Nonequilibrium thermodynamic fluctuations of black holes

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(Received 2 February 1987; revised manuscript received 14 December 1987)

With the aid of the Landau-Lifshitz theory for thermodynamic fluctuations we estimate and comment on the fluctuations in the rates of mass, angular momentum, and other relevant quantities of massive Schwarzschild and Kerr black holes.

I. INTRODUCTION

In two previous papers we dealt with equilibrium fluctuations of Schwarzschild[1] and Kerr[2] black holes. In both of them, in order to ensure thermodynamic equilibrium, the black hole was supposed to be enclosed in a large idealized box filled with radiation at the same temperature as the hole. The second moments in the fluctuations of the relevant quantities were calculated making use of the Bekenstein interpretation of black-hole entropy and the Einstein-Boltzmann formula for equilibrium fluctuations. Inspection of these correlations reveals, on the one hand, that a phase transition between the black hole and radiation occurs when the heat capacity of the whole system vanishes and, on the other hand, that the aforementioned Bekenstein interpretation works well in the envisaged situations. However, the situation considered in both papers seems rather artificial, no realistic black hole is enclosed in a box, so it would be more interesting to study the mesoscopic behavior of freely evaporating holes. In this case we definitely face a nonequilibrium situation in which the mass, angular momentum, and electric charge of the black hole are continuously decreasing. As is well known,[3] if the black hole is massive enough, i.e., its mass exceeds 10^{14} g, the loss rate of these quantities is almost constant and the evaporation may be approximated by a nonequilibrium steady-state process, provided that the time scale of any realistic measurement is much shorter than the scale of time of the process itself.

Our purpose in this paper is to analyze the nonequilibrium thermodynamic fluctuations of massive freely evaporating Schwarzschild and Kerr black holes. The correct way to do that should employ the fact that each normalized wave-packet mode of nearly constant frequency ω makes a roughly independent contribution to the fluctuation, which is $\langle(\delta N)^2\rangle=\langle N^2\rangle-\langle N\rangle^2$, for the number of particles emitted in the mode, where

$$\langle N\rangle=\Gamma/\{\exp[(\omega-m\Omega)/T]\pm 1\}$$

is the expected number in a mode of transmission coefficient Γ, energy ω, axial angular momentum m from a black hole of temperature T, and angular velocity Ω, with the upper sign for fermions and the lower sign for bosons. Units have been chosen so that $c=G=\hbar=k_B=1$. The number of orthonormal modes emitted per time is given by $(2\pi)^{-1}\sum\int d\omega$, where $\sum$ stands for the sum over the discrete quantum labeling the particle species. Thus if each particle carries away a certain quantity, say y_i, and the total carried by N particles is Y_i, so $\langle Y_i\rangle=\langle N\rangle y_i$, the expected emission rate of Y_i yields

$$\langle Y_i\rangle=(2\pi)^{-i}\sum\int d\omega\langle N\rangle y_i \ .$$

Consequently, the second moments in the fluctuations take the form

$$\langle\delta Y_i\delta Y_j\rangle=(2\pi)^{-1}\sum\int d\omega(\langle N^2\rangle-\langle N\rangle^2)y_iy_j \ .$$

However, because of the complicated dependence of $\langle N\rangle$ on ω and the discrete quantum numbers, particularly through Γ, the corresponding calculation is quite involved requiring numerical analysis. In view of that, we prefer to use a simpler method due to Landau and Lifshitz[4] which applies in classical nonequilibrium statistical theory. Although that theory, widely known as Landau-Lifshitz fluctuation hydrodynamic theory, was initially intended just for equilibrium situations its validity range was successfully extended[5,6] to include nonequilibrium steady-state situations also. It is well known that black holes comply with the laws of thermodynamics[7] and statistical physics.[8] Therefore it is natural to expect that, as for simple Newtonian systems, nonequilibrium Landau-Lifshitz fluctuation theory has some range of validity when applied to such collapsed objects. Strictly speaking, the Landau-Lifshitz theory is valid only for thermodynamic equilibrium situations, hence one may challenge its applicability to the situations we want to deal with. In this connection it is worth noting that the departure from thermodynamic equilibrium is measured by the magnitude of the entropy production $|\dot{S}|$, or equivalently by the magnitude of the dissipative fluxes. However, such as we shall make explicit below, for black holes whose masses are about 5×10^{14} g, and beyond, and angular momentum around $M^2/2$ the entropy production becomes quite negligible. So we expect that the theory mentioned will provide us, whether in the Schwarzschild case or in the Kerr one, with reasonable qualitative estimates at least. This is explicitly shown in Appendix A for a similar calculable problem: namely, the free emission of photons from a black body sphere.

