangulations of D_j. See [12].

Chapter 2

Hyperbolic Geometry

The hyperbolic geometry is a powerful tool in many mathematical and physical fields. Everyone is comfortable with the Euclidean geometry but we have to get familiar with the hyperbolic one because it can be very really useful when we are working with conformal mappings.

In this chapter we introduce the hyperbolic metric on $\mathbb{D}$ and we transfer it to other hyperbolic domains via the Uniformization Theorem. Then we prove the Schwarz-Pick Lemma, a key statement for the rest of this work. Finally, we present the special case of doubly connected domains and a proposition that compare hyperbolic and spherical distance, which will be needed to understand the last part of the Classification Theorem in Chapter 4.

2.1 Hyperbolic metric

Let $\Omega \subset \mathbb{C}$ be a domain. As a vector space, $\mathbb{C}$ has the standard $\mathbb{R}^2$ Euclidean metric $|dz|$ that defines infinitesimal displacements in the tangent space of any point. In fact, we can think $\mathbb{C}$ both as a topological space or as a a vector space. A Riemannian metric consists of an inner product that varies smoothly on the tangent space of each point. Thus it let us attribute length to tangent vectors.

Definition. A *conformal metric* is a Riemannian metric that is invariant under local Euclidean rotations, that is the length of tangent vectors to a point remain constant in a small neighbourhood. In that case, the metric is of the form $ds = \rho(z)|dz|$, where ρ is a smooth, positive function on Ω.

We will use the density function ρ to denote the metric in question by abuse of notation. Note they are locally equivalent to the Euclidean or chordal metric.

Definition. A conformal metric ρ_Ω on $\Omega \subset \hat{\mathbb{C}}$ is conformally invariant if

$$\rho_\Omega(f(z))|f'(z)||dz| = \rho_\Omega(z)|dz|$$

for every conformal automorphism of Ω.

In the same way, we define isometries.

Definition. If ρ_U and ρ_V are conformal metrics on two domains $U, V \subset \hat{\mathbb{C}}$ is an *isometry* if it sends tangents vectors to other tangents vectors of the same length, that is

$$\rho_V(f(z))|f'(z)v| = \rho_U(z)|v|$$

for every point $z \in U$ and vector $v \in \mathbb{C}$.

We can say a lot about the geometry of a metric space:

Definition. Let $\gamma : [0,1] \to \Omega$ be a smooth path joining z and w in Ω. The ρ_Ω-*length* of γ is defined as

$$l_\Omega(\gamma) = \int_\gamma \rho(\gamma(t))|\gamma'(t)|dt.$$

The ρ_Ω-*distance* from z to w is defined as

$$d_\Omega(z,w) = \inf_\gamma \rho_\Omega(\gamma),$$

where the infimum is taken over all possible smooth curves from z to w in Ω.

A *geodesic segment* in Ω is locally the shortest smooth path γ between two points.

A geodesic line, or simply a geodesic, is the same of a geodesic segment in Ω but globally. We also say that a closed geodesic is such that $\gamma(0) = \gamma(1)$ with the same tangent vector at this endpoint, and a simple closed geodesic is the one without self-intersections. One can prove that geodesics are solutions of a certain second-order ordinary differential equation. From the existence and uniqueness theorem for EDOs it follows that given $z \in \Omega$ and a vector $v \in \mathbb{C}$ there exits a unique geodesic passing through z and whose tangent vector at this point is v. Clearly isometries preserve the length of any path, so they map geodesics into geodesics. Moreover, we have a special metric with negative curvature:

Theorem 2.1. (Hyperbolic metric) *There exists a Riemannian metric on $\mathbb{D}$ whose isometry group is the one of conformal automorphisms of $\mathbb{D}$, which is conformal, complete (geodesics with infinite length) and with constant negative curvature. In fact, there is a unique such metric with curvature -1. The geodesics of this metric are circles orthogonal to the boundary.*

Such Riemannian metric on $\mathbb{D}$ is called the hyperbolic metric or Ponincaré metric on $\mathbb{D}$, which we are going to determinate in the following section. For a proof, see [*Well*].

2.2 The Poincaré metric on $\mathbb{D}$

Proposition 2.2. *The conformal automorphisms of $\mathbb{D}$ are maps of the form*

$$f_{\theta,a}(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z}, \quad a \in \mathbb{D},\ \theta \in \mathbb{R}$$

Proof. Since f is a conformal automorphism of $\mathbb{D}$, we may assume $f^{-1}(0) = a \in \mathbb{D}$. Then the Möbius transformation

$$g(z) = \frac{z-a}{1-\bar{a}z}$$

sends a to 0 and the unit circle to itself, so it sends $\mathbb{D}$ to itself. Thus $f \circ g^{-1}$ is a conformal automorphism of $\mathbb{D}$ sending 0 to 0. From the classical Schwarz's Lemma it follows that $f \circ g^{-1}$ is a rotation around 0, that is $f \circ g^{-1}(z) = \lambda z$ such that $|\lambda| = 1$, where $w = g(z)$. □

Proposition 2.3. *There is a unique (up to multiplication by positive real numbers) conformally invariant metric on* $\mathbb{D}$*, given by*

$$ds = \frac{2|dz|}{1-|z|^2}.$$

This is a very special metric called the *Poincaré metric*. By the Riemann Mapping Theorem, we can transfer the Poincaré metric to any proper, simply connected domain in $\mathbb{C}$. In fact, the Poincaré metric can be lifted to any simply connected hyperbolic domain by a conformal map. Moreover, we will discuss later that this is also possible for arbitrary hyperbolic domains by the Uniformization Theorem, although there is not a conformal map.

In fact, the hyperbolic metric on $\mathbb{D}$ can be lifted to a corresponding metric on any Riemann surface whose universal cover is $\mathbb{D}$. In general, if $\pi : \mathbb{D} \to X$ is a covering map, then we can push forward and backward the hyperbolic metric on $\mathbb{D}$ to get a metric on X that is locally isometric to the hyperbolic metric, which obviously is called the hyperbolic metric on X. But given a point $z \in X$, there is an ambiguity in the choice of the preimage of z to copy the metric from. However, this is an easy problem to solve with the hyperbolic metric on X because given any two preimages of z, there is a deck transformation $\phi : \mathbb{D} \to \mathbb{D}$ in such a way that the hyperbolic metric on $\mathbb{D}$ remains invariant as we defined in last chapter.

We note that although all geodesics in $\mathbb{D}$ go off to infinity, that is to $\partial\mathbb{D}$, a hyperbolic surface X can have closed geodesics that are projections to X of any geodesic of $\mathbb{D}$ passing through two points with the same image under the corresponding covering map π.

2.3 The upper half plane model $\mathbb{H}$

We may transfer the Poincaré metric on $\mathbb{D}$ to a hyperbolic metric on the upper half plane $\mathbb{H}$, which will be helpful to made some computations on hyperbolic distances in an easy way.

Proposition 2.4. *The conformal automorphism of* $\mathbb{H}$ *are the Möbius transformation of the form*

$$f(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{R}$ *and* $ad - bc = 1$.

Proof. The Riemann conformal map from $\mathbb{D}$ onto $\mathbb{H}$ is

$$h(z) = i\frac{1+z}{1-z}$$

and its inverse, from $\mathbb{H}$ onto $\mathbb{D}$,

$$h^{-1}(w) = \frac{w-i}{w+i}.$$

The conformal automorphisms of $\mathbb{H}$ should be of the form $f = hgh^{-1}$, where g is any conformal automorphism of $\mathbb{D}$, which we have explicitly found in last section. Then we can substitute the explicitly form of g from last preposition, and we get a conformal automorphism of $\mathbb{H}$ with the same form f given in the statement. □

Proposition 2.5. *There is a unique (up to multiplication by a positive constant) conformally invariant metric on* $\mathbb{D}$*, given by*

$$ds = \frac{|dz|}{Im(z)}.$$

Moreover, the geodesic lines of $\mathbb{H}$ *in its hyperbolic metric consist of circles or lines that are orthogonal to* $\partial\mathbb{H}$*.*

Since $\mathbb{H}$ is conformally equivalent to $\mathbb{D}$ via the conformal mapping given in last preposition, we can transfer the Poincaré metric on $\mathbb{D}$ to a unique Riemannian metric on $\mathbb{H}$ for which the above Möbius transformation are isometries. Alternatively, since translations and multiplication by real numbers are isometries of the hyperbolic metric, we can also deduce the same formula of the metric density $\rho_{\mathbb{H}}$.

Moreover, it is easy to check that the vertical lines are geodesics on $\mathbb{H}$. therefore, using the the metric density $\rho_{\mathbb{H}}$, we deduce the hyperbolic distance between two points in the imaginary axes, $z = ai$ and $w = bi$ ($b > a > 0$) is given by

$$d_{\mathbb{H}}(z, w) = |\log{(b/a)}|$$

From this we can derive a nice formula for the hyperbolic distance between any two points. First, let us consider the cross-ratio of four distinct points in the Riemann sphere, which is given by the expression

$$Cr(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_4)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z_4)}.$$

It can be easily proven that the cross ratio is preserved by the generators of the Möbius group and, consequently, by any Möbius transformation.

