Effects of friction on cosmic strings

Jaume Garriga
Tufts Institute of Cosmology, Department of Physics and Astronomy, Tufts University, Medford, Massachusets 02155

Maria Sakellariadou
Laboratoire de Modèles de Physique Mathématique, Faculté des Sciences, Parc de Grandmont, 37200 Tours, France
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We study the evolution of cosmic strings taking into account the frictional force due to the surrounding radiation. We consider small perturbations on straight strings, oscillation of circular loops, and small perturbations on circular loops. For straight strings, friction exponentially suppresses perturbations whose comoving scale crosses the horizon before the cosmological time $t_* \sim \mu^{-2}$ (in Planck units), where μ is the string tension. Loops with a size much smaller than t_* will be approximately circular at the time when they start the relativistic collapse. We investigate the possibility that such loops will form black holes. We find that the number of black holes which are formed through this process is well below present observational limits, so this does not give any lower or upper bounds on μ. We also consider the case of straight strings attached to walls and circular holes that can spontaneously nucleate on metastable domain walls.

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I. INTRODUCTION

Cosmic strings are topological defects that may have formed during phase transitions in the early Universe (see, e.g., [1]). Their properties and observational consequences, especially in connection with their possible role in the formation of large scale structure in the Universe, have been extensively studied during the past decade (see, e.g., [2]).

Cosmic strings of mass per unit length μ would have formed at cosmological time of order $t_0 \sim (G\mu)^{-1}t_{\rm Pl}$ (G is Newton's constant and $t_{\rm Pl}$ is the Planck time). It is well known that immediately after the phase transition the dynamics of strings would be dominated by the force of friction [3, 4]. This force is due to the scattering of thermal particles off the string. Friction would dominate the dynamics until a time of order $t_* \sim (G\mu)^{-2}t_{\rm Pl}$. In most of the investigations about cosmic strings, the effects of friction have been neglected. The reason is that if cosmic strings have to play a role in galaxy formation, then they have to form near the grand unification scale. In that case their mass per unit length is of order $G\mu \sim 10^{-6}$ and friction is important only for a very short period of time.

However, if strings have formed at later phase transitions, say, closer to the electroweak scale, their dynamics would be dominated by friction through most of the thermal history of the Universe. It is therefore of some interest to study the evolution of cosmic strings with friction in a quantitative way. The relativistic equation of motion for strings with friction was given by Vilenkin [4]. The main purpose of this paper is to solve this equation in a few simple cases, which should be representative of more complicated situations.

The plan of the paper is the following. In Sec. II we review the results of Ref. [4], in order to fix the notation. In Sec. III we study linearized perturbations on an infinite straight string. In Sec. IV we consider the dynamics of oscillating circular loops. The evolution of linearized perturbations on circular loops is discussed in Sec. V.

It is known that an exactly circular loop would form a black hole when it collapses under its tension [5, 6]. Since friction tends to erase perturbations whose comoving scale crosses the horizon before t_*, loops smaller than t_* will tend to be approximately circular. In Sec. VI we study the possibility that such loops form black holes. Note that these black holes can have masses of order $M \sim \mu t_*$, which can be considerably large if $G\mu$ is sufficiently small. We discuss possible observational consequences of these black holes.

Finally, in Sec. VII, we consider perturbations on strings which are attached to domain walls, and in particular, to circular holes which can spontaneously nucleate in a metastable wall. Our conclusions are summarized in Sec. VIII.

II. STRINGS WITH FRICTION

In this section we summarize the results of Ref. [4], where the equation of motion for a string with friction was found. A cosmic string can be represented as a two-dimensional world sheet in spacetime $x^\mu = x^\mu(\xi^a)$, where ξ^a $(a = 0, 1)$ are two arbitrary parameters on the world sheet and x^μ are spacetime coordinates $(\mu = 0, ..., 4)$. In flat space, the equation of motion for the strings is $\mu\Box x^\nu = 0$, where $\Box$ denotes the d'Alembertian on the world sheet. When one includes the effects of curvature of the spacetime and a force of friction F^ν, the equation of motion reads [4]

$$\mu\left[-\Box x^\nu + \Gamma^\nu_{\sigma\tau}x^\sigma_{,a}x^{\tau,a}\right] = F^\nu(u^\lambda_\perp, T, \sigma). \quad (1)$$