According to Landau and Lifshitz[4] if the flux $\dot{x}_i$ of a given thermodynamic quantity x_i, which varies in a general dissipative process, is governed by

$$\dot{x}_i = -\sum_j \Gamma_{ij} X_j + \delta \dot{x}_i \ , \quad (1)$$

and the entropy rate by

$$\dot{S} = -\sum_i X_i \dot{x}_i \quad (2)$$

then the second moments in the fluctuations of the fluxes read

$$\langle \delta \dot{x}_i \delta \dot{x}_j \rangle = k_B (\Gamma_{ij} + \Gamma_{ji}) \delta_{ij} \ . \quad (3)$$

Here and throughout the angular brackets mean average with respect to the steady state, and the fluctuations $\delta \dot{x}_i$ are considered spontaneous deviations from the steady-state value $\langle \dot{x}_i \rangle$ so $\langle \delta \dot{x}_i \rangle$ vanishes. The quantities Γ_{ij} and X_j stand for the phenomenological transport coefficients and the thermodynamic force conjugate to the flux $\dot{x}_i$, respectively. As usual k_B denotes the Boltzmann constant.

In Sec. II we study the fluctuations of the rate of mass, entropy, and temperature of a massive Schwarzschild black hole, whereas in Sec. III we extend our analysis to massive Kerr black holes. Finally, Sec. IV is devoted to comment on the behavior of the second moments in the limit $J \to M^2$.

II. THE SCHWARZSCHILD BLACK-HOLE CASE

As was derived by Page[3] the mass of freely evaporating black hole decreases with time according to

$$\dot{M} = -\alpha(M) M^{-2} \ , \quad (4)$$

where $\alpha(M)$ is a positive-definite numerical coefficient which depends on which species of particles can be emitted at a significant rate. For black-hole masses around 5×10^{14} g the hole preferably emits neutrinos, photons, and gravitons and $\alpha(M)$ may be replaced by a constant, α_0, of the order of 10^{-4}. Usually $\dot{M}$ is referred to as minus the luminosity of the hole.

Obviously if the measurement time is much shorter than the characteristic decay time ($\propto M^3$) we may consider that the right-hand side of Eq. (4) does not vary while the corresponding measurement is made. It is easy to check that for $M \geq 10^{14}$ g the characteristic decay time is of the order of the age of the Universe. So, for $M \geq 10^{14}$ g we can accurately say that we are dealing with a steady-state evaporation process. Then it is reasonable to study it in light of the nonequilibrium thermodynamic fluctuation theory sketched above. According to this we substitute $\alpha(M)$ by α_0 in expression (4) and we consider the stochastic equation $\dot{M} = -\alpha_0 M^{-2} + \delta \dot{M}$, where $\delta \dot{M}$ represents a spontaneous fluctuation of $\dot{M}$ around its steady-state value: namely, $\langle \dot{M} \rangle = -\alpha_0 M^{-2}$. Some care must be taken in using the Landau-Lifshitz method in conjunction with the Schwarzschild black-hole entropy expression $S = 4\pi M^2$ since that method was conceived for systems of common entropies, i.e., for systems whose entropy was supposed to be concave. In the case we are dealing with the entropy is not concave, and a slight modification in the Landau-Lifshitz procedure for obtaining the second moments is needed. For nonconcave entropies in the realm of black-hole thermodynamics see Refs. 9 and 10. The aforementioned modification consists in substituting expression (2) for the more general one

$$\dot{S} = \sum_i (\pm X_i \dot{x}_i) \ , \quad (5)$$

where the upper sign holds for contributions to the entropy rate which come from nonconcave parts of S, whereas the lower sign holds otherwise. In the case we are facing now S is nonconcave and the second moments for $\delta \dot{M}$ are found to be

$$\langle \delta \dot{M} \delta \dot{M} \rangle = \alpha_0 (4\pi M^3)^{-1} \ . \quad (6)$$

For black-hole masses about 5×10^{14} g the mass flux is of the order of 10^{-5} g/sec whereas the entropy and entropy production, $|\dot{S}| = (8\pi k_B G / ch) M \dot{M}$, are of the order of 10^{31} erg/K and 10^5 erg/K sec, respectively. In view of these figures we can say the system is close to the local thermodynamic equilibrium; hence we are confident that the results of the Landau-Lifshitz theory when applied to Schwarzschild black holes of mass $M \geq 5 \times 10^{14}$ g are at least qualitatively correct.

