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A MODEL OF CONTINUOUS TIME POLYMER ON THE LATTICE

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ABSTRACT. In this article, we try to give a rather complete picture of the behavior of the free energy for a model of directed polymer in a random environment, in which the polymer is a simple symmetric random walk on the lattice $\mathbb{Z}^d$, and the environment is a collection $\{W(t,x); t \geq 0, x \in \mathbb{Z}^d\}$ of i.i.d. Brownian motions.

1. Introduction

After two decades of efforts, the asymptotic behavior of polymer measures, either in a discrete [2, 4, 6, 16, 19, 21, 22, 24, 25] or continuous [3, 8, 13, 14, 17, 26] time setting, still remains quite mysterious. Furthermore, referring to the articles mentioned above, this problem is mainly tackled through the study of the partition function of the measure. It is thus natural to try to find a model for which a rather complete picture for the large time behavior of the partition function is available. In the current article, we shall show that one can achieve some sharp results in this direction for a model of continuous time random walk in a Brownian environment.

Indeed, this paper is concerned with a model for a d-dimensional directed random walk polymer in a Gaussian random medium which can be briefly described as follows: the polymer itself, in the absence of any random environment, will simply be modeled by a continuous time random walk $\{b_t, t \geq 0\}$ on $\mathbb{Z}^d$. This process is defined on a complete probability space $(\Omega_b, \mathcal{F}, \{P^x\}_{x\in\mathbb{Z}^d})$, where P^x stands for the measure representing the random walk starting almost surely from the initial condition x; we write the corresponding expectation as E_b^x. Let us recall that under P^x, the process is at x at time 0, it stays there for an exponential holding time (with parameter $\alpha = 2d$), and then jumps at one of the $2d$ neighbors of x in $\mathbb{Z}^d$ with equal probability. It stays there for an exponential holding time independent of everything else and so on. Notice that a given realization of b belongs to the space $\mathcal{D}$ of paths $y : \mathbb{R}_+ \to \mathbb{Z}^d$ such that $y_t = \sum_{i\geq 0} x_i \mathbf{1}_{[\tau_i,\tau_{i+1})}(t)$, for a given sequence of increasing positive times $(\tau_i)_{i\geq 0}$ and a given path $(x_i)_{i\geq 0}$ of a nearest neighbor random walk x. We will denote by $\mathcal{D}_{[0,t]}$ the restriction of such a space to $[0, t]$.

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The random environment in which our polymer lives is given by a family of independent Brownian motions $\{W(t,x);\ t \ge 0, x \in \mathbb{Z}^d\}$, defined on some probability space $(\Omega, \mathcal{G}, \mathbf{P})$ independent of $(\Omega_b, \mathcal{F}, \{P^x\}_{x\in\mathbb{Z}^d})$. More specifically, this means that W is a centered Gaussian process satisfying

$$\mathbf{E}\left[W(t,x)W(s,y)\right] = (s \wedge t)\delta(x-y),$$

where $\mathbf{E}$ denotes the expectation on $(\Omega, \mathcal{G}, \mathbf{P})$, and δ stands for the discrete Dirac measure, i.e. $\delta(z) = \mathbf{1}_0(z)$. We will also call $(\mathcal{G}_t)_{t\ge 0}$ the filtration generated by W.

Finally, our Gibbs type polymer measure is constructed as follows: for any $t > 0$, the energy of a given path b on $[0,t]$ is given by the Hamiltonian:

$$H_t(b) := \int_0^t W(ds, b_s).$$

Observe that for a given path b, the last quantity has to be understood as a Wiener integral with respect to W. In particular, it is a centered Gaussian variable with variance t. Now, for any $x \in \mathbb{Z}^d$, any $t \ge 0$ and a given $\beta > 0$ which represents the inverse of the temperature, we define the polymer measure by the formula:

$$dG_t^x(b) = \frac{e^{\beta H_t(b)}}{Z_t^x}\, dP^x(b), \qquad \text{with} \qquad Z_t^x = E_b^x\left[e^{\beta H_t(b)}\right]. \tag{1.1}$$

The normalisation constant Z_t^x above is called the partition function of the model. In the sequel, we will also have to consider the Gibbs average with respect to the polymer measure, defined as follows: for any $t \ge 0$, $n \ge 1$, and for any bounded measurable functional $f : \mathcal{D}^n_{[0,t]} \to \mathbb{R}$, we set

$$\langle f \rangle_t = \frac{E_b^x\left[f(b^1, \ldots, b^n)e^{\beta \sum_{l\le n} H_t(b^l)}\right]}{(Z_t^x)^n},$$

where $\{b^1, \ldots, b^n\}$ are understood as independent continuous time random walks.

Let us say now a few words about the partition function: the first thing one can notice, thanks to some invariance arguments, is that the asymptotic behavior of Z_t^x does not depend on the initial condition x. We shall thus consider generally $x = 0$, and set $Z_t^0 \equiv Z_t$. Moreover, it can be checked (see Section 2 for further explanations) that the following limits exist:

$$p(\beta) = \lim_{t\uparrow\infty} \frac{1}{t}\, \mathbf{E}\left[\log Z_t\right] = \mathbf{P} - \text{a.s.} - \lim_{t\uparrow\infty} \frac{1}{t}\ \log Z_t, \tag{1.2}$$

and it can also be shown that $p(\beta) \le \beta^2/2$ by an elementary Jensen's type argument. The quantity $p(\beta)$ is called the free energy of the system. It is then possible to separate a region of *weak disorder* from a region of *strong disorder* according to the value of $p(\beta)$, by saying that the polymer is in the strong disorder regime if $p(\beta) < \beta^2/2$, and in the weak disorder regime otherwise. It should be mentioned that this notion of strong disorder is rather called *very strong disorder* in [8], the exact concept of strong disorder being defined thanks to some martingale considerations e.g. in [13, 26]. It is however believed that strong and very strong disorder coincide (see [8] again). Furthermore, these notions have an interpretation in terms of localization [8] or diffusive behavior [14] of the polymer.

With these preliminaries in hand, we will see now that some sharp information on the partition function can be obtained for the model under consideration. Namely, to begin with, the weak and strong disorder regimes can be separated as follows:

Proposition 1.1. *Let $Z_t \equiv Z_t^0$ be the normalization constant given by formula (1.1), and define a $\mathcal{G}_t$-martingale $(M_t)_{t\geq 0}$ by $M_t = Z_t \exp(-\beta^2 t/2)$. Then:*

(1) *Whenever $d = 1, 2$ and $\beta > 0$, we have $\lim_{t\to\infty} M_t = 0$ in the $\mathbf{P}$-almost sure sense, which means that the polymer is in the strong disorder regime.*
(2) *For $d \geq 3$ and β small enough, the polymer is in the weak disorder regime, i.e. $\lim_{t\to 0} M_t > 0$, $\mathbf{P}$-almost surely.*
(3) *For any dimension d and $\beta > \beta_d$, the polymer is in the very strong disorder regime, which means that $p(\beta) < \beta^2/2$.*

This kind of separation for the weak and strong disorder regime has already been obtained for other relevant models, based on discrete time random walks [6] or Brownian motions [5, 14, 26]. However, the third point above can be sharpened substantially, and the following almost exact limit holds true in the continuous random walk context:

Theorem 1.2. *Let $p(\beta)$ be the quantity defined at (1.2), and ε_0 be a given arbitrary positive constant. Then, there exists $\beta_0 > 0$, depending on d and ε_0, such that*

$$C_0 \frac{\beta^2}{\log(\beta)}\,(1-\varepsilon_0) \leq p(\beta) \leq C_0 \frac{\beta^2}{\log(\beta)}\,(1+\varepsilon_0), \qquad \text{for } \beta \geq \beta_0,$$

where C_0 is a strictly positive positive constant which will be defined by relation (3.15).

