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Minimum-uncertainty states and pseudoclassical dynamics. II

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The set of initial conditions for which the pseudoclassical evolution algorithm (and minimality conservation) is verified for Hamiltonians of degrees N ($N > 2$) is explicitly determined through a class of restrictions for the corresponding classical trajectories, and it is proved to be at most denumerable. Thus these algorithms are verified if and only if the system is quadratic except for a set of measure zero. The possibility of time-dependent α-equivalence classes is studied and its physical interpretation is presented. The implied equivalence of the pseudoclassical and Ehrenfest algorithms and their relationship with minimality conservation is discussed in detail. Also, the explicit derivation of the general unitary operator which linearly transforms minimum-uncertainty states leads to the derivation, among others, of operators with a general geometrical interpretation in phase space, such as rotations (parity, Fourier).

I. INTRODUCTION

The relationship between classical and quantum problems, and the use of semiclassical approximations in quantum mechanics, are topics that pervade all branches of physics.[1] The formulation of quantum theory draws heavily on classical concepts. As classically both the position and the momentum are needed to specify the state of a system, it is desirable to define phase-space formulations of quantum theory. The implementation of such formulations can readily be carried out through the use of continuous representations such as that of the minimum-uncertainty states (MUS's).[2] These states provide the most classicallike description of a classical particle in quantum mechanics.[3] A particularly useful formulation of these states can be accomplished through the definition of a family of non-Hermitian operators depending on a real, positive (time-dependent or constant) parameter α,

$$\hat{a}(\alpha) = \alpha\hat{q} + \frac{i}{2\alpha}\hat{p}\,.$$

For each α the set of eigenstates $|z(\alpha)\rangle$ of the generalized annihilation operator $\hat{a}(\alpha)$ defines all the MUS's of momentum variance α.[4] This set, which is overcomplete, is called the α-equivalence class.[5] This fact and the calculational advantage of dealing with eigenstates of annihilation operators are two basic features of our approach.

In a recent paper[4] (hereafter referred to as I) the relevance of quadratic Hamiltonians has been proved for an appealing algorithm which expresses the nonrelativistic quantum mechanics of a spinless particle in a classicallike way: the pseudoclassical evolution algorithm. This algorithm is defined as follows: Given an initial MUS $|z_0\rangle$ at t_0, one studies which physical systems verify the condition that the MUS $|z_t\rangle$, which has for each $t > t_0$ maximum overlapping $|\langle z_t|z_0, t\rangle|^2$ with the evolved state $|z_0, t\rangle$, follows the classical trajectory in phase space:

$$\langle z_t|\hat{q}|z_t\rangle = q_{cl}(t)\,,$$
$$\langle z_t|\hat{p}|z_t\rangle = p_{cl}(t)\,.$$

We emphasize that this approach, and in fact all of our investigation, does not deal with any kind of approximation (e.g., the classical limit or the semiclassical ones).

The aim of this paper is to find the most general class of Hamiltonians that allows the description of the dynamics in terms of the pseudoclassical evolution algorithm emphasizing the study of time-dependent α-equivalence classes of the MUS's and the investigation of the cases in which the algorithm is verified not necessarily for all initial conditions. This program has already been completed for the special case of constant α and validity for any initial condition in I. In the general case we find here that the pseudoclassical condition is not restricted to quadratic Hamiltonians for some initial conditions. These are explicitly identified for each nonquadratic Hamiltonian. It is proved that they form a denumerable set, which is consequently of measure zero in the phase-space plane. This study is interwoven with the corresponding ones for the Ehrenfest algorithm and the conservation of minimality. By Ehrenfest algorithm we mean the condition that the expectation values of $\hat{q}$ and $\hat{p}$ verify the classical equations of motion, so that

$$\langle z_0, t|\hat{q}|z_0, t\rangle = q_{cl}(t)\,,$$
$$\langle z_0, t|p|z_0, t\rangle = p_{cl}(t)\,.$$

On the other hand, an explicit analysis of the Hamiltonians which conserve the α-equivalence classes leads to a natural way of calculating unitary operators which represent geometrical transformations in phase space.

The main body of this paper is divided into three sections. In Sec. II an explicit calculation is done of the transformation of the MUS's under linear unitary operators, that is

$$\hat{U}\hat{a}\hat{U}^{\dagger} = u\hat{a} + v\hat{a}^{\dagger} + w ,$$

which are related to the quadratic Hamiltonians of I. Particular cases of the derived general transformation operator are discussed in the Appendix. These results also clarify some misleading points in the literature. In Sec. III, the pseudoclassical and Ehrenfest algorithms and the conservation of minimality are explicitly resolved. These three problems are studied in the general case of time-dependent α's and when their validity is not demanded for all initial conditions. The higher-dimensional case is briefly outlined and, in particular, the external magnetic field is considered. We comment on the difference between the MUS's and the gauge-dependent coherent states usually defined. Finally in Sec. IV, the equivalence of the two algorithms and their relation to minimality conservation is discussed in detail. These results provide a deeper insight into quantum dynamics through classical notions.

II. LINEAR TRANSFORMATIONS IN PHASE SPACE

All one-particle position-momentum MUS's[4] are obtained as the eigenstates[6] of ($\hbar = 1$)

$$\hat{a}(\alpha) = \alpha\hat{q} + \frac{i}{2\alpha}\hat{p} , \tag{2.1}$$

where α is an arbitrary real, positive parameter. The set of eigenstates $|z(\alpha)\rangle$ for any fixed α is overcomplete, and it spans the whole complex plane z $[z = (q,p)]$. This leads us to consider the phase space defined by the continuous spectra of $\hat{q}$ and $\hat{p}$.[7] To any point $z = (q,p)$ in this space, there corresponds a MUS $|z(\alpha)\rangle$ in each α-equivalence class.[5] The parameter α measures the variance of the Gaussian along the p axis.

The different $\hat{a}(\alpha)$'s are related to each other by the linear relation

$$\hat{a}(\beta) = c(\alpha,\beta)\hat{a}(\alpha) + s(\alpha,\beta)\hat{a}^{\dagger}(\alpha) , \tag{2.2}$$

where

$$c(\alpha,\beta) = (\alpha^2 + \beta^2)/2\alpha\beta \equiv \cosh r , \tag{2.3a}$$

$$s(\alpha,\beta) = (\beta^2 - \alpha^2)/2\alpha\beta \equiv \sinh r , \tag{2.3b}$$

and

$$r = \ln(\beta/\alpha) . \tag{2.3c}$$

As $\hat{a}(\alpha)$ and $\hat{a}^{\dagger}(\alpha)$ verify the boson commutation relations for any α, it follows that all the $\hat{a}(\alpha)$'s are unitarily equivalent.

We wish to consider here a generalization of Eq. (2.2), that is, the most general linear canonical transformation on $\hat{a}(\alpha)$:

$$\hat{A}(u,v,w) = u\hat{a}(\alpha) + v\hat{a}^{\dagger}(\alpha) + w , \tag{2.4}$$

where

$$|u|^2 - |v|^2 = 1 . \tag{2.5}$$

In the following we shall restrict ourselves to the case $w = 0$, since the w term is induced simply by a displacement operator $\hat{D}(w)$; if

$$\hat{U}\hat{a}(\alpha)\hat{U}^{-1} = u\hat{a}(\alpha) + v\hat{a}^{\dagger}(\alpha) + w , \tag{2.6a}$$

$$\hat{V}\hat{a}(\alpha)\hat{V}^{-1} = u\hat{a}(\alpha) + v\hat{a}^{\dagger}(\alpha) , \tag{2.6b}$$

then

$$\hat{U} = \hat{D}(u^*w - vw^*)\hat{V} = \hat{V}\hat{D}(w) . \tag{2.7}$$

Unitarity means that the $\hat{A}$ eigenstates $|Z;u,v\rangle$ (eigenvalue Z) which we shall call linear transformed MUS's (LTM's)[8] verify the same kind of relations as the MUS's $|z(\alpha)\rangle$. In particular we have for an arbitrary operator $M(\,,\,)$:

$$\langle Z;u,v|M[\hat{A}^{\dagger}(u,v),\hat{A}(u,v)]|\psi\rangle = M\left(Z^*, \frac{Z}{2} + \frac{\partial}{\partial Z^*}\right)\langle Z;u,v|\psi\rangle \tag{2.8}$$

from where we can calculate the overlap integral

$$\langle z(\alpha)|Z;u,v\rangle = u^{-1/2}\exp\left[-\frac{|z(\alpha)|^2}{2} - \frac{|Z|^2}{2} - \frac{v}{2u}z^{*2}(\alpha) + \frac{v^*}{2u}Z^2 + \frac{1}{u}z^*(\alpha)Z\right]. \tag{2.9}$$

We remark that naturally a MUS is a LTM but a LTM is not always a MUS, since for the latter

$$\Delta_q\Delta_p = 2^{-1}|u^2 - v^2|^{1/2} \tag{2.10}$$

as it easily follows from the definitions.

A. Hamiltonians for the MUS → LTM evolution

We shall obtain the Hamiltonian which transforms any initial state $|z_0(\alpha_0)\rangle$ into the LTM $|Z;u(t),v(t)\rangle$ (Ref. 9), where

$$Z(t) = n(t)z_0(\alpha_0) + m(t) . \tag{2.11}$$

The LTM eigenvalue equation can be written as

$$\hat{U}_t^{\dagger}\hat{A}(t)\hat{U}_t|z_0(\alpha_0)\rangle = [n(t)z_0(\alpha_0) + m(t)]|z_0(\alpha_0)\rangle \tag{2.12}$$

and since the states $|z_0\rangle$ form a complete set, it follows that

$$\hat{A}(t) = n(t)\hat{a}_H(\alpha_0;-t) + m(t) , \tag{2.13}$$

where the subscript H refers to the Heisenberg picture (the absence of this subscript implies the Schrödinger picture). We rewrite this equation as

$$c(t)\hat{a}(\alpha_0)+s(t)\hat{a}^\dagger(\alpha_0)=\hat{a}_H(a_0;-t)+\Delta(t)\,, \tag{2.14}$$

where

$$c(t)\equiv u(t)/n(t)\,, \tag{2.15a}$$

$$s(t)\equiv v(t)/n(t)\,, \tag{2.15b}$$

$$\Delta(t)\equiv m(t)/n(t)\,. \tag{2.15c}$$

As both u, v and c, s define two different $\hat{A}$ operators, we conclude from Eq. (2.5) that $n(t)$ is a complex phase function.