At this point, the hyperbolic distance on $\mathbb{H}$ between the points ai and bi correspond to

$$d_{\mathbb{H}}(ai, bi) = \log(1 + Cr(0, ai, bi, \infty)).$$

Since any geodesic can be mapped to this vertical line by a proper isometry, we deduce by the invariance of the cross-ratio that the hyperbolic distance between any two point $z, w \in \mathbb{H}$ is equal to

$$d_{\mathbb{H}}(z, w) = \log(1 + Cr(z_\infty, z, w, w_\infty))$$

where z_∞ and w_∞ are the points at infinity associated to the endpoints of the geodesic between z and w in $\mathbb{H}$. Since all factors in this formula are invariant under conformal automorphisms by constructions of the hyperbolic metric, we have also successfully get the distance between any two points in $\mathbb{D}$.

2.4 The Schwarz-Pick Lemma

The classical Schwarz's Lemma is only invariant under conformal maps when they are interpreted in terms of hyperbolic geometry instead of Euclidean ones. in fact, in 1915 Pick stated a conformal invariant reformulation of Schwarz's Lemma. Moreover, he note that in this case the requirement that f has a fixed point in $\mathbb{D}$ is redundant.

Theorem 2.6. (Schwarz-Pick Lemma) *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic, and let $d_\mathbb{D}$ be the hyperbolic distance on $\mathbb{D}$. Then one of the following conditions hold:*

(1) f is a hyperbolic contraction, i.e., for any two different points $z, w \in \mathbb{D}$, we have

$$d_\mathbb{D}(f(z), f(w)) < d_\mathbb{D}(z, w).$$

(2) f is a hyperbolic isometry; that is, f is a conformal automorphism and for all z and w in $\mathbb{D}$,

$$d_\mathbb{D}(f(z), f(w)) = d_\mathbb{D}(z, w).$$

Proof. We distinguish between two cases:

On one hand, if f is an isometry, then we have directly condition (2) since the hyperbolic distance is invariant under conformal automorphisms of $\mathbb{D}$.

On the other hand, suppose f is not an isometry. Here our intention is to use the classical Schwarz Lemma and the fact that the hyperbolic plane is homogeneous. Let $z_0 \in \mathbb{D}$, and let

$$F = h \circ f \circ g,$$

where g and h are conformal automorphism of $\mathbb{D}$ (thus isometries) such that $g(0) = z_0$ and $h(w_0) = 0$ with $w_0 = h(z_0)$, so they are defined as the following Möbius transformations:

$$g(z) = \frac{z + z_0}{1 + \overline{z_0} z}, \qquad h(z) = \frac{z - w_0}{1 - \overline{w_0} z}.$$

Then F is a holomorphic self-map of $\mathbb{D}$ such that $F(0) = 0$ but it is not an isometry since f is not one. Therefore by the case (1) of Schwarz's Lemma, we have $|F'(0)| < 1$ and $|F(z)| < |z|$, or equivalently $d_\mathbb{D}(0, F(z)) < d_\mathbb{D}(0, z)$ for all $z \in \mathbb{D}$. Therefore, as g and h are hyperbolic isometries, and $F\circ$, we have

$$d_\mathbb{D}(f(z_0), f(z)) = d_\mathbb{D}(h \circ f(z_0), h \circ f(z)) = d_\mathbb{D}(F \circ g^{-1}\ (z_0), F \circ g^{-1}(z)),$$

Since $F(0) = 0$ and $|F(z)| < |z|$, or equivalently $d_\mathbb{D}(0, F(z)) < d_\mathbb{D}(0, z)$ for all $z \in \mathbb{D}$, we obtain

$$d_\mathbb{D}(0, F \circ g^{-1}(z)) < d_\mathbb{D}(0, g^{-1}(z)) = d_\mathbb{D}(g^{-1}(z_0), g^{-1}(z)) = d_\mathbb{D}(z_0, z),$$

for all $z_0, z \in \mathbb{D}$, and hence we are in condition (1). □

So if $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic map then either f strictly contracts the hyperbolic metric or it is an isometry.

Let us consider a holomorphic covering map $\pi : \mathbb{D} \to V$. Since the covering automorphism group is a subgroup of the group of Möbius transformations, which is a subgroup of the isometry group of the hyperbolic metric of $\mathbb{D}$, it follows that there exists a unique complete Riemann metric on V that is locally isometric to the hyperbolic metric on $\mathbb{D}$. This is the hyperbolic metric of V. We will show later that any domain of the Riemann sphere whose complement contains at least three points has a hyperbolic.

Theorem 2.7. (General Schwarz-Pick Lemma) *Let $f : R \to S$ be a holomorphic map between two hyperbolic Riemann surfaces. Then one of the following conditions hold:*

(1) f strictly decrease non-zero distances.

(2) f is a local isometry and a covering map. Moreover, if f is injective, then f is a conformal automorphism and a global isometry.

Proof. By the Uniformization Theorem, the universal covering spaces of any hyperbolic Riemann surface is $\mathbb{D}$. Then the lift of f is given by $\tilde{f} : \mathbb{D} \to \mathbb{D}$.

Now we apply the previous Schwarz-Pick Lemma to the lift of f, then $\tilde{f}$ is either a hyperbolic contraction or a hyperbolic isometry. In the first case, it follows that f must also strictly decrease non-zero hyperbolic distances. However, in the second case, f could be either also a covering automorphism or simply a covering map if it is not injective (and hence, a local isometry). □

This generalization of the Schwarz-Pick Lemma is the key tool to develop the proof of the Classification Theorem of Fatou components. Before we continue with the following section, let us present another important proposition for that Classification Theorem that compares the hyperbolic and spherical distances:

Proposition 2.8. *Let $U \subset \hat{\mathbb{C}}$ be a hyperbolic surface (a connected open subset of $\hat{\mathbb{C}}$ which omits at least 3 points). If $z_n \in U$ is a sequence all of whose accumulation points lie on ∂U, as z converge towards the boundary ∂U in $d_{\hat{\mathbb{C}}}$, then for all $r > 0$, the spherical diameter (i.e. the diameter in the spherical distance $d_{\hat{\mathbb{C}}}$) of the hyperbolic closed ball $B_U(z_n, r) = \{z \in U : d_U(z_n, z)\} \leq r$ tends to* 0.

2.5 The Hyperbolic metric on a doubly connected region

As we observed before, unlike the case of simply connected domains, a covering map $f : \mathbb{D} \to \Omega$ from the unit disc onto a multiply connected hyperbolic domain Ω (and hence, f is not injective) is only a local isometry, not a global isometry. That is, each point $z \in \Omega$ has a neighbourhood U such that $f|_U$ is an isometry. In that case, the hyperbolic metric is not invariant under general conformal mappings, but it is locally invariant under covering maps .

Here we present a study of doubly connected domains in $\hat{\mathbb{C}}$. We have that if Ω is a doubly connected region in $\hat{\mathbb{C}}$, then it is conformally equivalent to exactly one of: $\mathbb{C}^*$, $\mathbb{D}^*$, or an annulus $A(r,R)$. But first, we observe that Ω is itself the Riemann sphere with two punctures, and hence it is not hyperbolic, so there is not an associated hyperbolic metric.

In the second case, we have the following properties on $\mathbb{D}^*$:

Definition. The hyperbolic metric on the punctured disk $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$ is given by:

$$\rho_{\mathbb{D}^*} = \frac{|dz|}{|z|\log(1/|z|)}$$

It can be verified using the universal covering map from $\mathbb{H}$ onto $\mathbb{D}^*$, that is $f(z) = e^{iz}$. In that case, we have

$$\rho_{\mathbb{H}} = \rho_{\mathbb{D}^*}(f(z))|f'(z)| = \rho_{\mathbb{D}^*}(e^{iz})|e^{iz}|$$

By change of variable $w = e^{iz}$ and using the hyperbolic metric expression on $\mathbb{H}$, we obtain

$$\rho_{\mathbb{D}^*} = \frac{1}{|w|}\rho_{\mathbb{H}}(i\log(1/w)) = \frac{1}{2|w|\log(1/|w|)}.$$

Note that the points i and $i+\pi$ project down to the points $1/e$ and $-1/e$ under the covering map. Let $\gamma(t)$ be an arc of a circle orthogonal to the real axis, such that $\gamma(t)$ starts at i and ends at $i+\pi$. Set $\hat{\gamma}(t) = \gamma(t) + \pi$. Both $f(\gamma)$ and $f(\hat{\gamma})$ are geodesics in $\mathbb{D}^*$. Moreover, each realizes the distance between their endpoints $1/e$ and $-1/e$. Note we can find other geodesics joining these points that are longer than these and may have self intersections. Thus, in $\mathbb{D}^*$ geodesics joining pairs of points are not unique.