Putting together Proposition 1.1 and Theorem 1.2, we thus get a remarkably precise picture as far as the free energy of the system is concerned. It should also be mentioned that our method of proof heavily relies on sharp Gaussian estimates, especially concerning supremums of the field W over certain functional sets. We are thus happy to present a paper where Gaussian techniques are crucial in order to solve a physically relevant problem.

Remark 1.3. Many of our results would go through without much effort for a wide class of spatial covariance of the medium W, as done in [5]. We have sticked to the space-time white noise case in the current article for sake of simplicity.

Our paper is divided as follows: at Section 2, we recall some basic facts about the partition function of the polymer model. Section 3 is the bulk of our article, and is devoted to a sharp study of the free energy in the low temperature region, along the lines of the Lyapunov type result [9, 12, 20]. At Section 4, the first two items of Proposition 1.1 are shortly discussed.

2. Basic Properties of the Free Energy

Since it will be essential in order to show Theorem 1.2, we will first devote the current section to show briefly that Z_t converges almost surely to a constant $p(\beta)$, which can be done along the same lines as in [26]. First of all, some standard arguments yield the following asymptotic result:

Proposition 2.1. *For $t > 0$, define $p_t(\beta) = \frac{1}{t}\ \mathbf{E}\left[\log Z_t^x\right]$. Then, for all $\beta > 0$, there exists a constant $p(\beta) > 0$ such that*

$$p(\beta) \equiv \lim_{t\uparrow\infty} p_t(\beta) = \sup_{t\geq 0} p_t(\beta).$$

Furthermore, $p(\beta) \leq \beta^2/2$.

Proof. It can be proved e.g. as in [26]. More specifically, we should first show a Markov decomposition for Z_t^x as in [26, Lemma 2.4]; then we can argue as in [26, Proposition 2.5] in order to get the announced limit for $p_t(\beta)$. The bound $p(\beta) \leq \beta^2/2$ can be checked using Jensen's inequality. □

Remark 2.2. Due to the spatial homogeneity of W, the above limit does not depend on $x \in \mathbb{Z}^d$. Hence, from now on, we will choose $x = 0$ for our computations. Furthermore, in the sequel, when $x = 0$ we will write Z_t and E_b instead of Z_t^0 and E_b^0, respectively.

In order to get the almost sure convergence of $p_t(\beta)$, we will need some concentration inequalities which can be obtained by means of Malliavin calculus tools. Let us briefly recall here the main features of this theory, borrowed from [23]. First of all, let us notice that our Hamiltonian $H_t(b)$ can be written as follows:

$$H_t(b) := \int_0^t W(ds, b_s) = \int_0^t \sum_{x\in\mathbb{Z}^d} \delta_x(b_s) W(ds, x).$$

This leads to the following natural definition of an underlying Wiener space in our context: set $\mathcal{H} = L^2(\mathbb{R}_+ \times \mathbb{Z}^d)$, endowed with the norm

$$|h|_{\mathcal{H}}^2 = \int_{\mathbb{R}_+} \sum_{x\in\mathbb{Z}^d} |h(t,x)|^2 dt.$$

Then there exists a zero-mean isonormal Gaussian family $\{W(h); h \in \mathcal{H}\}$ defined by

$$W(h) = \int_{\mathbb{R}_+} \sum_{x\in\mathbb{Z}^d} h(t,x) W(dt, x),$$

and it can be shown that $(\Omega, \mathcal{H}, \mathbf{P})$ is an abstract Wiener space. Denote now by $\mathcal{S}$ the set of smooth functionals defined on this Wiener space, of the form

$$F = f(W(h_1), \ldots, W(h_k)), \quad \text{for} \quad k \geq 1,\ h_i \in \mathcal{H},\ f \in C_b^\infty(\mathbb{R}^k).$$

Then its Malliavin derivative is defined as

$$D_{t,x}F = \sum_{i=1}^k \partial_i f(W(h_1), \ldots, W(h_k)) h_i(t,x).$$

As usual, the operator $D : \mathcal{S} \longrightarrow \mathcal{H}$ is closable and we can build the family of Sobolev spaces $\mathbb{D}^{1,p}$, $p \geq 1$, obtained by completing $\mathcal{S}$ with respect to the norm $\|F\|_{1,p}^p = \mathbf{E}\left[|F|^p\right] + \mathbf{E}\left[|DF|_{\mathcal{H}}^p\right]$. The following chain rule is also available for $F \in \mathbb{D}^{1,p}$: if $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth function, then $\psi(F) \in \mathbb{D}^{1,q}$ for any $q < p$ and

$$D\psi(F) = \psi'(F) DF. \tag{2.1}$$

We are now ready to prove the almost sure limit of $\log(Z_t)/t$.

Proposition 2.3. *We have that*

$$\mathbf{P}-\text{a.s.}-\lim_{t\uparrow\infty}\frac{1}{t}\ \log Z_t = p(\beta).$$

Proof. It is easily shown that $Z_t \in \mathbb{D}^{1,2}$, and by differentiating in the Malliavin calculus sense, we obtain, if $s < t$:

$$D_{s,x}Z_t = \beta E_b\left[(D_{s,x}H_t(b))e^{\beta H_t(b)}\right] = \beta E_b\left[\delta_x(b_s)e^{\beta H_t(b)}\right].$$

Thus, if $U_t = \frac{1}{t}\ \log Z_t$, we have, thanks to the chain rule (2.1),

$$D_{s,x}U_t = \frac{D_{s,x}Z_t}{tZ_t} = \begin{cases} \beta\ \frac{\langle \delta_x(b_s)\rangle_t}{t}, & s \le t, \\ 0, & s > t. \end{cases}$$

So,

$$|DU_t|^2_{\mathcal{H}} = \frac{\beta^2}{t^2}\ \int_0^t \sum_{x\in\mathbb{Z}^d} \langle \delta_x(b^1_s)\delta_x(b^2_s)\rangle_t ds = \frac{\beta^2}{t^2}\ \int_0^t \langle \delta_0(b^1_s - b^2_s)\rangle_t ds \le \frac{\beta^2}{t}.$$

Now, since $|DU_t|^2_{\mathcal{H}}$ is bounded and tends to 0 as $t\to\infty$, we can prove the almost sure limit by means of a concentration inequality (see, for instance, [26, Proposition 2.1]) and a Borel-Cantelli type argument. □

3. Exact Limit for the Free Energy at Low Temperature

The aim of this section is to show our Theorem 1.2, by means of some Gaussian tools which have been already used for various models of polymers [5] or stochastic PDEs [9, 10, 11, 12, 15, 18]. It should be mentioned at this point that the reference [9] is especially relevant for us: in fact, our aim here is to clarify some of the arguments therein, and adapt them to the polymer context at the same time (our main result could be deduced from [9] by a simple scaling argument, but we have decided to adapt the proof for sake of clarity).