Here $\Gamma^{\nu}_{\sigma\tau}$ are the four-dimensional Christoffel symbols (greek indices run over spacetime coordinates). The force of friction F^{ν} will depend on the temperature of the surrounding matter T, the velocity of the fluid transverse to the world sheet $u^{\nu}_{\perp} \equiv u^{\nu} - x^{\nu}_{,a}x^{\sigma,a}u_{\sigma}$, and the type of interaction between the particles and the string, which we symbolically represent by σ. Vilenkin [4] found the form of F^{ν} for the case when friction is dominated by Aharonov-Bohm scattering of charged particles with the pure gauge field outside the string [7]

$$F^{\nu} = \beta T^3 u^{\nu}_{\perp}. \tag{2}$$

The numerical coefficient β is given by

$$\beta = 2\pi^{-2}\xi(3)\sum_a b_a \sin^2(\pi\nu_a),$$

where the summation is over all effectively massless degrees of freedom (i.e., mass $m \ll T$) $b_a = 1$ for bosons, $b_a = 3/4$ for fermions, and ν_a is the phase change experienced by a particle as it is transported around the string.

For the case of a Friedmann-Robertson-Walker (FRW) universe,

$$ds^2 = a^2(\tau)[-d\tau^2 + d\mathbf{x}^2], \tag{3}$$

with the four-velocity of the fluid given by $u^{\nu} = (a^{-1}, 0, 0, 0)$, and choosing the gauge $\xi^0 = \tau, \dot{\mathbf{x}}\cdot\mathbf{x}' = 0$, Eq. (1) reduces to

$$\ddot{\mathbf{x}} - \epsilon^{-1}\left(\frac{\mathbf{x}'}{\epsilon}\right)' + [2a(H+h)](1-\dot{\mathbf{x}}^2)\dot{\mathbf{x}} = 0, \tag{4}$$

$$\dot{\epsilon} + [2a(H+h)]\dot{\mathbf{x}}^2\epsilon = 0. \tag{5}$$

Here the overdot is exactly equal to $d/d\tau$, the prime is exactly equal to $d/d\xi^1$, $H \equiv \dot{a}/a^2$ is the Hubble parameter, and

$$h \equiv \frac{\beta T^3}{2\mu}.$$

The quantity $\epsilon \equiv [\mathbf{x}'^2/(1-\dot{\mathbf{x}}^2)]^{1/2}$ is related to the energy of the string [1]:

$$E = \mu a(\tau)\int d\xi^1 \epsilon. \tag{6}$$

[Using the definition of ϵ it can be easily seen that Eq. (5) is actually not independent of (4).] In the following sections we shall solve the dynamical equations (4) in a few idealized situations, which should be representative of more general cases.

As noted in Ref. [4], the effect of friction is to make the replacement

$$H \to H + h$$

in the equations of motion for a free string in an expanding universe. In what follows we will consider

$$a(\tau) \sim \tau^{\alpha}$$

($\alpha = 1$ in the radiation-dominated era and $\alpha = 2$ in the matter era). Since $T \sim a^{-1}$, the friction term $h(\tau)$ dominates at early times, while the expansion term $H(\tau)$ will dominate at late times. Defining τ_* as the time at which both terms are equal,

$$h(\tau_*) = H(\tau_*),$$

one expects that friction will be unimportant for $\tau \gg \tau_*$. For future reference, it is convenient to express this time in terms of cosmological time t defined by $dt = a(\tau)d\tau$.

Using Einstein's equations in the radiation era, $H^2 = (8\pi^3 G/90)\mathcal{N}T^4$, one finds [4]

$$t_* = A(G\mu)^{-2}t_{\rm Pl} \tag{7}$$

[for $G\mu \gtrsim (At_{\rm Pl}/t_{\rm eq})^{1/2}$]. Here $t_{\rm eq}$ is the time of equal matter and radiation densities and $A = [90\beta^{4/3}/32\pi^3\mathcal{N}]^{3/2}$, with $\mathcal{N}$ the effective number of massless degrees of freedom. The coefficient A can be rather small. Taking $\sin^2\pi\nu_a \sim 1/2$ in the expression for β one has

$$A \sim 4\times 10^{-4}\mathcal{N}^{1/2}.$$

This estimate should actually be taken as an upper bound for A, since some of the particle species may not interact with the string.