The initial conditions for the complex functions c, s, and Δ are 1, 0, and 0, respectively. Now the idea is to write down the equation of motion of (2.14), so that we find the relation between the coefficients of the Hamiltonian and the known functions c, s, and Δ.

If we consider the normal-ordered expansion of the Hamiltonian[10]

$$\hat{H}(t)=\sum_{n,m}f_{nm}(t)\hat{a}^{\dagger n}(\alpha_0)\hat{a}^m(\alpha_0) \tag{2.16}$$

we calculate

$$\begin{aligned}\frac{\partial}{\partial t}\hat{a}_H(\alpha_0;-t)+\dot{\Delta}(t)\\ =i\sum_{n,m}f_{nm}(t)[nc(t)\hat{a}^{\dagger n-1}(\alpha_0)\hat{a}^m(\alpha_0)\\ -ms(t)\hat{a}^{\dagger n}(\alpha_0)\hat{a}^{m-1}(\alpha_0)]+\dot{\Delta}(t)\end{aligned} \tag{2.17}$$

and as this equals

$$\frac{\partial}{\partial t}\hat{A}(t)=\dot{c}(t)\hat{a}(\alpha_0)+\dot{s}(t)\hat{a}^\dagger(\alpha_0)$$

we obtain from a term-by-term comparison

$$i(cf_{10}-sf_{01})+\dot{\Delta}=0\,, \tag{2.18a}$$

$$\dot{c}=i(cf_{11}-2sf_{02})\,, \tag{2.18b}$$

$$\dot{s}=i(2cf_{20}-sf_{11})\,, \tag{2.18c}$$

$$f_{nm}=0,\quad n+m>2\,. \tag{2.19a}$$

This leads us finally to the result

$$f_{10}=i(\dot{\Delta}c^*-\dot{\Delta}^*s)=f_{01}^*\,, \tag{2.19b}$$

$$f_{11}=i(c\dot{c}^*-s^*\dot{s})\,, \tag{2.19c}$$

$$f_{20}=\tfrac{1}{2}i(s\dot{c}^*-c^*\dot{s})=f_{02}^*\,. \tag{2.19d}$$

It should be noted that given u, v, m, and n (six mathematical degrees of freedom) we have defined c, s, and Δ (five degrees) and these determine f_{10}, f_{11}, and f_{20} (five degrees). This means that there exists an exhaustive (one-to-one) correspondence between the sets u, v, m, and n (c, s, and Δ) and f_{10}, f_{11}, and f_{20}. The liberty in the phase $n(t)$ explains the apparent disagreement between different results in the literature.[11]

As an application we study the case

$$\hat{A}(t)=\exp[i\varphi(t)]\hat{a}(\alpha_t)$$

in which u/v is real, so that $|Z(t);u(t),v(t)\rangle$ equals $|z_t(\alpha_t)\rangle$ and[12]

$$Z(t)=z_0(\alpha_0)+\Delta(t)=z_t(\alpha_t)\,.$$

Equations (2.19) lead us to the Hamiltonian [we write $\alpha_t\equiv\alpha(t)$]

$$\begin{aligned}\hat{H}_{\alpha(t)}(t)=&\tfrac{1}{2}\dot{\varphi}\cosh 2r\,\hat{a}^\dagger(\alpha_0)\hat{a}(\alpha_0)\\ &+\tfrac{1}{2}(\dot{\varphi}\sinh 2r-i\dot{r})a^{\dagger 2}(\alpha_0)\\ &+i(\dot{\Delta}e^{-i\varphi}\cosh r-\dot{\Delta}^*e^{i\varphi}\sinh r)\hat{a}^\dagger(\alpha_0)\\ &+\tfrac{1}{2}f_{00}+\text{H.c.}\,,\end{aligned} \tag{2.20a}$$

which can be expressed in terms of $a(\alpha_t)$ as[13]

$$\begin{aligned}\hat{H}_{\alpha(t)}(t)=&\tfrac{1}{2}\dot{\varphi}(1+\tfrac{1}{2}\sinh^2 2r)\hat{a}^\dagger(\alpha_t)\hat{a}(\alpha_t)\\ &-\tfrac{1}{4}i\dot{r}[\hat{a}^{\dagger 2}(\alpha_t)-\hat{a}^2(\alpha_t)]+i\dot{\Delta}e^{-i\varphi}\hat{a}^\dagger(\alpha_t)\\ &+\tfrac{1}{2}(f_{00}-\dot{\varphi}\sinh^2 r)+\text{H.c.}\end{aligned} \tag{2.20b}$$

We stress that both these results are expressed in the Schrödinger picture and that $\hat{a}(\alpha_t)$ is not to be identified with $\hat{a}_H(\alpha_0;t)$, as follows from Eq. (2.13).[14]

The physical meaning of $\hat{H}_{\alpha(t)}$ is that it conserves minimality with $\dot{\alpha}_t\neq 0$ ($r\neq 1$); it generalizes the $\dot{\alpha}_t=0$ case[15] ($\hat{H}_\alpha$) through the f_{20} (and f_{02}) terms. The relationship between both cases is expressed by the equations

$$\hat{q}_{2,H}(-t)=(\alpha_t/\alpha_0)\hat{q}_{1,H}(-t)\,, \tag{2.21a}$$

$$\hat{p}_{2,H}(-t)=(\alpha_0/\alpha_t)\hat{p}_{1,H}(-t)\,, \tag{2.21b}$$

where the subindex 1 (2) refers to $\dot{\alpha}_t\neq 0$ ($\dot{\alpha}_t=0$). Thus $\dot{\alpha}_t\neq 0$ induces a "canonical reciprocal scaling" of the q,p axes in phase space.

The presence in $\hat{H}_{\alpha(t)}$ of the term

$$\tfrac{1}{2}i\dot{r}(\hat{a}^2-\hat{a}^{\dagger 2})$$

(expressed in terms of any α) has the one and only role of changing α_t (the Δ_p of the MUS) in time and can be thus considered to be apart from the rest of the dynamics. This can be visualized as follows. Making use of the group property we can write

$$\hat{U}(T,0)=\lim\hat{U}(n\tau,(n-1)\tau)\cdots\hat{U}(\tau,0)\,,$$

where the limit is taken over $n\to\infty$ and $\tau\to 0$ (with $n\tau=T$, as usual), and where

$$\hat{U}(t+\tau,t)=1-i\hat{H}(t)\tau+O(\tau^2)\,.$$

At t the most general Hamiltonian which preserves the MUS's of the class α_t is $\hat{H}_{\alpha=\alpha_t}(t)$. Now at $t+\tau$ the evolved MUS should correspond to $\alpha_{t+\tau}$ and not α_t, so that we have to apply[16] $\hat{U}(\alpha_t\to\alpha_{t+\tau})$.

This repeated process, $\hat{U}_{\alpha=\alpha_t}(t+\tau,t)$ and $\hat{U}(\alpha_t\to\alpha_{t+\tau})$, can be understood as a unique evolution process. We shall prove that this combined process is precisely the one corresponding to $\hat{H}_{\alpha(t)}(t)$; if

$$\hat{U}(\alpha_t\to\alpha_{t+\tau})=\exp\left(\tfrac{1}{2}\sinh\frac{\alpha_{t+\tau}{}^2-\alpha_t{}^2}{2\alpha_t\,\alpha_{t+\tau}}\right)\times[\hat{a}^2(\alpha_t)-\hat{a}^{\dagger 2}(\alpha_t)]$$
$$=\hat{1}+\tfrac{1}{2}\dot{r}\,[\hat{a}^2(\alpha_t)-\hat{a}^{\dagger 2}(\alpha_t)]+O(\tau^2)\,,$$

$$\hat{U}_{\alpha=\alpha_t}(t+\tau,t)=\hat{1}-i\hat{H}_{\alpha=\alpha_t}(t)\tau+O(\tau^2)\,,$$

then

$$\hat{U}(\alpha_t\to\alpha_{t+\tau})\hat{U}_{\alpha=\alpha_t}(t+\tau,t)$$
$$=\hat{1}-i\tau\{\hat{H}_{\alpha=\alpha_t}(t)-\tfrac{1}{2}i\dot{r}\,[\hat{a}^{\dagger 2}(\alpha_t)-\hat{a}^2(\alpha_t)]\}+O(\tau^2)$$
$$=\hat{U}_{\alpha(t)}(t+\tau,t)\,.$$

Q.E.D.

Finally, we would like to point out that, although the free (or even driven) oscillator is included in $\hat{H}_\alpha$, a free oscillator with time-varying frequency is not included in $\hat{H}_{\alpha(t)}$.

B. Transformation operators – explicit derivation

In the last section we have found the Hamiltonians which make initial MUS's evolve into LTM's for all $t>0$ in a continuous manner. The calculation of the evolution operator of these $\hat{H}(t)$ cannot always be given in quadratures.[17] This is related to the fact that the set of differential equations (2.19), now considered with f_{ij} as data,[8] is not always explicitly integrable.

In this section we obtain the transformation operator $\hat{U}$ of the MUS's into the given LTM's:

$$\hat{U}|z_0(\alpha_0)\rangle=e^{i\phi}|z_0(\alpha_0)+\Delta;c,s\rangle\,, \tag{2.22}$$

where c, s, and Δ are given [via Eqs. (2.15) from the data $\boldsymbol{u}$, v, m, and $\boldsymbol{n}$] for all z_0 and α_0. Although this is not a time-evolution process, we propose to derive $\hat{U}$ by calculating the evolution operator $\hat{U}(T,0)$ of an appropriate constant Hamiltonian which transforms the MUS's at $t=0$ into the LTM's of interest at $t=T$ (the result being independent of T). The phase ϕ induced by this operator is then calculated with the aid of normal-ordering techniques.

If we solve Eqs. (2.19a) and (2.19b) for constant f_{ij}'s we arrive at

$$\ddot{c}+(f_{11}{}^2-4|f_{20}|^2)c=0\,.$$

This allows three kinds of solutions: (i) $f_{11}{}^2>4|f_{20}|^2$, (ii) $f_{11}{}^2<4|f_{20}|^2$, and (iii) $f_{11}{}^2=4|f_{20}|^2$.