The procedure we have followed allows us to define a sort of "phenomenological coefficient" associated to the spontaneous irreversible process of mass evaporation. That quantity reads $\Gamma_M = \alpha_0 (8\pi M^3)^{-1}$. Likewise we note, in passing, that the procedure trivially leads to establishing a kind of fluctuation-dissipation relation for this steady-state process: namely, $\langle \delta \dot{M} \delta \dot{M} \rangle = 2\Gamma_M$, relating the autocorrelation in that state to the above-defined phenomenological coefficient in close analogy with the usual fluctuation-dissipation expressions for equilibrium. In the case under study there is no gradient with which to associate, we have at our disposal a generalized "thermodynamical force" given by the inverse of the black-hole temperature, $T^{-1} = 8\pi M$.

Having derived expression (6) it is straightforward to obtain the following set of correlations:

$$\langle \delta \dot{S} \delta \dot{S} \rangle = 16\pi \alpha_0 M^{-1} \ , \quad (7)$$

$$\langle \delta \dot{T} \delta \dot{T} \rangle = \alpha_0 (256\pi^3 M^7)^{-1} \ , \quad (8)$$

$$\langle \delta \dot{S} \delta \dot{M} \rangle = 2\alpha_0 M^{-2} \ , \quad (9)$$

$$\langle \delta \dot{M} \delta \dot{T} \rangle = -\alpha_0 (32\pi^2 M^5)^{-1} \ , \quad (10)$$

$$\langle \delta \dot{S} \delta \dot{T} \rangle = -\alpha_0 (4\pi M^4)^{-1} \ . \quad (11)$$

Note that the right-hand side of Eq. (9) is positive while that of Eqs. (10) and (11) are negative. This is a direct consequence of both M and S decreasing with time and T increasing.

III. THE KERR BLACK-HOLE CASE

An evaporating Kerr black hole radiates away not only its mass but its angular momentum J also. The rates of both emission processes have been reported by Page[11] to be

$$\dot{M} = -f M^{-2} \ , \quad (12)$$

$$\dot{J} = -g J M^{-3} \ . \quad (13)$$

The parameter f is defined through $f=n_{1/2}f_{1/2}+n_1f_1+n_2f_2$—an analogous expression holds for the parameter g—where the n_i indicate the number of massless particle species with spin $\frac{1}{2}$, 1, and 2, respectively. Both parameters are positive definite. Page assumes $n_{1/2}=2$, $n_1=1$, and $n_2=1$. In their turn the f_i's and g_i's depend just on the ratio J/M^2, and they have numerically been calculated for different intervals of that ratio—see Table I in Ref. 11—ranging from a maximum rotating hole ($J=M^2$) to a nearly nonrotating one ($J=10^{-2}M^2$). For $J=0$ it is understood that f goes over α.

In view of the aforementioned table of values and bearing in mind the considerations of the previous section about the measurement time, we may safely say that Kerr black holes with masses beyond 5×10^{14} g will spend a lot of time with nearly the same value of M and J. Therefore, the interval of time used in any realistic experiment of measurement of $\dot{M}$ and $\dot{J}$ will be much shorter than the interval of time needed for the hole to emit any noticeable quantity of mass or angular momentum. Then, we feel it is not unwise to consider, in our approximation, f, g, M, and J constant. In this way we are facing a twofold steady-state process, in which $\langle \dot{M} \rangle$ and $\langle \dot{J} \rangle$ are fixed and given by the right-hand sides of Eqs. (12) and (13), respectively. Now we can consider spontaneous random fluctuations, $\delta\dot{M}$ and $\delta\dot{J}$, around these mean values, and apply the Landau-Lifshitz method as above.