If the strings are so light that $G\mu \lesssim (At_{\rm Pl}/t_{\rm eq})^{1/2}$ then $t_* > t_{\rm eq}$ and (7) is not valid. Then we have to take into account that $T \sim t^{-2/3}$ in the matter era and we have

$$t_* \approx A^{1/2}(G\mu)^{-1}\left(\frac{t_{\rm eq}}{t_{\rm Pl}}\right)^{1/2} t_{\rm Pl}. \tag{8}$$

Typically, this will apply only for strings with energy scale $G\mu \lesssim 10^{-29}$. For heavier strings t_* is in the radiation era and we should use (7).

The parameter t_* (or the corresponding conformal time τ_*) will play a central role in the following sections. As we shall see, the effect of friction on a given comoving scale will be essentially determined by the relative size of this scale with respect to τ_*.

III. LINEARIZED PERTURBATIONS ON A STRAIGHT STRING

Consider a straight string at rest, with trajectory $\mathbf{x} = (\xi, 0, 0)$, and introduce a small displacement in the plane transverse to the string:

$$\delta\mathbf{x} = e^{ik\xi}\mathbf{y}(\tau) = e^{ik\xi}(0, y_2(\tau), y_3(\tau)).$$

Substituting in (4) and keeping only terms linear in $\delta\mathbf{x}$ we have

$$\ddot{\mathbf{y}} + 2\alpha\left[\frac{1}{\tau} + \frac{\tau_*^{2\alpha-1}}{\tau^{2\alpha}}\right]\dot{\mathbf{y}} + k^2\mathbf{y} = 0, \tag{9}$$

where, as mentioned before, $\alpha = 1$ in the radiation era and $\alpha = 2$ in the matter era.

The case without friction corresponds to taking $\tau_* = 0$. Then (9) is just the modified Bessel equation. The corresponding solution that is well behaved as $\tau \to 0$ is given by

$$y = y_0\ 2^\nu \Gamma(\nu+1)(k\tau)^{-\nu} J_\nu(k\tau),$$

where $\nu = \alpha - 1/2$. At early times, $\tau \ll k^{-1}$, the solution tends to a constant,

$$y \approx y_0 \quad (\tau \ll k^{-1}), \tag{10}$$

whereas at late times, $\tau \gg k^{-1}$, the solution is oscillatory:

$$y \approx y_0\ (2\alpha-1)!! \frac{a(k^{-1})}{a(\tau)} \cos\left(k\tau - \frac{\alpha\pi}{2}\right) (\tau \gg k^{-1}). \tag{11}$$

Modes with wavelength larger than the horizon size ($\tau \ll k^{-1}$) are frozen in, and their physical amplitude $y_{\rm phys} \equiv a(\tau) y$ is conformally stretched by the expansion. Once the wavelength of a mode falls within the horizon ($\tau \gg k^{-1}$), the perturbation starts oscillating with constant physical amplitude [1].

When friction is included ($\tau_* \neq 0$), Eq. (9) can only be solved in the high frequency approximation ($k \gg \tau_*^{-1}$). This is actually the interesting regime, in which friction plays a role. In the opposite limit ($k \ll \tau_*^{-1}$), we just saw that perturbations are frozen in up to a time $\tau \sim k^{-1} \gg \tau_*$ due to the expansion term $2\alpha/\tau$ alone, and by the time they start oscillating the friction term has become negligible.

To solve for the case $k \gg \tau_*^{-1}$ we introduce the variable

$$\Psi \equiv \tau^\alpha \exp\left[-\frac{\alpha}{2\alpha-1}\left(\frac{\tau_*}{\tau}\right)^{2\alpha-1}\right] y. \tag{12}$$

Substituting in (9), one finds that Ψ satisfies the Schrödinger equation

$$-\ddot{\Psi} + V(\tau)\Psi = k^2 \Psi, \tag{13}$$

with the potential

$$V \equiv \frac{\alpha}{\tau^2}\left[(\alpha-1) + \alpha\left(\frac{\tau_*}{\tau}\right)^{4\alpha-2}\right].$$

Now Eq. (13) can be solved in the WKB approximation. This is done in Appendix A.