In the first case (i) we obtain $[\lambda\equiv(f_{11}{}^2-4|f_{20}|^2)^{1/2}]$

$$c(t)=\cos\lambda t+i(f_{11}/\lambda)\sin\lambda t\,, \tag{2.23a}$$

$$s(t)=2i(f_{20}/\lambda)\sin\lambda t\,, \tag{2.23b}$$

so that from (2.20c) there follows

$$\Delta(t)=(2f_{20}f_{01}-f_{11}f_{10})(\cos\lambda t-1)/\lambda^2-i(f_{10}/\lambda)\sin\lambda t\,. \tag{2.23c}$$

We have to distinguish two possibilities: $s=0$, $s\neq 0$. If $s=0$ (and $|c|=1$) we have

$$f_{11}=(\ln c)/iT=\arg c/T\,, \tag{2.24a}$$

$$f_{20}=0\,, \tag{2.24b}$$

$$f_{10}=i\Delta\ln c/T(c-1)=\Delta\arg c/T(1-c)\,, \tag{2.24c}$$

and

$$\hat{U}=\exp(-iT\hat{H})=\exp(-i\arg c)\left[\hat{a}^\dagger(\alpha_0)\hat{a}(\alpha_0)+\frac{\Delta}{1-c}\hat{a}^\dagger(\alpha_0)+\frac{\Delta^*}{1-c}\hat{a}(\alpha_0)\right], \tag{2.25}$$

where for simplicity we take $f_{00}=0$. The trivial case $c=1$ is considered in the Appendix (displacement operator).

When $s\neq 0$, from Eqs. (2.23) and after some straightforward algebra we infer

$$f_{11}=\frac{1}{T}\frac{\operatorname{Im}c}{[1-(\operatorname{Re}c)^2]^{1/2}}\cos^{-1}(\operatorname{Re}c)\,, \tag{2.26}$$

$$f_{20}=-if_{11}s/2\operatorname{Im}c\,, \tag{2.27a}$$

$$f_{10}=\frac{f_{11}}{2\operatorname{Im}c}\{[\operatorname{Re}\Delta(\operatorname{Im}c+\operatorname{Im}s)-\operatorname{Im}\Delta(1+\operatorname{Re}c-\operatorname{Re}s)]+i[\operatorname{Re}\Delta(1+\operatorname{Re}c-\operatorname{Re}s)+\operatorname{Im}\Delta(\operatorname{Im}c-\operatorname{Im}s)]\}\,, \tag{2.27b}$$

and

$$\hat{U}=\exp\left(\frac{-iTf_{11}}{2\operatorname{Im}c}\right)\{2\operatorname{Im}c\,\hat{a}^\dagger(\alpha_0)\hat{a}(\alpha_0)+i[s^*a^2(\alpha_0)-s\hat{a}^{\dagger 2}(\alpha_0)]$$
$$+[\operatorname{Re}\Delta(\operatorname{Im}c+\operatorname{Im}s)-\operatorname{Im}\Delta(1+\operatorname{Re}c-\operatorname{Re}s)][\hat{a}^\dagger(\alpha_0)+\hat{a}(\alpha_0)]$$
$$+i[\operatorname{Re}\Delta(1+\operatorname{Re}c-\operatorname{Re}s)+\operatorname{Im}\Delta(\operatorname{Im}c-\operatorname{Im}s)][\hat{a}^\dagger(\alpha_0)-\hat{a}(\alpha_0)]\}\,. \tag{2.28}$$

In the second case (ii) we obtain the same result (2.28) except for

$$f_{11}=\frac{1}{T}\frac{\mathrm{Im}c}{[(\mathrm{Re}c)^2-1]^{1/2}}\cosh^{-1}(\mathrm{Re}c) \tag{2.29}$$

the condition reading $|\mathrm{Re}c|>1$ (so necessarily $s\neq 0$).

In the last case (iii) the only modification is

$$f_{11}=\mathrm{Im}c/T\,, \tag{2.30}$$

the condition of applicability being $|\mathrm{Re}c|=1$ and $s\neq 0$.

Thus we have explicitly calculated $\hat{U}$. There only remains to calculate the phase it induces. This can be done by taking the scalar product of Eq. (2.22) with $\langle z_0(\alpha_0)|$,

$$\langle z_0(\alpha_0)|\hat{U}|z_0(\alpha_0)\rangle=e^{i\phi}\langle z_0(\alpha_0)|z_0(\alpha_0)+\Delta;c,s\rangle.$$

The left-hand side is evaluated by normal ordering,[18] and the right-hand side through Eq. (2.10). The result is

$$\begin{aligned}i\phi=&|z_0|^2\left(A+1-\frac{1}{c}\right)+z_0^2\left(B-\frac{s^*}{2c}\right)\\&+z_0^{*2}\left(C+\frac{s}{2c}\right)+z_0\left[D+\frac{\Delta^*}{2}-\frac{s^*\Delta}{c}\right]\\&+z_0^*\left[E-\Delta\left(\frac{1}{c}-\frac{1}{2}\right)\right]+\left(F+\frac{|\Delta|^2+\ln c}{2}-\frac{s^*\Delta^2}{c}\right),\end{aligned} \tag{2.31}$$

where

$$\begin{aligned}&A=J^{-1}-1\,,\\&B=(y\sinh j)(jJ)^{-1}\,,\\&C=(z\sinh j)(jJ)^{-1}\,,\\&D=j^{-2}(\tau A+2\mu B)\,,\\&E=j^{-2}(\mu A+2\tau B)\,,\\&F=-\tfrac{1}{2}\ln J-\tfrac{1}{2}x+j^{-2}(\varphi-x\delta\epsilon)+j^{-4}\tau\mu A\\&\qquad+j^{-5}J^{-1}(4yz\varphi-x^2\varphi-8xyz\delta\epsilon)\sinh\gamma\end{aligned} \tag{2.32a}$$

with

$$\begin{aligned}&J=\cosh j-\gamma^{-1}x\sinh j\,,\\&j=(x^2-4yz)^{1/2},\quad x=-iTf_{11}\,,\\&\varphi=y\epsilon^2+z\delta^2,\quad y^*=z=-iTf_{20}\,,\\&\tau=x\delta-2y\epsilon,\quad \epsilon=-\delta^*=-iTf_{10}\,,\\&\mu=x\epsilon-2z\delta\,.\end{aligned} \tag{2.32b}$$

In conclusion, when $s=0$ the result is given by Eqs. (2.25), (2.31), and (2.32); when $s\neq 0$, by (2.28), (2.31), (2.32) and by (2.26) if $|\mathrm{Re}c|<1$, by (2.29) if $|\mathrm{Re}c|>1$, and by (2.30) if $|\mathrm{Re}c|=1$.

This general result is applied in the Appendix for the derivation of the following particular transformations:

Case $s=0$. The displacement operator and the rotation operator about an arbitrary point in phase space. This includes the Fourier operator and parity, which is closely related to the Wigner distribution function. As $s=0$ the equivalence class of MUS's is conserved. This means that the actuation of $\hat{U}$ on an arbitrary state conserves the same geometrical interpretation in phase space as it has for the MUS's (in terms of which it is originally defined).

Case $s\neq 0$ *and* $|\mathrm{Re}c|<1$. Rotations of the MUS's.

Case $s\neq 0$ *and* $|\mathrm{Re}c|>1$. The change of α-equivalence class with an added displacement. We comment here on the $r\to\pm\infty$ limit of the MUS's.

III. QUANTUM DYNAMICS IN PHASE SPACE – CLASSICAL ALGORITHMS

The question envisaged here is: When and how does classical "intuition" work in the quantum-mechanical dynamics of a nonrelativistic, spinless particle? To begin with, the classical picture of a particle as a point in phase space is best translated by a MUS with the same localization.[19] Thus the initial condition of the quantal systems' evolution will always be taken as MUS's in our discussion.

Given the Hamiltonian $\hat{H}$ as a function $H(\hat{q},\hat{p},t)$ and the initial MUS located at z_0, we can determine the corresponding classical problem. Three different algorithms are studied below in order to see when and in what way does the time evolved state follow the classical trjectory in phase space: Ehrenfest, minimality conservation, and pseudoclassical. The analysis is done in two cases: for any initial MUS, and not for all initial MUS's; in each case both $\dot{\alpha}=0$ and $\dot{\alpha}\neq 0$ are considered.

A. The Ehrenfest algorithm

In this well-known problem,[20] the question is when do the expectation values $(\langle\hat{q}\rangle_t,\langle\hat{p}\rangle_t)\equiv z_E(t)$ follow the classical trajectory $z_{\rm cl}(t)$. [We write $\langle\hat{A}\rangle_t$ as shorthand for $\langle\psi(t)|\hat{A}|\psi(t)\rangle$.] The answer is if and only if[21]

$$-i\langle[\hat{q},\hat{H}]\rangle_t=\partial H_{\rm cl}(\langle\hat{q}\rangle_t,\langle\hat{p}\rangle_t,t)/\partial\langle\hat{p}\rangle_t, \tag{3.1a}$$

$$i\langle[\hat{p},\hat{H}]\rangle_t=\partial H_{\rm cl}(\langle\hat{q}\rangle_t,\langle\hat{p}\rangle_t,t)/\partial\langle\hat{q}\rangle_t. \tag{3.1b}$$

1. *For all* $|z_0\rangle$. Equations (3.1) are verified for a general quadratic operator $\hat{H}_Q$, and for all initial states $|\psi(t=t_0)\rangle$ (not necessarily MUS's). But in general $z_E(t)$ cannot be identified with the coordinates of the classical particle: The classical trajectory, solution of a second-order differential equation, is determined by the initial conditions,

while the expectation values $\langle\cdots\rangle_t$ depend upon the initial wave function. Given this function, $\langle\hat{q}\rangle_{t_0}$ and $\langle\hat{p}\rangle_{t_0}$ are determined, but they are not independent. Thus if, for example, $|\psi(t_0)\rangle$ is an energy eigenstate of an oscillator, $\langle\hat{q}\rangle_t$ is zero for all t. This fact also underlines the importance of dealing with the MUS's as initial conditions.