Since each hyperbolic geodesic in $\mathbb{D}^*$ is the image of a hyperbolic geodesic in $\mathbb{H}$ under the covering map $f(z) = e^{iz}$, we have that every radial segment $\gamma_{r,R}$ from $re^{i\theta}$ to $Rei\theta]$ is part of a hyperbolic geodesic. Since the hyperbolic metric density does not depend on θ, we can compute the hyperbolic length of this geodesic segment as

$$l_{\mathbb{D}^*}(\gamma_{r,R}) = \int_{[r,R]} \rho_{\mathbb{D}^*}(z)|dz| = \int_r^R \frac{dt}{t\log t} = \log\left|\frac{\log R}{\log r}\right|.$$

In the case that Ω is conformally equivalent to an annulus, we may deduce its hyperbolic metric by two steps:

(1) We compute the hyperbolic metric for the band strip $L = \{z \in \mathbb{C} : 0 < \mathrm{Im}(z) < \lambda\}$ from the hyperbolic metric on $\mathbb{H}$:

The conformal map $f(z) = \frac{\lambda \log z}{\pi}$ takes the upper half plane $\mathbb{H}$ to L. Therefore, as an isometry we have

$$\rho_{\mathbb{H}}(z) = \rho_L(f(z))|f'(z)| = \rho_L\left(\frac{\gamma \log z}{\pi}\right)\frac{\lambda}{\pi|z|}$$

By change of variable $w = f(z)$ and by the inverse map of f given by $z = e^{\frac{\pi w}{\lambda}}$, we have that

$$\rho_L(w) = \frac{\pi|e^{\frac{\pi w}{\lambda}}|}{\lambda}\frac{1}{2\,\mathrm{Im}(\frac{\pi w}{\lambda})} = \frac{\pi}{2\lambda\sin(\frac{\pi}{\lambda}\,\mathrm{Im}(w))}.$$

(2) Then we calculate the hyperbolic metric on the corresponding annulus:

Note that the conformal map $g(w) = e^{iw}$ maps the strip L to the annulus $A_{e^{-\lambda}} = \{t \in \mathbb{C} : a < |t| < 1\}$. Therefore, for $a = e^{-\lambda}$, we have

$$\rho_{A_a}(g(w))|g'(w)| = \rho_l(w)$$

If we set $t = g(w)$ and substitute we get

$$\rho_{A_a}(t) = \frac{\pi}{2|t|\lambda \sin\left(\frac{\pi}{\lambda}\log(1/|t|)\right)}.$$

We can check that

$$\rho_{A_a} \to \rho_{\mathbb{D}^*}$$

when $\lambda \to \infty$, i.e. the hyperbolic metric on the annulus A_a tends to the one on the punctured disc $\mathbb{D}^*$, as we could have expected.

As we can find in $[McM]$, the modulus of an annulus $A(1, R) = 1 < |z| < R$ is defined by

$$mod(A(1, R)) = \frac{\log(R)}{2\pi}.$$

Then for a general round annulus $A(r, R)$ with $0 < r < R$, the modulus of $A(r, R)$ correspond to

$$mod(A(r, R)) = \log(R/r)$$

This characteristic number is important because it is known that two annuli $A(r_1, R_1)$ and $A(r_2, R_2)$ are conformally equivalent if and only if

$$R_1/r_1 = R_2/r_2,$$

that is, if and only if they have equal moduli.

Moreover, thanks to that modulus we have the following property

Proposition 2.9. *The core curve γ of an annulus $A(1, R)$ of finite modulus is its unique geodesic with hyperbolic length*

$$l_A(\gamma) = \frac{\pi}{mod(A)}$$

as can be checked by considering the circle $|z| = \sqrt{R}$.

The annulus $A(1, R)$ is conformally equivalent to $A(1/R, 1)$. Then for teh annulus in the previous example $A_a = A(a, 1)$, by the change of variable $a = 1/R$, the modulus is $2\pi/\log(1/a)$, and thus scaling, we obtain that the core curve of the annulus A_a in the example is the circumference $|z| = \sqrt{a}$.

Chapter 3

Dynamics of rational functions

In this chapter we present the most important topics on the dynamics of rationals maps in order to define properly the Fatou and Julia sets.

Here we are interested in the iteration of a rational maps $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. The sequence of iterates of such maps is denoted as

$$f, \quad f^2 = f \circ f, \quad ... \quad , \quad f^n = f \circ f^{n-1} = f^{n-1} \circ f, \quad ...,$$

where f^n represent the n-th iterate by composition $\circ$. It should not be confusion with the ordinary power, which would be explicitly written as $(f(z))^n$. By convention, $f^0 = id$.

In section 1 we present the basic definitions and results about rationals maps related to critical points. Then we tall about its fixed and period points in section 2.

In section 3 we introduce the notions of normality and equicontinuity, and then we prove the Montel's theorem, a normality criterion.

Finally, in section 4 we give a formal definition of the Fatou and Julia sets, and some proofs of the properties of these sets thanks to Montel's theorem.

3.1 Rational maps

A *rational map* is a map of the form

$$f(z) = \frac{P(z)}{Q(z)}$$

where $P(z)$ and $Q(z)$ are polynomials with complex coefficients. We may assume without loss of generality that these polynomials are coprime, that is they have no common factors. Then the *degree* of $f(z)$ is defined as

$$d = \deg(f) = \max\{\deg(P), \deg(Q)\}.$$

In this work, we assume $d \geq 2$ since these types of rational maps are the Möbius transformations studied in Chapter 1, and their dynamics are almost trivial. We should remind a useful property of the degree:

$$\deg(f \circ g) = \deg(f)\deg(g) \quad \Rightarrow \quad \deg(f^n) = (\deg(f))^n.$$

We recall that in Chapter 1 we show that rational maps are the only holomorphic maps of the Riemann sphere $\hat{\mathbb{C}}$ to itself, and we consider that f is holomorphic if the map $z \to f(1/z)$ is well-defined and holomorphic.

It is well known that for any point $w \in \hat{\mathbb{C}}$, the set

$$f^{-1}(w) = \{w \in \hat{\mathbb{C}} : f(z) = w\} = \{z_1, ..., z_d\}$$

consists of exactly d elements, counting multiplicity.

A point $w \in \hat{\mathbb{C}}$ for which $f^{-1}(w)$ has less than d different elements is called a *critical value* of f. Then at least one $z_i \in f^{-1}(w)$ will have multiplicity greater than one. In that case z_i is called a *critical point* of f.

In fact, critical points will be zeros of the derivative $f'(z)$ or poles of $f(z)$ with multiplicity greater than one, where f fails to be injective in any neighbourhood of the point. By a *pole* of f we mean a point $z_0 \in \mathbb{C}$ such that $f(z_0) = \infty$.

By direct application of the Riemann-Hurwitz Formula of Chapter 1, we obtain an upper bound of the number of critical points.

Corollary 3.1. *A rational map of positive degree d has at most* $2d - 2$ *critical points in* $\hat{\mathbb{C}}$.

3.2 Fixed and periodic points

In order to study the iteration of rational maps $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, we introduce some concepts:

Definition. The *orbit* (or forward orbit) of f at z_0 is the sequence of iterates

$$O^+(z_0) = \{f^n(z_0)\}_{n \in \mathbb{N}}.$$

The *backward orbit* of f at z_0 is defined as

$$O^-(z_0) = \cup_{n \in \mathbb{N}} R^{-n}(z_0) = \cup_{n \in \mathbb{N}} \{w \in \hat{\mathbb{C}} : R^n(w) = z_0\}.$$

Definition. A *periodic point* of f of period $p \geq 1$ is a point $z_0 \in \hat{\mathbb{C}}$ such that $f^p(z_0) = z_0$ and $f^n(z_0) \neq z_0$ for every $0 < n < p$.

A *fixed point* of f is a point $z_0 \in \hat{\mathbb{C}}$ such that $f(z_0) = z_0$, that is a periodic point of period 1. The *multiplier* of f at such a fixed point is the complex value $\lambda = f'(z_0)$.

We classify fixed points according to their multiplier:

Definition. A fixed point z_0 of f is called:

- *Superattracting* if $\lambda = 0$.

- *Attracting* if $|\lambda| < 1$.
- *Repelling* if $|\lambda| > 1$.
- *Indifferent* or *neutral* if $|\lambda| = 1$, that is $\lambda = e^{2\pi i\theta}$ for some $\theta \in \mathbb{R}$.
  - *Rationally indifferent* or *parabolic* if $\lambda^n = 1$ for some $n \in \mathbb{N}$.
  - *Irrationally indifferent* if $\lambda^n \neq 1$ for all $n \in \mathbb{N}$.

The same classification applies to a periodic point z_0 of period p with a multiplier $\lambda = (f^p)'(z_0)$.