With the boundary condition that the perturbations are frozen in with the expansion at early times, the solution is

$$y(\tau) \approx y_0 \qquad (\tau \ll \tau_k), \tag{14}$$

$$y(\tau) \approx 2y_0 \frac{a(\tau_k)}{a(\tau)} e^{W(\tau/\tau_*)} e^{-\gamma k \tau_k} \cos(k\tau + \phi_k) \quad (\tau \gg \tau_k). \tag{15}$$

Here

$$W(\tau/\tau_*) = \frac{\alpha}{2\alpha-1}\left(\frac{\tau_*}{\tau}\right)^{2\alpha-1},$$

$$\gamma = \frac{1}{4\alpha-2} B\left(\frac{1+2\alpha}{4\alpha}, \frac{1}{2}\right),$$

where B is Euler's β function and τ_k is the classical turning point of the corresponding Schrödinger problem $V(\tau_k) = k^2$. The phase ϕ_k is just a constant. In a radiation-dominated universe,

$$\tau_k = (k^{-1}\tau_*)^{1/2},$$

$$\gamma \approx 1.2.$$

In general

$$\tau_k \approx [\alpha k^{-1} \tau_*^{2\alpha-1}]^{1/2\alpha}. \tag{16}$$

It can be checked that the conditions for the validity of the WKB approximation are satisfied provided that $k \gg \tau_*^{-1}$ and $\alpha > 1/2$.

Comparing (15) with (11) we can summarize as follows. Without friction perturbations of wave number k are conformally stretched up to a time $\tau \sim k^{-1}$, after which they start oscillating with constant physical amplitude

$$y_{\rm phys} \approx (2\alpha-1)!! a(k^{-1}) y_0. \tag{17}$$

With friction included, short wavelength perturbations $k \gg \tau_*^{-1}$ are conformally stretched up to a time τ_k given by (16), hence the factor $a(\tau_k)$ in (15). Since $\tau_k > k^{-1}$, friction contributes to increase the amplitude of the perturbation by a factor $(\tau_k k)^\alpha$, from time $\tau \sim k^{-1}$ up to time τ_k. After τ_k, the perturbation starts oscillating and losing energy to friction. This is represented by the two exponential factors in (14). By the time $\tau \sim \tau_*$ the amplitude of the perturbations has been damped by a factor $\exp(-\gamma k \tau_k)$. For $\tau \gg \tau_*$ the modes do not lose any more energy to friction, the first exponential in (15) has reached its asymptotic value of unity, and the string oscillates with constant physical amplitude:

$$y_{\rm phys} \approx 2a(\tau_k) e^{-\gamma k \tau_k} y_0. \tag{18}$$

The basic feature that distinguishes (18) from (17) is the exponential factor, which strongly suppresses short wavelength perturbations $k \gg \tau_k$. This means that, as a result of friction, the string will become very smooth on comoving scales much smaller than τ_*.

As expected, friction plays no role after τ_*. Also, the exponential suppression in (18) becomes less and less dramatic as k approaches τ_* (i.e., $k^{-1} \sim \tau_k \sim \tau_*$), in agreement with the intuitive expectation that friction does not affect wavelengths of comoving size comparable or larger than τ_*.

As mentioned before, the WKB approximation is only valid for $\alpha > 1/2$. The case $\alpha \leq 1/2$ is markedly different because friction never "switches off." As a result, the amplitude of perturbations is eventually damped to zero for all wavelengths. The simplest example is the flat space case ($H = 0$, $h =$ const). In this case the equation for y is that of a damped harmonic oscillator and all excitations disappear after a characteristic lifetime which is given by $\tau \sim h^{-1}$ for $k \gg h$, and by $\tau \sim 2hk^{-2}$ for $k \ll h$ (taking $a = 1$).

