Electromagnetic phenomena induced by weak gravitational fields. Foundations for a possible gravitational wave detector

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In the context of the weak-field approximation to the De Rham wave-vector equation, and applying the standard Lorentz gauge to both electromagnetism and gravitation, a linearized expression for the De Rham operator is derived. The derived equation is applied to solve (1) the interaction between plane, monochromatic, and polarized electromagnetic and gravitational waves and (2) the interaction between a transverse electric ($_0$T E_n) mode, propagating in a rectangular wave guide, and a gravitational wave. The second result permits us to consider the possibility of a gravitational-wave antenna based on purely electromagnetic means, because new transverse electric and magnetic modes are generated, whose characteristics depend on the gravitational wave. Evidently this detector is completely different from that based on electromagnetic gravitational resonance designed by Braguinski and Manoukine. Also, in the optic limit of the De Rham equation the known expression for the equivalent index of refraction is obtained and other laws for the amplitude and polarization are derived.

I. LINEARIZATION OF THE DE RHAM WAVE-VECTOR OPERATOR

The equation for the vector potential in curved space-time for a free electromagnetic field is

$$(\Delta_{\rm dR}A)^\mu \equiv -A^{\mu;\alpha}{}_\alpha + R^\mu{}_\alpha A^\alpha = 0\,, \tag{1.1}$$

where we follow the nomenclature and sign convention of Ref. 1 and assume the Lorentz gauge $A^\alpha{}_{;\alpha}=0$.

Our objective is to obtain an equivalent expression for (1.1) suited for applying the linearized theory. After a lengthy calculation we obtain

$$\begin{aligned}(\Delta_{\rm dR}A)^\mu = -g^{\alpha\nu}(A^\mu{}_{,\alpha\nu} + \Gamma^\mu{}_{\beta\nu}A^\beta{}_{,\alpha} \\ + \Gamma^\mu{}_{\gamma\alpha}A^\gamma{}_{,\nu} - \Gamma^\tau{}_{\nu\alpha}A^\mu{}_{,\tau}) \\ - g^{\alpha\nu}\Gamma^\mu{}_{\nu\alpha,\beta}A^\beta + R^\mu{}_\beta A^\beta = 0\,.\end{aligned} \tag{1.2}$$

Now, linearizing, i.e., making the usual weak-field approximation

$$g_{\alpha\nu} = \eta_{\alpha\nu} + h_{\alpha\nu}\,, \quad |h_{\alpha\nu}| \ll 1 \tag{1.3}$$

where $\eta_{\alpha\nu}$ is the Minkowski metric of signature +2 we get

$$\begin{aligned}({}_1\Delta_{\rm dR}A)^\mu = -(\eta^{\alpha\beta} - h^{\alpha\beta})(A^\mu{}_{,\alpha\beta} + {}_1\Gamma^\mu{}_{\nu\beta}A^\nu{}_{,\alpha} \\ + {}_1\Gamma^\mu{}_{\gamma\alpha}A^\gamma{}_{,\beta} - {}_1\Gamma^\gamma{}_{\beta\alpha}A^\mu{}_{,\gamma}) \\ - (\eta^{\alpha\beta} - h^{\alpha\beta})\,{}_1\Gamma^\mu{}_{\beta\alpha,\nu}A^\nu + 2\,{}_1R^\mu{}_\nu A^\nu = 0\,.\end{aligned} \tag{1.4}$$

In this equation the subscript 1 stands for linearized terms. Using the Hilbert-Lorentz gauge

$$\eta^{\alpha\beta}\,{}_1\Gamma^\mu{}_{\alpha\beta} = \bar{h}^{\mu\nu}{}_{,\nu} = 0\,, \tag{1.5}$$

where $\bar{h}^{\mu\nu} \equiv h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h^\alpha{}_\alpha$, we finally obtain the desired equation

$$\begin{aligned}({}_1\Delta_{\rm dR}A)^\mu = -A^\mu{}_{,\alpha}{}^\alpha + h^{\alpha\nu}A^\mu{}_{,\alpha\nu} \\ - 2\,{}_1\Gamma^\mu{}_{\beta\nu}A^{\beta,\nu} + 2\,{}_1R^\mu{}_\beta A^\beta = 0\,.\end{aligned} \tag{1.6}$$