In fact, a cycle of length p defined by such a periodic point is

$$\alpha = \{z_0, f(z_0), ..., f^{p-1}(z_0)\}$$

Its multiplier $\lambda = \lambda(\alpha)$ is just defined to be the multiplier of the periodic point z_0, and by the Chain Rule,

$$(f^p)'(z_0) = \prod_{j=0}^{p-1} f'(f^j(z_0)),$$

so λ depends only on α. Its elements are also called periodic points.

Although we are not to analyse the local theory of fixed points (see [*Mil*]), which gives a description of f near a fixed point up to conjugacy, we give some information. An attracting fixed point has a local inverse near it, but not if it is a superattracting fixed point since it is a critical point of f. For an indifferent fixed point z_0 the best linear approximation to f near it is a rotation about z_0, that could have finite or infinite order if it is rationally or irrationally indifferent, respectively.

At this point, we add another definition to our list:

Definition. Let $\alpha = \{z_0, ..., z_{p-1}\}$ be an attracting cycle of length p. The basin of attraction of $\mathcal{C}$ is defined as

$$\mathcal{A}(\alpha) = \{z \ \in \hat{\mathbb{C}} : f^n(z) \to z_j \text{ as } n \to \infty, \quad 0 \leq j \leq p-1\}.$$

The union of the connected components of $\mathcal{A}(\alpha)$ which contains the cycle is denoted by $\mathcal{A}^*(\alpha)$ and it is called the immediate basin of (attraction) of α.

At this point, we introduce conjugacy, the natural notion of conformal automorphism for dynamical systems:

Definition. Let f and g be rational maps. We say that f and g are conjugate if there exists a Möbius transformation h such that

$$g \circ h = h \circ f$$

In that case, its diagram commutes:
Conjugate maps have the same dynamical behaviour since

$$g^n \circ h = h \circ f^n,$$

that is their iterates are also conjugate.

Therefore we can transfer conformally a dynamical problem about f to a possibly easier one about a conjugate map g of it. For example, h sends fixed points of f to fixed points of g, periodic points of f to periodic points of g, or critical values of f to critical values of g, and vice versa.

Moreover, the degree is a conjugation invariant property, and it respect the multipliers if they are not 0 or ∞.

Finally, we state an important theorem that classify a fixed point under certain conditions, which will be needed to prove the Classification Theorem in next chapter.

Theorem 3.2. (Snail Lemma) *Let $f(z) = \lambda z + a_2 z^2 + a_3 z^3 + ...$ be a map which is defined and holomorphic in some neighbourhood V at the origin, and which has a fixed point with multiplier λ at $z_0 = 0$. Let $p : [0, \infty) \to V \setminus \{0\}$ be a path which converges to the origin , that is a continuous map such that $p(t) \to 0$ as $t \to 0$, and which is mapped into itself by f such that $f(p(t)) = p(t+1)$. Then either the origin is an attracting ($|\lambda| < 1$) or parabolic ($\lambda = 1$) fixed point.*

For a proof, see [9]. It is called the Snail Lemma since in its proof appear a sequence of nested images of a certain simply connected domain that tend to an irrationally indifferent fixed point so slowly that the union of them would fill a neighbourhood of the fixed point, just like a trace of a snail does.

3.3 Montel's Theorem

Before state a definition of the Fatou set and the Julia set of a rational map, we must introduce two fundamental concepts in order to extend the notion of continuity for families of functions. Remind that $d_{\hat{\mathbb{C}}}$ is the spherical metric on $\hat{\mathbb{C}}$.

Definition. Let $\Omega \subset \mathbb{C}$ be a domain, and let $\mathcal{F}$ be a family of holomorphic functions from Ω to $\hat{\mathbb{C}}$. The family $\mathcal{F}$ is *equicontinuous* at $z_0 \in \Omega$ if given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_{\hat{\mathbb{C}}}(z, z_0) < \delta \quad \Rightarrow \quad d_{\hat{\mathbb{C}}}(f(z), f(z_0)) < \epsilon,$$

for all $z \in \Omega$ and every $f \in \mathcal{F}$.

Then if $\mathcal{F} = \{f^n\}$ is equicontinuous at z_0, we have that the orbit of a point near z_0 remains always close to the orbit of z_0.

Definition. Let $\Omega \subset \mathbb{C}$ be a domain, and let $\mathcal{F}$ be a family of holomorphic functions from Ω to $\hat{\mathbb{C}}$. The family $\mathcal{F}$ is *normal* at z_0 if every infinite sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$ contains a subsequence $\{f_{n_j}\}_{j\in\mathbb{N}}$ that converges locally uniformly on Ω.

Note that locally uniform convergence (with respect the spherical metric) on Ω means uniform convergent on compact subsets of Ω, so it is sufficient to check normality on open discs inn Ω. Moreover by the well-know Weierstrass Theorem, the limit function of the convergent subsequence is a holomorphic map, but it could not belong to the family such as the $z \mapsto \infty$ constant map.

In fact, both notions, equicontinuity and normality, are equivalent by the Arzela-Ascoli Theorem that we can present in the following way since $\hat{\mathbb{C}}$ is compact. For a proof, see for example [1].

Theorem 3.3. (Arzela-Ascoli Theorem) *Let $\Omega \subset \mathbb{C}$ be a domain, and let $\mathcal{F}$ be a family of holomorphic functions from Ω to $\hat{\mathbb{C}}$. The family $\mathcal{F}$ is normal if and only if it is locally equicontinuous on Ω.*

Before present the so-called Montel's Theorem, we present a little version of it that use this equivalence of both concepts. It was present in Montel's thesis of 1907.

Theorem 3.4. *Let $\Omega \subset \mathbb{C}$ be a domain, and let $\mathcal{F}$ be a family of holomorphic functions from Ω to $\hat{\mathbb{C}}$. If the family $\mathcal{F}$ is uniformly bounded, then $\mathcal{F}$ is a normal family.*

Proof. Since normality (or equicontinuity) is a local property we only need to prove it on an open disc, and so by an appropriate translation and a homothety we may assume that it is the unit disc.

Then by Cauchy's Integral Formula, we have

$$f(z) - f(w) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} f(\zeta)\left(\frac{1}{\zeta - z} - \frac{1}{\zeta - w}\right) = \frac{z-w}{2\pi i}\int_{\partial\mathbb{D}} \frac{f(\zeta)}{(\zeta - z)(\zeta - w)}$$

for every $z, w \in \mathbb{D}$.

In particular, we can take the points z, w in the open disc $D_r \in \mathbb{D}$ with radius $r < 1$ and centre 0 and $\mathcal{F}$, such that

$$|\zeta - z| > 1 - r, \quad |\zeta - w| > 1 - r.$$

Moreover, since the family $\mathcal{F}$ is uniformly bounded, i.e. there is a positive constant M such that $|f(z)| \leq M$ for all $z \in \Omega$ and every $f \in \mathcal{F}$. Therefore, we obtain that

$$f(z) - f(w) = \frac{|z-w|}{2\pi}\frac{2\pi}{(1-r)^2}M.$$

Finally, since $C = \frac{M}{(1-r)^2}$ does not depend on f, we have that f is locally equicontinuous on Ω, and then by the Arzela-Ascoli Theorem, $\mathcal{F}$ is a normal family. $\square$

At this point, we have all the tools to prove the Montel's Theorem (1911) that, as it was said in the Introduction, is a key theorem to characterise the Fatou and Julia sets.

Theorem 3.5. **(Montel's Theorem)** *Let $\Omega \subset \mathbb{C}$ be a domain, and let $\mathcal{F}$ be a family of holomorphic functions from Ω to $\hat{\mathbb{C}}$. If there are three points in $\mathbb{C}$ that are omitted by every $f \in \mathcal{F}$, then $\mathcal{F}$ is a normal family.*

Proof. Since normality is a local property we may assume again that Ω is the unit disc without loss of generality. Morevor, via a Möbius transformation, we may assume that the functions in $\mathcal{F}$ omits the values 0, 1 and ∞. Since $U = \mathbb{C} \setminus \{0, 1\}$ is a hyperbolic domain, by the Uniformization Theorem, its universal covering space is conformally equivalent to $\mathbb{D}$. Observe that the set of lifts $\tilde{f}$: $\mathbb{D} \to \mathbb{D}$ of all $f \in \mathcal{F}$, self-maps of the unit disk, form a normal family $\mathcal{F}'$ due to the previous Theorem 3.4. However a sequence of functions $\phi \circ \tilde{f}$ may contain a boundary point in U, where $\phi : \mathbb{D} \to U$ is the covering map. Therefore we have to find some way to avoid that.