The entropy and temperature of a rotating black hole read

$$S=2\pi M^2(1+b)\ , \tag{14}$$

$$T=(4\pi M)^{-1}b(1+b)^{-1}\ , \tag{15}$$

with $b=[1-(J^2/M^4)]^{1/2}$. For black-hole masses around 5×10^{14} g and angular momentum, say, $M^2/2$ the corresponding fluxes of mass and angular momentum, in Planck units, are of the order of 10^{-41} and 10^{-22}, respectively. Likewise, the entropy and the entropy production,

$$|\dot{S}|=(\pi/2M)(1+b)[b^{-1}(g-2f)-g]\ ,$$

are of the order of 10^{40} and 10^{-22}, respectively. Once again, we see that the system is so close to the local thermodynamic equilibrium that it is quite reasonable to expect at least good qualitative predictions. Besides, inspection of the above expressions for $|\dot{S}|$ reveals that for $M\geq5\times10^{14}$ g the entropy production is negligible for rotating holes with angular momentum quite close to M^2 [for instance, if $J=(1-10^{-3})M^2$, $|\dot{S}|$ becomes of the order of 10^{-18}]. In this view the Landau-Lifshitz theory seems to be qualitative valid even in that extreme situation. Of course, if $J=M^2$, $|\dot{S}|$ diverges and that theory is no longer reliable.

Taking into account that the entropy part corresponding to M is nonconcave (i.e., $\partial^2S/\partial M^2>0$) whereas the part corresponding to J is concave (i.e., $\partial^2S/\partial J^2<0$) and proceeding along the same lines as in Sec. II we obtain

$$\langle \delta\dot{M}\delta\dot{M} \rangle = f(2\pi)^{-1}bM[(b+b^2)M^4+J^2]^{-1}\ , \tag{16}$$

$$\langle \delta\dot{J}\delta\dot{J} \rangle = gb(\pi M)^{-1}\ . \tag{17}$$

These second moments may be interpreted as fluctuation-dissipation expressions for nonequilibrium steady-state evaporation, where the associated dissipative coefficients are nothing but one-half of the corresponding right-hand side. A quick inspection of the right-hand side of Eq. (16) reveals that a Schwarzschild black hole will experience fluctuations in the mass flux greater than a rotating hole of the same mass. For vanishing angular momentum $\langle \delta\dot{M}\delta\dot{M} \rangle$ reduces to its Schwarzschild value as it should. However, $\langle \delta\dot{J}\delta\dot{J} \rangle$ does not vanish since the parameter g remains above zero (see Table II in Ref. 11). This should not be surprising since the angular momentum flux can fluctuate around its vanishing average value. This kind of feature is also present in equilibrium fluctuations of dissipation quantities in electric and hydrodynamic systems. On the other hand, it should be realized that the minus sign in Eq. (13) indicates that the black hole preferably emits angular momentum of the same sign as the hole. Then, it is possible that the hole emits angular momentum of the opposite sign increasing in this way its angular momentum. In this light it is easier to understand expression (17).

As mentioned above, for extreme Kerr black holes, $J=M^2$, the system ceases to be close to local thermodynamic equilibrium for the entropy production diverges. Therefore, the application of the Landau-Lifshitz theory to that situation is not justified in principle, and, at best, it can be viewed as an extrapolation. However, as noted before, since one can go quite close to maximum rotating black-hole situations without the system deviating significantly from local thermodynamic equilibrium it is tempting to speculate about which physical consequences may be drawn in the limit $J\to M^2$. As we shall see these consequences will prove to be quite reasonable.

For $J\to M^2$ both second moments, expressions (16) and (17), tend to vanish. This is so because J cannot go beyond M^2, otherwise the event horizon would disappear and the singularity inside the hole would attain causal connection with the external world. Hence, the constraint $J\leq M^2$ represents a boundary condition, and it is well known, for instance, in hydrodynamic systems,[6] the autocorrelations vanish when a relevant parameter in the fluctuating system reaches a boundary value. Likewise, it can be easily understood that if M and J were able to fluctuate when $J=M^2$ then either J could surpass M^2 or equivalently M^2 could go over values below J.