2. *Not for all* $|z_0\rangle$. We can write

$$-i[\hat{q},\hat{H}]=(2\alpha_0 i)^{-1}\{(c^*-s^*)[\hat{A},\hat{H}]+(c-s)[\hat{A}^\dagger,\hat{H}]\}\equiv\sum_{r,s}F_{rs}\hat{A}^{\dagger r}\hat{A}^s$$

$$=\sum_{r,s}F_{rs}\left(\alpha_0(c^*+s^*)\hat{q}+\frac{i}{2\alpha_0}(s^*-c^*)\hat{p}\right)^r\left(\alpha_0(c+s)\hat{q}+\frac{i}{2\alpha_0}(c-s)\hat{p}\right)^s\equiv\sum_{r,s}F_{rs}P_{rs}(\hat{q},\hat{p}),\tag{3.2}$$

where F_{rs} and P_{rs} are defined in the second and fourth equalities. The expectation value of (3.2) equals $\sum_{rs}F_{rs}P_{rs}(\langle\hat{q}\rangle_t,\langle\hat{p}\rangle)_t$ if and only if $\langle\hat{A}^{\dagger r}(t)\hat{A}^s(t)\rangle$ factorizes. This necessity stems from the fact that the $\hat{A}$'s, linear in $\hat{q}$ and $\hat{p}$, provide the most general realization of the commutation relations. The factorization is trivial when the maximum of $r+s$ is one (for $\hat{H}_Q$). When this is not the case, the necessary and sufficient condition is that $|z_0(\alpha_0);t\rangle$ be a LTM with $c(t)$ and $s(t)$ (see the Appendix and Ref. 35 for a comment on the case of $\langle\hat{a}^{\dagger r}\hat{a}^s\rangle$ and the MUS's; the proof for $\langle\hat{A}^{\dagger r}\hat{A}^s\rangle$ and the LTM's is a straightforward generalization). We shall prove below that for Hamiltonians more than quadratic, conservation of the LTM's can be verified, but only for some initial conditions. In particular, we shall obtain a similar result for minimality conservation (both for $\dot{\alpha}_t=0$ and $\dot{\alpha}_t\neq 0$).

B. Conservation of the LTM's and MUS's (minimality)

We are interested in finding the conditions under which a particular initial LTM state $|z_0;c_0,s_0\rangle$ is an eigenstate of $\hat{A}(t)$ for all times t with an eigenvalue z_t. A first-order calculation leads us to

$$[\hat{A}(t),\hat{H}(t)]\,|z_t;c(t),s(t)\rangle$$
$$=i[\dot{z}_t+(\dot{s}s^*-\dot{c}c^*)\hat{A}(t)+(\dot{c}s-\dot{s}c)\hat{A}^\dagger(t)]\times|z_t;c(t),s(t)\rangle,\tag{3.3}$$

where $|z_t;c(t),s(t)\rangle$ is the time-evolved state from $|z_0;c_0,s_0\rangle$ (except for a phase). In the Heisenberg picture we have

$$[\hat{A}_{0,H}(t),\hat{H}_H(t)]\,|z_0;c_0,s_0\rangle$$
$$=i[\dot{z}_t+(\dot{s}s^*-\dot{c}c^*)\hat{A}_{0,H}(t)+(\dot{c}s-\dot{s}c)\hat{A}^\dagger_{0,H}(t)]\times|z_0;c_0,s_0\rangle$$

so that the argument that follows is valid in both pictures.

Making use of the normal-ordered expansion of $\hat{H}$ in terms of $\hat{A}(t)$,

$$\hat{H}(t)=\sum_{n,m}F_{nm}(t)\hat{A}^{\dagger n}(t)\hat{A}^m(t)\tag{3.4}$$

and taking the scalar product with an arbitrary $\langle z_1;c(t),s(t)|$, Eq. (3.3) leads us to

$$\sum_{n,m}nF_{nm}(t)z_1^{*\,n-1}z_t^m$$
$$=i[\dot{z}_t+(\dot{s}s^*-\dot{c}c^*)z_t+(\dot{c}s-\dot{s}c)z_1^*].\tag{3.5}$$

The arbitrariness of z_1 implies

$$\sum_m F_{1m}(t)z_t{}^m=i[\dot{z}_t+(\dot{s}s^*-\dot{c}c^*)z_t],\tag{3.6a}$$

$$\sum_m F_{2m}(t)z_t{}^m=\tfrac{1}{2}i(\dot{c}s-\dot{s}c),\tag{3.6b}$$

$$\sum_m F_{nm}(t)z_t{}^m=0,\quad n>2.\tag{3.6c}$$

If $\hat{H}$ is a polynomial of degree N (the maximum, over all terms, of the sum of exponents of $\hat{q}$ and $\hat{p}$), an upper bound to the number of possible z_0's will be $N-2$, which is the degree of the $N-2$ polynomial equations (3.6c). From (3.6) we conclude that, given F_{ij}, $N-2$ solutions z_t are possible although their existence is not guaranteed. On the other hand, given z_0, there exist many $\hat{H}$'s which verify (3.6).

As we are interested in minimality conservation where c and s are $\cosh r(t)$ and $\sinh r(t)$, respectively, from (3.6) we obtain

$$\sum_m F_{1m}(t)[z_t(\alpha_t)]^m=i\dot{z}_t(\alpha_t),\tag{3.7a}$$

$$\sum_m F_{2m}(t)[z_t(\alpha_t)]^m=-i\dot{\alpha}_t/2\alpha_t,\tag{3.7b}$$

$$\sum_m F_{nm}(t)[z_t(\alpha_t)]^m=0,\quad n>2,\tag{3.7c}$$

where in this case the F_{nm} correspond to the normal-ordered expansion of $\hat{H}$ in terms of $\hat{a}(\alpha_t)$. In the particular case $\dot{\alpha}_t=0$, the right-hand side of (3.7b) becomes zero.[22]

If we restrict the condition of validity for only a particular initial condition, to validity for any initial condition, Eq. (3.6) leads us to

$$F_{nm}=0, \quad n+m>2, \tag{3.8a}$$

$$F_{20}=\tfrac{1}{2}i(\dot{c}s-\dot{s}c)=F_{02}^*, \tag{3.8b}$$

$$F_{11}=i(\dot{s}s^*-\dot{c}c^*)=F_{11}^*, \tag{3.8c}$$

$$F_{10}=i\dot{\Delta}=F_{01}^*, \tag{3.8d}$$

where

$$z_t=z_0+\Delta(t).$$

When the corresponding f_{nm} of (2.16) are calculated, we naturally reobtain (2.19), with the only difference that the initial conditions (for LTM conservation) for c and s are c_0 and s_0 instead of 1 and 0. In particular when minimality is conserved, we recover[23] $\hat{H}_{\alpha(t)}$ if $\dot{\alpha}_t \neq 0$ and $\hat{H}_\alpha$ if $\dot{\alpha}_t=0$ (see Sec. II A). For this reason any $|z_0(\alpha_0)\rangle$ was valid.

We note that when z_1 in Eq. (3.5) is taken as z_t, we get precisely the classical evolution for $\langle\hat{q}\rangle_t$ and $\langle\hat{p}\rangle_t$ (Ehrenfest algorithm). The explicit time dependence of $\hat{A}(t)$, of which the evolved LTM is an eigenstate, reflects itself in the right-hand side of (3.5). In the case of minimality conservation, the effect is a dilatation (if $\dot{\alpha}_t>0$) of the q axis (at a rate $\dot{\alpha}_t/\alpha_t$) and a reciprocal contraction of the p axis. These renormalizations of the phase-space axes are due to the $\frac{1}{2}i\dot{r}(\hat{a}^2-\hat{a}^{\dagger 2})$ term in $\hat{H}_{\alpha(t)}$, and are in agreement with the interpretation given in Sec. II A.

C. The pseudoclassical evolution algorithm

As stated above, once the quantum problem is defined [Hamiltonian $\hat{H}$ and initial condition $|z_0(\alpha_0)\rangle$ at t_0], we can immediately calculate the solution to the corresponding classical problem $z_{\text{cl}}(t)$. The pseudoclassical evolution[4] is verified when $z_{\text{cl}}(t)$ equals a $z_p(t)$ defined for the quantum problem as follows. Although there is a one-to-one correspondence between $|z_0(\alpha_0)\rangle$ and phase space (for each α_0), this is not the case for the time evolved $|z_0(\alpha_0);t\rangle$ since this state is not in general a MUS. Taking into account

$$|z_0(\alpha_0);t\rangle=\pi^{-1}\int d^2z(\alpha)\langle z(\alpha)|z_0(\alpha_0);t\rangle|z(\alpha)\rangle\,, \tag{3.9}$$

we define for $|z_0(\alpha_0);t\rangle$ the phase-space point $z_p(t)$, which corresponds to the maximum overlap:

$$z_p(t)=\max_z\;|\langle z(\alpha)|z_0(\alpha_0);t\rangle|^2\,, \tag{3.10}$$

since, of the set $\{|z(\alpha)\rangle\}$, it is $|z_p(t)\rangle$ which best describes $|z_0(\alpha_0);t\rangle$. This is the idea behind the pseudoclassical condition: That the best (of the most classicallike[24]) position-momentum measurements lie along the classical trajectory in phase space.

1. $\dot{\alpha}_t=0$

We shall first consider the same $\alpha(=\alpha_0)$ for all t. The point z_p, which corresponds in fact to the maximum of the antinormal distribution function,[7] verifies[4]

$$\langle[z_p(t)](\alpha_0)|\hat{a}(\alpha_0)|z_0(\alpha_0);t\rangle =[z_p(t)](\alpha_0)\langle[z_p(t)](\alpha_0)|z_0(\alpha_0);t\rangle, \tag{3.11}$$

where $|[z_p(t)](\alpha_0)\rangle$ is the α_0 equivalence class MUS of eigenvalue $z_p(t)$. To first order in time this is

$$z_p(t_0+\tau)=z_E(t_0+\tau)+O(\tau^2)\,, \tag{3.12}$$

where z_E denotes the expectation-values (Ehrenfest) phase space point. So the verification of the pseudoclassical condition implies the Ehrenfest condition in first order. On the other hand, the Ehrenfest condition, if it is required for any t_0, is equivalent to the Ehrenfest condition (not in first order). This necessary condition is also sufficient as we prove below.