We note that the covering map ϕ transfer the Poincaré metric on $\mathbb{D}$ to a metric in U where the three omitted points are pairwise infinitely distant boundary points. Each lift $\tilde{f}$ does not increase hyperbolic distances as state the Schwarz-Pick Lemma, and therefore it is also the case of each f. Since there is a positive number d such that the hyperbolic distance between any two neighbourhoods of each omitted point exceed d, the image of a small disc D in U has diameter at most d, that is the image $\phi \circ \tilde{f}(D)$ can meet at most one of these neighbourhoods. Then there is an infinite subsequence $\{\tilde{\mathrm{f}}_n\}$ in $\mathcal{F}'$ such that each disc $\phi \circ \tilde{\mathrm{f}}_n(D)$ is disjoint from one particular of those neighbourhoods, say the one about 0. If g is the Möbius transformation that maps this neighbourhood onto the exterior of $\mathbb{D}$, then $f = g \circ \phi \circ f_n'$ form a normal family in D, thus $\mathcal{F}$ is a normal family. □

3.4 The Fatou and Julia sets

We consider a rational map $f(z)$ and partition the Riemann sphere into two disjoint invariant sets: the Fatou set, where $f(z)$ is well-behaved, and the Julia set, where $f(z)$ has a chaotic behaviour.

In what follows, unless it is explicitly explained, we will assume the family $\mathcal{F}$ is the one given by the iterates of a rational map $f : U \to \hat{\mathbb{C}}$ on a certain domain $U \subset \mathbb{C}$, that is

$$\mathcal{F} = \{f^n|_U\}_{n\in\mathbb{N}} = \{f|_U, f^2|_U, ...\}.$$

Definition. The *Fatou set* $\mathcal{F}(f)$ of f is the set of points $z \in \hat{\mathbb{C}}$ such that the family $\{f^n\}_{n\in\mathbb{N}}$ is normal in some neighbourhood of z. The *Julia set* $\mathcal{J}(f)$ of f is the complement of the Fatou set, that is

$$\mathcal{J}(f) = \hat{\mathbb{C}} \setminus \mathcal{F}(f).$$

By definition, the Fatou set is the largest open set of normality, and hence the Julia set is closed and also compact (since $\hat{\mathbb{C}}$ is compact). Here we have defined the Fatou set as the normality set of a family of the family of iterates of a rational map, but we could also have done as the equicontinuity set of this family. Both notions are fully interchangeably.

A property common to Fatou and Julia sets is the complete invariance. Recall that a non-empty set $E \subset \hat{\mathbb{C}}$ is invariant if $f(E) \subset R$, and backward invariant if $f^{-1}(E) \subset E$. If both hold, i.e. if $f(E) = E = f^{-1}(E)$, then E is called *completely invariant*.

Theorem 3.6. *The Fatou and Julia sets are completely invariant under f.*

The completely invariance of the Fatou set follows from the definition of it as an open set, the continuity of f and the fact that rational maps are open.

We also have that $\mathcal{F}(f) = \mathcal{F}(f^p)$ and $\mathcal{J}(f) = \mathcal{J}(f^p)$, which is obtained from the fact that a finite union of families of holomorphic maps is normal if and only if each of these families is normal.

In this section, we are going to see how useful the Montel's theorem is as we explore the structure of the Julia set. Our first task is to check that Julia set can be non-empty. The case when the rational map R has degree one is trivial and of little interest, but in all other cases, J is non-empty as we are going to show.

Proposition 3.7. *Let f be a rational map of degree $d \geq 2$. $\mathcal{J}(f)$ is non-empty, that is $\mathcal{J}(f) \neq \emptyset$.*

Proof. Let f be a rational map of degree $d \geq 2$.

Suppose that the Julia set $\mathcal{J}(f)$ is empty, that is the family $\{f^n\}_{n \in \mathbb{N}}$ are normal in the whole Riemann sphere. Then any sequence in the family would contain some uniformly convergent subsequence such that $f^{n_j}(z) \to g(z)$ as $j \to \infty$, whose limit $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is holomorphic, so it would be either the constant map ∞ or else a rational map.

If g is constant, then the images of f^{n_j} are eventually contained in a small neighbourhood of the constant value ∞, which is impossible since f^n covers the whole $\hat{\mathbb{C}}$.

If g is not constant, the sequence f^{n_j} with limit g eventually has the same number of zeros as f by the Argument Principle, which is also impossible since $d \geq 2$ and

$$\deg(f^{n_j}) = d^{n_j} \to \infty,$$

so the Julia set should be empty. $\square$

At this point, we need to introduce th exceptional set of a rational map f in order to prove that the Julia set is infinite.

Montel's theorem has many applications in theory of rational maps. For example, it implies that any invariant hyperbolic domain, i.e. its complements has at least two points, is contained in the Fatou set. Also any backward invariant hyperbolic domain and compact set containing at least three points in fact contains the Julia set. Then we have that $\mathcal{J}(f)$ is the smallest completely invariant closed set containing at least three points. This is because the complement of a completely invariant closed set containing at least three points is an open completely invariant set omitting at most three points, hence contained in the Fatou set by Montel's Theorem.

Moreover, for any domain Ω intersecting the $\mathcal{J}(f)$, the set $E(D) = \hat{\mathbb{C}} \setminus O^+(\Omega)$ contains at most two points.

In fact, the points of $E(\Omega)$ are called exceptional points and play an important dynamic role, and $E(\Omega)$ is the exceptional set of f.

The next theorem is concerned with the structure of the exponential set $E(\Omega)$. It shows that $E(\Omega)$ may be non-empty only in particular, albeit important, cases.

Theorem 3.8. *If $\Omega \cap \mathcal{J}(f) \neq \emptyset$, and if the exceptional set $E(\Omega)$ contains exactly one point, then f is conjugate to some polynomial. If $E(\Omega)$ contains two points, then f is conjugate to one of the functions $z \to z^{\pm d}$.*

Proof. Since $\mathcal{J}(f) \neq \emptyset$, let $z \in \mathcal{J}(f)$ and let U be any neighbourhood of z. By Montel's theorem, the sequence $\{f^n\}_{n \in \mathbb{N}}$ on U omits a set E_z containing at most two points.

By definition $f^{-1}(E_z) \subset E_z$. If E_z is one point $a \in \hat{\mathbb{C}}$, then $f(a)$ is conjugate to some polynomial. In this case we may assume, by conjugacy, that $E_z = a = \infty$. Since $f^{-1}(\infty) = \infty$, there are no other poles, and f is a polynomial. Otherwise f would have a finite pole z_0 not contained in

$$\cup_{n \in \mathbb{N}} f^n(\Omega) = \hat{\mathbb{C}} \setminus \{\infty\}.$$

Clearly E_z is independent of z.

The second case is studied in the same way. If E_z consists of two points, we may assume without loss of generality that these are 0 and ∞, and either $f(0) = 0$ and $f(\infty) = \infty$, or $f(0) = \infty$ and $f(\infty) = 0$. In the first situation, f is a polynomial with 0 as its only zero, so $f(z) = Cz^d$. Similarly, $f(z) = Cz^{-d}$ in the second situation since f has all its zeros and poles in $E(\Omega) = \{0, \infty\}$. Both of them are conjugate to the map $z \to z^{\pm d}$. □

We note that there is non-dependence of $E(\Omega)$ on Ω. Obviously $E(\Omega)$ belongs to the Fatou set, since the points of $E(\Omega)$ either are superattracting fixed points or else form a superattracting cycle.

Another useful consequence of the Riemann-Hurwitz Formula is the next one:

Corollary 3.9. *Let f be a rational map of degree $d \geq 2$. If E is a finite completely invariant subset of $\hat{\mathbb{C}}$, then E contains at most two points.*

Now we are ready to prove the next corollary of the last proposition about $\mathcal{J}(f)$.

Corollary 3.10. *$\mathcal{J}(f)$ is an infinite set.*

Proof. Since $\mathcal{J}(f)$ is non-empty, we have that $\mathcal{J}(f)$ contains some point $z_0 \in \hat{\mathbb{C}}$.

If we suppose that $\mathcal{J}(f)$ is finite, as we know that $\mathcal{J}(f)$ is the completely invariant, the only possibilities up to conjugacy are $\{\infty\}$ or $\{0, \infty\}$, so we have that z_0 must be an exceptional point. This is not possible as exceptional point lie in $\mathcal{F}(f)$; thus $\mathcal{J}$ is infinite. □

Another important property that we will use for the Classification Theorem's proof is the following one:

Proposition 3.11. *$\mathcal{J}(f)$ is perfect, that is it is closed with no isolated points.*

Proof. Let J_0 be the set of accumulation points of $\mathcal{J}(f)$, then J_0 is non-empty as $\mathcal{J}(f)$ is infinite, closed by definition, and completely invariant because f is continuous, open and of finite degree.

But J_0 cannot be finite since it would then be exceptional, and hence contained in $\mathcal{F}(f)$. Therefore, $J_0 = \mathcal{J}(f)$ since $\mathcal{J}(f)$ is the smallest completely invariant closed set containing at least three points. □

A condition for a cycle to belong to the Fatou or to the Julia set could be the following one:

Proposition 3.12. *If α be an (super)attracting cycle of length p, then $\mathcal{A}(\alpha) \subset \mathcal{F}(f)$. But if α is a repelling cycle of length p, then $\alpha \subset \mathcal{J}(f)$.*

Proof. As $\mathcal{F}(f) = \mathcal{F}(f^p)$ it suffices to consider fixed points, and by a proper Möbius transformation, we may assume the fixed point $z_0 = 0$.

