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Stochastic semiclassical equations for weakly inhomogeneous cosmologies

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Semiclassical Einstein-Langevin equations for arbitrary small metric perturbations conformally coupled to a massless quantum scalar field in a spatially flat cosmological background are derived. Use is made of the fact that for this problem the in-in or closed time path effective action is simply related to the Feynman-Vernon influence functional which describes the effect of the "environment," the quantum field which is coarse grained here, on the "system," the gravitational field which is the field of interest. This leads to identify the dissipation and noise kernels in the in-in effective action, and to derive a fluctuation-dissipation relation. A tensorial Gaussian stochastic source which couples to the Weyl tensor of the spacetime metric is seen to modify the usual semiclassical equations which can be viewed now as mean field equations. As a simple application we derive the correlation functions of the stochastic metric fluctuations produced in a flat spacetime with small metric perturbations due to the quantum fluctuations of the matter field coupled to these perturbations.

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I. INTRODUCTION

The semiclassical equations of gravity are the generalization of Einstein equations when the matter source is described quantically. The source of these equations is the vacuum expectation value of the stress tensor of the matter fields. This theory, which assumes that the gravitational field is classical, should work well outside the Planck scales and when the quantum fluctuations are small because the matter source is an average term. But as has been emphasized by several authors [1] one should expect that, if the fluctuations above the average are large—and this depends on the quantum state of the fields—the semiclassical equations should not give a correct description. In fact, one expects that a better approximation should describe the gravitational field in a probabilistic way. In other words, the semiclassical equations should be substituted by some Langevin-type equation with a stochastic source which describes classically the quantum fluctuations. A significant step in this direction was made by Hu [2] who proposed to view the semiclassical back-reaction problem in the framework of quantum open systems, where the quantum fields are seen as the "environment" and the gravitational field is seen as the "system." Following this proposal a systematic study of the connection between quantum field theory and statistical mechanics has resulted in the development of a consistent mathematical framework in which the relevant semiclassical Einstein-Langevin equations can be derived [3–5]. The key to these results is the influence functional method [6] used in nonequilibrium statistical mechanics to describe the environment-system interaction when only the state of the system is of interest. Here, the environment is the quantum fields whose degrees of freedom are not of interest in the semiclassical back-reaction problem and are traced out, i.e., coarse grained, and the system is the gravitational field which is the field of interest. The influence functional provides information about the dissipation and the noise suffered by the system and the corresponding fluctuation-dissipation relation. In this framework it is seen that external stochastic sources with a given probability distribution appear in the semiclassical equation for the gravitational field. The origin of the noise acting on the system is the coarse-grained quantum fields whose degrees of freedom have been traced out.

In this paper we derive the semiclassical Einstein-Langevin equations for arbitrary linear metric perturbations conformally coupled to a massless quantum scalar field in a spatially flat cosmological model. Semiclassical Einstein-Langevin equations have been recently derived by Hu and Sinha [4] for small anisotropies conformally coupled to massless fields in a spatially homogeneous background working in the framework of quantum cosmology and by Hu and Matactz [5] who derived the semiclassical equations for the scale factor in a spatially flat universe due to the coupling of different quantum scalar fields.

We work in the "in-in" or closed time path (CTP) effective action framework. The CTP effective action technique which was first proposed by Schwinger [7] is an effective action method adapted to compute expectation values of quantum operators rather than matrix elements as in the usual "in-out" effective action method. It has

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been adapted by Jordan [8] and Calzetta and Hu [9] to a curved background and from it real evolution equations which admit an initial value formulation can be derived. It has been used to derive the semiclassical equations for small anisotropies in an homogeneous background due to quantum matter fields [9,10] and it was also used in a previous paper [11], referred from now on as paper I, to derive the semiclassical equations for arbitrary linear perturbations in homogeneous backgrounds due to massless conformally coupled matter fields. In this last case the CTP effective action method leads in a very direct way to the evaluation of the vacuum expectation value of the field stress tensor which had been obtained previously by other methods [12]. The CTP effective action method is also currently used to study dissipative effects produced by the inflaton oscillations around a true vacuum [13]. These effects are of great interest to understand the reheating mechanism in the inflationary cosmological scenario.

The connection between the CTP effective action and the influence functional was established by Calzetta and Hu [3] in the semiclassical context. It turns out that in this case there is a very simple linear relation between the so-called influence action and the CTP effective action. That relation is not so simple if one quantizes the gravitational field or one wants the field equations for the mean quantum field on a given background [14] since then one has to advocate decoherence to go from a quantum to a classical description. This, of course, is of great interest for the study of structure formation in the early universe. In our semiclassical case, however, the CTP effective action already partially derived in paper I gives all the necessary information not only on the dissipation effects but also on the noise due to quantum fluctuations. All we need to do is to identify the corresponding dissipation and noise kernels.

The plan of the paper is the following. In Sec. II we briefly review the CTP functional formalism and derive the complete CTP effective action for arbitrary linear metric perturbations in a homogeneous background. Note that in paper I the CTP effective action was only partially given since the complete action was not necessary to derive the (mean field) semiclassical field equations. In Sec. III the influence action is derived, the dissipation and noise kernels are obtained, and the fluctuation-dissipation relation is given. In Sec. IV we derive the semiclassical Einstein-Langevin equation and identify the stochastic source. We show that a tensorial Gaussian stochastic source which couples to the Weyl tensor of the spacetime metric is the origin of noise in this case. Finally, in Sec. V we apply the Einstein-Langevin equation in a noncosmological background and derive the correlation functions of the stochastic metric fluctuations produced in a flat spacetime with small metric perturbations as a consequence of the coupling of the quantum matter field to these perturbations.

II. CTP EFFECTIVE ACTION FOR WEAK INHOMOGENEITIES

First, let us sketch the CTP functional formalism for the evaluation of the CTP effective action; for a more detailed exposition see Refs. [8,9,11,15]. Let us consider the quantization of a scalar field $\phi(x)$; the usual in-out effective action is based in the generating functional $W[J]$ which is related to the vacuum persistence amplitude in the presence of some classical source $J(x)$ [16]. From this functional one generates matrix elements of the field $\phi(x)$ by functional derivation with respect to $J(x)$. Now in order to work with expectation values rather than matrix elements one defines a new generating functional whose dynamics is determined by two different external classical sources J_+ and J_- by letting the in vacuum evolve independently under these sources, i.e., $\exp\{iW[J_+,J_-]\} = \sum_\alpha \langle 0,\text{in}|\alpha,T\rangle_{J_-}\langle\alpha,T|\text{in},0\rangle_{J_+}$, where we have assumed that $\{|\alpha,T\rangle\}$ is a complete basis of eigenstates of the field operator $\phi(x)$ at some future time T. This generating functional admits a path integral representation which we may write in a compact form as

$$e^{iW[J_+,J_-]} = \int \mathcal{D}[\phi_+]\mathcal{D}[\phi_-]e^{i(S[\phi_+]-S[\phi_-]+J_+\phi_+-J_-\phi_-)}, \tag{2.1}$$

where $J_\pm\phi_\pm$ stands for $\int d^nx J_\pm(x)\phi_\pm(x)$ and it is understood that we sum over all fields ϕ_+ and ϕ_- with negative and positive frequency modes, respectively, in the remote past (these are the in boundary conditions) but which coincide at time $t=T$ (in practice one usually takes $T\to\infty$, the remote future). Thus this path integral can be thought of as the path sum of two different fields evolving in two different time branches: one going forward in time in the presence of J_+ from the in vacuum to a time T and the other backward in time in the presence of J_- from time T to the in vacuum, with the constraint that $\phi_+(T,\mathbf{x}) = \phi_-(T,\mathbf{x})$. For that reason this formalism is called CTP functional formalism. Note that we use an arbitrary number n of spacetime dimensions in order to perform dimensional regularization.