IV. CIRCULAR LOOPS

A circular loop can be parametrized as

$$\mathbf{x}(\tau,\xi) = R(\tau) \cdot (\cos(\xi/R_0), \sin(\xi/R_0), 0), \tag{19}$$

where R_0 is a constant and $\xi \in [0, 2\pi R_0]$. The equations of motion (4) and (5) read (see Appendix B)

$$\ddot{R} + 2a(H+h)(1-\dot{R}^2)\dot{R} + \frac{(1-\dot{R}^2)}{R} = 0, \tag{20}$$

$$\dot{\epsilon} + 2a(H+h)\dot{R}^2\epsilon = 0, \tag{21}$$

with

$$\epsilon = \frac{R}{R_0(1-\dot{R}^2)^{1/2}}. \tag{22}$$

In the case of flat space and no friction, $H = h = 0$, ϵ is a constant, so Eq. (4) is linear and one finds the well-known oscillatory solution

$$R(t) = R_0 \cos(t/R_0). \tag{23}$$

Note that, as a consequence of exact circular symmetry, this solution collapses to a point. Of course the same will happen when we include friction and expansion. In reality, when the string shrinks to a point it would form a black hole [5, 8]. We will consider this process in more detail in Sec. VI, but for now we shall ignore it, and continue the solutions beyond the singular points. The reason is that here we are interested in the energy loss of a loop due to friction. Although we consider the circular loop for simplicity, similar results would apply to nearly circular loops which do not shrink to a point.

In the following subsections we shall study the evolution of circular loops when friction and expansion are included. In the generic case $H + h \neq 0$ we cannot find exact analytic solutions, so we have to use numerical solutions and analytic approximations.

A. Loops with friction but no expansion

It is instructive to start by considering the case $H = 0$ and $h = \text{const} \neq 0$. Consider a loop initially at rest and whose initial radius R_0 is sufficiently large. It is clear that as long as $aR \gg h^{-1}$ the motion of the loop will be overdamped, with characteristic velocity

$$\dot{R} \approx (2ahR)^{-1}$$

[i.e., we neglect the $\ddot{R}$ term in (20)]. That this is a good approximation can be checked by computing $\ddot{R}$ from the previous equation and comparing it to the other terms in (21). On the contrary, for $aR \ll h^{-1}$ the damping term can be neglected and the string undergoes a relativistic collapse similar to the frictionless one. This suggests that the energy of the string at the moment of its first collapse will be independent of R_0 (for $R_0 \gg h^{-1}$), and will be roughly equal to the energy of a loop of physical size h^{-1}:

$$E \sim \mu h^{-1}.$$

This estimate will be of some importance later on, since the energy determines the size of the black hole that forms as a result of the collapse of the loop.

The estimate can be made more rigorous by using a simple scaling argument. Introducing dimensionless variables $p \equiv \dot{R}$, $u \equiv 2haR$, and $v \equiv 2haR_0\epsilon$, the differential equations (20) and (21) can be reduced to

$$\frac{dp}{du} = \left[\frac{(1-p^2)^{1/2}}{pv} + (1-p^2)\right],$$

$$\frac{dy}{du} = -pv,$$

where there is no reference to time or to h. The initial conditions $R = R_0$ and $\dot{R} = 0$ become

$$v(u_0) = u_0, \quad p(u_0) = 0,$$

where $u_0 \equiv 2haR_0$. The energy of the string at the moment of first collapse is

$$E = 2\pi\mu a R_0 \epsilon = \chi\pi\mu h^{-1}. \tag{24}$$

The coefficient $\chi \equiv v(u = 0)$ can be found by numerical integration. Although $v(u = 0)$ depends on u_0, the result rapidly saturates to a constant

$$\chi \approx 0.57$$

for u_0 larger than 1.

After the first collapse the loop will undergo a series of oscillations, losing energy to friction in each one of them. Let us estimate this energy loss. Since the behavior of energy is controlled by Eq. (21) we will be interested in the quantity $\langle\dot{R}^2\rangle$, where the angular brackets denote the temporal average between two consecutive collapses of the loop. We have

$$\begin{aligned}\langle\dot{R}^2\rangle &= -\langle R\ddot{R}\rangle = \langle(1-\dot{R}^2)[1 + 2ahR\dot{R}]\rangle \\ &= 1 - \langle\dot{R}^2\rangle - 2ah\langle R\dot{R}^3\rangle,\end{aligned}$$

or $\langle\dot{R}^2\rangle = (1 - 2ah\langle R\dot{R}^3\rangle)/2$, where in the first step we have integrated by parts and in the second we have used the equation of motion (20). Repeating similar steps for the calculation of $\langle R\dot{R}^3\rangle$ one can generate a perturbative expansion in powers of ahR. The first terms are

$$\langle\dot{R}^2\rangle = \frac{1}{2} - \frac{1}{8}a^2h^2\langle R^2\rangle + O(a^4h^4R^4).$$