If $|f'(0)| < 1$, then there exists some arbitrarily small neighbourhood U of 0 such that $f(U) \subset U$, and hence $f^n(U) \subset U$ for $n \in \mathbb{N}$. This proves the normality of $\{f^n\}$. Since U contains the origin and, in fact tend to it, we have that any iterate of a point in the basin of attraction will fall in U.

On the other hand, if $|f'(0)| > 1$, then $(f^n)'(0) = (f'(0))^n \to \infty$ as $n \to \infty$. Therefore, there is no subsequence f^{n_k} that converge uniformly to a holomorphic limit function ϕ since, as $\phi(0) = 0$, $(f^{n_k})'(0)$ should tend to the finite value $\phi'(0)$, and this in not our case. □

Chapter 4

Classification of Fatou Components

4.1 Properties of Fatou components for rational maps

The Fatou set $\mathcal{F}(f)$ of a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree $d \geq 2$ is the largest open set of normality, and hence it can be expressed as a countable union of connected components called *Fatou components*, that is a maximal domain of normality of the iterates of f.

In fact, they are stable domains as the following important property says for a rational map f.

Proposition 4.1. *If U is a Fatou component, then $f(U)$ is also a Fatou component.*

Proof. Suppose two points z_1, z_2 in a Fatou component U are mapped by f to $f(z_1)$ and $f(z_2)$ in two distinct Fatou components U_1 and U_2. Since U is connected there is a smooth path between z_1 and z_2 that is entirely contained in U. Then the image of this path should intersect the boundary of each component, as the rational map f is continuous. But this is a contradiction since the path should lie in $\mathcal{F}(f)$ but the boundary point is in $\mathcal{J}(f)$ since Fatou components are connected open subsets. Therefore, we have $f(U)$ is contained in only one Fatou component U'.

At this point, suppose $f(U) \subsetneq U'$. Let $w \in U'$ be a point in the boundary of $f(U)$. Consider a sequence $\{w_n\}$ of points in $f(U)$ converging to $w \in \partial f(U)$, and the sequence $\{v_n\}$ of the preimages of w_n in U. If v is any accumulation point of the sequence of v_n, then $f(v) = w$. In that case, $v \in \partial U \subset \mathcal{J}(f)$, which contradicts with the fact that $w \in U' \subset \mathcal{F}(f)$. In conclusion, we have that $f(U) = U'$ is another Fatou component. $\square$

Therefore the domains, images of a certain Fatou component U, of the sequence

$$U, f(U), ..., f^n(U), ...$$

are also Fatou components. In that direction, we present some definitions.

Definition. In general, we say that a Fatou component U is a:

- *Periodic Fatou component* of period $p \geq 1$ if $f^p(U) = U$ and $f^n(U) \neq U$ for all $n < p$.
- *Preperiodic Fatou component* if $f^m(U) = U$ is a periodic Fatou component for some $m \geq 1$, that is eventually periodic Fatou components.
- *Fixed Fatou component* if $f(U) = U$, that is a periodic Fatou component of period 1.
- *Wandering domain* if all $f^n(U)$ are distinct.

But for rationals maps, its that the wandering domain case is not a possibility due to Sullivan's No wandering Domain Theorem (1985).

Theorem 4.2. Sullivan's No Wandering Domain Theorem *Every Fatou component of a rational map is eventually periodic.*

This is a key theorem to complete the classification of Fatou components for rational maps, although it is beyond the scope of this project because it requires quasiconformal surgery's theory. For more details, see [3].

Finally, we determine the possible number of components of the Fatou set of a rational map.

Proposition 4.3. *Let f be a rational map of degree $d \geq 2$. The Fatou set of f contains at most two simply connected completely invariant components.*

Proof. By the Riemann Mapping Theorem, such components are conformally equivalent to the unit disc $\mathbb{D}$. Then the restriction of f to $\mathbb{D}$ is a d-to-1 mapping.

From the Riemann-Hurwitz Formula, we have that f has

$$n_c = d - 1$$

critical points in $\mathbb{D}$, counting multiplicity. Since in last chapter we deduced a common upper bound of $2d - 2$ critical points for a rational maps, we may have at most two such Fatou componets, i.e. at most two simply connected completely invariant components of $\mathcal{F}(f)$. □

Proposition 4.4. *The Fatou set of a rational map f has either 0, 1, 2 or infinitely many components.*

Proof. For $d = 1$ this is trivial. If $d \geq 2$, we are going to apply the last proposition for the case that $\mathcal{F}(f)$ has only finitely many components U_1, ..., U_k. Here f must act as a permutation of them, so there is an integer m such that $g = f^m$ maps each U_i to itself, i.e. g is the identity map of the Fatou set into itself. Since $\mathcal{F}(f) = \mathcal{F}(f^m)$, we have that each U_i is completely invariant for g.

Moreover, since $\mathcal{J}(f)$ is the smallest closed completely invariant closed set under g, we have $\partial U_i = \mathcal{J}(f)$, so the sequence $\{f^n\}$ omits the open set U_i on $\hat{\mathbb{C}} \setminus \overline{U_i}$. Then $\{f^n\}$ is normal there by Montel's theorem, and hence $\hat{\mathbb{C}} \setminus \overline{U_i} \subset \mathcal{F}(f)$. Since U_i is connected, each other component of $\hat{\mathbb{C}} \setminus \overline{U_i}$ is simply connected. And in a similar way U_i is also simply connected. Then by the preceding proposition there are at most two such components. □

Definition. A stable fixed domain V of f is called a Fatou domain, if the sequence $\{f^n\}_{n\in\mathbb{N}}$ converges to a fixed point $a \in \overline{V}$, locally uniformly in V. More precisely, V is called a

- *Attracting basin* (or Schröder domain) if $a \in V$ and $0 < |\lambda| < 1$.
- *Superattracting basin* (or Böttcher domain) if $a \in V$ and $\lambda = 0$.
- *Parabolic basin* (or Leau domain) if $a \in \partial V$ and $\lambda = 1$.

where λ is the multiplier of a.

A stable fixed domain is called a *rotation domain* if none of the limit functions of the sequence $\{f^n|_V\}_{n\in\mathbb{N}}$ is constant. According to its connectivity, V is called a

- *Siegel disc* if V is simply connected and it contains an indifferent fixpoint.
- *Herman ring* if V is doubly connected.

The same terminology is used for corresponding fixed domains of f^p, i.e. for periodic domains of f.

For the sake of completeness, we present the Shishikura's sharp inequality for rational maps, which seems related to a critical points on each type of Fatou components (any cicle of Herman ring even needs two critical points) but this theorem also needs quasiconformal surgery.

Theorem 4.5. *Let f be a rational map of degree $d \geq 2$. The number of distinct cycles of different types of periodic Fatou components satisfy the inequalities:*

$$n_{superattr} + n_{attr} + n_{indiff} + 2n_{AH} \leq 2(d-1)$$

and

$$n_{AH} < d - 1,$$

where $n_{superattr}$, n_{attr}, n_{indiff} and n_{AH} are the number of cycles of periodic superattractive basins, attractive basins, parabolic basins plus Siegel discs (indifferent cases), and Herman rings, respectively.

4.2 Classification Theorem of periodic Fatou components

Finally, we are ready to give a proof of the Classification Theorem of periodic Fatou components for rationals maps. Here we apply all tools developed throughout these notes such as the general Schwarz-Pick Lemma, the Uniformization Theorem, Montel's theorem, or the Snail Lemma, among other propositions related to hyperbolic geometry and theory of rational maps, specially the one that compare the hyperbolic and spherical distances.

Theorem 4.6. (Classification of periodic Fatou components) *A component U of period p in the Fatou set of a rational map f is of exactly one of the following five types:*

(1) An attractive basin: there is a point z_0 in U, fixed by f^p, with $0 < |(f^p)'(z_0)| < 1$, attracting all points of U under iteration of f^p.

(2) A superattractive basin: as above, but z_0 is a critical point of f^p, so $(f^p)'(z_0) = 0$.

(3) A parabolic basin: there is a point z_0 in ∂U with $(f^p)'(z_0) = 1$, attracting all points of U.

(4) A Siegel disc: U is conformally equivalent to the unit disk, and f^p is conjugate to an irrational rotation.

(5) A Herman ring: U is conformally equivalent to an annulus, and f^p is conjugate again to an irrational rotation.

Proof. Replacing f by f^p, we can assume U is a fixed Fatou component, that is $f(U) = U$. Since the Julia set is infinite, U is a hyperbolic Riemann surface. By the the Schwarz-Pick Lemma, f does not increase the hyperbolic distance d_U on U.