II. GRAVITATIONAL CORRECTION TO FREE ELECTROMAGNETIC FIELDS

We shall now assume that the vector potential A^μ can be split into two parts. Although this assumption is different from that of the gauge-invariant Hamiltonian perturbation theory by Moncrief,[2] we shall follow Bertotti's point of view.[3]

$$A^\mu = {}_0A^\mu + {}_1A^\mu\,, \tag{2.1}$$

where ${}_0A^\mu$ is a solution of the flat-space-time wave equation and ${}_1A^\mu$ is the correction to ${}_0A^\mu$ due to the presence of a weak gravitational field. If (2.1) is introduced into (1.6) and quadratic terms in the corrections ${}_1A^\mu$ and $h^{\alpha\beta}$ are neglected we obtain

$$\begin{aligned}[{}_1\Delta_{\rm dR}({}_0A + {}_1A)]^\mu = -{}_0A^\mu{}_{,\alpha}{}^\alpha - {}_1A^\mu{}_{,\alpha}{}^\alpha \\ + h^{\alpha\nu}{}_0A^\mu{}_{,\alpha\nu} - 2\,{}_1\Gamma^\mu{}_{\beta\nu0}A^{\beta,\nu} \\ + 2\,{}_1R^\mu{}_{\beta0}A^\beta = 0\,.\end{aligned} \tag{2.2}$$

For free fields ${}_0A^\mu{}_{,\alpha}{}^\alpha = 0$, and therefore (2.2) reduces to

$${}_1A^\mu{}_{,\alpha}{}^\alpha = h^{\alpha\nu}{}_0A^\mu{}_{,\alpha\nu} - 2\,{}_1\Gamma^\mu{}_{\beta\nu0}A^{\beta,\nu} + 2\,{}_1R^\mu{}_{\beta0}A^\beta\,. \tag{2.3}$$

This is a wave equation which gives us the linear correction to the free electromagnetic field ${}_0A^\mu$ due to an external applied gravitational field and/or the gravitational field generated by ${}_0A^\mu$.

Formally (2.3) may be written defining an effective-current vector

$$J^\mu_{\rm eff} \equiv \frac{1}{4\pi}\left(h^{\alpha\nu}{}_0A^\mu{}_{,\alpha\nu} - 2\,{}_1\Gamma^\mu{}_{\beta\nu0}A^{\beta,\nu} + 2\,{}_1R^\mu{}_{\beta0}A^\beta\right) \tag{2.4}$$

as

$$ {}_1A^{\mu}{}_{,\alpha}{}^{\alpha} = 4\pi J^{\mu}_{\rm eff} . \qquad (2.5)$$

But the effective current (2.4) is not conservative:

$$J^{\mu}_{{\rm eff},\mu} \neq 0 \qquad (2.6)$$

and therefore has no direct physical meaning from an electromagnetic point of view.

The integral solution of Eq. (2.5) is

$$ {}_1A^{\mu} = \int d^4x' D_R(x-x') J^{\mu}_{\rm eff}(x') , \qquad (2.7)$$

where $D_R(x-x')$ is the known retarded Green's function

$$D_R(x-x') = -\frac{1}{4\pi}\delta[(x-x')^2][1+\epsilon(x-x')] . \qquad (2.8)$$

The integration of (2.7) is extended all over Minkowskian inobservable background space-times.[4]