From this generating functional expectation values can be obtained. Let us define

$$\frac{\delta W[J_+,J_-]}{\delta J_\pm} \equiv \pm\bar{\phi}_\pm[J_+,J_-], \tag{2.2}$$

and assume that these equations can be reversed; then, the CTP effective action is the Legendre transform of the above functional

$$\Gamma_{\rm CTP}[\bar{\phi}_+,\bar{\phi}_-] = W[J_+,J_-] - J_+\bar{\phi}_+ + J_-\bar{\phi}_-, \tag{2.3}$$

where it is understood that the external sources are functionals of the fields $\bar{\phi}_+$ and $\bar{\phi}_-$ through the definitions (2.2). From this by functional derivation with respect to $\bar{\phi}_\pm$ we get the equations for the expectation values $\bar{\phi}_\pm[J_+,J_-]$. The equation for the vacuum expectation value of the field $\bar{\phi}[0] \equiv \bar{\phi}_\pm[0,0] = \langle 0,\text{in}|\phi(x)|\text{in},0\rangle$ is then simply obtained imposing that $J_\pm = 0$:

$$\left.\frac{\delta\Gamma_{\rm CTP}[\bar{\phi}_+,\bar{\phi}_-]}{\delta\bar{\phi}_+}\right|_{\bar{\phi}_\pm=\bar{\phi}[0]} = 0. \tag{2.4}$$

To simplify the notation it is useful at this stage to introduce a more compact notation $\mathcal{S}[\phi_a] = S[\phi_+] - S[\phi_-]$, $\phi_a(x) = (\phi_+, \phi_-)$, and $J_a(x) = (J_+, J_-)$ where a and b take the two values $+$ and $-$, and introduce the metric $c_{ab} = \mathrm{diag}(1, -1) = c^{ab}$ to lower and raise a, b indices.

Let us now proceed to the evaluation of the effective action $\Gamma_{\rm CTP}[\bar{\phi}_a]$ up to one-loop order, which corresponds to the first order expansion of $W[J_a]$ in powers of $\hbar$. As usual [16] we solve (2.1) by the steepest descent method, let $\phi_a^{(0)}(x)$ be the solutions of the classical field equations, and expand the exponent in (2.1) about these background fields up to the second derivative of $\mathcal{S}[\phi_a]$. The integration in (2.1) is now Gaussian and we can write, to this one-loop order,

$$e^{iW[J_a]} \simeq e^{iW^{(0)}[J_a]} \left[\det A_{ab}(x,y)\right]^{-1/2}, \qquad (2.5)$$

where $W^{(0)}[J_a] = \mathcal{S}[\phi_a^{(0)}] + \int d^n x J^a \phi_a^{(0)}$, $A_{ab}(x,y)$ is a 2×2 matrix defined by $A_{+-}(x,y) = A_{-+}(x,y) \equiv 0$, and

$$A_{++}(x,y) \equiv \left.\frac{\delta^2 S[\phi_+]}{\delta\phi_+(x)\delta\phi_+(y)}\right|_{\phi_+=\phi_+^{(0)}},$$
$$A_{--}(x,y) \equiv -\left.\frac{\delta^2 S[\phi_-]}{\delta\phi_-(x)\delta\phi_-(y)}\right|_{\phi_-=\phi_-^{(0)}}. \qquad (2.6)$$

In terms of the propagator $G(x,y) = A^{-1}(x,y)$ which is a functional of the background fields $\phi_a^{(0)}(x)$ we can write (2.5) as

$$W[J_a] \simeq W^{(0)}[J_a] - \frac{i}{2}\mathrm{Tr}(\ln G). \qquad (2.7)$$

The effective action, which is a functional of $\bar{\phi}_a$, can now be explicitly found to the same order. Using (2.2), (2.3), and the fact that $\bar{\phi}_a$ differs from $\phi_a^{(0)}$ by a term of order $\hbar$ one can show that $W^{(0)}[J_a] \simeq \mathcal{S}[\bar{\phi}_a] + \int d^n x J^a \bar{\phi}_a$, and we have finally

$$\Gamma_{\rm CTP}[\bar{\phi}_a] \simeq \mathcal{S}[\bar{\phi}_a] - \frac{i}{2}\mathrm{Tr}(\ln G). \qquad (2.8)$$

This formalism can be extended to curved spacetimes without difficulties if the spacetime is globally hyperbolic [8]. The hypersufaces of constant time are now Cauchy hypersurfaces, the in and out states are defined in the Cauchy hypersurfaces corresponding to the far past and far future respectively, and the spacetime integral must be taken now with the correct volume element.

Let us now compute explicitly the CTP effective action for a conformally coupled massless real scalar field in a nearly conformally flat spacetime. Since the detailed calculations were explained in paper I, here we give only a summary of the main results. Our spacetime background is a spatially flat Friedmann-Lamaitre-Robertson-Walker (FLRW) universe with small perturbations

$$\tilde{g}_{\mu\nu}(x) = e^{2\omega(\eta)}\left[\eta_{\mu\nu} + h_{\mu\nu}(x)\right] \equiv e^{2\omega(\eta)} g_{\mu\nu}, \qquad (2.9)$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, \cdots, +1)$, $\exp[2\omega(\eta)]$ is the conformal factor, η is the conformal time which is related to the cosmological time t by $dt = \exp[\omega(\eta)]d\eta$, $h_{\mu\nu}(x)$ is a symmetric tensor which represents arbitrary small metric perturbations, and we have also introduced the nearly flat metric $g_{\mu\nu}$ which is conformally related to $\tilde{g}_{\mu\nu}$. The classical action for a free massless conformally coupled real scalar field $\Phi(x)$ is given by $S_m[\tilde{g}_{\mu\nu}, \Phi] = -\frac{1}{2}\int d^n x\sqrt{-\tilde{g}}\left[\tilde{g}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \xi(n)\tilde{R}\Phi^2\right]$, where $\xi(n) = (n-2)/[4(n-1)]$ and $\tilde{R}$ is the Ricci scalar for the metric $\tilde{g}_{\mu\nu}$. Because of the conformal coupling we can define a new field $\phi(x) \equiv \exp[(n/2-1)\omega(\eta)]\Phi(x)$ and the action S_m after one integration by parts takes the form

$$S_m[g_{\mu\nu}, \phi] = -\frac{1}{2}\int d^n x\sqrt{-g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \xi(n)R\phi^2\right], \qquad (2.10)$$

which is the action for a free massless conformally coupled real scalar field $\phi(x)$ in a spacetime with metric $g_{\mu\nu}$, i.e., a nearly flat spacetime in our case. Although the physical field is $\Phi(x)$, the fact that it is related to the field $\phi(x)$ by a power of the conformal factor implies that a positive frequency mode of the field $\phi(x)$ in flat spacetime will correspond to a positive frequency mode in the conformally related space. Thus nontrivial quantum effects will be due to the breaking of conformal flatness which in this case is produced by the perturbations $h_{\mu\nu}(x)$. Let us expand the above action in terms of these perturbations:

$$S_m[h_{\mu\nu}, \phi] = \sum_{i=0}^{\infty} S_m^{(i)}[h_{\mu\nu}, \phi], \qquad (2.11)$$

where the first term ($i = 0$) is simply the action for the field ϕ in flat spacetime.

To the classical action for the matter fields S_m we have to add the action of the gravitational field $S_g[\tilde{g}_{\mu\nu}]$, i.e., the Einstein-Hilbert action. In order to renormalize the effective action later on we need to add appropriate terms quadratic in the Riemann tensor [see Eq. (3.6) in paper I]. We can also expand this action as $S_g[\tilde{g}_{\mu\nu}] \equiv \sum_{i=0}^{\infty} S_g^{(i)}[\omega, h_{\mu\nu}]$, where we note that $S_g^{(0)}[\omega]$ depends on ω only.

We now introduce two scalar fields $\phi_+(x)$ and $\phi_-(x)$ which coincide at some future time T, $\phi_+(T,\mathbf{x}) = \phi_-(T,\mathbf{x})$ and which evolve into two different geometries given by $h_{\mu\nu}^+$ and $h_{\mu\nu}^-$ such that $h_{\mu\nu}^+(T,\mathbf{x}) = h_{\mu\nu}^-(T,\mathbf{x})$. The CTP effective action for the gravitational and matter action can be written as

$$\Gamma_{\rm CTP} = S_g[\omega, h_{\mu\nu}^+] - S_g[\omega, h_{\mu\nu}^-] + \Gamma_m[h_{\mu\nu}^\pm, \bar{\phi}_\pm], \qquad (2.12)$$

where Γ_m contains the quantum effects of the scalar field which one can compute using (2.8).