(i) For all $|z_0\rangle$. In this case the Ehrenfest condition is verified by (and only by)

$$\hat{H}_Q(t)=c_{20}(t)\hat{p}^2+c_{02}(t)\hat{q}^2+\tfrac{1}{2}c_{11}(t)[\hat{p},\hat{q}]_+ +c_{10}(t)\hat{p}+c_{01}(t)\hat{q}+c_{00}(t). \tag{3.13}$$

These Hamiltonians[25] verify, in addition, the pseudoclassical evolution. Equation (3.10) now reads

$$z_p(t)=\max_z|\langle z(\alpha_0)|z_0(\alpha_0)+\Delta(t);c(t),s(t)\rangle|^2 \tag{3.14}$$

and this gives

$$[z_p(t)](\alpha_0)=c^*(t)[z_0(\alpha_0)+\Delta(t)]-s(t)[z_0(\alpha_0)+\Delta(t)]^*. \tag{3.15}$$

The initial condition is correct, and so are the evolution equations: From (2.18) it follows

$$\dot{z}_p(t)=-if_{11}(t)z_p(t)-2if_{20}(t)z_p^*(t)-if_{10}(t) \tag{3.16}$$

which is the same as the classical equation of motion

$$\dot{q}_{\text{cl}}(t)=2c_{20}(t)p_{\text{cl}}(t)+c_{11}(t)q_{\text{cl}}(t)+c_{10}(t), \tag{3.17a}$$

$$\dot{p}_{\text{cl}}(t)=-c_{11}(t)p_{\text{cl}}(t)-2c_{02}(t)q_{\text{cl}}(t)-c_{01}(t) \tag{3.17b}$$

taking into account the relation between the f_{nm} and c_{ij}. This proof completes the one given in I.

(ii) Not for all $|z_0\rangle$. The Ehrenfest condition is equivalent to LTM conservation, Eqs. (3.6). But from (2.9) it can be shown that the MUS $|z(\alpha)\rangle$ with maximum overlap with the LTM $|Z;c,s\rangle$ is defined

by

$$z(\alpha) = c^* Z - s Z^*,$$

so that its localization $z=(q,p)$ in phase space is precisely the LTM-expected values of $\hat{q}$ and $\hat{p}$. The conjunction of these conditions implies the pseudoclassical algorithm.

2. $\dot{\alpha}_t \neq 0$

We next approach the same problem when the α-equivalence class chosen in each time varies continuously as α_t. This extra liberty in α_t could generalize the $\dot{\alpha}_t = 0$ results. That this is not the case can be seen as follows.

(i) *For all* $|z_0\rangle$. The equivalent to (3.11) is now

$$\langle [z_p(t)](\alpha_t) | \hat{a}(\alpha_t) | z_0(\alpha_0); t\rangle = [z_p(t)](\alpha_t) \langle [z_p(t)](\alpha_t) | z_0(\alpha_0); t\rangle . \quad (3.18)$$

Nevertheless, a first-order calculation based on the relations

$$\begin{aligned}
\alpha_{t+\tau} &= \alpha_t + \mathring{\alpha}_t \tau + O(\tau^2),\\
z(t+\tau) &= [q(t+\tau), p(t+\tau)]\\
&= z(t) + \mathring{z}(t)\tau + O(\tau^2),\\
[z(t+\tau)](\alpha_{t+\tau}) &= \alpha_{t+\tau} q(t+\tau) + (i/2\alpha_{t+\tau}) p(t+\tau)\\
&= [z(t)](\alpha_t) + (\mathring{\alpha}_t/\alpha_t)[z(t)]^*(\alpha_t)\tau\\
&\quad + [\mathring{z}(t)](\alpha_t)\tau + O(\tau^2),\\
|[z(t)](\alpha_t); t+\tau\rangle &= |[z(t)](\alpha_t)\rangle - i\tau \hat{H}(t) |[z(t)](\alpha_t)\rangle\\
&\quad + O(\tau^2),\\
\hat{a}(\alpha_{t+\tau}) &= \hat{a}(\alpha_t) + (\mathring{\alpha}_t/\alpha_t)\tau \hat{a}^\dagger(\alpha_t) + O(\tau^2)
\end{aligned} \quad (3.19)$$

also leads to Eq. (3.11), so that the result cannot be more general than $\hat{H}_Q$.

In order to explicitly prove the sufficiency of $\hat{H}_Q$ we translate the nonconstant-α problem into a constant-α one for each t, by writing

$$|z_0(q_0) + \Delta(t); c(t), s(t)\rangle = |\bar{z}_0(\alpha_t) + \Delta(t); \bar{c}(t), \bar{s}(t)\rangle , \quad (3.20)$$

where

$$\bar{z}_0(\alpha_t) = z_0(\alpha_0), \quad (3.21a)$$

$$\bar{c}(t) = c(t) c(\alpha_0, \alpha_t) - s(t) s(\alpha_0, \alpha_t), \quad (3.21b)$$

$$\bar{s}(t) = -c(t) s(\alpha_0, \alpha_t) + s(t) c(\alpha_0, \alpha_t). \quad (3.21c)$$

Equation (3.10) is now formally a problem with fixed α ($=\alpha_t$, for each t), and the solution is [see Eq. (3.15)]

$$\begin{aligned}
[z_p(t)](\alpha_t) = &[c^*(t)c(\alpha_0, \alpha_t) - s^*(t)s(\alpha_0, \alpha_t)]\\
&\times [z_0(\alpha_0) + \Delta(t)]\\
&- [s(t)c(\alpha_0, \alpha_t) - c(t)s(\alpha_0, \alpha_t)]\\
&\times [z_0(\alpha_0) + \Delta(t)]^* .
\end{aligned} \quad (3.22)$$

The equation of motion for $z_p(t)$ calculated from Eqs. (2.20) and from

$$\mathring{c}(\alpha_0, \alpha_t) = (\mathring{\alpha}_t/\alpha_t) s(\alpha_0, \alpha_t) \quad (3.23a)$$

$$\mathring{s}(\alpha_0, \alpha_t) = (\mathring{\alpha}_t/\alpha_t) c(\alpha_0, \alpha_t) \quad (3.23b)$$

is

$$\frac{d}{dt}\{[z_p(t)](\alpha_t)\} = c(\alpha_0, \alpha_t)\mathring{z}_t(\alpha_0) + s(\alpha_0, \alpha_t)\mathring{z}_t^*(\alpha_0) + (\mathring{\alpha}_t/\alpha_t)[z_p^*(t)](\alpha_t), \quad (3.24)$$

where

$$z_t(\alpha_0) = c^*(t)[z_0(\alpha_0) + \Delta(t)] - s(t)[z_0(\alpha_0) + \Delta(t)]^* . \quad (3.25)$$

Equation (3.24) implies

$$\mathring{z}_p(t) = \mathring{z}_t .$$

The physical significance of z_t is the pseudoclassical phase-space point in the case $\mathring{\alpha}_t = 0$, Eq. (3.15). As z_t equals $z_{cl}(t)$, so does $z_p(t)$.

(*ii*) *Not for all* $|z_0\rangle$. A similar argument to the one for the $\mathring{\alpha}_t = 0$ case leads to Eqs. (3.6) as necessary and sufficient conditions.

D. The 3-dimensional case

The generalization of the previous results to higher dimensions is straightforward. We define a vectorlike operator $\hat{\vec{a}}(\vec{\alpha})$ whose three components are

$$\hat{a}_j(\alpha_j) = \alpha_j \hat{q}_j + (i/2\alpha_j)\hat{p}_j, \quad j = 1, 2, 3 . \quad (3.26)$$

The commutability of these component operators allows the definition of common eigenstates $|\vec{z}(\vec{\alpha})\rangle$ as the direct product of three component MUS's $|z_i(\alpha_i)\rangle$ eigenstates of $\hat{a}_i(\alpha_i)$.

All proofs and results go through to the 3-dimensional case, the only qualitative difference being the appearance in all the Hamiltonians of crossed $\hat{q}_i \hat{p}_j$ terms for $i \neq j$; as $\hat{q}_i$ and $\hat{p}_j$ commute, their appearance is natural.

The appearance of these crossed terms implies that a charged particle in a magnetic field $\vec{\mathcal{H}}(t)$,

$$\hat{H}(t) = (2m)^{-1}[\hat{\vec{p}} - \vec{A}(\hat{\vec{q}})]^2 \quad (3.27)$$

with

$$\vec{\mathcal{H}}(t) = \vec{\nabla} \times \vec{A}(t)$$

conserves minimality (and verifies the pseudoclassical condition) if and only if $\vec{\mathcal{H}}$ is uniform, as follows from the previous sections. Obviously this result is independent of the gauge chosen. In the usual treatment of the uniform magnetic field, a class of coherent states is defined which depends upon the gauge. This coherence is conserved by (3.27). We remark that due to this gauge dependence not all these classes are MUS's. The usually considered coherent states,[26] the symmetrical gauge ones, are MUS's. As the unitary gauge transformation operator is

$$\hat{G} = \exp[iwf(\hat{\mathrm{q}})],$$

where w is a real constant, we conclude that it corresponds to minimality-conserving transformations only in the case of a linear f function. In fact $\hat{G}$ has to be restricted to a displacement operator. So the gauge function f referred to the symmetrical gauge[27] can only be linear in $\hat{\mathrm{q}}$. We conclude that, in this sense, coherence and minimality are different matters.

IV. CONCLUDING REMARKS

The previous discussion leads us to establish the equivalence between the pseudoclassical and Ehrenfest algorithms, for both the $\dot{\alpha} = 0$ and $\dot{\alpha} \neq 0$ cases. Furthermore, the requirement of LTM conservation is verified by the same class of Hamiltonians. This is true in the two cases when all or only some particular initial conditions are studied. We note that minimality conservation implies the above algorithms, but the converse is not true. We interpret this in the sense that MUS conservation is a restriction on mean values but also on the second moments' relation. The significance of the subindexes α and $\alpha(t)$ ($\equiv \alpha_t$) of $\hat{H}_\alpha$, $\hat{H}_{\alpha(t)}$ is the reference made to which of the α-equivalence classes does the conservation of minimality refer to.