III. INTERACTION BETWEEN ELECTROMAGNETIC AND GRAVITATIONAL WAVES

We shall consider a plane linear polarized monochromatic electromagnetic wave that interacts obliquely with a plane gravitational wave. Let α be the angle between the two waves, and the direction of propagation of the gravitational wave defines the z axis. The nonperturbed electromagnetic wave has the expression

$$ {}_0A^{\mu} = {}_0A(-\cos\alpha\,\delta^{\mu}{}_y + \sin\alpha\,\delta^{\mu}{}_z) e^{-ik^{\sigma}x_{\sigma}} , \qquad (3.1)$$

where $\delta^{\alpha}{}_{\beta}$ is the Kronecker δ and the propagation vector is

$$k^{\sigma} = (\omega, 0, \omega\sin\alpha, \omega\cos\alpha) . \qquad (3.2)$$

Trivally (3.1) satisfies the Lorentz gauge

$$ {}_0A^{\beta}{}_{,\beta} = 0 . \qquad (3.3)$$

For the gravitational wave we choose the TT gauge (transverse-traceless gauge) and + polarization:

$$\begin{aligned} h^{\rm TT}_{\mu\nu} &= A_{\mu\nu} e^{-iK^{\sigma}x_{\sigma}} \\ &= A_+(\delta^x{}_{\mu}\delta^x{}_{\nu} - \delta^y{}_{\mu}\delta^y{}_{\nu}) e^{-iK^{\sigma}x_{\sigma}} , \end{aligned} \qquad (3.4)$$

where the propagation vector is

$$K^{\sigma} = (W, 0, 0, W) . \qquad (3.5)$$

Equation (2.3), which now reduces to

$$ {}_1A^{\mu}{}_{,\alpha}{}^{\alpha} = h^{\alpha\nu}{}_0A^{\mu}{}_{,\alpha\nu} - 2\,{}_1\Gamma^{\mu}{}_{\beta\nu}\,{}_0A^{\beta,\nu} \qquad (3.6)$$

because as we neglect the electromagnetic field as the source of the gravitational field ${}_1R^{\mu}{}_{\beta}=0$, will give us the correction to the elctromagnetic wave (3.1) due to the presence of the external gravitational wave (3.4).

Applying (3.1) and (3.4) to (3.6), we obtain

$$\begin{aligned} {}_1A^{\mu}{}_{,\alpha}{}^{\alpha} = (&-A_{\alpha\beta}k^{\alpha}k^{\beta}{}_0A^{\mu} - A_{\alpha\beta 0}A^{\alpha}k^{\beta}K^{\mu} \\ &+ A^{\mu}{}_{\alpha 0}A^{\alpha}k_{\beta}K^{\beta} + A^{\mu}{}_{\beta}k^{\beta}K_{\alpha 0}A^{\alpha}) e^{-i(K^{\alpha}+k^{\alpha})x_{\alpha}} , \end{aligned} \qquad (3.7)$$

which after an easy calculation reduces to

$$\begin{aligned} {}_1A^0{}_{,\alpha}{}^{\alpha} &= -A_+A_0\omega W\sin\alpha\cos\alpha e^{-i(K^{\alpha}+k^{\alpha})x_{\alpha}} , \\ {}_1A^x{}_{,\alpha}{}^{\alpha} &= 0 , \\ {}_1A^y{}_{,\alpha}{}^{\alpha} &= -A_+A_0\omega[\omega\sin^2\alpha\cos\alpha \\ &\quad + W\cos\alpha(1-\cos^2\alpha) \\ &\quad + W\sin^3\alpha] e^{-i(K^{\alpha}+k^{\alpha})x_{\alpha}} , \\ {}_1A^z{}_{,\alpha}{}^{\alpha} &= A_+A_0\omega(\omega\sin^3\alpha - W\sin\alpha\cos\alpha) e^{-i(K^{\alpha}+k^{\alpha})x_{\alpha}} . \end{aligned} \qquad (3.8)$$

Define the vector ξ^{μ} by its components,

$$\begin{aligned} \xi^0 &= -A_+A_0\omega W\sin\alpha\cos\alpha , \\ \xi^x &= 0 , \\ \xi^y &= -A_+A_0\omega[\omega\sin^2\alpha\cos\alpha \\ &\quad + W\cos\alpha(1-\cos\alpha) + W\sin^2\alpha] , \\ \xi^z &= A_+A_0\omega\sin\alpha(\omega\sin^2\alpha - W\cos\alpha) . \end{aligned}$$

The effective current (2.4) is written

$$J^{\mu}_{\rm eff}(x') = \xi^{\mu} e^{-i\kappa^{\sigma}x'_{\sigma}} , \qquad (3.9)$$

where we have defined $\kappa^{\sigma} \equiv K^{\sigma} + k^{\sigma}$.