3.1. Strategy. In order to understand how the free energy will be computed, let us introduce first some additional useful notations: let P_n be the set of paths of a discrete time random walk of length n starting at 0, and $S_{n,t}$ be the set of possible times of the jumps of the continuous time random walk b in $[0,t]$, namely:

$$S_{n,t} = \{\mathbf{s} = (s_0,\dots,s_n);\ 0 = s_0 \le s_1 \le \dots \le s_n \le t\}.$$

For $\mathbf{s}\in S_{n,t}$ and $\mathbf{x} = (x_0,\dots,x_n)\in P_n$, we also define

$$B_n(\mathbf{s},\mathbf{x}) = \sum_{j=0}^{n}\left[W(s_{j+1},x_j) - W(s_j,x_j)\right].$$

For any positive t, let N_t be the number of jumps of b in $[0,t]$, which is known to be a Poisson process with intensity $2d$. Then one can decompose the Hamiltonian

$H_t(b)$ according to the values of N_t in order to obtain:

$$\begin{aligned} Z_t &= \sum_{n=0}^{\infty} E_b\left[e^{\beta\int_0^t W(ds,b_s)}\ \mathbf{1}_{\{N_t=n\}}\right] \\ &= \sum_{n=0}^{\infty} \frac{(2dt)^n}{n!}\ e^{-2dt}\ E_b\left[e^{\beta\int_0^t W(ds,b_s)}|N_t=n\right] \\ &= \sum_{n=0}^{\infty} e^{-2dt}\sum_{\mathbf{x}\in P_n}\int_{S_{n,t}} e^{\beta B_n(\mathbf{s},\mathbf{x})}ds_1\cdots ds_n. \end{aligned} \tag{3.1}$$

With these preliminaries in mind, we can now sketch the strategy we shall follow in order to obtain the lower and upper bounds on $p_t(\beta)$ announced at Theorem 1.2. Indeed, the basic idea is that one should find an equilibrium between two constraints:

(i) The more the random walk jumps, the more it will be able to see the peaks of the energy, represented by $\sup_{j\le n,\mathbf{s}\in S_{j,t},\mathbf{x}\in P_j} B_j(\mathbf{s},\mathbf{x})$. We shall see that, roughly speaking, $\sup_{j\le rt,\mathbf{s}\in S_{j,t},\mathbf{x}\in P_j} B_j(\mathbf{s},\mathbf{x})$ is of order $r^{1/2}t$ for a given $r>0$.

(ii) The number of jumps of the random walk has an entropy cost, which is represented in formula (3.1) by the area of $S_{n,t}$. It is a well known fact that this area decreases as $n!$.

After reducing our calculations to this optimization problem, it will be easily seen that the accurate choice for r is of order $(\beta/\log(\beta))^2$, a fact which has already been outlined in [5]. This means that, under the influence of the environment when β is large enough, the random walk is allowed to jump substantially more -the typical number of jumps before t is of order $(\beta/\log(\beta))^2t$- than in the free case (for which this typical number is of order $2dt$). By elaborating this kind of considerations, we shall obtain our bound $C_0\beta^2/\log(\beta)$ for $p(\beta)$.

We are now ready to perform our first technical step, which is the control on the Gaussian random field B_n.

3.2. Control of the random field B_n. Let us mention that, in order to control the supremum of the random Gaussian fields we will meet, we shall use the following classical results, borrowed from [1].

Theorem 3.1. *Let $G(t)$ be a mean zero Gaussian field over a set T with the associated pseudo-metric*

$$d(t_1,t_2)=\left(\mathbf{E}\left[G(t_1)-G(t_2)\right]^2\right)^{1/2},$$

and assume that the metric space (T,d) is compact. Let $\mathcal{N}(\eta)$ be the metric entropy associated with d, i.e. the smallest number of closed d-balls of radius η needed to cover T. Then, there exists a universal constant K such that

$$\mathbf{E}\sup_T G(t)\le K\int_0^\infty \sqrt{\log\mathcal{N}(\eta)}\ d\eta,$$

provided that the right hand side is finite.

This useful theorem for the computation of the mean of $\sup_{t\in T} G(t)$ has generally to be completed by a control on the fluctuations of Gaussian fields, given by the following result:

Theorem 3.2. *Let $G(t)$ be a mean zero Gaussian field over a set T, and suppose that the sample paths of G are bounded almost surely. Then, we have $\mathbf{E}\sup_T G(t) < \infty$, and for all $y > 0$,*

$$\mathbf{P}\left\{\left|\sup_T G(t) - \mathbf{E}\sup_T G(t)\right| > y\right\} \le 2e^{-y^2/2\tau^2} ,$$

where $\tau^2 = \sup_T \mathbf{E}[G(t)^2]$.

With these Gaussian tools in hand, we can now control B_n as follows:

Proposition 3.3. *For $r \ge 0$, let T_r be the space of paths of the continuous time random walk starting at $x = 0$ with no more than rt jumps. Then, we have*

$$\mathbf{E} \sup_{(n,\mathbf{s},\mathbf{x})\in T_r} B_n(\mathbf{s},\mathbf{x}) \le C\sqrt{[rt]t}, \tag{3.2}$$

for some positive constant C, where $[u]$ stands for the integer part of a real number u. Moreover,

$$\lim_{t\to+\infty} \frac{1}{t}\, \mathbf{E} \sup_{(n,\mathbf{s},\mathbf{x})\in T_r} B_n(\mathbf{s},\mathbf{x}) \equiv F(r) < +\infty. \tag{3.3}$$

Finally, the following scaling identity holds true for the function F: for any $r > 0$, we have $F(r) = \sqrt{r}\ F(1)$.

Remark 3.4. Notice that F coincides with function F defined in [9].

Remark 3.5. Observe that, in order to describe an element of T_r, we have to know the number of jumps n, the times of jumps $\mathbf{s} \in S_{n,t}$ and the paths $\mathbf{x} \in P_n$. Hence, the family $B_n(\mathbf{s},\mathbf{x})$ can be considered as a Gaussian field over T_r. Moreover, the random variable $\sup_{(n,\mathbf{s},\mathbf{x})\in T_r} B_n(\mathbf{s},\mathbf{x})$ and its expectation only depend on the parameters r and t.

Before giving the proof Proposition 3.3, let us also state the following corollary, which can be proved along the same lines.

Corollary 3.6. *Let T_r^ϱ be the space of paths of the continuous time random walk starting at $x = 0$ with no more than rt jumps, with the additional constraint that the jumps are separated from each other and from the endpoints of the interval $[0,t]$ by a distance of at least 2ϱ. Then, the limit*

$$\lim_{t\to+\infty} \frac{1}{t}\, \mathbf{E} \sup_{(n,\mathbf{s},\mathbf{x})\in T_r^\varrho} B_n(\mathbf{s},\mathbf{x}) \equiv F^\varrho(r) < +\infty$$

exists and, moreover, $F^\varrho(r) = \sqrt{r}\ F^{r\varrho}(1)$.

Proof of Proposition 3.3: In order to obtain the bound (3.2) we will use Theorem 3.1, which involves the entropy of the Gaussian field $B_n(\mathbf{s},\mathbf{x})$ over T_r. Let us then estimate this entropy: for $n \ge 0$, $\mathbf{s},\mathbf{s}' \in S_{n,t}$ and $\mathbf{x} \in P_n$, we define the distance between $(n,\mathbf{s},\mathbf{x})$ and $(n,\mathbf{s}',\mathbf{x})$ as

$$d((n,\mathbf{s},\mathbf{x}),(n,\mathbf{s}',\mathbf{x})) = \sqrt{\mathbf{E}\left[|B_n(\mathbf{s},\mathbf{x}) - B_n(\mathbf{s}',\mathbf{x})|^2\right]};$$

and let $\mathcal{N}(\eta)$ be the entropy function related with $B_n(\mathbf{s},\mathbf{x})$ and T_r. Since $\mathbf{E}\left[B_n(\mathbf{s},\mathbf{x})^2\right] = t$, the diameter of T_r is smaller than $2\sqrt{t}$. Assume then that $\eta \le 2\sqrt{t}$. It is not difficult to check (see [18] for a similar computation) that

$$d^2((n,\mathbf{s},\mathbf{x}),(n,\mathbf{s}',\mathbf{x})) \le 2\sum_{j=0}^{n} \left|s_j - s_j'\right|.$$