For the cross correlations one has

$$\langle \delta\dot{M}\delta\dot{J} \rangle = 0\ . \tag{18}$$

This result can be understood from different points of view. On the one hand, one may consider the microscopic stochastic emission of quanta from the black hole. The energy and angular momentum, around the hole axis, of each individual quanta leaving the black hole, in the process of evaporation, are not related to each other. Therefore, it is quite natural that the flux of energy and angular momentum are, on the average, completely uncorrelated. On the other hand, employing the Gibbs equation for black holes, $T\delta\dot{S}=\delta\dot{M}-\Omega\delta\dot{J}$, it can be guessed that $\delta\dot{M}$ and $\delta\dot{J}$ are uncorrelated, much in the same way as δM and δJ are in equilibrium fluctuations of Kerr black holes.[2]

It is straightforward, with the use of Eqs. (14)–(18), to determine any second moment involving whichever of the relevant fluxes. Some of them are listed below while some others are collected in Appendix B in order not to burden the main text:

$$\langle \delta\dot{S}\delta\dot{M} \rangle = \frac{2f}{M^2} , \tag{19}$$

$$\langle \delta\dot{S}\delta\dot{J} \rangle = -\frac{bgM}{2\pi M^3} , \tag{20}$$

$$\langle \delta\dot{S}\delta\dot{S} \rangle = \frac{4\pi}{bM^5}[2fb(1+b)M^4+(2f+g)J^2] . \tag{21}$$

The fact of S being an increasing function of M and a decreasing one of J is reflected in that the right-hand side of relation (19) is positive whereas that one of relation (20) is negative. Note that $\langle \delta\dot{S}\delta\dot{M} \rangle$ goes over to its Schwarzschild value when f is substituted by α. The cross correlation $\langle \delta\dot{S}\delta\dot{J} \rangle$ vanishes both for nonrotating as well as for maximum rotating black holes. The first case, $J=0$, is an obvious consequence of the fact that Schwarzschild entropy does not involve angular momentum, and the second one, $J=M^2$, is a direct consequence of the above-mentioned boundary restriction. For vanishing angular momentum $\langle \delta\dot{S}\delta\dot{S} \rangle$ reduces to its Schwarzschild values and for $J=M^2$ diverges. This last result agrees with that one by Page[11] stating that the rate at which the event horizon area varies diverges as J approaches M^2.

Other relevant second moments are

$$\langle \delta\dot{T}\delta\dot{T} \rangle = \frac{1}{16\pi^3 M^{11} b(1+b)^4}\left[\frac{f[b(1+b)M^4-2J^2]^2}{2[b(1+b)M^4+J^2]}+g^{J2}\right] , \tag{22}$$

$$\langle \delta\dot{\Omega}\delta\dot{J} \rangle = \frac{8bg}{\pi M^4(1+b)}\left[1-\frac{J}{2M^2}\right] , \tag{23}$$

$$\langle \delta\dot{\Omega}\delta\dot{\Omega} \rangle = \frac{64b}{\pi M^7(1+b)^2}\left[\frac{f\{4(1+b)M^2+3J[16\pi(1+b)M^4+J^2]\}^2}{9(1+b)^2M^3[2(b+b^2)M^4+J^2]}+g\left[1-\frac{J}{2M^2}\right]^2 b\right] , \tag{24}$$

$$\langle \delta\dot{S}\delta\dot{T} \rangle = -\frac{1}{2\pi M^8 b(1+b)^2}\{f[b(1+b)M^4-2J^2]gMJ\} , \tag{25}$$

where the angular velocity of the hole equals $\pi J(MS)^{-1}$. For extreme Kerr black holes $\langle \delta\dot{T}\delta\dot{T} \rangle$ and $\langle \delta\dot{S}\delta\dot{T} \rangle$ diverge, whereas both $\langle \delta\dot{\Omega}\delta\dot{J} \rangle$ and $\langle \delta\dot{\Omega}\delta\dot{\Omega} \rangle$ vanish. The latter result is, once again, a consequence of the boundary condition $J\leq M^2$. The former one will be interpreted below in the light of the other divergences occurring in the limit $J\to M^2$. As can be checked the right-hand side of Eqs. (22) and (25) reduce for nonrotating black holes to their Schwarzschild value $\alpha_0(256\pi^3M^7)^{-1}$ and $-\alpha_0(4\pi M^4)^{-1}$, respectively. Note that the right-hand side of Eq. (23) cannot become negative, it reflects the fact that Ω is an always increasing function of J. For vanishing angular momentum $\langle \delta\dot{\Omega}\delta\dot{J} \rangle$ and $\langle \delta\dot{\Omega}\delta\dot{\Omega} \rangle$ do not vanish though J, $\dot{J}$, Ω, and $\dot{\Omega}$ vanish. This corroborates once again that the rates of the angular quantities fluctuate around their vanishing average values.