These equivalences are synthetized in the table below. But first we would like to emphasize the conceptual distinction[4] between the α_t's which characterize the equivalence class of the MUS's and the arbitrary α's in terms of whose $\hat{a}(\alpha)$, $\hat{a}^\dagger(\alpha)$ one can express the Hamiltonians. This is exemplified by $\hat{H}_{\alpha(t)}$ which allocates $|z_0(\alpha_0);t\rangle$ in the α_t-equivalence class: $\hat{H}_{\alpha(t)}$ can be expressed in terms of $\hat{a}(\alpha_t), \hat{a}^\dagger(\alpha_t)$ [see Eq. (2.24b)] but also in terms of $\hat{a}(\alpha_0), \hat{a}^\dagger(\alpha_0)$ [see Eq. (2.24a)] and in terms of any $\hat{a}(\beta), \hat{a}^\dagger(\beta)$. This distinction plays a central role in our discussions, as is highlighted by the study of the rotation operator in the Appendix. On the other hand and also in relation with $\hat{H}_{\alpha(t)}$, we remark that the time dependence of $\hat{a}(\alpha_t)$ is an explicit one, and it does not correspond to the dynamics in the Heisenberg picture.[14] The same comment is relevant for the time dependence of the α_t-equivalence class considered in the pseudoclassical equations, and of the $\hat{A}(t)$ considered in the LTM conservation.

The relevant features of these Hamiltonians are the following (all the coefficients are time-dependent functions):

(i)

$$\hat{H}_Q(t) = c_{20}\hat{p}^2 + c_{02}\hat{q}^2 + \tfrac{1}{2}c_{11}[\hat{p},\hat{q}]_+ + c_{10}\hat{p} + c_{01}\hat{q} + c_{00}. \tag{4.1}$$

(ii) $\hat{H}_{\alpha(t)}$ equals $\hat{H}_Q$ with

$$c_{02}/c_{20} = 4\alpha_t^{\,4}, \quad c_{11} = i\dot{\alpha}_t/\alpha_t \tag{4.2}$$

(iii) $\hat{H}_\alpha$ equals $\hat{H}_{\alpha(t)}$ with $\dot{\alpha}_t = 0$.

(iv)

$$\hat{H}_L(t) = \sum_{ij} \tfrac{1}{2}c_{ij}[\hat{p}^i,\hat{q}^j]_+ \equiv \sum_{nm} F_{nm}\hat{A}^{\dagger n}(t)\hat{A}^m(t) \tag{4.3}$$

TABLE I. Equivalence of the pseudoclassical and Ehrenfest algorithms with LTM conservation, and their relation with minimality conservation.

Initial condition	Algorithms	Applicability	Valid for all initial conditions	Valid only for some initial conditions
MUS	Minimality conservation	$\dot{\alpha} = 0$	$\hat{H}_{\alpha_0}$	$\hat{H}_1$
	$\lvert z_0(\alpha_0);t\rangle$ is a α_t MUS	$\dot{\alpha} \neq 0$	$\hat{H}_{\alpha(t)}$	$\hat{H}_2$
MUS	Pseudoclassical	$\dot{\alpha} = 0$		
	$z_P(t) = z_{\mathrm{cl}}(t)$	$\dot{\alpha} \neq 0$	$\hat{H}_Q$	$\hat{H}_L$
MUS	Ehrenfest			
	$z_E(t) = {}_{\mathrm{cl}}(t)$			
LTM	LTM conservation			
(or MUS)	$\hat{U}(t,t_0)\lvert z_0;C_0,S_0\rangle$ is a LTM			

with

$$L_1(t)=i\{\dot{z}_{c1}(\alpha_0)+[\dot{s}s^*-\dot{c}c^*]z_{c1}(\alpha_0)\}\ , \tag{4.4a}$$

$$L_2(t)=\tfrac{1}{2}i\,[\dot{c}(t)s(t)-\dot{s}(t)c(t)]\ , \tag{4.4b}$$

$$L_n(t)=0\ ,\quad n>2\ , \tag{4.4c}$$

where

$$L_n(t)\equiv\sum_m F_{nm}(t)\{\alpha_0[c(t)+s(t)] + (i/2\alpha_0)[c(t)-s(t)]p_{c1}(t)\}^m \tag{4.5}$$

and

$$\hat{A}(t)=c(t)\hat{a}(\alpha_0)+s(t)\hat{a}^\dagger(\alpha_0)\ ,$$

$$z_{c1}(t)=[q_{c1}(t),p_{c1}(t)]\ .$$

(v) $\hat{H}_2(t)$ equal $\hat{H}_L(t)$ with $c(t)$ and $s(t)$ being $\cosh r(t)$ and $\sinh r(t)$:

$$\hat{H}_2(t)=\sum_{nm} f_{nm}\hat{a}^{\dagger n}(\alpha_t)\hat{a}^m(\alpha_t)\ , \tag{4.6}$$

$$L_1(t)=i[\dot{z}_{c1}(t)](\alpha_t)\ , \tag{4.7a}$$

$$L_2(t)=i\dot{\alpha}_t/2\alpha_t\ , \tag{4.7b}$$

$$L_n(t)=0,\quad n>2 \tag{4.7c}$$

and

$$L_n(t)\equiv\sum_m nf_{nm}(t)\{[z_{c1}(t)](\alpha_t)\}^m\ . \tag{4.8}$$

(vi) $\hat{H}_1(t)$ equals $\hat{H}_2(t)$ with $\dot{\alpha}_t=0$.

Although Eqs. (4.4) must be verified for all t $(t>t_0)$, they express restictions on the initial conditions if $\hat{H}$ is given; conversely, they restrict a class of $\hat{H}$ if the initial conditions are given. These equations essentially restrict the quantum class of physical systems in which the algorithms are verified, to those in which the classical trajectories comply with a number of restrictions. In conclusion, we have shown that for Hamiltonians of degree $N\,(N>2)$, the pseudoclassical algorithm is verified at most by $N-2$ classical trajectories. In this respect, the main conclusion of our analysis can be thus stated as follows:

Theorem. The set of initial conditions for which the pseudoclassical evolution algorithm [minimality conservation] is verified for Hamiltonians more than quadratic is defined by (4.4) [(4.7)] and is denumerable. Thus, the pseudoclassical algorithm [minimality conservation] is verified if and only if the Hamiltonian is at most a general [restricted] quadratic form in the dynamical variables (4.1) [(4.2)], except for a set of measure zero.

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APPENDIX

1. Change of α-equivalence class. *$\hat{U}_1$ verifies*

$$\hat{U}_1\,|z(\alpha)\rangle=e^{i\phi_1}\,|(z'+\Delta')(\beta)\rangle$$

with

$$(z'+\Delta')(\beta)=z'(\beta)+\Delta'(\beta)\ ,$$

$$z'(\beta)=z(\alpha)\ ,$$

$$\Delta'(\beta)=\beta\Delta_1'+\frac{i}{2\beta}\ \Delta_2'=\Delta\ ,$$

so that

$$\hat{U}_1\hat{a}(\alpha)\hat{U}_1^{-1}=\hat{a}(\beta)+\Delta\ .$$

This case corresponds to $|\mathrm{Re}\,c\,|\rangle\,|$ and we obtain

$$\hat{U}_1=\exp\left\{\tfrac{1}{2}r[\hat{a}^2(\alpha)-\hat{a}^{\dagger 2}(\alpha)] + \frac{r}{e^r-1}\,[\Delta\hat{a}^\dagger(\alpha)-\Delta^*\hat{a}(\alpha)]\right\}$$

with r defined in (2.3), so that c (s) equals $\cosh r$ ($\sinh r$). The phase is

$$i\phi_1=z\left(\frac{1-s-c}{2c}\,\Delta+\Delta^*\frac{c(c+s-1)-2s}{2c(c+s-1)}\right)-\Delta^2\frac{s}{c} +z^*\left(\Delta^*\frac{c-1}{c(c+s-1)}+\Delta\frac{(c-2)(c+s-1)+2s}{2c(c+s-1)}\right) +(\Delta^2-\Delta^{*2})\frac{cr-s}{2c(c+s-1)^2}+|\Delta|^2\left(\frac{1-c}{c(c+s-1)^2}+\frac{1}{2}\right).$$

When $\Delta=O$, we reobtain[28] the usual operator $\hat{U}(\alpha\to\beta)$ (and $\phi_1=0$),

$$\hat{U}(\alpha\to\beta)=\exp(r/2i)[\hat{q},\hat{p}]_+ \tag{A1}$$

which scales as follows

$$\hat{U}\hat{q}\hat{U}^{-1}=e^r\hat{q}=(\beta/\alpha)\hat{q}\ ,$$

$$\hat{U}\hat{p}\hat{U}^{-1}=e^{-r}\hat{p}=(\alpha/\beta)\hat{p}\ .$$

At this point we wish to comment on the limit of (A1) as $r\to\pm\infty$. It would seem that $\hat{H}_{\alpha(t)}$ (see Sec. II A) with α_t monotonically increasing (decreasing) towards $\infty(0)$ would transform an initial MUS into states with $\Delta_q\to 0$ $(\Delta_p\to 0)$, that is eigenstates of $\hat{q}$ ($\hat{p}$).[8] We note the following facts: (i) The norm of $\hat{U}(\alpha\to\beta)\,|z(\alpha)\rangle$ (α fixed) is 1, so $\lim\hat{U}(\alpha\to\beta)\,|z(\alpha)\rangle$ when $r\to\pm\infty$, if it exists, has norm 1; hence this state cannot be an eigenstate of $\hat{q}$ or $\hat{p}$. (ii) von Neumann's theorem[29] states that all representations of the commutation relations are unitarily equivalent within the same Hilbert space.[30] So if $\lim\hat{U}(\alpha\to\beta)$ exists it cannot drive us out of the original Hilbert space (L^2). (iii) The limit of $|\langle q'|\hat{U}(\alpha-\beta)|z(\alpha)\rangle|^2$ $[|\langle p'|\hat{U}(\alpha-\beta)|z(\alpha)\rangle|^2]$ for $r\to\infty$ $[r\to-\infty]$ is equal to $\delta(q'-q)$ $[\delta(p'-p)]$. (iv) The limit when $r\to\pm\infty$ of the $\hat{U}(\alpha-\beta)$ matrix elements is zero. From the above facts[31] we believe that this limit is not well defined.