Introducing (3.9) into (2.7) and solving the integral, we obtain the correction ${}_1A^{\mu}$ to ${}_0A^{\mu}$:

$$ {}_1A^{\mu} = -\xi^{\mu}\frac{1}{\kappa^0+\kappa} P\left(\frac{1}{\kappa^0-\kappa}\right) e^{-i\kappa^{\sigma}x_{\sigma}} . \qquad (3.10)$$

Here $P(1/(\kappa^0-\kappa))$ is the distribution Cauchy principal value of $1/(\kappa^0-\kappa)$. Because $(\kappa^0+\kappa)^{-1}$ is infinitely derivable, the product of distributions is well defined and (3.10) can be written

$$ {}_1A^{\mu} = -\xi^{\mu} P\left(\frac{1}{\kappa^{\sigma}\kappa_{\sigma}}\right) e^{-i\kappa^{\sigma}x_{\sigma}} , \qquad (3.11)$$

which in components is expressed as

$$\begin{aligned} {}_1A^0 &= -\tfrac{1}{2}A_+A_0 P\left(\frac{\sin\alpha\cos\alpha}{1-\cos\alpha}\right) e^{-i\kappa^{\sigma}x_{\sigma}} , \\ {}_1A^x &= 0 , \\ {}_1A^y &= -\tfrac{1}{2}A_+A_0 P\left(\frac{\omega}{W}\frac{\sin^2\alpha\cos\alpha}{1-\cos\alpha} + \cos\alpha + \frac{\sin^2\alpha}{1-\cos\alpha}\right) e^{-i\kappa^{\sigma}x_{\sigma}} , \\ {}_1A^z &= -\tfrac{1}{2}A_+A_0 P\left(\frac{\sin\alpha\cos\alpha}{1-\cos\alpha} - \frac{\omega}{W}\frac{\sin^3\alpha}{1-\cos\alpha}\right) e^{-i\kappa^{\sigma}x_{\sigma}} . \end{aligned} \qquad (3.12)$$

The vector κ^{σ} may be interpreted as the propagating vector of ${}_1A^{\mu}$. Its norm is

$$\kappa^\sigma \kappa_\sigma = -2\omega W(1-\cos\alpha)\,. \tag{3.13}$$

κ^σ is the timelike, except for $\alpha=0$ (longitudinal interaction) which is lightlike. The correction (3.12) propagates inside or on the light cone, which agrees with the Hadamard's solution to the wave equation in a curved space-time.[5,6]

In order to study the case $\alpha\to 0$, we shall make the gauge transformation defined by

$$\Psi \equiv \frac{-i}{\omega+W} A_0 A_+ P\left(\frac{1}{\alpha}\right)\exp\{i[(\omega+W)t-\omega\alpha y-(\omega+W)z]\}\,, \tag{3.14}$$

which in the limit $\alpha\to 0$ satisfies $\Psi_{,\beta}{}^{\beta}=0$, and consequently is compatible with the Lorentz gauge. Applying it to (3.12) and taking the limit $\alpha\to 0$, the correction ${}_1A^\mu$ is

$${}_1A^\mu = -\delta^\mu{}_y A_+ A_0\left(\frac{\omega}{W}+\frac{3}{2}+\frac{\omega}{\omega+W}\right)e^{i(t-z)(\omega+W)}\,, \tag{3.15}$$