Note that since our action is quadratic in the fields the one-loop order result (2.8) is exact in this case. Now the propagator G_{ab} cannot be found exactly; thus, we expand A_{ab} in the metric perturbations as $A_{ab} = A_{ab}^0 + V_{ab}^{(1)} + V_{ab}^{(2)} + \cdots$, where A_{ab}^0 corresponds to the Minkowski background and $V_{ab}^{(i)}$ are the inhomogeneous corrections. Their explicit values up to second order are given in Sec. III of paper I. Then we can write $G_{ab} = G^0_{ac}\left[\delta_{cb} - V^{(1)}_{cd}G^0_{db} - V^{(2)}_{cd}G^0_{db} + V^{(1)}_{cd}G^0_{de}V^{(1)}_{ef}G^0_{fb} + \cdots\right]$, where the unperturbed propagator, defined by $A^0_{ac}G^0_{cb} = \delta_{ab}$, is a matrix such that $G^0_{++} = \Delta_F$, $G^0_{--} = -\Delta_D$, $G^0_{+-} = -\Delta^+$, and $G^0_{-+} = \Delta^-$, where Δ_F and Δ_D are the Feynman and Dyson propagators, respectively, and $\Delta^\pm$ are the Wightman functions. Substituting this into (2.8) and expanding its logarithmic term we get, up to second order in the metric perturbations,

$$\Gamma_m[h^\pm_{\mu\nu},\bar\phi_\pm] \simeq \sum_{i=0}^{2}\left(S^{(i)}_m[h^+_{\mu\nu},\bar\phi_+] - S^{(i)}_m[h^-_{\mu\nu},\bar\phi_-]\right) - \frac{i}{2}\mathrm{Tr}(\ln G^0_{ab})$$
$$+\frac{i}{2}\mathrm{Tr}[V^{(1)}_+G^0_{++} - V^{(1)}_-G^0_{--} + V^{(2)}_+G^0_{++} - V^{(2)}_-G^0_{--}$$
$$-\frac{1}{2}V^{(1)}_+G^0_{++}V^{(1)}_+G^0_{++} - \frac{1}{2}V^{(1)}_-G^0_{--}V^{(1)}_-G^0_{--} + V^{(1)}_+G^0_{+-}V^{(1)}_-G^0_{-+}], \tag{2.13}$$

where we have defined $V^{(i)}_+ \equiv V^{(i)}_{++}$ and $V^{(i)}_- \equiv -V^{(i)}_{--}$. In paper I we did not write the terms which do not depend on the $\bar\phi_+$ field because such terms were not needed to derive the field equation for the mean field $\bar\phi[0]$; see (2.4).

The explicit computation of the different terms was given in paper I; the new term $-\frac{i}{4}\mathrm{Tr}(V^{(1)}_-\Delta_D V^{(1)}_-\Delta_D)$ can be easily evaluated following closely that reference. After dimensional regularization [17] of the divergent terms, renormalizing with the action of the gravitational field, and substituting the field equations for $\bar\phi_\pm$ we get the renormalized effective action for the gravitational field up to second order as

$$\Gamma^R_{\mathrm{CTP}}[\tilde g^\pm_{\mu\nu}] \equiv S^R_g[\tilde g^+_{\mu\nu}] - S^R_g[\tilde g^-_{\mu\nu}] + \frac{\alpha}{4}\int d^4x d^4y\left[3R^+_{\mu\nu\alpha\beta}(x)R^{+\mu\nu\alpha\beta}(y) - R^+(x)R^+(y)\right]K^+(x-y;\bar\mu)$$
$$-\frac{\alpha}{4}\int d^4x d^4y\left[3R^-_{\mu\nu\alpha\beta}(x)R^{-\mu\nu\alpha\beta}(y) - R^-(x)R^-(y)\right]K^-(x-y;\bar\mu)$$
$$+\frac{\alpha}{2}\int d^4x d^4y\left[3R^+_{\mu\nu\alpha\beta}(x)R^{-\mu\nu\alpha\beta}(y) - R^+(x)R^-(y)\right]K(x-y), \tag{2.14}$$

where $\alpha = (2880\pi^2)^{-1}$, $\bar\mu$ is a renormalization parameter,

$$K^\pm(x-y;\bar\mu) \equiv -\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot(x-y)}\ln\left[\frac{(p^2 \mp i\epsilon)}{\bar\mu^2}\right],$$
$$K(x-y) \equiv -\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot(x-y)}(2\pi i)\theta(-p^2)\theta(-p^0), \tag{2.15}$$

and $S^R_g[\tilde g^\pm_{\mu\nu}]$ are local terms coming from the gravitational field only [see (3.29) in paper I]:

$$S^R_g[\tilde g_{\mu\nu}] = \int d^4x[-\tilde g(x)]^{1/2}\left[\frac{\tilde R(x)}{16\pi G_N} - \frac{\alpha}{12}\tilde R^2(x)\right]$$
$$+2\alpha\int d^4x[-g(x)]^{1/2}\left[G^{\mu\nu}(x)\omega_{;\mu}\omega_{;\nu} + \Box_g\omega(\omega_{;\nu}\omega^{;\nu}) + \frac{1}{2}(\omega_{;\mu}\omega^{;\mu})^2\right]$$
$$+\alpha\int d^4x[-g(x)]^{1/2}\left[R_{\mu\nu\alpha\beta}(x)R^{\mu\nu\alpha\beta}(x) - R_{\mu\nu}(x)R^{\mu\nu}(x)\right]\omega(x). \tag{2.16}$$

Here terms such as $R_{\mu\nu\alpha\beta}$ refer to the metric $g_{\mu\nu}$ and are linear in $h_{\mu\nu}$, and terms with a tilde, $\tilde R_{\mu\nu\alpha\beta}$, refer to the metric $\tilde g_{\mu\nu}$.

It is remarkable that the Γ_{CTP} is explicitly gauge independent since it is given in terms of the curvature tensor; i.e., it is invariant under the infinitesimal coordinate change $x^\mu \to x^\mu + \zeta^\mu(x)$, for arbitrary fields $\zeta^\mu(x)$, which produce the change of $h_{\mu\nu}$ by $h_{\mu\nu} + 2\zeta_{(\mu,\nu)}$ in the metric perturbations. No gauge choice was made in the above calculation.

III. INFLUENCE ACTION FOR WEAK INHOMOGENEITIES

As explained in the Introduction we now consider the interaction of the scalar field $\phi(x)$ and the gravitational field from the point of view of a quantum open system. The gravitational field which is the field of interest to us will be the "system" and the quantum scalar field will be the "environment." Since we wish to know the effect of the environment on the system we will trace out the degrees of freedom of the scalar field.

We will now summarize the Calzetta-Hu [3] arguments leading to the relation between the CTP effective action and the influence action in the semiclassical case. We start by writing the generating functional $W[J_+, J_-]$ for a system in which we have the gravitational field, here symbolically called $g(x)$, and a scalar field $\phi(x)$. We can write the classical action for these two fields as $S_g[g] + S_m[\phi] + S_{\rm int}[g,\phi]$. Note that these terms can be easily identified in our case from the expressions of the matter and gravitational actions.

If we consider the field ϕ substituted by the two fields $g(x)$ and $\phi(x)$ in (2.1), we can write

$$e^{iW[J_+,J_-]} = \int \mathcal{D}[g^+]\mathcal{D}[g^-]e^{i\left(S_g[g^+]-S_g[g^-]+J_+g^+-J_-g^-+S_{\rm IF}[g^+,g^-,T]\right)}, \tag{3.1}$$

where we have defined

$$e^{iS_{\rm IF}[g^+,g^-,T]} = \int \mathcal{D}[\phi_+]\mathcal{D}[\phi_-]e^{i\left(S_m[\phi_+]-S_m[\phi_-]+S_{\rm int}[g^+,\phi_+]-S_{\rm int}[g^-,\phi_-]\right)}. \tag{3.2}$$

In (3.2) it is understood that we sum over all fields ϕ_+ and ϕ_- with negative and positive frequency modes, respectively, in the remote past which coincide at time T (remote future), and a similar interpretation is assumed in the path integration of (3.1). Since we are only interested in the gravitational field, we couple external sources $J_+(x)$, $J_-(x)$ to the fields $g(x)$ only and not to the scalar fields $\phi(x)$. Note that we are using a symbolic notation since the gravitational field is tensorial and so are the external sources $J(x)$.