Thanks to this identity, one can construct a η-net in T_r for the pseudo-metric d. It is simply based on all the paths of the random walk with any position vector $\mathbf{x}$, and j ump times in the subset $\hat{S}_{n,t}$ of $S_{n,t}$ defined as follows: $\hat{S}_{n,t}$ is the set of elements $\mathbf{s} = (s_0,\dots,s_n) \in S_{n,t}$ where all s_j are integer multiples of $\eta^2(4n)^{-1}$ (notice that some of the s_j's can be equal). It is readily checked that all these paths form a η-net in T_r, and furthermore, the cardinal of $\hat{S}_{n,t}$ can be bounded as:

$$\sharp\hat{S}_{n,t} = \frac{1}{n!}\left[\frac{4nt}{\eta^2}\right]^n \le \left(\frac{Ct}{\eta^2}\right)^n,$$

where in the last step we have used the inequality $n! \ge (n/3)^n$. So, owing to the fact that $\sharp P_n$ is bounded by $(2d)^n$, that $n \le [rt]$ and $\eta \le 2\sqrt{t}$, we obtain that

$$\mathcal{N}(\eta) \le \sum_{n=0}^{[rt]} (2d)^n \left(\frac{Ct}{\eta^2}\right)^n \le \left(\frac{Ct}{\eta^2}\right)^{[rt]} \le 1 \vee \left(\frac{Ct}{\eta^2}\right)^{[rt]}. \tag{3.4}$$

Thus, using Theorem 3.1, we end up with:

$$\begin{aligned}\mathbf{E}\sup_{(n,\mathbf{s},\mathbf{x})\in T_r} B_n(\mathbf{s},\mathbf{x}) &\le C\int_0^{+\infty}\sqrt{\log\mathcal{N}(\eta)}\,d\eta \le C\sqrt{[rt]}\int_0^{2\sqrt{t}}\sqrt{\log\left(\frac{Ct}{\eta^2}\right)}\,d\eta \\ &\le C\sqrt{[rt]t}\int_0^1\sqrt{-\log\eta^2}\,d\eta \le C\sqrt{[rt]t}.\end{aligned}$$

Our claim (3.3) is now easily verified thanks to a super-additive argument and, since W is a Wiener process in t, we also get, by Brownian scaling:

$$F(r) = \lim_{t\to+\infty}\frac{1}{t}\,\mathbf{E}\sup_{(n,\mathbf{s},\mathbf{x})\in T_r} B_n(\mathbf{s},\mathbf{x}) = \sqrt{r}\lim_{t\to+\infty}\frac{1}{t}\,\mathbf{E}\sup_{(n,\mathbf{s},\mathbf{x})\in T_1} B_n(\mathbf{s},\mathbf{x}) = \sqrt{r}F(1).$$

□

Notice that Corollary 3.6 depends on the discretization type parameter ϱ, which is useful for technical purposes. However, in order to get our final estimate on $p(\beta)$, we will need the following lemma, which relates $F^\varrho(1)$ and $F(1)$.

Lemma 3.7. *With the notations of Proposition 3.3 and Corollary 3.6 we have that*

$$F^\varrho(1) \underset{\varrho\to 0}{\longrightarrow} F(1).$$

Proof. Let us denote by ϕ and ϕ^ϱ the quantities defined by:

$$\phi^\varrho(r,t) = \mathbf{E}\sup_{(n,\mathbf{s},\mathbf{x})\in T_r^\varrho} B_n(\mathbf{s},\mathbf{x}), \quad \text{and} \quad \phi(r,t) = \mathbf{E}\sup_{(n,\mathbf{s},\mathbf{x})\in T_r} B_n(\mathbf{s},\mathbf{x}).$$

We will prove that, for an arbitrary $\varepsilon > 0$, we can find $\varrho_0 > 0$ such that

$$F(1) - \lim_{t\to\infty} \frac{\phi^{\varrho}(r,t)}{t} < \varepsilon, \qquad \text{for } \varrho \le \varrho_0. \tag{3.5}$$

Indeed, the triangular inequality implies that

$$\left|F(1) - \frac{\phi^{\varrho}(r,t)}{t}\right| \le \left|F(1) - \frac{\phi(r,t)}{t}\right| + \left|\frac{\phi(r,t)}{t} - \frac{\phi^{\varrho}(r,t)}{t}\right|. \tag{3.6}$$

Now, according to Proposition 3.3, there exists t_0 such that, for any $t \ge t_0$,

$$\left|F(1) - \frac{1}{t}\ \phi(r,t)\right| \le \frac{\varepsilon}{2}. \tag{3.7}$$

Furthermore, for a fixed t_0, since $\phi^{\varrho}(r,t_0) \to \phi(r,t_0)$ when $\varrho \to 0$, there exists $\varrho_0 > 0$ such that

$$\left|\frac{1}{t_0}\left[\phi(r,t_0) - \phi^{\varrho}(r,t_0)\right]\right| \le \frac{\varepsilon}{2} \qquad \text{for } \varrho \le \varrho_0. \tag{3.8}$$

Finally, a super-additivity type argument easily yields the fact that $\phi^{\varrho}(r,t)/t$ is increasing in t. This property, together with (3.6)-(3.8) implies (3.5), which ends our proof. □

Let us now complete the information we have obtained on the expected value of $\sup B_n(\mathbf{s},\mathbf{x})$ by a study of the fluctuations in $\mathbf{s}$ of the field $B_n(\mathbf{s},\mathbf{x})$. To this purpose, let us introduce a little more notation: for $r,\rho > 0$, let $Y_{r,\rho} \subseteq T_r \times T_r$ be the set defined as:

$$Y_{r,\rho} = \Big\{((n,\mathbf{s},\mathbf{x}),(n',\mathbf{s}',\mathbf{x}')) \in T_r \times T_r;\ n = n' \le [rt],\ \mathbf{x} = \mathbf{x}', \\ |s_j - s'_j| \le \rho \text{ for } 1 \le j \le n\Big\}. \tag{3.9}$$

Proposition 3.8. *For $r,\rho > 0$, let $\Upsilon(r,\rho) = \limsup_{t\to\infty} \frac{1}{t}\ \mathbf{E}\sup_{Y_{r,\rho}} A_n(\mathbf{s},\mathbf{s}',\mathbf{x})$, where $A_\cdot(\cdot,\cdot,\cdot)$ is a fluctuation Gaussian field defined on $Y_{r,\rho}$ as follows:*

$$A_n(\mathbf{s},\mathbf{s}',\mathbf{x}) = B_n(\mathbf{s},\mathbf{x}) - B_n(\mathbf{s}',\mathbf{x}). \tag{3.10}$$

Then, $\Upsilon(r,\rho) = \sqrt{r}\ \Upsilon(1,r\rho)$ and

$$\lim_{\rho\to 0} \Upsilon(r,\rho) = 0. \tag{3.11}$$

Proof. The scaling property $\Upsilon(r,\rho) = \sqrt{r}\ \Upsilon(1,r\rho)$ is easily shown. Let us concentrate then on relation (3.11): The bound (3.4) implies that the entropy for the field $B_n(\mathbf{s},\mathbf{x})$ over T_1 is bounded as follows:

$$\mathcal{N}_B(\eta) \le 1 \vee \left(\frac{Ct}{\eta^2}\right)^t. \tag{3.12}$$

Hence, the entropy for the field $A_n(\mathbf{s},\mathbf{s}',\mathbf{x})$ over $Y_{1,\rho}$ also satisfies:

$$\mathcal{N}_A(\eta) \le \mathcal{N}_B^2\big(\frac{\eta}{2}\big). \tag{3.13}$$

The second ingredient to start the proof of (3.11) is a bound on the diameter of $Y_{1,\rho}$ in the canonical metric associated to B:

$$\mathrm{Diam}(Y_{1,\rho}) \le 2 \sup_{Y_{1,\rho}} \mathbf{E}^{1/2}\left[A_n^2(\mathbf{s},\mathbf{s}',\mathbf{x})\right] \le 2\sqrt{\rho t}. \tag{3.14}$$

Then, Theorem 3.1 together with (3.12), (3.13) and (3.14) yield

$$\begin{aligned}\mathbf{E}\sup_{Y_{r,\rho}} A_n(\mathbf{s},\mathbf{s}',\mathbf{x}) &\le C\int_0^{2\sqrt{\rho t}} \sqrt{\log \mathcal{N}_A(\eta)}\, d\eta \le C\int_0^{2\sqrt{\rho t}} \sqrt{\log\left(\frac{4Ct}{\eta^2}\right)^{2t}}\, d\eta \\ &= 2C\sqrt{2t}\int_0^{2\sqrt{\rho t}} \sqrt{\log\left(\frac{4Ct}{\eta^2}\right)}\, d\eta \le Ct\int_0^{\sqrt{\rho/C}} \sqrt{-2\log\zeta}\, d\zeta.\end{aligned}$$

The proof is now finished along the same lines as for Proposition 3.3. □

It is worth mentioning at this point that the function F emerging at relation (3.3) is the one which allows us to define the constant C_0 in Theorem 1.2. Indeed, this constant is simply given by

$$C_0 = \frac{1}{8}F(1)^2. \tag{3.15}$$

With these preliminaries in hand, we are now ready to proceed to the proof of our Theorem 1.2. This proof will be divided between the lower and the upper bound, for which the techniques involved are slightly different.