2. *Displacement in phase space.* A displacement $\Delta=(a,b)$ of $|z(\alpha)\rangle$ means that $\langle\hat{q}\rangle$ and $\langle\hat{p}\rangle$ are displaced by a and b. This is

$$\hat{U}_2\,|\,z(\alpha)\rangle = e^{i\phi_2}\,|(z+\Delta)(\alpha)\rangle$$

and corresponds to the case $s=0$. As here $c=1$, Eqs. (2.24) reduce to

$$f_{11}=f_{20}=0,\ \ f_{10}=i\Delta/T$$

and we obtain

$$\hat{U}_2=\exp[\Delta(\alpha)a^\dagger(\alpha)-\Delta^*(\alpha)a(\alpha)]$$
$$=\exp i(b\hat{q}-a\hat{p}),$$
$$i\phi_2=2^{-1}[\Delta(\alpha)\,z^*(\alpha)-\Delta^*(\alpha)z(\alpha)]\,.$$

This is the well-known Weyl operator.

3. *Rotation in phase space.* A rotation θ of $|z(\alpha)\rangle$ means that $\langle\hat{q}\rangle$ and $\langle\hat{p}\rangle$ are rotated an angle θ about the origin:

$$c=\cos\theta-i\sin\theta(\alpha^2+1/4\alpha^2)\,, \tag{A2a}$$

$$s=+i\,\sin\theta(\alpha^2-1/4\alpha^2)\,, \tag{A2b}$$

$$\Delta=0\,. \tag{A2c}$$

When $\alpha=2^{-1/2}$ we apply the case $s=0$ and obtain

$$\hat{U}_\theta(\alpha=2^{-1/2})=\exp[(+i\theta)\hat{a}^\dagger(2^{-1/2})\hat{a}(2^{-1/2})]$$
$$=\exp[(+\tfrac{1}{2}i\theta)(\hat{p}^2+\hat{q}^2-1)]\,, \tag{A3a}$$

$$i\phi_\theta(\alpha=2^{-1/2})=-\tfrac{1}{2}i\theta\,. \tag{A3b}$$

This is the evolution operator for a harmonic oscillator with $m=1$, $w=1$ at time $t=-\theta$. For this system the trajectories in phase space of $\langle\hat{q}\rangle_t$ and $\langle\hat{p}\rangle_t$ are circles, as should be. The structure of $\hat{U}_\theta(\alpha=2^{-1/2})$ tells us that it conserves the $\alpha=2^{-1/2}$ MUS's (of course, $s=0$). When $\theta=-\pi$ we have an implementation of the parity operator.[32]

For any α, when $\theta=-\pi$ we apply also case $s=0$:

$$\hat{U}_{-\pi}=\exp[-i\pi\hat{a}^\dagger(\alpha)\hat{a}(\alpha)]$$
$$=\exp\left[(-i\pi)\left(\alpha^2\hat{q}^2+\frac{1}{4\alpha^2}\,\hat{p}^2-\tfrac{1}{2}\right)\right], \tag{A4a}$$

$$i\phi_{-\pi}=\tfrac{1}{2}i\pi\,. \tag{A4b}$$

We verify that this is the parity operator $\hat{P}$: From the overlap (2.9) we have

$$|z(\alpha);-1,0\rangle=\pi^{-1}\int d^2z'\langle z'(\alpha)\,|z(\alpha);-1,0\rangle\,|z'(\alpha)\rangle$$
$$=-i\,|(-z)(\alpha)\rangle$$

and as

$$\langle q\,|-z\rangle=\langle -q\,|z\rangle$$

we obtain

$$\langle q\,|\hat{U}_{-\pi}|\,z(\alpha)\rangle=i\langle q\,|z(\alpha);\,-1,0\rangle$$
$$=\langle q\,|(-z)(\alpha)\rangle$$
$$=\langle -q\,|z(\alpha)\rangle$$

so that

$$\hat{U}_{-\pi}=\hat{P}\,.$$

In the general case when $\alpha\neq 2^{-1/2}$ and $\theta\neq-\pi$ in (A2), we apply case $|\mathrm{Re}c|<1$ ($s\neq 0$). The result $\hat{U}_\theta, i\phi_\theta$ turns out to be

$$\hat{U}_\theta=\exp(+i\theta)\left[\left(\alpha^2+\frac{1}{4\alpha^2}\right)\hat{a}^\dagger(\alpha)\hat{a}(\alpha)\right.$$
$$\left.-\tfrac{1}{2}\left(\alpha^2-\frac{1}{4\alpha^2}\right)[\hat{a}^{\dagger 2}(\alpha)+\hat{a}^2(\alpha)]\right]$$
$$=\exp(+\tfrac{1}{2}i\theta)\left[\hat{q}^2+\hat{p}^2-\left(\alpha^2+\frac{1}{4\alpha^2}\right)\right], \tag{A5a}$$

$$i\phi_\theta=-\tfrac{1}{2}i\theta\left(\alpha^2+\frac{1}{4\alpha^2}\right). \tag{A5b}$$

As in (A3) and (A4), the independent term of the $(\hat{q},\hat{p})$ form is directly responsible for the phase ϕ. Obviously $\hat{U}_\theta$ does not conserve the minimality of the MUS's in the α-equivalence class. This is only valid for $\alpha=2^{-1/2}$. The special role played by this α is due to the circular trajectories of the corresponding oscillator in phase space (for general α they are ellipses).

For an arbitrary state $|\psi\rangle$, a rotation θ in phase space implies the corresponding rotation of expected values $\langle\hat{q}\rangle$, $\langle\hat{p}\rangle$. It is clear that in the case $\hat{U}_\theta(\alpha=2^{-1/2})$ the conservation of minimality assures us that no other changes (see below) have been entailed in the $|\psi\rangle$ probability distribution. This leads us to define the rotation operator as

$$\hat{R}_\theta\equiv\hat{U}_\theta(\alpha=2^{-1/2})\,.$$

In this way, $\hat{R}_{\theta=0}$ is the unity operator $\hat{1}$ and $\hat{R}_{\theta=-\pi}$ is the parity operator $\hat{P}$. We can obtain $\hat{P}$ continuously from $\hat{1}$ because we are dealing with the rotation group in an even-dimensional space (phase space).

The state $\hat{R}_\theta\,|z(\alpha)\,\rangle$ has the mean-square deviations

$$\Delta_{q,\theta}{}^2=(4\alpha^2)^{-1}\cos^2\theta+\alpha^2\sin^2\theta\,,$$
$$\Delta_{p,\theta}{}^2=\alpha^2\cos^2\theta+(4\alpha^2)^{-1}\sin^2\theta$$

which allows us to visualize the wave function ${}_\theta\langle q\,|z(\alpha)\rangle$ (where $|q\rangle_\theta$ denotes $\hat{R}_\theta\,|q\rangle$) as a Gaussian probability distribution of variance $\Delta_{q,\theta}$ for each θ over phase space. This Gaussian over phase space is rigidly rotated around the origin by $\hat{R}_\theta$: This rigidity was what was meant by no other changes. In this sense it should be stressed that $\hat{R}_\theta\,|z(\alpha)\rangle$ $(\alpha \neq 2^{-1/2})$ is not equal to $|(e^{i\theta}z)(\alpha)\rangle$ nor to $|z_\theta(\alpha)\rangle$ with $z_\theta(\alpha)=e^{i\theta}z(\alpha)$: these states are the MUS's of the α-equivalence class, while $\hat{R}_\theta\,|z(\alpha)\rangle$ is clearly not.

The previous analysis can be done for the rotation operator around the phase-space point $\Delta=(a,\,b)$:

$$\hat{R}_\theta(a,b)=\exp(+i)\Big(\tfrac{1}{2}\theta(\hat{q}^2+\hat{p}^2)+\frac{1}{\theta}\,[a(-\tfrac{1}{2}\theta^2+\cos\theta-1)+b(\sin\theta-\theta)]\hat{q}$$
$$+\frac{1}{\theta}\,[b(-\tfrac{1}{2}\theta^2+\cos\theta-1)-a(\sin\theta-\theta)]\hat{p}+\tfrac{1}{6}\,\theta(a^2+b^2-3)+\frac{a^2+b^2}{\theta}\,(1-\cos\theta)\Big).$$

The transformation it induces is

$$\hat{R}_\theta(a,b)\,|z(\alpha)\rangle=e^{-i\theta/2}|z(\alpha):c,s,w\rangle$$

with

$$c=\cos\theta-i\sin\theta\left(\alpha^2+\frac{1}{4\alpha^2}\right)\quad,$$

$$s=+i\sin\theta\left(\alpha^2-\frac{1}{4\alpha^2}\right),$$

$$w=(1-\cos\theta)\left(\alpha a+\frac{i}{2\alpha}\,b\right)+\sin\theta\left(\alpha b-\frac{i}{2\alpha}\,a\right).$$

The presence of w only alters our scheme in that now $\Delta-w$ plays the role of Δ. The parity about Δ is $\hat{R}_{-\pi}(a,b)$ and

$$\hat{R}_{-\pi}(a,b)\,|z(\alpha)\rangle=\hat{D}(2\Delta)\,|(-z)(\alpha)\rangle\,.$$

This operator has a special interest[33] since its expectation value, that is the overlap of the wave function with its mirror image about (a,b), is the Wigner distribution function at (a,b).

4. *Fourier operator.* The $-\pi/2$ geometrical rotation in phase space interchanges the roles of the coordinate and momentum axes. This is in accordance with the performance of $\hat{R}_{-\pi/2}$; it is easily verified that

$$\langle q'\,|\hat{R}_{-\pi/2}|z(\alpha)\rangle=\langle p'\,|z(\alpha)\rangle\,,$$

where $|q'\rangle(|\hat{p}'\rangle)$ is an eigenstate of $\hat{q}(\hat{p})$ with $q'=p'$. This $\hat{R}_{-\pi/2}$ is the Fourier operator.[32] This result gives us an insight into the physical interpretation of $\hat{R}_\theta$. $|\langle q'\,|\psi\rangle|^2dq'$ $[|\langle p'\,|\psi\rangle\,|^2dp']$ is the probability distribution in a vertical [horizontal] slit of phase space (with the q' axis as abscissa axis) through the phase-space point $(q',0)$ $[(0,p')]$. As $\langle p'\,|\psi\rangle$ is $\langle q'|\hat{R}^\dagger_{\pi/2}\,|\psi\rangle$ (with eigenvalues $p'=q'$), we conclude[34] that $|\langle q'|\hat{R}^\dagger_\theta(a,b)\,|\psi\rangle|^2\times dq'_\theta$ is the probability distribution in a tilted (by an angle $\theta+\pi/2$) slit of phase space which includes the point rotated of $(q',0)$ about (a,b) by an angle θ.