The interesting point here is that (3.2) under the above interpretation is exactly the influence functional at time T, $\mathcal{F}[g^+,g^-,T]$ as defined in [6], and thus $S_{\rm IF}[g^+,g^-,T]$ is the influence action. It is the action one has to add to the classical actions $S_g[g^+]-S_g[g^-]$ to compute any quantum probability for the transition from an initial state g at $t\to-\infty$ to a final state at the future time T. It is also the essential ingredient for the evolution operator, from initial to final time, for the reduced density matrix in this case [3]. In this context the fields $+$ and $-$ and the sign difference appear as a result of the double integration needed to go from a transition amplitude to a probability.

We do not attempt to quantize the gravitational field even though the formal path integration in (3.1) seems to indicate so. Such quantization is, of course, a highly nontrivial problem, we do not know the measure in (3.1) and even in the linear case we have to deal with the gauge freedom. But if we restrict ourselves to the classical approximation, it is easy to find the CTP effective action $\Gamma_{\rm CTP}[g^+,g^-]$ from the generating functional $W[J_+,J_-]$ in (3.1) using the Legendre transformation (2.3). We just need to follow the steps which lead from (2.1) to (2.8) but to zero order in $\hbar$; thus, we have simply $\Gamma_{\rm CTP}[g^+,g^-] = S_g[g^+]-S_g[g^-]+S_{\rm IF}[g^+,g^-,T]$. On the other hand the $\Gamma_{\rm CTP}[g^+,g^-]$ in this equation is just the renormalized CTP action $\Gamma^R_{\rm CTP}[\tilde{g}^\pm]$ found in (2.14) since that action follows from the path integration (3.2) to one loop order for the matter fields (including scalar sources) and the explicit substitution of the field equations for these quantum fields. The divergences are removed by appropriate terms in the classical gravitational action S_g as has been shown in Sec. II; note that these counterterms are implicitly assumed in (3.1) to remove the divergencies of (3.2). Thus we have [3]

$$\Gamma^R_{\rm CTP}[\tilde{g}^\pm] = S^R_g[\tilde{g}^+] - S^R_g[\tilde{g}^-] + S^R_{\rm IF}[h^+,h^-,T], \tag{3.3}$$

where we have written $h^\pm$ in $S^R_{\rm IF}$ to emphasize that it depends on the metric perturbations only. Comparing this expression with (2.14) we can write the influence action as

$$\begin{aligned} S^R_{\rm IF}[h^\pm_{\mu\nu}] = {} & \frac{3\alpha}{2}\int d^4x d^4y C^+_{\mu\nu\alpha\beta}(x)C^{+\mu\nu\alpha\beta}(y)K^+(x-y;\bar{\mu}) - \frac{3\alpha}{2}\int d^4x d^4y C^-_{\mu\nu\alpha\beta}(x)C^{-\mu\nu\alpha\beta}(y)K^-(x-y;\bar{\mu}) \\ & +3\alpha\int d^4x d^4y C^+_{\mu\nu\alpha\beta}(x)C^{-\mu\nu\alpha\beta}(y)K(x-y). \end{aligned} \tag{3.4}$$

Here we have introduced the Weyl tensor $C_{\mu\nu\alpha\beta}(x)$, after using the following expressions, which are easily shown to be satisfied for an arbitrary function $f(x-y)$:

$$\int d^4x d^4y f(x-y)[R_{\mu\nu\alpha\beta}(x)R^{\mu\nu\alpha\beta}(y) - 4R_{\mu\nu}(x)R^{\mu\nu}(y) + R(x)R(y)] = O(h^3_{\mu\nu}),$$

$$\int d^4x d^4y f(x-y)[C_{\mu\nu\alpha\beta}(x)C^{\mu\nu\alpha\beta}(y) - R_{\mu\nu\alpha\beta}(x)R^{\mu\nu\alpha\beta}(y) + 2R_{\mu\nu}(x)R^{\mu\nu}(y) - \tfrac{1}{3}R(x)R(y)] = O(h^3_{\mu\nu}), \tag{3.5}$$

where the metrics may be different at the points x and y; i.e., one may have $h^+_{\mu\nu}(x)$ and $h^-_{\mu\nu}(y)$, respectively.

Let us now write the influence action $S^R_{\rm IF}[h^\pm_{\mu\nu}]$ in a form which is more appropriate for the analysis of fluctuations [6]. We first decompose $S^R_{\rm IF}$ into its real and imaginary parts, because to the quadratic order in the metric perturbations the real part gives information on the dissipation whereas the imaginary part is related to noise:

$$S^R_{\rm IF}[h^\pm_{\mu\nu}] = \hat\Gamma_{\rm IF}[h^\pm_{\mu\nu}] + i\tilde\Gamma_{\rm IF}[h^\pm_{\mu\nu}]. \tag{3.6}$$

For this we need to decompose the kernels $K^\pm(x-y;\bar\mu)$ and $K(x-y)$ of (2.15) into their real and imaginary parts also:

$$\begin{aligned} K^\pm(x-y;\bar\mu) &= \hat\gamma_e(x-y;\bar\mu) \pm i\tilde\gamma_e(x-y), \\ K(x-y) &= \hat\gamma_o(x-y) - i\tilde\gamma_e(x-y), \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} \hat\gamma_e(x-y;\bar\mu) &= -\frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\cos p(x-y)\ln\frac{|p^2|}{\bar\mu^2}, \\ \tilde\gamma_e(x-y) &= \frac{\pi}{2}\int \frac{d^4p}{(2\pi)^4}\cos p(x-y)\theta(-p^2), \\ \hat\gamma_o(x-y) &= \frac{\pi}{2}\int \frac{d^4p}{(2\pi)^4}\sin p(x-y)\theta(-p^2){\rm sgn}(-p^0). \end{aligned} \tag{3.8}$$

From the above definitions it is easy to show that these kernels verify the symmetry relations

$$\hat\gamma_e(x-y;\bar\mu) = \hat\gamma_e(y-x;\bar\mu), \quad \tilde\gamma_e(x-y) = \tilde\gamma_e(y-x), \quad \hat\gamma_o(x-y) = -\hat\gamma_o(y-x). \tag{3.9}$$

It is also convenient to introduce two new kernels $\hat\gamma(x-y;\bar\mu)$ and $H(x-y;\bar\mu)$ as

$$\begin{aligned} \hat\gamma(x-y;\bar\mu) &= \hat\gamma_e(x-y;\bar\mu) - \hat\gamma_o(x-y){\rm sgn}(x^0-y^0) = \hat\gamma(y-x;\bar\mu), \\ H(x-y;\bar\mu) &= \hat\gamma_e(x-y;\bar\mu) + \hat\gamma_o(x-y). \end{aligned} \tag{3.10}$$

With these definitions the real and imaginary parts of (3.6) may be written, respectively, as

$$\hat\Gamma_{\rm IF}[h^\pm_{\mu\nu}] = \frac{3\alpha}{2}\int d^4x d^4y \left[C^+_{\mu\nu\alpha\beta}(x) - C^-_{\mu\nu\alpha\beta}(x)\right] H(x-y;\bar\mu)\left[C^{+\mu\nu\alpha\beta}(y) + C^{-\mu\nu\alpha\beta}(y)\right], \tag{3.11}$$

$$\tilde\Gamma_{\rm IF}[h^\pm_{\mu\nu}] = \frac{3\alpha}{2}\int d^4x d^4y \left[C^+_{\mu\nu\alpha\beta}(x) - C^-_{\mu\nu\alpha\beta}(x)\right] \tilde\gamma_e(x-y)\left[C^{+\mu\nu\alpha\beta}(y) - C^{-\mu\nu\alpha\beta}(y)\right]. \tag{3.12}$$