which represents a plane electromagnetic wave propagating along the z axis with the velocity of light. This agrees with the only privileged direction of the problem and with the world line of the photons associated with ${}_1A^\mu$, because the Christoffel symbols ${}_1\Gamma^\mu{}_{\beta\gamma}$ for $\alpha=0$ are null and the geodesic equation is reduced to

$$\kappa^\alpha{}_{;\beta}\kappa^\beta = \frac{d\kappa^\beta}{d\lambda} \tag{3.16}$$

in accord with a result stated in Bertotti's paper.[4] [It must be remarked that (3.15) is a solution of ${}_1A^\mu{}_{,\alpha}{}^{\alpha}=0$ and therefore there is no contradiction with $\xi^\mu=0$, $\alpha=0$; then $\alpha=0$ is a spurious singular case.]

From $A^\mu = {}_0A^\mu + {}_1A^\mu$ we can calculate the electromagnetic field tensor $F^{\mu\nu}=A^{\nu,\mu}-A^{\mu,\nu}$ and the impulse-energy tensor, whose T_{00} component gives us the energy density. A remarkable consequence is the modulation of this component by the frequency of the gravitational wave W when it is verified that $\omega \gg W$.

IV. INTERACTION BETWEEN A TE MODE PROPAGATING INSIDE AN IDEAL RECTANGULAR WAVEGUIDE AND A GRAVITATIONAL WAVE: DETECTOR OF GRAVITATIONAL WAVES

At the meeting in Erice (Italy, 1975) it was established that the most promising future detectors of gravitational waves would be Weber-type resonant bars, Michelson interferometers using laser beams, and Doppler tracking of interplanetary spacecraft. [See the lecture by Thorne of Ref. 7 which was in principle confirmed at the 8th International Conference on General Relativity and Gravitation, at Waterloo University (Canada, 1977).] The results below dare us to consider the possibility of a detector based on the perturbations of a TE mode propagating in a guide. Let us consider a wave guide built up of two superconducting plates parallel to the X-Y coordinate plane, and separated a distance d. We suppose that one of the plates is the X-Y plane. The potential vector corresponding to an unperturbed ${}_0\mathrm{TE}_n$ mode propagating inside the guide is[8]

$${}_0A^\mu = \delta^\mu{}_x A_0 \sin\frac{n\pi z}{d}\sin(\omega t - k_y y)\,. \tag{4.1}$$

Trivially the boundary conditions and the Lorentz gauge are satisfied.

For a gravitational wave we choose one propagating along the x axis, plane, monochromatic, and + polarization, which in the transverse-traceless gauge has the components

$$h^{TT}_{\mu\nu} = A_+(\delta^y{}_\mu\delta^y{}_\nu - \delta^z{}_\mu\delta^z{}_\nu)\cos K^\sigma x_\sigma\,. \tag{4.2}$$

The components of the propagation vector K^σ are

$$K^\sigma = W(\delta^\sigma{}_0 + \delta^\sigma{}_x)\,. \tag{4.3}$$

As in the paragraph above, we neglect the electromagnetic ${}_0\mathrm{TE}_n$ mode as a source of the gravitational field. Calculating J^μ_{eff} and resolving (2.7) we obtain after a lengthy but simple computation the correction ${}_1A^\mu$ to ${}_0A^\mu$:

$$\begin{aligned}
{}_1A^0 &= 0\,,\\
{}_1A^x &= \tfrac{1}{8}A_0A_+\frac{1}{\omega W}\left[-k_y{}^2+\left(\frac{n\pi}{d}\right)^2\right](\cos\kappa^\sigma x_\sigma - \cos\bar\kappa^\sigma x_\sigma)\,,\\
{}_1A^y &= \tfrac{1}{8}A_0A_+\frac{1}{\omega}k_y(\cos\kappa^\sigma x_\sigma - \cos\bar\kappa^\sigma x_\sigma)\,,\\
{}_1A^z &= \tfrac{1}{8}A_+A_0\frac{1}{\omega}\frac{n\pi}{d}(\cos\kappa^\sigma x_\sigma - \cos\bar\kappa^\sigma x_\sigma)\,,
\end{aligned} \tag{4.4}$$