3.3. Proof of Theorem 1.2: the lower bound. Recall that we wish to prove that given an arbitrary positive constant ε_0, we have that for β large enough $p(\beta) \ge C_0\frac{\beta^2}{\log(\beta)}(1-\varepsilon_0)$. Now, since $p(\beta)$ exists and is non-random, we only need to prove that, for $\beta \ge \beta_0$ and for t large enough,

$$\mathbf{P}\left\{\frac{1}{t\beta^2}\log\left[\sum_{n=0}^{[r(\beta)t]} e^{-2dt}\sum_{\mathbf{x}\in P_n}\int_{S_{n,t}} e^{\beta B_n(\mathbf{s},\mathbf{x})} ds_1\cdots ds_n\right] \ge \frac{C_0(1-\varepsilon_0)}{\log(\beta)}\right\} \ge \frac{1}{2}, \tag{3.16}$$

where, as mentioned at Section 3.1, the parameter $r(\beta)$ will be chosen as:

$$r(\beta) = \frac{C_0}{2}\frac{\beta^2}{\log^2(\beta)}.$$

Step 1: Reduction of the problem. Observe that, on the one hand, we have

$$\begin{aligned}&\frac{1}{t\beta^2}\log\left[\sum_{n=0}^{[r(\beta)t]} e^{-2dt}\sum_{\mathbf{x}\in P_n}\int_{S_{n,t}} e^{\beta B_n(\mathbf{s},\mathbf{x})} ds_1\cdots ds_n\right] \\ &\quad\ge \frac{1}{t\beta^2}\log\left[\sup_{n\le[r(\beta)t]} e^{-2dt}\sum_{\mathbf{x}\in P_n}\int_{S_{n,t}} e^{\beta B_n(\mathbf{s},\mathbf{x})} ds_1\cdots ds_n\right] \\ &\quad\ge -\frac{2d}{\beta^2} + \frac{1}{t\beta^2}\log\left[\sup_{n\le[r(\beta)t]}\sup_{\mathbf{x}\in P_n}\int_{S_{n,t}} e^{\beta B_n(\mathbf{s},\mathbf{x})} ds_1\cdots ds_n\right].\end{aligned} \tag{3.17}$$

On the other hand, for $\beta \geq \beta_1$ and β_1 large enough, we have $\frac{2d}{\beta^2} \leq C_0 \frac{\varepsilon_0}{2} \frac{1}{\log(\beta)}$. Furthermore, since $n \leq \frac{C_0}{2} \frac{\beta^2}{\log^2(\beta)} t$, we obtain:

$$\frac{2n\log(\beta)}{t\beta^2} \leq C_0 \frac{1}{\log(\beta)}, \quad \text{and hence} \quad \frac{\log(\beta^{-2n})}{t\beta^2} \geq -C_0 \frac{1}{\log(\beta)}.$$

Plugging this inequality into (3.17), we get that in order to show (3.16), it is sufficient to prove that, for β, t large enough, the following relation holds:

$$\mathbf{P}\left\{ \frac{1}{t\beta^2} \log\left[\sup_{n\leq[r(\beta)t]} \sup_{\mathbf{x}\in P_n} \int_{S_{n,t}} \beta^{2n} e^{\beta B_n(\mathbf{s},\mathbf{x})} ds_1 \cdots ds_n \right] \geq C_0(2-\varepsilon_0/2)\frac{1}{\log(\beta)} \right\} \geq \frac{1}{2}. \tag{3.18}$$

In order to show relation (3.18) set first $\hat{r}(\beta) = (2\beta^2)^{-1}$. For $r, \varrho > 0$, define also a set $\hat{Y}_{r,\varrho}$ by

$$\hat{Y}_{r,\varrho} = \left\{ ((n,\mathbf{s},\mathbf{x}),(n,\mathbf{s}',\mathbf{x})) \in Y_{r,\varrho};\ |s_{j+1} - s_j| \geq 2\varrho, \text{ for } j = 1\dots n \right\}, \tag{3.19}$$

where $Y_{r,\varrho}$ is defined by (3.9). Finally, for $((n,\mathbf{s},\mathbf{x}),(n,\mathbf{s}',\mathbf{x})) \in Y_{r(\beta),\hat{r}(\beta)}$, we set

$$\xi_\beta(\mathbf{s}') = \{\mathbf{s};\ |s_j - s_j'| \leq \hat{r}(\beta),\ 1 \leq j \leq n\}.$$

Using the definition of these sets, and recalling that the field $A_\cdot(\cdot,\cdot,\cdot)$ has been defined by (3.10), we have that, for any $(n,\mathbf{s}',\mathbf{x}) \in T_{r(\beta)}^{2\hat{r}(\beta)}$,

$$\begin{aligned}\int_{S_{n,t}} \beta^{2n} e^{\beta B_n(\mathbf{s},\mathbf{x})} ds_1 \cdots ds_n &\geq \beta^{2n} e^{\beta B_n(\mathbf{s}',\mathbf{x})} \int_{S_{n,t}} e^{\beta A_n(\mathbf{s},\mathbf{s}',\mathbf{x})} ds_1 \cdots ds_n \\ &\geq \beta^{2n} e^{\beta B_n(\mathbf{s}',\mathbf{x})} \int_{\xi_\beta(\mathbf{s}')} e^{\beta A_n(\mathbf{s},\mathbf{s}',\mathbf{x})} ds_1 \cdots ds_n \\ &\geq e^{\beta B_n(\mathbf{s}',\mathbf{x})} \exp\left(-\sup\{\beta A_n(\mathbf{s},\mathbf{s}',\mathbf{x});\ \mathbf{s} \in \xi_\beta(\mathbf{s}')\}\right).\end{aligned} \tag{3.20}$$

Since (3.20) is true for any $(n,\mathbf{s}',\mathbf{x}) \in T_{r(\beta)}^{2\hat{r}(\beta)}$, and with (3.18) in mind, we obtain:

$$\frac{1}{t\beta^2} \log\left[\sup_{n\leq[r(\beta)t]} \sup_{\mathbf{x}\in P_n} \int_{S_{n,t}} \beta^{2n} e^{\beta B_n(\mathbf{s},\mathbf{x})} ds_1 \cdots ds_n \right] \geq \frac{1}{t\beta} \sup_{T_{r(\beta)}^{2\hat{r}(\beta)}} B_n(\mathbf{s}',\mathbf{x}) - \frac{1}{t\beta} \sup_{\hat{Y}_{r(\beta),\hat{r}(\beta)}} A_n(\mathbf{s},\mathbf{s}',\mathbf{x}). \tag{3.21}$$

Step 2: Study of the term involving B_n. Let us consider $\varepsilon_1 > 0$. According to Corollary 3.6, there exists a constant $\beta_2 > 0$ such that for $\beta \geq \beta_2$, we can find