Note that the integrand of the imaginary part depends (quadratically) only on the differences of the field at the points x and y as one expects of a term which is going to be the origin of noise. In order to write the influence functional in the standard Feynman-Vernon form [6] it is convenient to change the limits in the time integrations of the above expressions so that $x^0 > y^0$ always in the dissipation and fluctuation terms. For this we use the following further symmetries of the above kernels:

$$\begin{aligned} \hat\gamma_e(x^0-y^0,\mathbf{x}-\mathbf{y};\bar\mu) &= \hat\gamma_e(y^0-x^0,\mathbf{x}-\mathbf{y};\bar\mu), \quad \tilde\gamma_e(x^0-y^0,\mathbf{x}-\mathbf{y}) = \tilde\gamma_e(y^0-x^0,\mathbf{x}-\mathbf{y}), \\ \hat\gamma(x^0-y^0,\mathbf{x}-\mathbf{y};\bar\mu) &= \hat\gamma(y^0-x^0,\mathbf{x}-\mathbf{y};\bar\mu), \quad \hat\gamma_o(x^0-y^0,\mathbf{x}-\mathbf{y}) = -\hat\gamma_o(y^0-x^0,\mathbf{x}-\mathbf{y}), \end{aligned} \tag{3.13}$$

and that, for an arbitrary function $f(x^0,y^0)$, the integral $\int_{-\infty}^\infty dx^0\int_{-\infty}^\infty dy^0 f(x^0,y^0)$ is $2\int_{-\infty}^\infty dx^0\int_{-\infty}^{x^0} dy^o f(x^0,y^0)$ if $f(x^0,y^0)=f(y^0,x^0)$ and is zero if $f(x^0,y^0)=-f(y^0,x^0)$. Since the kernels $\hat\gamma_o(x-y)$ and $\tilde\gamma_e(x-y)$ will be related respectively to dissipation and noise, we will introduce the dissipation and noise kernels as

$$\begin{aligned} D(x-y) &= -3\alpha\hat\gamma_o(x-y), \\ N(x-y) &= 3\alpha\tilde\gamma_e(x-y). \end{aligned} \tag{3.14}$$

Finally we can write the influence functional in the standard form as

$$\begin{aligned} S^R_{\rm IF}[h^\pm_{\mu\nu}] = {}& \frac{3\alpha}{2}\int d^4x d^4y\,\hat\gamma(x-y;\bar\mu)\left[C^+_{\mu\nu\alpha\beta}(x)C^{+\mu\nu\alpha\beta}(y) - C^-_{\mu\nu\alpha\beta}(x)C^{-\mu\nu\alpha\beta}(y)\right] \\ & - \int_{-\infty}^\infty dx^0\int_{-\infty}^{x^0} dy^0\int d^3\mathbf{x}d^3\mathbf{y}\,\Delta C_{\mu\nu\alpha\beta}(x)D(x-y)\{C^{\mu\nu\alpha\beta}(y)\} \\ & + i\int_{-\infty}^\infty dx^0\int_{-\infty}^{x^0} dy^0\int d^3\mathbf{x}d^3\mathbf{y}\,\Delta C_{\mu\nu\alpha\beta}(x)N(x-y)\Delta C^{\mu\nu\alpha\beta}(y), \end{aligned} \tag{3.15}$$

where we have used the notation

$$\Delta C(x) \equiv C^+(x) - C^-(x), \qquad \{C(x)\} \equiv C^+(x) + C^-(x), \tag{3.16}$$

for the difference and sum of the + and − fields at the same spacetime point.

The first term in the influence functional is nonlocal and will contribute to give a nonlocal stress tensor in the semiclassical equations but does not mix the + and − fields. On the other hand, in the second and third terms, which are also nonlocal, there is mixing of the + and − fields. The second will contribute to the dissipation and the third to noise as we will see in the next section. We should remark also on the explicit gauge-invariant form of the influence functional above.

A. Fluctuation-dissipation relation

It is easy to derive a relation between the dissipation and noise kernels in our case in analogy with the quantum Brownian model of Ref. [18] or the case of a quantum scalar field in an anisotropic Bianchi I cosmology [4].

We first note that the dissipation kernel $D(x)$ of (3.14) can be written as a time derivative:

$$D(x) = \frac{\partial}{\partial \eta}\gamma(x), \tag{3.17}$$

where

$$\gamma(x) = \frac{3\pi\alpha}{2}\int \frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot x}}{|p^0|}\theta(-p^2), \tag{3.18}$$

where $1/|p^0|$ in this integral must be understood as its Hadamard's finite part distribution. Then the fluctuation-dissipation relation takes the form

$$N(x) = \int d^4x' K_{\rm FD}(x-x')\gamma(x'), \tag{3.19}$$

where the fluctuation-dissipation kernel $K_{\rm FD}(x-x')$ is given by the distribution

$$K_{\rm FD}(x-x') = \delta^3(\mathbf{x}-\mathbf{x}')\int_0^\infty \frac{dq^0}{\pi} q^0 \cos q^0(\eta-\eta'). \tag{3.20}$$

To check this fluctuation-dissipation relation one may simply substitute (3.20) into the right-hand side of (3.19), perform the integrations, and the result is the noise kernel defined by (3.8) and (3.14). This relation is important since it gives a direct connection between the effect of quantum fluctuations of the environment on the dissipation of the gravitational field inhomogeneities.

IV. EFFECTIVE INFLUENCE ACTION AND STOCHASTIC SEMICLASSICAL EQUATIONS

In this section we obtain from the influence action of Sec. III the effective influence action. From that effective action the explicit semiclassical Einstein-Langevin equations will follow.

We start by recalling some well-known relations of frequent use in statistical physics. The path integral Gaussian identity [6] $\int \mathcal{D}\xi \exp[-\frac{1}{2}(\xi, L\xi) + (b,\xi) + c] = (\det L)^{-1/2}\exp[\frac{1}{2}(b, L^{-1}b) - c]$, where L is a linear operator acting on the field $\xi(x)$ and the brackets in the exponents stand for $(\xi,\zeta) = \int \xi(x)\zeta(x)d^nx$, can be used to show that (change L by A^{-1}, b by ik, and take $c=0$)

$$\Phi[k] \equiv e^{-\frac{1}{2}(k,Ak)} = \int \mathcal{D}\xi\mathcal{P}[\xi]e^{i(k,\xi)}, \tag{4.1}$$

where $\mathcal{P}[\xi]$ is given by

$$\mathcal{P}[\xi] = \frac{e^{-\frac{1}{2}(\xi,A^{-1}\xi)}}{\int \mathcal{D}\xi e^{-\frac{1}{2}(\xi,A^{-1}\xi)}}\ . \tag{4.2}$$

If we interpret $\mathcal{P}[\xi]$ as the probability distribution functional of the field $\xi(x)$, then $\Phi[k]$, which is its functional Fourier transform, is the characteristic functional. The mean value of a given functional $\mathcal{A}[\xi]$ is defined by

$$\langle \mathcal{A}\rangle_\xi = \int \mathcal{D}\xi\mathcal{P}[\xi]\mathcal{A}[\xi], \tag{4.3}$$

and the n-correlation functions of the field $\xi(x)$ may be derived from the characteristic functional as

$$\langle \xi(x^1)\cdots\xi(x^n)\rangle_\xi = \frac{1}{i^n}\left.\frac{\delta^n\Phi[k]}{\delta k(x^1)\cdots\delta k(x^n)}\right|_{k=0}. \tag{4.4}$$

Since in this case the characteristic functional is Gaussian, we have that $\langle\xi(x)\rangle_\xi = 0$ and that the two-point correlation function is $\langle\xi(x)\xi(x')\rangle_\xi = A(x,x')$.