It is traditional to split the square total energy of a rotating hole in two pieces, $M^2=M_{\rm irr}^2+M_{\rm rot}^2$. The first one, $M_{\rm irr}^2=(4\pi)^{-1}S$ is proportional to the entropy and corresponds to that part of the total energy that cannot be drained from the hole by a series of Penrose processes.[12] The second one, $M_{\rm rot}^2=4\pi J^2S^{-1}$, corresponds to that part of M that can be obtained from the black hole through that process. It is clear that in the steady-state evaporating regime the fluctuations of $\dot{M}$, $\dot{M}_{\rm irr}$, and $\dot{M}_{\rm rot}$ have to be related to each other much in the same way as in the equilibrium[2] the fluctuations of M, $M_{\rm irr}$, and $M_{\rm rot}$ are related. A brief calculation leads to the expressions

$$\langle \delta\dot{M}\delta\dot{M}_{\rm irr} \rangle = \frac{f}{2\pi M^3[2(1+b)]^{1/2}} , \tag{26}$$

$$\langle \delta\dot{M}\delta\dot{M}_{\rm rot} \rangle = -\frac{fJ}{2^{1/2}\pi M^5(1+b)^{3/2}} , \tag{27}$$

$$\langle \delta\dot{M}_{\rm irr}\delta\dot{M}_{\rm rot} \rangle = -\frac{J}{16\pi^2M^7b^2}\left[gb+\frac{4\pi}{M^2}[2f(b+b^2)M^4+(2f+g)J^2]\right] . \tag{28}$$

As can be checked, for nonrotating holes $\langle \delta\dot{M}\delta\dot{M}_{\rm irr} \rangle$ reduces to its Schwarzschild counterpart $\alpha_0(4\pi M^3)^{-1}$, while the right-hand sides of Eqs. (27) and (28) vanish as expected in accordance with Eq. (18). For extreme Kerr black holes the two former right-hand sides remain finite but expression (28) diverges. The positive-definite character of Eq. (26) and the negative-semidefinite of Eqs. (27) and (28) correspond to the fact that the black-hole energy increases with the entropy and decreases with the angular momentum.

Inspection of all second moments reveals that given two black holes with the same ratio J/M^2 the fluctuations will be smaller in that one with the higher mass, or equivalently with the higher angular momentum. So loosely speaking, we may say that both J and M tend to diminish the strength of the fluctuations. On the other hand, $\langle \delta\dot{J}\delta\dot{J} \rangle$ is greater than $\langle \delta\dot{M}\delta\dot{M} \rangle$. This is in accordance with the fact that black holes emit their angular momentum faster than their mass,[11] so they are expected to experience fluctuations in $\dot{J}$ greater than in $\dot{M}$.