where κ^σ and $\bar\kappa^\sigma$ are timelike vectors whose components are

$$\kappa^\sigma = \left(\omega+W,\, W,\, k_y,\, \frac{n\pi}{d}\right), \tag{4.5}$$

$$\bar\kappa^\sigma = \left(\omega+W,\, W,\, k_y,\, -\frac{n\pi}{d}\right), \tag{4.6}$$

and both have the same norm:

$$\kappa^\sigma\kappa_\sigma = \bar\kappa^\sigma\bar\kappa_\sigma = -2\omega W - \left[\omega^2 - k_y{}^2 - \left(\frac{n\pi}{d}\right)^2\right]. \tag{4.7}$$

Taking into account the allowed frequencies of ${}_0A^\mu$ in the guide,

$$\omega^2 = k_y{}^2 + \left(\frac{n\pi}{d}\right)^2 \tag{4.8}$$

is reduced to

$$\kappa^\sigma\kappa_\sigma = \bar\kappa_\sigma\bar\kappa_\sigma = -2\omega W < 0\,. \tag{4.9}$$

Trivially the correction ${}_1A^\mu$, (4.4), verifies the boundary conditions that in principle allow its propagation along the guide. In order to interpret the correction ${}_1A^\mu$, we shall calculate the electromagnetic field tensor ${}_1F^{\mu\nu}$, whose components rearranged in a suitable order are

$$
\begin{aligned}
{}_1E^x &= -\tfrac{1}{4}A_+A_0\left[-k_y{}^2+\left(\frac{n\pi}{d}\right)^2\right]\left\{\sin\frac{n\pi z}{d}\cos Wx\cos[-(\omega+W)t+k_y y]-\sin\frac{n\pi z}{d}\sin Wx\sin[-(\omega+W)t+k_y y]\right\}\frac{W+\omega}{W\omega}\,,\\
{}_1H^y &= -\tfrac{1}{4}A_+A_0\frac{1}{\omega W}\frac{n\pi}{d}\left[W^2-k_y{}^2+\left(\frac{n\pi}{d}\right)^2\right]\left\{\cos\frac{n\pi z}{d}\cos Wx\sin[-(\omega+W)t+k_y y]+\sin\frac{n\pi z}{d}\sin Wx\cos[-(\omega+W)t+k_y y]\right\},\\
{}_1H^z &= \tfrac{1}{4}A_+A_0\frac{1}{\omega W}k_y\left[-W^2-k_y{}^2-\left(\frac{n\pi}{d}\right)^2\right]\left\{\sin\frac{n\pi z}{d}\cos Wx\cos[-(\omega+W)t+k_y y]-\sin\frac{n\pi z}{d}\sin Wx\sin[-(\omega+W)t+k_y y]\right\},
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
{}_1E^y &= -\tfrac{1}{4}A_+A_0\frac{W+\omega}{\omega}k_y\left\{\sin\frac{n\pi z}{d}\cos Wx\cos[-(\omega+W)t+k_y y]-\sin\frac{n\pi z}{d}\sin Wx\sin[-(\omega+W)t+k_y y]\right\},\\
{}_1E^z &= \tfrac{1}{4}A_+A_0\frac{W+\omega}{\omega}\frac{n\pi}{d}\left\{\cos\frac{n\pi z}{d}\cos Wx\sin[-(\omega+W)t+k_y y]+\cos\frac{n\pi z}{d}\sin Wx\cos[-(\omega+W)+k_y y]\right\},\\
{}_1H^x &= \tfrac{1}{2}A_+A_0\frac{k_y}{\omega}\frac{n\pi}{d}\left\{\cos\frac{n\pi z}{d}\cos Wx\sin[-(\omega+W)t+k_y y]+\cos\frac{n\pi z}{d}\sin Wx\cos[-(\omega+W)t+k_y y]\right\}.
\end{aligned}
\tag{4.11}
$$