$t' = t'(\varepsilon_1, \beta)$ satisfying, for $t \geq t'$:

$$\begin{aligned}
\frac{1}{\beta}\mathbf{E}\left[\sup_{T_{r(\beta)}^{2\hat{r}(\beta)}} B_n(\mathbf{s}', \mathbf{x})\right] &\geq \frac{t}{\beta}\left[F^{2\hat{r}(\beta)}(r(\beta)) - \frac{\varepsilon_1\sqrt{r(\beta)}}{2}\right] \\
&= \frac{t}{\beta}\left[\sqrt{r(\beta)}F^{2\hat{r}(\beta)r(\beta)}(1) - \frac{\varepsilon_1\sqrt{r(\beta)}}{2}\right] \\
&= \frac{t}{\beta}\sqrt{r(\beta)}\left[F^{2\hat{r}(\beta)r(\beta)}(1) - \frac{\varepsilon_1}{2}\right].
\end{aligned}$$

Notice that we have chosen $r(\beta)$ and $\hat{r}(\beta)$ so that

$$\lim_{\beta\to\infty} r(\beta)\hat{r}(\beta) = 0. \tag{3.22}$$

Hence, applying now Lemma 3.7, we get that there exists $\beta_3 > 0$ such that for any $\beta \geq \beta_3$, it holds that

$$\frac{1}{\beta}\mathbf{E}\left[\sup_{T_{r(\beta)}^{2\hat{r}(\beta)}} B_n(\mathbf{s}', \mathbf{x})\right] \geq \frac{t}{\beta}\sqrt{r(\beta)}\left[F(1) - \varepsilon_1\right].$$

Thus, Theorem 3.2 for $\tau^2 = \frac{t}{\beta^2}$ and $y = \frac{t}{\beta}\varepsilon_1\beta\sqrt{r(\beta)}$ implies, for t large enough,

$$\mathbf{P}\left\{\sup_{T_{r(\beta)}^{2\hat{r}(\beta)}} \frac{1}{\beta}B_n(\mathbf{s}', \mathbf{x}) < \frac{t}{\beta}\sqrt{r(\beta)}\left[F(1) - 2\varepsilon_1\right]\right\} \leq 2\exp\left\{-\frac{t\varepsilon_1^2 r(\beta)}{2}\right\} \leq \frac{1}{4}. \tag{3.23}$$

Step 3: Study of the term involving A_n. Invoking Proposition 3.8, we have

$$\limsup_{t\to\infty}\frac{1}{t}\,\mathbf{E}\sup_{Y_{r(\beta),\hat{r}(\beta)}} A_n(\mathbf{s}, \mathbf{s}', \mathbf{x}) < \Upsilon(r(\beta), \hat{r}(\beta)) + \frac{\varepsilon_1}{2}.$$

Then, since (3.22) holds true, and using (3.11), we can choose $\beta_4 > 0$ such that for t large enough and $\beta \geq \beta_4$ we have:

$$\frac{1}{\beta}\mathbf{E}\left[\sup_{Y_{r(\beta),\hat{r}(\beta)}} \beta A_n(\mathbf{s}, \mathbf{s}', \mathbf{x})\right] \leq \frac{t}{\beta}\sqrt{r(\beta)}\left(\Upsilon(1, r(\beta)\hat{r}(\beta)) + \frac{\varepsilon_1}{2}\right) \leq \varepsilon_1 t\beta\sqrt{r(\beta)}.$$

Hence, Theorem 3.2 for $\tau^2 \leq 2\frac{t}{\beta^2}$ and $y = \varepsilon_1\frac{t}{\beta}\sqrt{r(\beta)}$ yields, for t large enough,

$$\mathbf{P}\left\{\sup_{Y_{r(\beta),\hat{r}(\beta)}} \frac{1}{\beta}A_n(\mathbf{s}, \mathbf{s}', \mathbf{x}) > 2\varepsilon_1\frac{t}{\beta}\sqrt{r(\beta)}\right\} \leq 2\exp\left\{-\frac{t\varepsilon_1^2 r(\beta)}{4}\right\} \leq \frac{1}{4}. \tag{3.24}$$

Step 4: Conclusion. Choose $\beta_0 = \beta_1 \vee \beta_2 \vee \beta_3 \vee \beta_4$, and consider $\beta \geq \beta_0$. Define also the sets:

$$\begin{aligned}
\Omega_1 &= \left\{\sup_{Y_{r(\beta),\hat{r}(\beta)}} \frac{1}{\beta}A_n(\mathbf{s}, \mathbf{s}', \mathbf{x}) > 2\varepsilon_1\frac{t}{\beta}\sqrt{r(\beta)}\right\}, \\
\Omega_2 &= \left\{\sup_{T_{r(\beta)}^{2\hat{r}(\beta)}} \frac{1}{\beta}B_n(\mathbf{s}', \mathbf{x}) < \frac{t}{\beta}\sqrt{r(\beta)}\left[F(1) - 2\varepsilon_1\right]\right\},
\end{aligned}$$

for a constant $\varepsilon_1 = \varepsilon_0\sqrt{2C_0}/16$. Then, (3.23) and (3.24) yield $P(\Omega_1)\vee P(\Omega_2) \le 1/4$. Moreover, inequality (3.21) implies

$$\mathbf{P}\left\{\frac{1}{t\beta^2}\ \log\left[\sup_{n\le[r(\beta)t]}\sup_{\mathbf{x}\in P_n}\int_{S_{n,t}}\beta^{2n}e^{B_n(\mathbf{s},\mathbf{x})}ds_1\cdots ds_n\right] \ge C_0(2-\varepsilon_0/2)\frac{1}{\log(\beta)}\right\}$$

$$\ge \mathbf{P}\left\{\Omega_1^c\cap\Omega_2^c\right\} = \mathbf{P}\left\{(\Omega_1\cup\Omega_2)^c\right\} \ge 1-\mathbf{P}\left(\Omega_1\right)-\mathbf{P}\left(\Omega_2\right) \ge \frac{1}{2},$$

which proves (3.18), and thus the lower bound of our Theorem 1.2.

3.4. Proof of Theorem 1.2: the upper bound. As in the lower bound section, let us recall that we wish to prove that, for β large enough and an arbitrary positive constant ε_0, we have $p(\beta) \le C_0\frac{\beta^2}{\log(\beta)}\,(1+\varepsilon_0)$. Again, since $p(\beta)$ exists and is non-random, we only need to prove that, for $\beta \ge \beta_5$ and for t large enough

$$\mathbf{P}\left\{\frac{1}{t}\ \log\left[\sum_{n=0}^{\infty}e^{-2dt}\sum_{\mathbf{x}\in P_n}\int_{S_{n,t}}e^{\beta B_n(\mathbf{s},\mathbf{x})}ds_1\cdots ds_n\right] \le \frac{C_0(1+\varepsilon_0)\beta^2}{\log(\beta)}\right\} \ge \frac{1}{2}.$$

Actually, we will prove the equivalent inequality

$$\mathbf{P}\left\{\frac{1}{t}\ \log\left[\sum_{n=0}^{\infty}e^{-2dt}\sum_{\mathbf{x}\in P_n}\int_{S_{n,t}}e^{\beta B_n(\mathbf{s},\mathbf{x})}ds_1\cdots ds_n\right] \ge \frac{C_0(1+\varepsilon_0)\beta^2}{\log(\beta)}\right\} \le \frac{1}{2}. \tag{3.25}$$