First, we are going to show that if one orbit tends to infinity in U (i.e. if it eventually escapes any compact subset $K \subset U$, or in other words, the set of accumulation points of the orbit of z_0 satisfies $A(z_0) \subset \partial U$), then all orbits tend to infinity in U. For this purpose, suppose the orbit $z_n = f^n(z_0)$ of $z_0 \in U$, tends to infinity in U, and let $w_n = f^n(w_0)$ be another orbit in U. Since f is non-increasing the hyperbolic distance on U,

$$d_U(z_n, w_n)) \leq d_U(z_0, w_0)$$

for all $n \in \mathbb{N}$.

Since the hyperbolic distance $d_U(z_n, w_n)$ is bounded by the constant distance $r = d_U(z_0, w_0)$ between the initial points and the orbit z_n has all his accumulation points on ∂U, we can apply the proposition that compares hyperbolic and spherical distances. This one conclude that the spherical diameter of the hyperbolic closed ball $B_U(z_n, w_n)$ tends to 0, that is the orbit w_n should also tends to infinity in U since convergence properties are defined with the spherical metric on $\hat{\mathbb{C}}$. We should keep in mind this arguments because they will be used several times throughout this proof, so we are not going to give all details next time.

Since we have shown that if one orbit tend to infinity, then the other orbits should do the same, we can distinguish the following two cases:

(1) *Every orbit tend to infinity in U, i.e.* $A(z) \subseteq \partial U$, $\forall z \in U$.

Let $z \in U$, and let $z_n = f^n(z_0)$ be its orbit. We apply the same arguments as above: the Schwarz-Pick Lemma implies that

$$d_U(z_n, z_{n+1}) \leq d_U(z_0, z_1),$$

then by the Prop. 2.8 we have

$$d_{\widehat{\mathbb{C}}}(z_n, z_{n+1}) \to 0.$$

This fact tells us that $A(z_0)$, the set of accumulation points of the orbit of z_0, consists of fixed points of f lying on ∂U. Moreover, it is a connected set. Otherwise, $A(z_0)$ would have at least 2 components A_1 and A_2, and then both of them would be disjoint compact subsets of $\widehat{\mathbb{C}}$, separated by a spherical distance $\delta > 0$. But we know that $d_{\widehat{\mathbb{C}}}(z_n, z_{n+1}) < \delta$ for n large (as we have seen it tends to 0), so this contradicts the hypothesis that the orbit accumulates on both components.

Furthermore, we can show to show that the attracting basin of z_0 is a singleton, i.e. $A(z_0) = p$, where p is an indifferent fixed point, as follows:

Since f is not the identity map, and $A(z_0)$ is a discrete subset that consists of fixed points, the only possible connected set with these conditions is a singleton $A(z_0) = p$. Since, by the Schwarz-Pick Lemma,

$$d_U(f^n(w), z_n) \leq d_U(w, z_0),$$

we have that

$$d_{\widehat{\mathbb{C}}}(f^n(w), z_n) \to 0$$

due to Prop. 2.8. Then as p is a fixed point and we have shown that $f^n(w) \to p$, we conclude that $p \in \partial U$ attracts all points $w \in U$.

Since the fixed point p attracts all points in U, its multiplier should satisfy $|\lambda| = |f'(p)| \leq 1$. As we have that $p \in \partial U \subset \mathcal{J}(f)$ and we know that all fixed points with $|\lambda| < 1$ are in $\mathcal{J}(f)$, the fixed point p must be an indifferent fixed point with $|\lambda| = 1$.

Finally, we can choose any path $\alpha : [0,1) \to U$ from z_0 to $z_1 = f(z_0)$, and we extended it to a path $p : [0,\infty) \to U$ by setting $p(t+1) = f(p(t))$, that is we construct p by glueing all pieces of path α, $f(\alpha)$,..., together, taking a unit of time for each piece. Then if we apply the Snail Lemma in that case, we conclude that $\lambda = f'(p) = 1$, i.e. U is a parabolic basin (case 3).

(2) *None of the orbits tend to infinity in U, i.e.* $A(z) \subseteq U$, $\forall z \in U$.

Let $z_n = f^n(z_0)$ be the orbit of $z_0 \in U$ with an accumulation point $p \in U$. By definitions, there is a subsequence $\{z_{n_j}\}_{j\in\mathbb{N}}$ such that $z_{n_j} \to p$.

We consider the family of rational maps

$$\{g_j := f^{n_{j+1}-n_j}\}_{j\in\mathbb{N}},$$

with the property

$$g_j(z_{n_j}) = f^{n_{j+1}-n_j}(z_{n_j}) = f^{n_{j+1}}(z_0) = z_{n_{j+1}}.$$

Since U is hyperbolic, Montel's Theorem says that there is a subsequence $\{g_{j_l}\}_{l\in\mathbb{N}}$ that converges locally uniformly to a limit function $g : U \to U$ that is holomorphic.

Furthermore, since $z_{n_{j+1}} = g_j(z_{n_j})$ and $z_{n_j} \to p$, taking limits we have $g(p) = p$, so p is a fixed point of g in the two following cases.

According to the general Schwarz Lemma, one of the following two conditions hold:

(2a) *f is a hyperbolic contraction: $d_U(f(z), f(w)) < d_U(z, w), \quad \forall z, w \in U\ (z \neq w)$.*
Since f strictly decreases non-zero Poincaré distances, g must also decrease non-zero Poincaré distances because

$$d_U(g(z), g(w)) \leq d(f(z), f(w)) < d(z, w),$$

for all $z, w \in U\ (z \neq w)$.
Since g is the limit function of the iterates of f, and due to the relation

$$g_j \circ f = f \circ g_j,$$

we have that f and g commute. Therefore, f must map fixed points of g, such as $p = g(p)$, to fixed points of g, that is

$$f(p) = f(g(p)) = g(f(p)).$$

But since g decreases all non-zero hyperbolic distances, g cannot have 2 distinct fixed points. And hence, $p = f(p)$ is the only fixed point of g in U, and then also the only one of f.
At the end, since $f^n(z) \to p$ for all $z \in U$, we have that $p \in U$ is an attracting fixed point with $|\lambda| < 1$, i.e. U is an attractive or superattractive basin (case 1 or 2).

(2b) *f is a local isometry: $d_U(f(z), f(w)) = d_U(z, w), \quad \forall z, w \in U$.*
By a local isometry, we mean that the lift of $f : U \to U$ to its universal covering space (that is conformally conjugate to $\mathbb{D}$ due to the Uniformization Theorem) is a hyperbolic isometry.
Since f preserves the hyperbolic distance, f is either a conformal automorphism or just a covering map.
First, we have that the lift $G : \mathbb{D} \to \mathbb{D}$ of g is a conformal automorphism with the corresponding fixed point $\phi(p)$, where $g : U \to U$ has the fixed point $p \in U$, and $\phi : U \to \mathbb{D}$ is the map such that $G \circ \phi = \phi \circ g$. Thus G should be a rotation about the fixed point $\phi(p)$ by the Schwarz Lemma.
Then we have a sequence $g^{n_j} \rightrightarrows id_U$, and since $f^{n_{j+1}-n_j} \rightrightarrows g$ locally, there is a sequence $f^{m_j} \rightrightarrows id_U$ locally, i.e. converges uniformly on compact subsets of U.
Therefore f is one-to-one, and as it is a covering map, we conclude that f is a conformal automorphism.

Now we can prove that U has not two different simple closed hyperbolic geodesics:

Indeed, if U has a non-trivial simple closed hyperbolic geodesic γ, then $f^n(\gamma) = \gamma$ for sufficiently large n, since any isometry of U sufficiently close to id_U must map γ to itself.

If U has two simple closed hyperbolic geodesics that intersect, then f^n must fix all intersection points, and hence must fix both geodesics pointwise, since f is an isometry and $f^n \rightrightarrows id_U$. This fact is due to the properties of isometries: they send geodesics to geodesics, so they map closed geodesics to closed geodesics when they are arbitrarily close to identity, and since all closed geodesics are isolated, we have that f^n must map γ to itself.

Therefore f^n is also the identity for n large, but this is a contradiction because $deg(f^n) = deg(f)^n \geq 2$ so f cannot have finite order globally, and particularly in U.

Finally we suppose that U has two disjoint closed hyperbolic geodesics. Let γ' be a shortest geodesic segment connecting these two closed geodesics. Then again f must fix γ' pointwise, so we apply the same arguments and we arrive to the same type of contradiction.

In conclusion, since U has at most one simple closed hyperbolic geodesic, and the components of $\mathcal{F}(f)$ cannot contain isolated punctures (as $\mathcal{J}(f)$ does not contain isolated points), we have that U is conformally equivalent to $\mathbb{D}$ or A_R with finite modulus. In fact, we have computed their hyperbolic metrics and geodesics in chapter 2.