The above expressions show us that the correction ${}_1F^{\mu\nu}$ can be split into two modes: a transverse electric mode ${}_1\mathrm{TE}_m$ defined by (4.10) and a transverse magnetic mode ${}_1\mathrm{TE}_m$ defined by (4.11), both with frequency $\omega+W$, propagating along the y axis and with amplitude modulated by the frequency of the gravitational wave. Also, although the phase velocity of these modes is different from that of ${}_0\mathrm{TE}_n$, its group velocity calculated from the dispersion relation (4.7) is equal to the group velocity of ${}_0\mathrm{TE}_m$: $v_g=[1-(n\pi/d\omega)^2]^{1/2}$. These characteristics of ${}_1F^{\mu\nu}$ in principle make its experimental detection possible.

It must be remarked that the case studied above corresponds to orthogonal interaction between the confined electromagnetic field and the gravitational wave. In the case of parallel interaction only a ${}_1\mathrm{TM}_n$ transverse magnetic mode is generated with the same characteristics; then the proposed antenna has directional properties.

V. OPTIC LIMIT

Finally we shall now apply the optic limit to the linearized DeRham equation (1.6). The vector potential in this case adopts the expansion

$$A_\mu=\mathcal{R}(a_\mu+\epsilon b_\mu+\epsilon^2 c_\mu+\cdots)e^{ik^\sigma x_\sigma/\epsilon} \tag{5.1}$$

(see Ref. 1, p. 572). If we introduce (5.1) into (1.6) we obtain for the order ϵ^{-2} and ϵ^{-1} the following equations:

$$a^\mu(k_\alpha k^\alpha - h^{\alpha\beta}k_\alpha k_\beta)=0,\quad O(\epsilon^{-2}) \tag{5.2}$$

$$a^\mu{}_{,\nu}k^\nu+{}_1\Gamma^\mu{}_{\beta\nu}a^\beta k^\nu+\tfrac{1}{2}k^\alpha{}_{,\alpha}a^\mu-\tfrac{1}{2}h^{\alpha\nu}k_{\alpha,\nu}a^\mu-a^\mu{}_{,\alpha}k_\nu h^{\alpha\nu}=0,\quad O(\epsilon^{-1})\,. \tag{5.3}$$

Applying the Lorentz gauge ${}_1\nabla_\mu A^\mu=0$ we obtain

$$a^\mu k_\mu=0 \tag{5.4}$$

and therefore the amplitude is perpendicular to the wave vector. The meaning of Eqs. (5.2) and (5.3) is that (5.2) can be written

$$(\eta^{\alpha\beta}-h^{\alpha\beta})k_\alpha k_\beta=g^{\alpha\beta}k_\alpha k_\beta=0\,, \tag{5.5}$$

which for the one-body problem of mass-energy M and metric

$$ds^2=-\left(1-\frac{2M}{r}\right)dt^2+\left(1+\frac{2M}{r}\right)(dx^2+dy^2+dz^2) \tag{5.6}$$

trivially gives the equivalent refractive index

$$n(r)=\frac{|\vec{k}|}{k^0}=1+\frac{2M}{r}\,, \tag{5.7}$$

which implies the deflection of light and radar time delay. (5.3) after reordering terms and applying the linearized equation $k_{\mu,\alpha}=\frac{1}{2}h^\beta{}_{\mu,\alpha}k_\beta$ of the ray's trajectories, reduces to

$$(a^\mu{}_{,\alpha}k^\alpha+{}_1\Gamma^\mu{}_{\alpha\beta}k^\alpha a^\beta)+\tfrac{1}{2}(k^\alpha{}_{,\alpha}+{}_1\Gamma^\alpha{}_{\beta\alpha}k^\beta)a_\mu = h^{\alpha\nu}k_\nu a^\mu{}_{,\alpha}+\tfrac{1}{2}(h^{\alpha\nu}k_\nu)_{,\alpha}a^\mu\,, \tag{5.8}$$