Step 1: Setup. Let $\nu = C_0(1+\varepsilon_0)\beta^2/\log(\beta)$. For t large enough, the probability defined in (3.25) can be estimated from above by the sum of the probabilities of disjoint sets as follows:

$$\mathbf{P}\left\{\sum_{n=0}^{\infty}e^{-2dt}\sum_{\mathbf{x}\in P_n}\int_{S_{n,t}}e^{\beta B_n(\mathbf{s},\mathbf{x})}ds_1\cdots ds_n \ge e^{\nu t}\right\} \le \sum_{l=1}^{\infty}\mathbf{P}\left(\Lambda_{a_l,b_l}(\nu_l)\right), \tag{3.26}$$

where the sets $\Lambda_{a_l,b_l}(\nu_l)$ are defined by:

$$\Lambda_{a_l,b_l}(\nu_l) = \left\{\sum_{m=\left[\frac{a_l t\beta^2}{\log^2(\beta)}+1\right]}^{\left[\frac{b_l t\beta^2}{\log^2(\beta)}\right]} e^{-2dt}\sum_{\mathbf{x}\in P_m}\int_{S_{m,t}}e^{\beta B_m(\mathbf{s},\mathbf{x})}ds_1\cdots ds_m \ge e^{\nu_l t}\right\},$$

and for $l \ge 1$, the quantities a_l, b_l, ν_l are of the form

$$a_l = (l-1)\varrho_1, \qquad b_l = l\varrho_1, \qquad \nu_l = \left(C_0(1+\varepsilon_0/2) - l\varrho\right)\frac{\beta^2}{\log(\beta)},$$

for two positive constants ϱ and ϱ_1 which will be chosen later on, at relation (3.32). Notice that the first set $\Lambda_{a_1,b_1}(\nu_1)$ starts at $m = 0$ instead of $m = \left[a_1 t\beta^2/\log^2(\beta)+1\right]$. However, this set can be handled along the same lines as the other ones. Observe also that the estimate (3.26) holds true due to the identity $\sum_{l=1}^{\infty}e^{\nu_l t} < e^{\nu t}$, satisfied for t large enough.

Step 2: Study the sets $\Lambda_{a_l,b_l}(\nu_l)$. Computing the number of terms in each sum and the area of $S_{m,t}$, we obtain that

$$\begin{aligned} Q_l \equiv & \sum_{m=\left[\frac{a_l t\beta^2}{\log^2(\beta)}+1\right]}^{\left[\frac{b_l t\beta^2}{\log^2(\beta)}\right]} e^{-2dt} \sum_{\mathbf{x}\in P_m} \int_{S_{m,t}} e^{\beta B_m(\mathbf{s},\mathbf{x})} ds_1 \cdots ds_m \\ & \leq \frac{t(b_l-a_l)\beta^2}{\log^2(\beta)}\; e^{-2dt}(2d)^{\frac{b_l t\beta^2}{\log^2(\beta)}} \exp\left\{ \sup_{T_{b_l(\beta)}} \beta B_m(\mathbf{s},\mathbf{x}) \right\} \\ & \qquad \times \sup_{\left[\frac{a_l t\beta^2}{\log^2(\beta)}+1\right]\leq m \leq \left[\frac{b_l t\beta^2}{\log^2(\beta)}\right]} \frac{t^m}{m!}, \end{aligned} \tag{3.27}$$

where we have set $b_l(\beta) = b_l\beta^2/\log^2(\beta)$. Furthermore, since $x \leq (2d)^x$, we have that

$$\frac{t(b_l-a_l)\beta^2}{\log^2(\beta)} \leq (2d)^{\frac{b_l t\beta^2}{\log^2(\beta)}},$$

and hence the logarithm of Q_l in (3.27) is bounded by

$$\begin{aligned} \log(Q_l) \quad \leq \quad & \frac{2b_l t\beta^2}{\log^2(\beta)} \log(2d) - 2dt + \sup_{T_{b_l(\beta)}} \beta B_m(\mathbf{s},\mathbf{x}) \\ & \qquad + \log \sup_{\left[\frac{a_l t\beta^2}{\log^2(\beta)}+1\right]\leq m \leq \left[\frac{b_l t\beta^2}{\log^2(\beta)}\right]} \frac{t^m}{m!}. \end{aligned} \tag{3.28}$$

Let us find now an estimate for $\sup t^m/m!$ in the expression above: denote by K a generic constant which depends only on d and can change at each step of our computations. Owing to the bound $(m/3)^m \leq m!$, notice that for $l \geq 2$ and β large enough this supremum is attained at the initial point $\left[\frac{a_l t\beta^2}{\log^2(\beta)}+1\right]$. So, we have

$$\begin{aligned} \log \sup_{\left[\frac{a_l t\beta^2}{\log^2(\beta)}+1\right]\leq m \leq \left[\frac{b_l t\beta^2}{\log^2(\beta)}\right]} \frac{t^m}{m!} & \leq \frac{a_l t\beta^2}{\log^2(\beta)} \left[\log t - \log\left(\frac{a_l t\beta^2}{\log^2(\beta)} \right) + K \right] \\ & = \frac{a_l t\beta^2}{\log^2(\beta)} \left[K - \log(a_l) - 2\log(\beta) + 2\log(\log(\beta)) \right], \end{aligned}$$

for any $l \geq 2$. Using this fact, the trivial bound $b_l \leq 2a_l$ and plugging the last considerations into (3.28) and (3.27), we end up with

$$\log(Q_l) \leq \sup_{T_{b_l(\beta)}} \beta B_m(\mathbf{s},\mathbf{x}) + \frac{a_l t\beta^2}{\log^2(\beta)} \left[K - 2\log(\beta) - \log(a_l) + 2\log(\log(\beta)) \right].$$

With this inequality in mind and recalling the definition of $\Lambda_{a_l,b_l}(\nu_l)$, it is readily checked that:

$$\mathbf{P}\left(\Lambda_{a_l,b_l}(\nu_l)\right) \leq \tag{3.29}$$
$$\mathbf{P}\left\{\sup_{T_{b_l(\beta)}} B_m(\mathbf{s},\mathbf{x}) \geq t\left[\frac{\nu_l}{\beta}+\frac{a_l\beta}{\log^2(\beta)}\left[2\log(\beta)+\log(a_l)-K-2\log(\log(\beta))\right]\right]\right\},$$

for any $l \geq 2$.

In order to get an accurate bound on $\mathbf{P}(\Lambda_{a_l,b_l}(\nu_l))$, we will now use the information about B_n we have gathered at Section 3.2: notice for instance that Proposition 3.3 asserts that

$$\lim_{t\to+\infty}\frac{1}{t}\,\mathbf{E}\sup_{(n,\mathbf{s},\mathbf{x})\in T_{b_l(\beta)}} B_n(\mathbf{s},\mathbf{x}) = F\left(\frac{b_l\beta^2}{\log^2(\beta)}\right) = \frac{\sqrt{b_l}\beta}{\log(\beta)}\ F(1).$$

Then Theorem 3.2 applied to $\tau^2 = t$ together with (3.29) and the last relation imply that, for $l \geq 2$, the probability of $\Lambda_{a_l,b_l}(\nu_l)$ can be bounded as

$$\mathbf{P}\left(\Lambda_{a_l,b_l}(\nu_l)\right) \leq 2\exp\left\{-\frac{t}{2}\frac{\beta^2}{\log^2(\beta)}\Xi_l^2\right\}, \tag{3.30}$$

where, plugging the definitions of a_l, b_l and ν_l, we can write:

$$\begin{aligned}
\Xi_l &= \frac{\log\beta}{\beta}\left(\frac{\nu_l}{\beta}+\frac{a_l\beta}{\log^2(\beta)}\left[2\log(\beta)+\log(a_l)-K-2\log(\log(\beta))\right]-\frac{\sqrt{b_l}\beta}{\log(\beta)}\ F(1)\right)\\
&= C_0\left(1+\frac{\varepsilon_0}{2}\right)-l\varrho+\frac{(l-1)\varrho_1}{\log(\beta)}\left[2\log(\beta)+\log((l-1)\varrho_1)-K-2\log(\log(\beta))\right]\\
&\qquad -\sqrt{l\varrho_1}F(1)\\
&\geq C_0\left(1+\frac{\varepsilon_0}{2}\right)-l\varrho+2l\varrho_1-2\varrho_1+\frac{(l-1)\varrho_1}{\log(\beta)}\left[\log(\varrho_1)-K-2\log(\log(\beta))\right]\\
&\qquad -\sqrt{l\varrho_1}F(1).
\end{aligned}$$

Observe that inequality (3.30) has been obtained for $l \geq 2$. The same kind of calculations are also valid for $l = 1$, except for the bound on $t^m/m!$. Indeed, in this latter case, owing to the facts that the maximum of the function $f(x) = \left(\frac{3t}{x}\right)^x$ is attained at $x = \frac{3t}{e}$ and that $2d > 3/e$, we obtain that

$$\log\sup_{0\leq m\leq\left[\frac{b_1 t\beta^2}{\log^2(\beta)}\right]}\frac{t^m}{m!} \leq \frac{3t}{e},$$

which yield a coefficient Ξ_1 of the form:

$$\Xi_1 := C_0\left(1+\frac{\varepsilon_0}{2}\right)-\varrho+\varrho_1\frac{K}{\log(\beta)}-\sqrt{\varrho_1}F(1).$$

Going back to inequality (3.25), it is also worth mentioning that all the previous considerations only make sense if $\Xi_l \geq 0$ for all $l \geq 1$.