Let us go back now to the influence functional derived in Sec. III:

$$\mathcal{F}_{\rm IF}[h^\pm_{\mu\nu}] \equiv e^{iS^R_{\rm IF}[h^\pm_{\mu\nu}]} = e^{i\hat{\Gamma}_{\rm IF}[h^\pm_{\mu\nu}]}e^{-\tilde{\Gamma}_{\rm IF}[h^\pm_{\mu\nu}]}, \tag{4.5}$$

where the real and imaginary parts of the influence action $\hat{\Gamma}_{\rm IF}$ and $\tilde{\Gamma}_{\rm IF}$, respectively, are given in (3.15). We recall now the observation that the imaginary part of the influence action depends on the difference between the + and − fields only; more precisely, it can be written as $\tilde{\Gamma}_{\rm IF}[h^\pm_{\mu\nu}(x)] = \tilde{\Gamma}_{\rm IF}[\Delta C^{\mu\nu\alpha\beta}(x)]$. This is the signal [6] that the effect of the environment on the system, given by this part of the influence functional, is equivalent to a classical stochastic source $\xi_{\mu\nu\alpha\beta}(x)$ (a tensor field in this case) whose probability distribution has such part of the influence functional as its characteristic functional. In fact, using (4.1) we see that (4.5) can be written as

$$\mathcal{F}_{\rm IF}[h^{\pm}_{\mu\nu}] = \int \mathcal{D}\xi \mathcal{P}\left[\xi\right] \exp i \left\{ \hat{\Gamma}_{\rm IF}[h^{\pm}_{\mu\nu}] + \int d^4x \xi_{\mu\nu\alpha\beta}(x) \Delta C^{\mu\nu\alpha\beta}(x) \right\}, \tag{4.6}$$

where $\mathcal{P}[\xi]$ is the Gaussian probability distribution given by [see (4.2) and (3.15)]

$$\mathcal{P}\left[\xi\right] = \frac{\exp\left\{-\frac{1}{2}\int d^4x d^4y \xi(x)[N(x-y)]^{-1}\xi(y)\right\}}{\int \mathcal{D}\xi \exp\left\{-\frac{1}{2}\int d^4x d^4y \xi(x)[N(x-y)]^{-1}\xi(y)\right\}}. \tag{4.7}$$

Therefore using (4.3) we can interprete the influence functional as the mean value

$$\mathcal{F}_{\rm IF}[h^{\pm}_{\mu\nu}] = \langle e^{iS^{\rm eff}_{\rm IF}[h^{\pm}_{\mu\nu},\xi]}\rangle_{\xi}, \tag{4.8}$$

where the effective influence action is defined by

$$S^{\rm eff}_{\rm IF}[h^{\pm}_{\mu\nu},\xi] = \hat{\Gamma}_{\rm IF}[h^{\pm}_{\mu\nu}] + \int d^4x \xi_{\mu\nu\alpha\beta}(x) \Delta C^{\mu\nu\alpha\beta}(x). \tag{4.9}$$

The effect of the environment (quantum fields) on the system (the gravitational field) is completely characterized by this effective action; the tensor $\xi_{\mu\nu\alpha\beta}(x)$ plays the role of a stochastic source with the Gaussian probability distribution given by (4.7). This tensor has the symmetries of the Weyl tensor; i.e., it has the symmetries of the Riemann tensor and vanishing trace in all its indices. The kernel N, which appears in (4.7), can thus be interpreted as the noise kernel in our problem. Since the probability distribution is Gaussian, the noise kernel is the two-point correlation function of the stochastic source. This source, in fact, is completely characterized by the relations

$$\begin{aligned} \langle \xi_{\mu\nu\alpha\beta}(x)\rangle_{\xi} &= 0, \\ \langle \xi_{\mu\nu\alpha\beta}(x)\xi_{\rho\sigma\lambda\theta}(y)\rangle_{\xi} &= T_{\mu\nu\alpha\beta\rho\sigma\lambda\theta}N(x-y), \end{aligned} \tag{4.10}$$

where the explicit form of the tensor $T_{\mu\nu\alpha\beta\rho\sigma\lambda\theta}$ is given in the Appendix; it is the product of four metric tensors, in such a combination that the right-hand sides of the equation satisfy the Weyl symmetries of the two stochastic fields on the left-hand side. It is easy to obtain these relations using (4.4); note that the characteristic functional has the form $\Phi[k_{\mu\nu\alpha\beta}(x)]$, where $k_{\mu\nu\alpha\beta}(x)$ has the symmetries of the Weyl tensor. It should be now clear also that D is the dissipation kernel for this problem since it is related to the noise by the fluctuation-dissipation relation (3.19). It is worth mentioning that the association of the imaginary terms of the effective action (or the influence action) as terms coming from the interaction of the field with stochastic sources has been used also in the context of scalar fields in interaction with other fields, or self-interacting, in order to study the nonequilibrium dynamics of these quantum fields [19].

A. Stochastic semiclassical equations

We are now in the position to derive the semiclassical equations for the gravitational field due to a quantum scalar field. We recall that the effect of the quantum field on the gravitational field is given by the effective influence action (4.9). Therefore the total effective action, which includes the action of the gravitational field plus the previous effective influence action, is given by

$$S_{\rm eff}[\tilde{g}^{\pm},\xi] = S^R_g[\tilde{g}^+_{\mu\nu}] - S^R_g[\tilde{g}^-_{\mu\nu}] + S^{\rm eff}_{\rm IF}[h^{\pm}_{\mu\nu},\xi], \tag{4.11}$$

where $S^R_g[\tilde{g}_{\mu\nu}]$ is the renormalized action of the gravitational field (2.16). Since this is an effective action for the metric perturbations, the field equations for the metric can be derived in a similar way as (2.4). It is usually convenient to introduce new variables $\bar{h}_{\mu\nu} \equiv \left(h^+_{\mu\nu} + h^-_{\mu\nu}\right)/2$, $\Delta h_{\mu\nu} \equiv h^+_{\mu\nu} - h^-_{\mu\nu}$, the average field and the difference field, respectively. Then the field equations are $\frac{\delta}{\delta \Delta h_{\mu\nu}}\left(S_{\rm eff}\left[\omega, \bar{h}_{\mu\nu}, \Delta h_{\mu\nu}, \xi\right]\right)\Big|_{\Delta h_{\mu\nu}=0} = 0$, or in an equivalent form, which is of more practical use to us,

$$\frac{\delta}{\delta h^+_{\mu\nu}}\left(S^R_g[\tilde{g}^+_{\mu\nu}] + \hat{\Gamma}_{\rm IF}[h^{\pm}_{\mu\nu}] + \int d^4x \xi_{\mu\nu\alpha\beta}(x) C^{+\mu\nu\alpha\beta}(x)\right)\Bigg|_{h^+_{\mu\nu}=h^-_{\mu\nu}} = 0. \tag{4.12}$$

The functional derivations needed in the first two terms can all be found in Appendix E of paper I. For the new term, we can easily prove that

$$\int d^4x \xi^{\alpha\sigma\tau\rho}(x) C^+_{\alpha\sigma\tau\rho}(x) = -2\int d^4x \partial_\sigma\partial_\rho \xi^{\alpha\sigma\tau\rho}(x) h^+_{\alpha\tau}(x), \tag{4.13}$$

where in this expression we have assumed that the tensor field $\xi_{\mu\nu\alpha\beta}$ has the symmetries of the Weyl tensor.

Finally the semiclassical equation (4.12) can be written as

$$e^{6\omega}\left[-\frac{1}{16\pi G_N}\left(\tilde{G}^{\mu\nu}_{(0)}+\tilde{G}^{\mu\nu}_{(1)}\right)-\frac{\alpha}{12}\left(\tilde{B}^{\mu\nu}_{(0)}+\tilde{B}^{\mu\nu}_{(1)}\right)+\frac{\alpha}{2}\left(\tilde{H}^{\mu\nu}_{(0)}+\tilde{H}^{\mu\nu}_{(1)}\right)-\alpha\tilde{R}^{(0)}_{\alpha\beta}\tilde{C}^{\mu\alpha\nu\beta}_{(1)}\right]$$
$$+\frac{3\alpha}{2}\left[-4(C^{\mu\alpha\nu\beta}_{(1)}\omega)_{,\alpha\beta}+\int d^4y A^{\mu\nu}_{(1)}(y)H(x-y;\bar{\mu})\right]+F^{\mu\nu}[\xi]=O(h^2_{\mu\nu}), \tag{4.14}$$

where the (0) and (1) subindices mean the zero and one orders, respectively, in terms of the perturbation $h_{\mu\nu}$. Terms with and without a tilde refer to tensors obtained with metrics $\tilde{g}_{\mu\nu}$ and $g_{\mu\nu}$, respectively. $G^{\mu\nu}(x)$ is the Einstein tensor; $B^{\mu\nu}(x)$, $A^{\mu\nu}(x)$, and $H^{\mu\nu}(x)$ are given in paper I as

$$\begin{aligned}B^{\mu\nu}(x)&\equiv\tfrac{1}{2}g^{\mu\nu}R^2-2RR^{\mu\nu}+2R^{;\mu\nu}-2g^{\mu\nu}\,\Box_g R,\\ A^{\mu\nu}(x)&\equiv\tfrac{1}{2}g^{\mu\nu}C_{\alpha\beta\rho\sigma}C^{\alpha\beta\rho\sigma}-2R^{\mu\alpha\beta\rho}R^{\nu}{}_{\alpha\beta\rho}+4R^{\mu\alpha}R_{\alpha}{}^{\nu}-\tfrac{2}{3}RR^{\mu\nu}-2\,\Box_g R^{\mu\nu}+\tfrac{2}{3}R^{;\mu\nu}+\tfrac{1}{3}g^{\mu\nu}\,\Box_g R,\\ H^{\mu\nu}(x)&\equiv-R^{\mu\alpha}R_{\alpha}{}^{\nu}+\tfrac{2}{3}RR^{\mu\nu}+\tfrac{1}{2}g^{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}-\tfrac{1}{4}g^{\mu\nu}R^2.\end{aligned} \tag{4.15}$$