From (24) it is clear that after the first collapse $ahR \lesssim 0.3$, so the second term is a very small correction (at most of order 10^{-2}) to the first and we have

$$\langle\dot{R}^2\rangle \approx 1/2.$$

From (21) the fraction of the energy lost in between two consecutive collapses is given by $\Delta E/E \approx -2\langle\dot{R}^2\rangle hT$, where T is the time spent in the oscillation. Therefore $E \propto \exp(-ht)$ and the loop will disappear in a characteristic lifetime of order $\sim h^{-1}$ after the first collapse.

In this subsection we have ignored the expansion of the Universe. This would have the effect of providing a time dependent h. As we shall see in the next subsection, by the time the loop initiates the relativistic collapse, the expansion rate of the Universe is slow compared with the period of oscillation of the loop, so some of the results presented here will be useful in understanding the general case.

B. Loops with friction and expansion

The case *without* friction in an expanding universe has been previously studied in the literature [1]. Loops are conformally stretched by the expansion until they cross the horizon. This happens at time t_c defined by

$$H^{-1}(t_c) = a(t_c)R_0 \equiv r_c.$$

Here r_c is the physical radius of the loop at horizon crossing. After that, the loop oscillates with constant physical amplitude. This behavior is illustrated in Fig. 1(a), which corresponds to the numerical solution of (20) with $h = 0$ and for a radiation-dominated universe ($\alpha = 1$). The result is plotted in terms of dimensionless cosmological time t/t_c. The upper line represents E/E_c, where E is the energy of the loop and $E_c = 2\pi\mu r_c$. In the same figure, we plot r/r_c, where $r = aR$ is the physical radius, and $\dot{R}$ the velocity of the string with respect to the cosmological fluid.

With friction included, for loops such that $r_c \gg t_*$, the evolution is not much different from the one just described. The reason is that by the time these loops cross the horizon and start oscillating, we already have $h \ll H$.

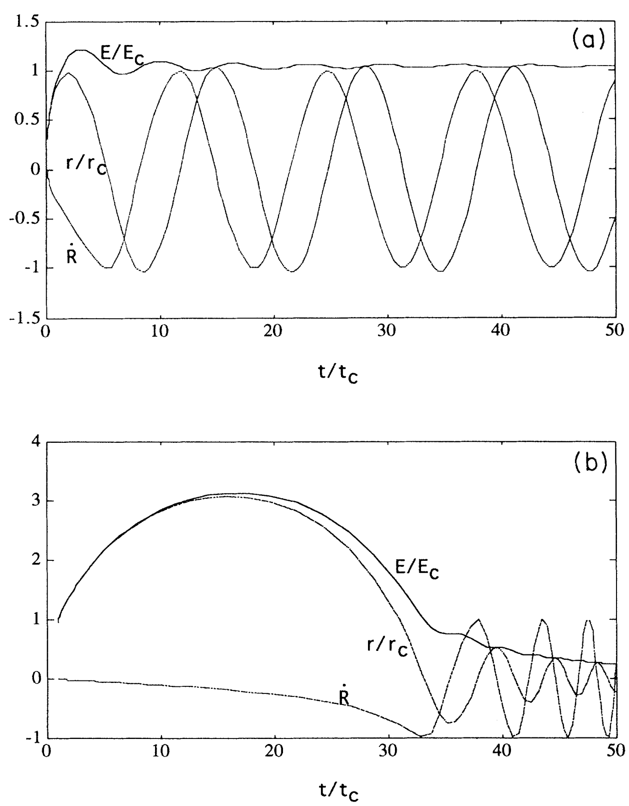

FIG. 1. (a) Evolution of a circular loop without friction in a radiation-dominated universe, as a function of cosmological time t. Here t_c is the time at which the loop crosses the horizon, r is the physical radius of the loop, and r_c is the radius at horizon crossing. The energy of the loop E rapidly approaches the value $E_c \equiv 2\pi\mu r_c$ after the loop crosses the horizon, and remains approximately constant. We also plot $\dot{R}$, the velocity of the string with respect to the cosmological fluid. (b) Evolution of a circular loop with friction in a radiation-dominated universe, with $r_c = 10^{-3}t_*$. [Same conventions as in (a).]