IV. BEHAVIOR OF THE FLUCTUATIONS IN THE $J \to M^2$ LIMIT

As noted above, some second moments, namely, $\langle \delta\dot{S}\delta\dot{S} \rangle$, $\langle \delta\dot{T}\delta\dot{T} \rangle$, $\langle \delta\dot{S}\delta\dot{T} \rangle$, $\langle \delta\dot{M}_{\rm irr}\delta\dot{M}_{\rm rot} \rangle$, $\langle \delta\dot{\Omega}\delta\dot{T} \rangle$, $\langle \delta\dot{M}_{\rm irr}\delta\dot{T} \rangle$, and $\langle \delta\dot{M}_{\rm rot}\delta\dot{T} \rangle$ (see Appendix B for the last three) tend to diverge in the $J \to M^2$ limit. From statistical thermodynamics it is well known that phase transitions are usually accompanied by the occurrence of divergences in some relevant correlations.[13] Hence, one is tempted to speculate that these divergencies may correspond to a nonequilibrium phase transition from extreme to nonextreme Kerr black holes. This conjecture seems reasonable since, on the one hand, no divergences occur in any of the second moments for values of J other than M^2; which agrees with the fact that no radical change affecting the black-hole properties is seen for values of J lower than M^2. On the other hand, extreme Kerr black holes have some properties radically different from nonextreme ones. In particular, their temperature vanishes—see Eq. (15)—hence, no Hawking thermal radiation coming from the extreme hole is observed at infinity. In fact they can radiate only through superradiant scattering. Besides, the inner horizon region degenerates into a timelike one. Therefore we might say that the aforementioned divergences closely correspond to the sudden change in properties from extreme to nonextreme Kerr black hole. Our result is in accordance with that found by Curir[14] who exploits the analogy between rotating black holes and paramagnetic systems. These divergences in the second moments in the limit $J \to M^2$ can be attributed to the same mechanism that makes the rates of the event-horizon area and surface gravity diverge in that limit [see Sec. IV and Eqs. (54) and (56) in Ref. 11]. At this point it is convenient to keep in mind that the event-horizon area and the surface gravity are proportional, respectively, to the entropy and temperature of the hole. That mechanism is nothing but the superradiant (stimulated) scattering of quanta by the black hole. At J values close to M^2 the Kerr hole emits particles essentially via superradiant scattering, whereas at lower values of J the spontaneous (thermal) Hawking radiance dominates over the superradiant emission. Lastly, at $J=0$ no superradiant scattering is expected. So, on physical grounds, it should not be surprising that the divergences dS/dt and dT/dt, encountered by Page,[11] are accompanied by divergences in some second moments, all of them involving either $\delta\dot{S}$, $\delta\dot{T}$, or both. This may be seen as giving support to our conjecture which states that these divergences in the second moments seem to correspond to a phase transition between extreme and nonextreme Kerr black holes.

Some additional support to that conjecture can be obtained from quantum optics. On the one hand, it is well known that the stimulated emission of quanta by rotating black holes resembles that occurring in atomic physics.[15] In particular, superradiant scattering by black holes is a laserlike phenomenon.[16] Suppose we have a gas of atoms possessing two energy levels and we prepare the gas so that initially most atoms are in the excited state. If electromagnetic radiation is incident on these atoms transitions from the upper level to the lower one will primarily be induced resulting in stimulated emission. In this way the transmitted electromagnetic radiation becomes amplified. This is nothing but the basic mechanism of a laser gas. On the other hand, it was established long ago (see, e.g., De Giorgio and Scully[17]) that when the frequency of a laser crosses the laser threshold a thermodynamic second-order phase transition takes place. The laser behaves quite differently above than from below the threshold. Above it their atoms oscillate in phase at a single resonant frequency while below the threshold these atoms emit radiation with random phases like a conventional lamp. In view of this analogy between lasers and rotating black holes it seems quite reasonable to expect that a phase transition also takes place when a Kerr black hole, which may be understood as an excited state of a Schwarzschild hole, passes from the superradiant scattering regime to that one dominated by spontaneous thermal radiation. It is to say, when it passes from J values close to M^2 to lower J values.

Some underlying resemblance seems to exist between the results of this paper and those of equilibrium situations.[1,2] In the latter ones a phase transition also happens but in those cases the transition involves the disappearance of the black hole which evaporates away altogether. It occurs when the effective heat capacity of the whole system, black-hole plus radiation, vanishes as it leads some second moments to diverge. In the case we are considering the phase transition does not entail black-hole disappearance as the event horizon persists. The divergence of some second moments indicating the phase transition is due to the vanishing of parameter b which measures the separation of the hole from the extreme Kerr case.

The reasonableness of our results, is to say that the satisfactory behavior of the correlations seems to indicate that despite the fact that Landau-Lifshitz theory represents a crude approximation to this problem rotating black holes might be viewed as dissipative systems amenable to a judicious treatment along the lines of that theory. This in turn corroborates that one quarter of the area of the event horizon plays also the role of black-hole entropy in nonequilibrium processes. A parallel study to that carried out here could be done for electrically charged black holes by resorting to the equation for the rate of charge emission.[18]

Some time ago, Davies,[19] in his analysis of the equilibrium between a Kerr black hole and rotating thermal radiation, claimed to have discovered a phase transition at $J=(2\sqrt{3}-3)^{1/2}M^2$. For that value of J the heat capacity of the hole, $T(\partial S/\partial T)$, suffers an infinite discontinuity which, according to Davies, would indicate the occurrence of a second-order phase transition. However, such a claim was refuted by Sokolowski and Mazur[20] who were able to show that, in reality, such a phase transition is purely geometrical in origin lacking actual physical meaning inasmuch as the Kerr black-hole behavior does not experience any sudden change. In particular, its internal state remains unaffected. This greatly contrasts with the nonequilibrium phase transition reported here since the divergences in several correlations are accompanied by important changes in the black-hole demeanor.