The tensor $F^{\mu\nu}(x)$ results from the functional variation of the stochastic term; it can be read off directly from (4.13),

$$F^{\mu\nu}(x)=-2\partial_\alpha\partial_\beta\xi^{\mu\alpha\nu\beta}(x), \tag{4.16}$$

and it is symmetric and traceless, i.e., $F^{\mu\nu}(x)=F^{\nu\mu}(x)$ and $F^{\mu}_{\mu}(x)=0$, which means that there is no stochastic correction to the trace anomaly.

From Eqs. (4.14) we may now define the effective stress tensor $T^{\mu\nu}_{\text{eff}}$, which is the term to write on the right-hand side of the Einstein equations; i.e., let us write (4.14) as

$$\begin{aligned}\tilde{G}^{\mu\nu}(x)&=8\pi G_N\left(T^{\mu\nu}_c+T^{\mu\nu}_{\text{eff}}\right),\\ T^{\mu\nu}_{\text{eff}}&\equiv\langle T^{\mu\nu}\rangle_q+2e^{-6\omega}F^{\mu\nu}[\xi],\end{aligned} \tag{4.17}$$

where $\langle T^{\mu\nu}\rangle_q$ is the (quantum) vacuum expectation value of the stress tensor of the quantum field up to first order in $h_{\mu\nu}$ and we have added a classical stress tensor $T^{\mu\nu}_c$ to include the case in which there is also a classical source (this was not considered from the beginning for simplicity). The quantum stress tensor is given by

$$\begin{aligned}\langle T^{\mu\nu}_{(0)}\rangle_q&=\alpha\left[\tilde{H}^{\mu\nu}_{(0)}-\frac{1}{6}\tilde{B}^{\mu\nu}_{(0)}\right],\\ \langle T^{\mu\nu}_{(1)}\rangle_q&=\alpha\left[(\tilde{H}^{\mu\nu}_{(1)}-2\tilde{R}^{(0)}_{\alpha\beta}\tilde{C}^{\mu\alpha\nu\beta}_{(1)})-\frac{1}{6}\tilde{B}^{\mu\nu}_{(1)}+3e^{-6\omega}\left(-4(C^{\mu\alpha\nu\beta}_{(1)}\omega)_{,\alpha\beta}+\int d^4yA^{\mu\nu}_{(1)}(y)H(x-y;\bar{\mu})\right)\right].\end{aligned} \tag{4.18}$$

This tensor was already given in paper I and was first computed by other means in Ref. [12]. Now we have derived a stochastic correction to this tensor which accounts for the noise due to the fluctuations of the quantum field.

If we now take the mean value of Eq. (4.17) with respect to the stochastic source ξ, we find that, as a consequence of (4.10),

$$\langle T^{\mu\nu}_{\text{eff}}\rangle_\xi=\langle T^{\mu\nu}\rangle_q \tag{4.19}$$

and we recover the semiclassical Einstein equations of paper I.

The stochastic correction to the stress tensor has vanishing divergence to first order in the metric perturbations. In fact, using that $\tilde{g}_{\mu\nu}=e^{2\omega}g_{\mu\nu}$ and that $F^{\mu\nu}$ is symmetric and traceless, it is easy to see that $\tilde{\nabla}_\nu\left(e^{-6\omega}F^{\mu\nu}\right)=e^{-6\omega}\nabla_\nu F^{\mu\nu}$; then, from (4.16) and the symmetries of ξ we obtain that $\nabla_\nu F^{\mu\nu}=O(h_{\mu\nu})$. It is thus consistent to write this term on the right-hand side of the Einstein equations and consider it as a correction of order higher than $\langle T^{\mu\nu}_{(0)}\rangle_q$ [note that $\tilde{\nabla}_\nu\langle T^{\mu\nu}_{(0)}\rangle_q=O(h^2_{\mu\nu})$].

To summarize, the stochastic semiclassical equations (4.17) can be called the semiclassical Einstein-Langevin equations for weakly inhomogeneous spatially flat cosmologies in the presence of a conformally coupled massless scalar field. The usual semiclassical equations can be seen as the mean value of these equations with respect to a tensor field source ξ with Gaussian probability distributions (4.7). This stochastic source couples to the Weyl (conformal) tensor of the spacetime metric in the form given by (4.9); this means that it has the symmetries of that tensor and thus that it has only ten independent components at each spacetime point.

The fact that the stochastic source couples to the conformal tensor should not come as a surprise since we expect that, for a conformal quantum field, nontrivial quantum effects should be a consequence of breaking the conformal symmetry of the spacetime, which is characterized by the conformal tensor. For instance, it is known that the probability density of pair creation in this case, or in the presence of only small anisotropies, is determined by the square of the Weyl tensor [20]. Thus as has been emphasized in [3] there is a direct relation between particle creation and noise.

V. METRIC FLUCTUATIONS IN FLAT SPACETIME

In this section we make a simple application of the semiclassical Einstein-Langevin equations obtained in the previous section to the case in which the background spacetime is not cosmological, i.e., when $\omega = 0$ in (2.9). This restriction simplifies considerably the semiclassical equations. We will take here a perturbative approach in which the semiclassical corrections to Einstein's equations are seen as analytic perturbations (in $\hbar$) to the classical Einstein's equations; see Simon [21] for a justification of this point of view.

Let us assume that we have some weak stress tensor source $T^{c}_{\mu\nu}$ in flat spacetime. For instance, we could have a cosmic string or a bulk of Newtonian matter. Such a source will produce some classical linear inhomogeneities $h^{c}_{\mu\nu}$ and the spacetime metric will be

$$g_{\mu\nu} = \eta_{\mu\nu} + h^{c}_{\mu\nu}. \tag{5.1}$$

If we have a massless conformally coupled quantum scalar field in this background, the stress tensor acting on the spacetime will now be $T^{c}_{\mu\nu}+T^{\rm eff}_{\mu\nu}$. The semiclassical equations up to first order in the metric perturbations can be written as

$$G_{\mu\nu}(x) = 8\pi G_N \left[T^{c}_{\mu\nu}(x) + T^{\rm eff}_{\mu\nu}(x)\right], \tag{5.2}$$

where $T^{\rm eff}_{\mu\nu}$ follows from (4.17) and (4.18) with $\omega = 0$; note that in this case $\langle T^{\mu\nu}_{(0)}\rangle_q = 0$.

The stochastic term $2F^{\mu\nu}$ will produce a stochastic contribution $h^{\rm st}_{\mu\nu}$ to the spacetime inhomogeneity, i.e. $h_{\mu\nu} = h^{c}_{\mu\nu} + h^{\rm st}_{\mu\nu}$. Let us now consider such contribution.