5. *Comment.* We would finally like to comment on a generalization of the rotation operator which does not transform a MUS into a LTM. $\hat{R}_\theta$ can be written as

$$\hat{R}_\theta=\exp(+i\theta\hat{a}^\dagger\hat{a})\,,$$

where α is understood to be $2^{-1/2}$, and

$$\hat{R}_\theta\;|z\rangle=\sum_n e^{+in\theta}\langle n\,|z\rangle\,|n\rangle\,.$$

If the phases are allowed to become arbitrary (i.e., if each component $|n\rangle$ rotates a different angle), one obtains

$$|\phi\rangle=\sum e^{+i\theta_n}\langle n\,|\,z\rangle\;|n\rangle\,. \tag{A6}$$

These states (A6) (Ref. 36) are the most general ones which satisfy

$$\langle\phi\,|\hat{a}^{\dagger n}\hat{a}^n\,|\phi\rangle=\langle\phi\,|\hat{a}^\dagger\,|\phi\rangle^n\langle\phi\,|\hat{a}\,|\phi\rangle^n\,. \tag{A7}$$

In fact, (A7) is verified for any $\hat{a}(\alpha)$ when $|\phi\rangle_\alpha$ is defined in (A6) for the corresponding α $[\,|z(\alpha)\rangle$ and $|n\rangle_\alpha]$. Nevertheless, the most general states $|\psi\rangle_\alpha$ which factorize

$${}_\alpha\langle\psi\,|\hat{a}^{\dagger n}(\alpha)\hat{a}^m(\alpha)\,|\psi\rangle_\alpha={}_\alpha\langle\psi\,|\hat{a}^\dagger\,|\psi\rangle^n_\alpha\;{}_\alpha\langle\psi\,|\hat{a}\,|\psi\rangle^m_\alpha \tag{A8}$$

for all n,m are the MUS's of the corresponding α-equivalence class.[35] From this factorization property which testifies to the absence of certain quantum correlations in these states, there follows their relevance in quantum noise and communication theory.[37]

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[1] J. Klander, in *Path Integrals and their Applications in Quantum, Statistical and Solid State Physics*, proceedings of the NATO A.S.I., Antwerpen, Belgium, 1977 (Plenum, New York, to be published).

[2] The spectrum of MUS's is continuous but the set of MUS's is overcomplete. In fact there exists a discrete subset which spans the whole of the MUS's Hilbert space (L^2), as was first noted by von Neumann. See,

J. Klauder and E. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968), p. 115; A. Barut and L. Girardello, Comm. Math. Phys. 21, 41 (1971).

[3]G. Patsakos, Am. J. Phys. 44, 158 (1976).

[4]V. Canivell and P. Seglar, Phys. Rev. D 15, 1050 (1977).

[5]D. Stoler, Phys. Rev. D 1, 3217 (1970).

[6]A. Messiah, *Mécanique Quantique* (Dunod, Paris, 1969), Vol. 1. Chap. 8, p. 254.

[7]Although a classical phase space has no meaning in quantum mechanics [see, e.g., P. Dirac, *The Principles of Quantum Mechanics* (Oxford Univ. Press, London, 1970), 4th ed., p. 132], distribution functions in the space of eigenvalues of $\hat{q}$ and $\hat{p}$ can formally act as classical probabilities for conveniently ordered operators. See G. Agarwal and E. Wolf, Phys. Rev. D 2, 2161 (1970).

[8]H. Yuen, Phys. Rev. A 13, 2226 (1976). See Sec. III: Our LTM's are essentially the two-coherent states of quantum optics, which appear in the study of two-photon lasing processes.

[9]We obviate the consideration of the complex phase function which is necessary to guarantee the correct solution of the evolution equation. This phase only contributes to the determination of the c-number term in the Hamiltonian, which does not affect the relevant physics here. Nevertheless, when this phase is overlooked one can be led in some problems to uncorrect results. This has been pointed out by H. Letz [Phys. Lett. 60A, 399 (1977)] in reference to a recent paper.

[10]In this case we know that $\hat{H}$ is quadratic (for a linear transformation). For future reference we note that normal expansions are appropriate (in the sense of weak convergence over the set of MUS's) under very general conditions, more than for antinormal or symmetrical ones. See K. Cahill and R. Glauber, Phys. Rev. 177, 1857 (1969).

[11]This liberty is the link between Yuen's results (Ref. 8) that correspond in fact to the choice $n(t)=1$ [which does not allow the case $u(t)=1$, $v(t)=0$], and previous results in the literature for quadratic Hamiltonians related to the conservation of minimality [C. Mehta *et al.*, Phys. Rev. 157, 1198 (1967)] and others [A. Holz, Lett Nuovo Cimento 4, 1319 (1970); I. Malkin *et al.*, J. Math. Phys. 14, 576 (1973)] which differ from Yuen because $n(t)\neq 1$.

[12]Even for $\Delta=0$, as $z_1(\alpha_1)=z_2(\alpha_2)$ implies $q_1=(\alpha_2/\alpha_1)q_2$ and $p_1=(\alpha_1/\alpha_2)p_2$, this unitary transformation moves the MUS localization z in phase space.

[13]The relevant feature of this result is the f_{20} (and f_{02}) term, which was first obtained by D. Trifonov [Phys. Lett. 48A, 165 (1974)], although with a changed sign, in a first-order calculation.

[14]This identification has inadvertently been done in the literature: D. Stoler, Phys. Rev. D 11, 3033 (1975). This is misleading since it led Stoler to conclude incorrectly the nonvalidity of Trifonov's results (Ref. 13) in the Schrödinger picture.

[15]See comments and references quoted in I.

[16]See Ref. 5 and I. Also, this operator will be deduced in the Appendix.

[17]A sufficient condition is for the commutator of $\hat{H}(t)$ and $\hat{H}(t')$ to be a c number: see J. Schwinger, Phys. Rev. 75, 651 (1949).

[18]R. Wilcox, J. Math. Phys. 8, 962 (1967); C. Mehta, *ibid.* 18, 404 (1977).

[19]Any pure state that minimizes the position-momentum uncertainty product is a MUS. No mixed state minimizes it: see D. Stoler and S. Newman, Phys. Lett. 38A, 443 (1972).

[20]See Ref. 6, Chap. 6, p. 182.

[21]If not originally so, a Hamiltonian can be modified to be infinitely differentiable without altering the observable dynamics of the system. See G. Rosen, *Formulations of Classical and Quantum Dynamical Theories* (Academic, New York, 1969), p. 12.

[22]This result includes the special $N=3$ Hamiltonian studied by L. Mista, Phys. Lett. 25A, 646 (1967), and it corrects the scheme of Y. Kano, *ibid.* 56A, 7 (1976); see also D. Trifonov and V. Ivanov, Phys. Lett. 64A, 269 (1977).

[23]We point out that the obtainment of $\hat{H}_\alpha$ through a unitary transformation in Ref. 14 is incomplete: As the "ground-state-preserving" Hamiltonian $\hat{H}_g$ (which contains only terms f_{nm} with nonzero subindex m) conserves the $|0\rangle$ as such—that is, only one state—then $\hat{D}(Z_t)\hat{H}_g\hat{D}^\dagger(Z_t)-\hat{D}(Z_t)\,\partial\hat{D}(Z_t)/\partial t$ is only guaranteed to conserve this particular $|Z_t\rangle$. Owing to this fact and in order to obtain $\hat{H}_{\rm coh}$ (our $\hat{H}_\alpha$) one has to arbitrarily restrict $\hat{H}_g$ to $\hat{a}^+\hat{a}$. On the other hand, the step from $\hat{H}_{\rm coh}$ to $\hat{H}_{\rm min}$ (our $\hat{H}_{\alpha(t)}$) is consistent, since $\hat{H}_{\rm coh}$ is valid for any $|z_0\rangle$.

[24]B. d'Espagnat, *Conceptual Foundations of Quantum Mechanics* (Benjamin, Menlo Park, Cal., 1971), p. 238.

[25]The crossed term $[\hat{p},\hat{q}]_+$ was erroneously not included in Eq. (3.11) of I.

[26]See, e.g., I. Malkin and I. Manko, Zh. Eksp. Teor. Fiz. 59, 1746 (1970) [Sov. Phys. JETP 32, 949 (1971)].

[27]W. Tam, Physica 54, 557 (1971).

[28]R. Haag, in *Lectures in Theoretical Physics*, edited by W. Brittin and A. Barut (Interscience, New York, 1961), Vol. 3, p. 358; see also Ref. 5.

[29]C. Putnam, *Commutation Properties of Hilbert Space Operators and Related Topics* (Springer, Berlin, 1967), pp. 88, 89.

[30]It should be borne in mind that we are dealing with a finite number of degrees of freedom. The possibility of inequivalent representations appears in the case of infinite degrees: see G Guralnik, C Hagen, and T Kibble, in Advances in Particle Physics, edited R. Cool and R Marshak (Interscience, New York, 1968), Vol. 2, p 567.

[31]In disagreement with Ref. 8 and with D. Stoler, Phys. Rev. D 4, 1925 (1971).

[32]S. Epstein, Am. J. Phys. 33, 375 (1965).

[33]A. Royer, Phys. Rev. A 15, 449 (1977).

[34]This is in accordance with J. Snygg, Am. J. Phys. 45, 58 (1977) for rotations about the origin of phase space.

[35]The proofs for the MUS's follow exactly those for coherent states in quantum optics: U. Titulaer and R. Glauber, Phys. Rev. 145, 1041 (1966). Naturally the same α must be considered throughout.

[36]D. Stoler, Phys. Rev. D 4, 2309 (1971).

[37]O. Hirota *et al.*, IECE of Japan Trans. E No. 1, 1978 (unpublished). See also Ref. 8 and references therein.