which gives the propagation equation that governs the evolution of the amplitude. If we write (5.8) in the equivalent form

$${}_1\nabla_K a^\mu+\tfrac{1}{2}{}_1\nabla\cdot k a^\mu=h^{\alpha\nu}k_\nu a^\mu{}_{,\alpha}+\tfrac{1}{2}(h^{\alpha\nu}k_\nu)_{,\alpha}a^\mu, \tag{5.8$'$}$$

it can be compared with the corresponding equation in curved space-time:

$$\nabla_k a^\mu+\tfrac{1}{2}\nabla\cdot k a^\mu=0\,. \tag{5.9}$$

The incomplete correspondence is due to the fact that the wave vector in linearized theory is timelike and not lightlike as in curved space-time.

Defining $a^\mu\equiv af$ where $a_\mu\bar{a}^\mu=a^2$ and $f_\mu\bar{f}^\mu=1$ (the bar means complex conjugation) and introducing this definition into (5.8), we obtain separate equations for amplitude a and polarization f:

$$(\eta^{\alpha\nu} - h^{\alpha\nu})a_{,\alpha}k_\nu + \tfrac{1}{2}k^\alpha{}_{,\alpha}a = 0\,, \qquad (5.10)$$

$$(\eta^{\alpha\nu} - h^{\alpha\nu})\bar{f}_\mu(f^\mu{}_{,\alpha}k_\nu + {}_1\Gamma^\mu{}_{\alpha\beta}f^\beta k_\nu) = 0\,. \qquad (5.11)$$

Evidently the method outlined in this paper is suited for calculating first corrections to classical electromagnetism, induced by weak gravitational fields. If the gravitational field generated by a "nonperturbed" electromagnetic field has to be taken into account, the linearized Einstein-Maxwell equation would have to be resolved with the impulse-energy tensor associated with the "nonperturbed" electromagnetic field, in order to obtain the $h_{\mu\nu}$.

APPENDIX

In order to apply Green's theorem, we shall determine the adjoint of the DeRham operator. In operator form (1.6) is expressed as

$$({}_1\Delta_{dR})^\mu{}_\beta = (-\partial_\sigma\partial^\sigma + h^{\sigma\nu}\partial_\sigma\partial_\nu)\eta^\mu{}_\beta + (-2{}_1\Gamma^\mu{}_{\beta\nu}\partial^\nu + 2{}_1R^\mu{}_\beta) \qquad (1.6')$$

and its adjoint is

$$({}_1\Delta_{dR})^{*\mu}{}_\beta = (-\partial_\sigma\partial^\sigma + \partial_\sigma\partial_\nu h^{\sigma\nu})\eta^\mu{}_\beta + (2\partial^\nu{}_1\Gamma^\mu{}_{\beta\nu} + 2{}_1R^\mu{}_\beta)\,. \qquad (1.6'')$$

Let ϕ^α and A^β be arbitrary differentiable vector fields; then the Green's theorem reads

$$\phi^\alpha({}_1\Delta_{dR})_{\mu\beta}A^\beta - A^\beta({}_1\Delta_{dR})^*{}_{\mu\beta}\phi^\alpha = -\partial_\sigma\xi^{\sigma\alpha}{}_\mu\,,$$

where

$$\xi^{\sigma\alpha}{}_\mu = \eta_{\mu\beta}[(\phi^\alpha\partial^\sigma A^\beta - A^\beta\partial^\sigma A^\alpha) - (\phi^\alpha h^{\sigma\nu}\partial_\nu A^\beta - A^\beta\partial_\nu h^{\nu\sigma}\phi^\alpha)] + 2\phi^\alpha{}_1\Gamma_{\mu\beta}{}^\sigma A^\beta\,.$$

This will be useful after integration and application of the Gauss theorem to the second member for studying the Cauchy problem of the linearized De Rham equation.

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