Substituting into (5.2) and taking into account that $h^{c}_{\mu\nu}$ already satisfies the classical equation, we obtain a linear equation for the stochastic term $h^{\rm st}_{\mu\nu}$, which we may write as

$$\Box h^{\rm st}_{\mu\nu} = 16\pi G_N S^{\rm st}_{\mu\nu},$$
$$S^{\mu\nu}_{\rm st} = 2F^{\mu\nu} = -4\partial_\alpha\partial_\beta\xi^{\mu\alpha\nu\beta}, \tag{5.3}$$

together with the harmonic gauge condition $(h^{\rm st}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{\rm st})^{,\nu} = 0$, which was used to write the previous equation. The solution of these equations as a Cauchy problem with boundary conditions $h^{\rm st}_{\mu\nu}(-\infty,\mathbf{x}) = \partial_t h^{\rm st}_{\mu\nu}(-\infty,\mathbf{x}) = 0$ is given by

$$h^{\rm st}_{\mu\nu}(t,\mathbf{x}) = 16\pi G_N \int_{-\infty}^{t} dt' \int_{R^3} d^3\mathbf{x} D_R(t-t',\mathbf{x}-\mathbf{x}') S^{\rm st}_{\mu\nu}(t',\mathbf{x}'), \tag{5.4}$$

where D_R is the retarded Green's function,

$$D_R(x-x') = -\frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}\delta(t-t'-|\mathbf{x}-\mathbf{x}'|). \tag{5.5}$$

Note that after imposing the above boundary conditions for $h^{\rm st}_{\mu\nu}$, the gauge has been completely fixed because two metric components within the harmonic gauge can differ by $2\zeta_{(\mu,\nu)}$ where ζ_μ is a harmonic vector field but this vector field is zero when such boundary conditions are imposed. Since $S^{\rm st}_{\mu\nu}$ is linear in the stochastic source, it is obvious from (5.4) that $\langle h^{\rm st}_{\mu\nu}(x)\rangle_\xi = 0$.

Let us now compute the two-point correlation function of the stochastic metric fluctuations $h^{\rm st}_{\mu\nu}$:

$$\langle h^{\rm st}_{\mu\nu}(t,\mathbf{x}) h^{\rm st}_{\lambda\theta}(s,\mathbf{y})\rangle_\xi = (4G_N)^2 \int_{R^3} d^3\mathbf{x}' \int_{R^3} d^3\mathbf{y}' \frac{\langle S^{\rm st}_{\mu\nu}(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}') S^{\rm st}_{\lambda\theta}(s-|\mathbf{y}-\mathbf{y}'|,\mathbf{y}')\rangle_\xi}{|\mathbf{x}-\mathbf{x}'||\mathbf{y}-\mathbf{y}'|}; \tag{5.6}$$

here, we made use of (5.4) and performed two time integrations. Now since $S^{\rm st}_{\mu\nu}$ has the form of a linear operator acting on the tensorial stochastic source ξ, the correlation function of $S^{\rm st}_{\mu\nu}$ can be written in terms of the correlation function of that source, i.e., in terms of the noise kernel N. After some simple manipulations we obtain

$$\langle S^{\mu\nu}_{\rm st}(x) S^{\lambda\theta}_{\rm st}(y)\rangle_\xi = \frac{3\pi\alpha}{2}\Theta^{\mu\nu\lambda\theta}\int \frac{d^4p}{(2\pi)^4} e^{ip\cdot\sigma}\theta(-p^2), \tag{5.7}$$

where we have defined $\sigma^\mu \equiv x^\mu - y^\mu$ and $\Theta^{\mu\nu\lambda\theta}$ is the operator

$$\Theta^{\mu\nu\lambda\theta} \equiv \frac{2}{3}\left[3\hat{P}^{\lambda(\mu}\hat{P}^{\nu)\theta} - \hat{P}^{\mu\nu}\hat{P}^{\lambda\theta}\right], \tag{5.8}$$

with $\hat{P}^{\mu\nu} = \eta^{\mu\nu}\Box - \partial^\mu\partial^\nu$, where all derivatives are with respect to σ^μ. Substituting this into (5.6) leads to

$$\langle h^{\rm st}_{\mu\nu}(t,\mathbf{x}) h^{\rm st}_{\lambda\theta}(s,\mathbf{y})\rangle_\xi = 6\pi^7\alpha\,(32G_N)^2\,\Theta_{\mu\nu\lambda\theta}\int \frac{d^4p}{(2\pi)^4}\theta(-p^2) e^{ip\cdot\sigma}\Delta^+(p)\Delta^-(p), \tag{5.9}$$

where $\Delta^\pm$ are the Wightman functions. Introducing a suitably regularized kernel $K_{\rm reg}(x)$,

$$K_{\rm reg}(x) = \int d^4p\, e^{ip\cdot x} \Delta^+(p)\Delta^-(p), \tag{5.10}$$

after some further manipulations the correlation function of $h^{\rm st}_{\mu\nu}(x)$ can be written as

$$\langle h^{\rm st}_{\mu\nu}(t,\mathbf{x}) h^{\rm st}_{\lambda\theta}(s,\mathbf{y})\rangle_\xi = (16\pi G_N)^2 \Theta_{\mu\nu\lambda\theta} \int d^4z\, K_{\rm reg}(\sigma - z) N(z). \tag{5.11}$$

Since the noise kernel is nonlocal, [see Eqs. (3.8) and (3.14)], the noise is colored. A simple order of magnitude estimate of the above result gives that it is of the order of (Planck time/time interval)^4, as one should expect of quantum fluctuations. Thus, if we measure the gravitational potential at a given point at different time intervals, we should find variations in this field of the order $\sqrt{\langle h^{\rm st}_{00} h^{\rm st}_{00}\rangle} \sim$ (Planck time/time interval)^2.

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APPENDIX

The $T_{\mu\nu\alpha\beta\rho\sigma\lambda\theta}$ tensor is given by

$$\begin{aligned} T_{\mu\nu\alpha\beta\rho\sigma\lambda\theta} \equiv \frac{1}{24} \Big\{ & 8\left[\eta_{\rho[\mu}\eta_{\nu]\sigma}\eta_{\lambda[\alpha}\eta_{\beta]\theta} + \eta_{\rho[\alpha}\eta_{\beta]\sigma}\eta_{\lambda[\mu}\eta_{\nu]\theta} + \eta_{\alpha[\mu}\eta_{\nu]\beta}\eta_{\lambda[\rho}\eta_{\sigma]\theta}\right] \\ & +4\left[\eta_{\rho[\mu}\eta_{\beta]\sigma}\eta_{\lambda[\alpha}\eta_{\nu]\theta} + \eta_{\rho[\mu}\eta_{\alpha]\sigma}\eta_{\lambda[\nu}\eta_{\beta]\theta} + \eta_{\rho[\nu}\eta_{\alpha]\sigma}\eta_{\lambda[\beta}\eta_{\mu]\theta} + \eta_{\rho[\beta}\eta_{\nu]\sigma}\eta_{\lambda[\alpha}\eta_{\mu]\theta}\right] \\ & -3\Big[\eta_{\mu\alpha}\left(\eta_{\rho\lambda}\eta_{\sigma(\nu}\eta_{\beta)\theta} + \eta_{\sigma\theta}\eta_{\rho(\nu}\eta_{\beta)\lambda} - \eta_{\sigma\lambda}\eta_{\rho(\nu}\eta_{\beta)\theta} - \eta_{\rho\theta}\eta_{\sigma(\nu}\eta_{\beta)\lambda}\right) \\ & \quad +\eta_{\nu\beta}\left(\eta_{\rho\lambda}\eta_{\sigma(\mu}\eta_{\alpha)\theta} + \eta_{\sigma\theta}\eta_{\rho(\mu}\eta_{\alpha)\lambda} - \eta_{\sigma\lambda}\eta_{\rho(\mu}\eta_{\alpha)\theta} - \eta_{\rho\theta}\eta_{\sigma(\mu}\eta_{\alpha)\lambda}\right) \\ & \quad -\eta_{\nu\alpha}\left(\eta_{\rho\lambda}\eta_{\sigma(\mu}\eta_{\beta)\theta} + \eta_{\sigma\theta}\eta_{\rho(\mu}\eta_{\beta)\lambda} - \eta_{\sigma\lambda}\eta_{\rho(\mu}\eta_{\beta)\theta} - \eta_{\rho\theta}\eta_{\sigma(\mu}\eta_{\beta)\lambda}\right) \\ & \quad -\eta_{\mu\beta}\left(\eta_{\rho\lambda}\eta_{\sigma(\nu}\eta_{\alpha)\theta} + \eta_{\sigma\theta}\eta_{\rho(\nu}\eta_{\alpha)\lambda} - \eta_{\sigma\lambda}\eta_{\rho(\nu}\eta_{\alpha)\theta} - \eta_{\rho\theta}\eta_{\sigma(\nu}\eta_{\alpha)\lambda}\right)\Big]\Big\}. \end{aligned} \tag{A1}$$

It comes from the functional derivative of the Weyl tensor with respect to itself taking into account all its symmetries.

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