Therefore we can distinguish these two last cases:

(2b1) *U is simply connected.*
Since f is a conformal automorphism and it has a fixed point $p \in U$, we have that f is conformally conjugate on U to a rotation $R_\theta : z \to e^{2\pi i\theta} z$. If $\theta \in \mathbb{Q}$, then $f^n = Id_U$ for some $n \in \mathbb{N}$, which is a contradiction because $deg(f) \geq 2$. Hence $\theta \in \mathbb{R} \setminus \mathbb{Q}$, i.e. p is a rationally indifferent fixed point and U is a Siegel disc (case 4).

(2b2) *U is not simply connected.*
Then it must be an annulus of finite modulus, that is a doubly connected domain. An orientation-preserving isometry of the hyperbolic metric of an annulus can only be a rotation, or an inversion which reverses the direction of the circle, that is conformal automorphisms of the form $z \to e^{2\pi i\theta} z$ or $z \to \frac{r}{z}$.
The return map cannot be periodic, or else by analytic continuation the entire rational map would be periodic. Moreover, if the return map reversed the orientation of the circle, a second return map would be the identity. Therefore, the return map can only be an irrational rotation of the annulus, i.e. U is a Herman ring.

$\square$

4.3 Some examples

In this last section we present some beautiful pictures of Fatou components that we have classified in last chapter.

We have used also other types of holomorphic maps such as for example Blaschke factors that suppose the only explicit examples known to have an Arnol'd-Herman ring, and they are of the form

$$f(z) = e^{i\alpha} \prod_{j=0}^{2p} \frac{z - a_j}{1 - \overline{a_j} z}$$

In the case of iteration of entire functions, if a domain V is unbounded and if f is holomorphic in $\overline{V} \setminus \{\infty\}$, then it may arise that $f^n \to \infty$, locally uniformly in V. Such domain is called Baker domain, a special type for entire maps.

In fact, entire maps are just holomorphic functions over the whole complex plane such as polynomials, exponentials, sinus or cosinus.

An interesting subtype of them are the transcendental entire map, which are entire map different than a polynomial. They have an essential singularity at ∞.

This picture invite us to continue the research on complex dynamics of transcendental entire maps among others.

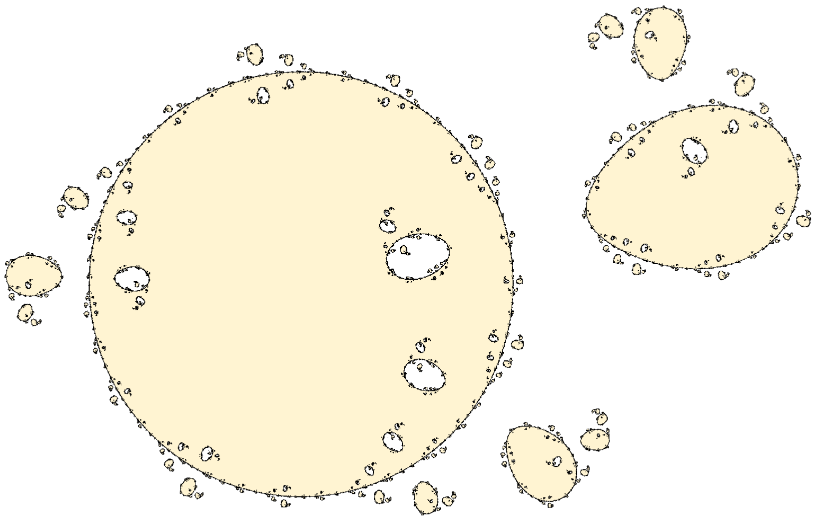
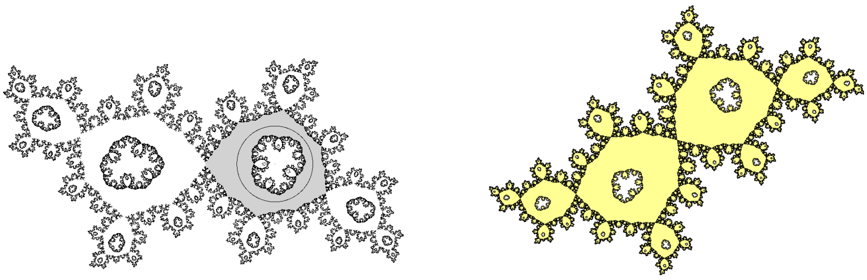


Figure 4.1: Rational function with infinite Fatou components. The yellow disc represent the Fatou component of the attracting point $z = 0$. The other yellow components are preimages of this one (preperiodic components).



(a) (b)

Figure 4.2: One of the yellow annuli are invariant components is a Herman ring. All other yellow annuli are preimages of this one. It is a rational map.

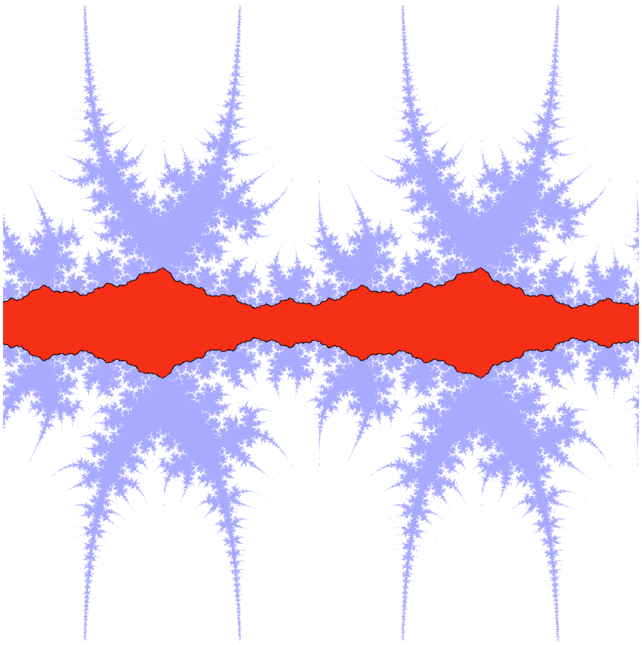


Figure 4.3: Baker domain for the function $z \mapsto z + \alpha + \beta \sin(z)$ for proper values of α and β. In the red domain all points go to the right.

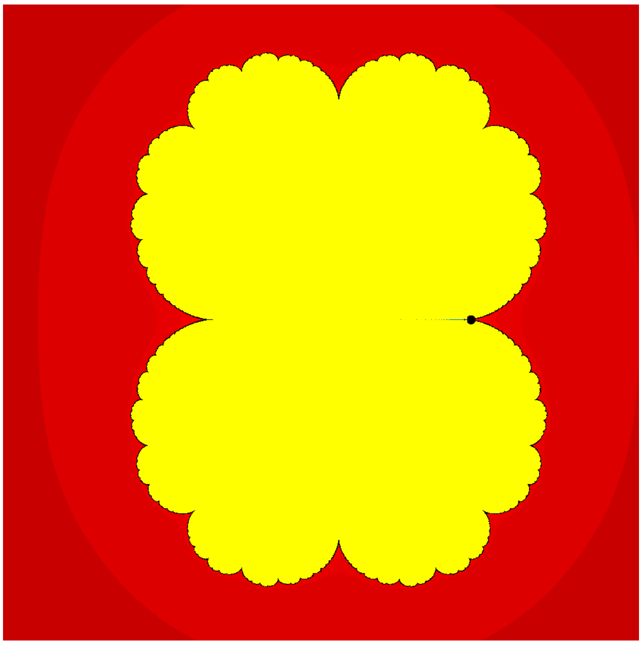


Figure 4.4: Function $z^2 + 0.25$. The point $z = 0.5$ is a parabolic point.

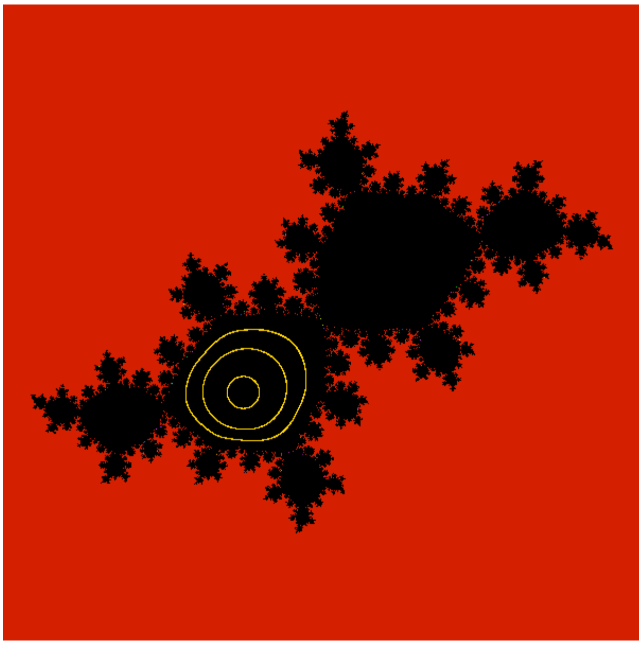


Figure 4.5: $z^2 + c$. Some Siegel discs orbits are represented.

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