Step 3: Conclusion. In order to prove (3.25) and finish the proof of the upper bound, according to (3.26) and (3.30), we only need to show, for t large enough, that

$$\sum_{l=1}^{\infty} 2\exp\left\{-\frac{t}{2}\frac{\beta^2}{\log^2(\beta)}\Xi_l^2\right\} < \frac{1}{2}, \tag{3.31}$$

with the additional restriction $\Xi_l \geq 0$. Now, in order to satisfy this latter condition, we choose $C_0 = \frac{1}{8}\, F(1)^2$, which allows to get rid of the term $-\sqrt{l\varrho_1}F(1)$ as follows:

$$\begin{aligned} \Xi_l \quad \geq \quad & \left(\sqrt{C_0\left(1+\frac{\varepsilon_0}{4}\right)} - \sqrt{\frac{2l\varrho_1}{1+\frac{\varepsilon_0}{4}}}\right)^2 + \left(\frac{C_0\varepsilon_0}{4} - 2\varrho_1\right) \\ & + l\left(2\varrho_1\left(\frac{\varepsilon_0}{4+\varepsilon_0}\right) - \varrho\right) + \varrho_1\Psi_l, \end{aligned}$$

where we have set $\Psi_1 := \frac{K}{\log(\beta)}$, and for any $l \geq 2$:

$$\Psi_l := \frac{(l-1)}{\log(\beta)}\left[\log(\varrho_1) - K - 2\log(\log(\beta))\right].$$

Let us insist at this point on the fact that $C_0 = \frac{1}{8}\, F(1)^2$ is the largest value of C_0 allowing such a decomposition. So, choosing

$$\varrho_1 := \frac{C_0\varepsilon_0}{16}, \quad \text{and} \quad \varrho := \varrho_1\frac{\varepsilon_0}{4+\varepsilon_0}, \tag{3.32}$$

we clearly have that

$$\Xi_l \geq \frac{C_0\varepsilon_0}{8} + l\frac{C_0\varepsilon_0^2}{16(4+\varepsilon_0)} + \frac{C_0\varepsilon_0}{16}\Psi_l.$$

Thus, there exists β_5 such that for any $\beta \geq \beta_5$ and for any $l \geq 1$, Ξ_l is strictly positive and the following bound holds true:

$$\Xi_l \geq \frac{C_0\varepsilon_0}{16} + l\frac{C_0\varepsilon_0^2}{32(4+\varepsilon_0)}.$$

Inequality (3.31), which ends our proof, follows now easily.

4. Weak and Strong Disorder Regimes

This section is devoted to the proof of Proposition 1.1, starting from the result on weak disorder:

Lemma 4.1. *Assume $d \geq 3$. Then, for β in a neighborhood of 0, the polymer is in the weak disorder regime.*

Proof. Similarly to [13, 26], it suffices to show that

$$\mathbf{E}[Z_t^2] \leq K_1(\mathbf{E}[Z_t])^2. \tag{4.1}$$

Indeed, recall that the martingale M has been defined by $M_t = Z_t\exp(-\frac{\beta^2}{2}t)$. Then, inequality (4.1) yields that M is a martingale bounded in L^2 with $\mathbf{E}[M_t] = 1$, and hence $M_\infty = \lim_{t\to\infty} M_t > 0$ on a set of full probability in Ω, which corresponds to our definition of weak disorder.

In order to check (4.1), let us consider b^1 and b^2 two independent random walks. Then we get

$$\begin{aligned}\mathbf{E}\left[Z_t^2\right] &= E_b\left[\exp\left(\beta^2\left(t+\int_0^t \delta_0(b_s^1-b_s^2)ds\right)\right)\right]\\ &= \left(\mathbf{E}\left[Z_t\right]\right)^2 E_b\left[\exp\left(\beta^2\int_0^t \delta_0(b_s^1-b_s^2)ds\right)\right]\\ &\le \left(\mathbf{E}\left[Z_t\right]\right)^2 E_{\hat b}\left[\exp\left(\beta^2 l_\infty(\hat b)\right)\right],\end{aligned}$$

where $\hat b = b^1 - b^2$ and $l_\infty(\hat b) = \lambda(\{t \ge 0, \hat b_t = 0\})$ with λ the Lebesgue measure. Notice that $\hat b$ is again a continuous time random walk on $\mathbb{Z}^d$ with exponential holding times of parameter $4d$. Following our notation of Section 1, $\hat b$ is described by its jump times $(\hat\tau_i)_{i\ge 0}$ and its positions $(\hat x_i)_{i\ge 0}$. Then, if $\beta < \sqrt{4d}$, introducing the obvious notation $E_{\hat x}$ and $E_{\hat\tau}$, we end up with:

$$\begin{aligned}&E_{\hat b}\left[\exp\left(\beta^2 l_\infty(\hat b)\right)\right] = E_{\hat b}\left[\exp\left(\beta^2\sum_{i=0}^\infty(\hat\tau_{i+1}-\hat\tau_i)\delta_0(\hat x_i)\right)\right]\\ &= E_{\hat x}\left[\prod_{i=0}^\infty E_{\hat\tau}\left[\exp\left(\beta^2(\hat\tau_{i+1}-\hat\tau_i)\delta_0(\hat x_i)\right)\right]\right] = E_{\hat x}\left[\prod_{i=0}^\infty \frac{4d}{4d-\beta^2\delta_0(\hat x_i)}\right]\\ &= E_{\hat x}\left[\left(\frac{4d}{4d-\beta^2}\right)^{L_\infty(\hat x)}\right] = E_{\hat x}\left[\exp\left(\gamma L_\infty(\hat x)\right)\right],\end{aligned}$$

where we have set $\gamma = \gamma(\beta) = \log(4d/(4d-\beta^2))$ and $L_\infty(\hat x) = \#\{j \le n, \hat x_j = 0\}$, which is the local time at $x = 0$ (and $n = \infty$) of the discrete time random walk induced by $\hat x$. It is now a well known fact that, in dimension $d \ge 3$ and for γ small enough, we have $E_{\hat x}\left[\exp\left(\gamma L_\infty(\hat x)\right)\right] < \infty$, since $L_\infty(\hat x)$ is a geometric random variable. Furthermore, $\lim_{\beta\to 0}\gamma(\beta) = 0$, from which our proof is easily finished. $\square$

Let us now say a few words about the remainder of Proposition 1.1: point (3) is a direct consequence of our stronger Theorem 1.2. As far as point (2) is concerned, we refer to [7] for a complete proof of this fact. Like in [13, 26], it is based on an application of Itô's formula to the medium W, which allows to prove that $\lim_{t\to\infty}\mathbf{E}[M_t^\theta] = 0$ for any $\theta \in (0,1)$.

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