Nonequilibrium black-hole fluctuations have also been studied by Bekenstein.[21] His analysis makes no use of Page's Eqs. (11) and (12) nor of the Landau-Lifshitz method. However, his approach, unlike ours, seems to be able to infer consequences for black-hole luminosities in the small-mass regime. Nonetheless, cross correlations are not calculated and no mention about the possible thermodynamic phase transition is made.

ACKNOWLEDGMENT

Limited financial support was provided by the Comision Asesora de Investigatión Cientifica y Técnica of the Spanish Government under Grant No. PB85-0381.

APPENDIX A

Let us consider the blackbody emission of photons from a sphere of radius $R \gg T^{-1}$ freely radiating into vacuum space. For the dominant frequencies, $\omega \gg R^{-1}$, T is close to unity for most of the roughly $2R^2w^2$ modes having impact parameter equal to or less than R, and close to zero for nearly all the other modes. So one has

$$\langle \dot{E} \rangle \approx 2R^2 \int_0^\infty \frac{w^2}{2\pi(e^{w/T}-1)} dw = \frac{\pi^3}{15} R^2 T^4 , \qquad \text{(A1)}$$

$$\langle \delta\dot{E} \delta\dot{E} \rangle \approx 2R^2 \int_0^\infty \frac{w^4 e^{w/T}}{2\pi(e^{w/T-1})^2} dw = 4T \langle \dot{E} \rangle , \qquad \text{(A2)}$$

where $\dot{E}$ denotes the energy emitted per unit of time. The Landau-Lifshitz theory when applied to this problem yields $\langle \delta\dot{E}\delta\dot{E} \rangle = 2T\langle \dot{E} \rangle$, which shows good qualitative agreement.

APPENDIX B

In this appendix we collect the following correlations:

$$\langle \delta\dot{M} \delta\dot{T} \rangle = -\frac{f[b(1+b)M^4-2J^2]}{[8\pi^2M^5(1+b)^2][b(1+b)M^4+2J^2]} , \qquad \text{(B1)}$$

$$\langle \delta\dot{J} \delta\dot{T} \rangle = \frac{gJ}{4\pi^2M^6(1+b)^2} , \qquad \text{(B2)}$$

$$\langle \delta\dot{M} \delta\dot{\Omega} \rangle = -\frac{4fb\{4(1+b)M^2+3J[16\pi(1+b)M^4+J^2]\}}{3\pi M^5(1+b)^2[b(1+b)M^4+J^2]} , \qquad \text{(B3)}$$

$$\langle \delta\dot{M}_{\text{irr}} \delta\dot{J} \rangle = -\frac{gJ}{2^{3/2}M^4(1+b)^{1/2}} , \qquad \text{(B4)}$$

$$\langle \delta\dot{M}_{\text{rot}} \delta\dot{J} \rangle = \frac{gb}{2\pi^{1/2}S^{3/2}M^3}(4SM^2+J^2) , \qquad \text{(B5)}$$

$$\langle \delta\dot{\Omega} \delta\dot{T} \rangle = \frac{2}{\pi^2M^9(1+b)^3}\left[b(1+b)M^4-2J^2] \left[\frac{4(1+b)M^2+3J}{6(1+b)M^2} \right] \frac{f}{b(1+b)M^4+J^2} + J\left[1-\frac{J}{2M^2}\right]g \right] , \qquad \text{(B6)}$$

$$\langle \delta\dot{M}_{\text{irr}} \delta\dot{T} \rangle = -\frac{f[b(1+b)M^4-2J^2]+gJ^2}{16\pi^2bM^8M_{\text{irr}}(1+b)^2} , \qquad \text{(B7)}$$

$$\langle \delta\dot{M}_{\text{rot}} \delta\dot{T} \rangle = \frac{1}{2\pi^{3/2}b(1+b)^2M^8S^{3/2}}\{gbM^2S+\pi J[f[b(1+b)M^4-2J^2]-gMJ]\} . \qquad \text{(B8)}$$

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