The effect will be important for loops that cross the horizon well before t_* (i.e., $r_c \ll t_*$). In Fig. 1(b) we plot the time evolution of a loop with $r_c = 10^{-3}t_*$, in a radiation-dominated era [same conventions as in Fig. 1(a)]. Not surprisingly, friction delays the time at which the loop first collapses. Initially, this increases the energy of the loop, since it is stretched up to a radius much larger than t_c. Later, as the loop shrinks, it loses energy to friction. It is interesting to observe that both effects roughly compensate each other, in the sense that at the time of first collapse t_f we have $E(t_f) \approx E_c$, just like in the frictionless case. After t_f the loop keeps losing energy during each oscillation, up to a time t_*, when friction switches off. From that time on, the loop oscillates with constant physical amplitude.

It is possible to give an approximate analytical description of this evolution. At early times the motion of the loop is overdamped. Neglecting the $\ddot{R}$ term in (20) we have

$$R\dot{R} \approx \frac{-1}{2a(H+h)}. \tag{25}$$

Before t_*, H can be neglected and we can integrate (25) to find

$$R^2 = R_0^2 - \frac{1}{\alpha(2\alpha+1)}\frac{\tau^{2\alpha+1}}{\tau_*^{2\alpha-1}}. \tag{26}$$

The loop will first collapse at a value of conformal time which is of order

$$\tau \sim \tau_f \equiv [\alpha(2\alpha+1)\tau_*^{2\alpha-1}R_0^2]^{1/(2\alpha+1)}. \tag{27}$$

Actually, Eq. (26) is only valid as long as $\dot{R} \ll 1$. After that, the $\ddot{R}$ term in (20) becomes comparable to the others. Since $R_0 \ll \tau_f$, the effective friction coefficient ah will not appreciably change during the relativistic collapse (which occurs at a conformal time close to τ_f). From (24), the energy of the loop at the moment of first collapse is $E_f =\approx 0.57\pi\mu/h(\tau_f)$. This has to be compared to $E_c = 2\pi\mu r_c \approx 2\pi\mu R_0 a(R_0)$. Using (27) for τ_f we have

$$\frac{E_f}{E_c} \approx \left(\frac{\tau_*}{R_0}\right)^{\frac{2\alpha^2-3\alpha+1}{2\alpha+1}}.$$

In the radiation era ($\alpha = 1$) we have $E_f \approx E_c$ independently of R_0, in good agreement with the numerical results. In the matter-dominated era, for $R_0 \ll \tau_*$ we have $E_f \gg E_c$, so by the time τ_f loops have actually more energy than they would have had in the absence of friction.

After t_f, the radius of the loop is much smaller than the horizon and the quantity $2a(H+h)$ is slowly varying compared with the period of oscillation. From the arguments of the previous subsection we have $\langle \dot{R}^2 \rangle \approx 1/2$, and from (21) the change in ϵ during one oscillation is $\Delta\epsilon/\epsilon = -a(H+h)\Delta\tau$. Since $\Delta\epsilon \ll \epsilon$, one can take the continuous limit

$$\frac{\dot{\epsilon}}{\epsilon} \approx -\frac{\dot{a}}{a} - \frac{\alpha \tau_*^{2\alpha-1}}{\tau^{2\alpha}},$$

and since $E \propto a\epsilon$, the energy of the loop is given by

$$E \approx E_f \exp\left[-\int_{\tau_f}^{\tau} \alpha \frac{\tau_*^{2\alpha-1}}{\tau^{2\alpha}} d\tau\right]. \tag{28}$$

In the case without friction ($\tau_* = 0$) we have $E \approx$ const, in agreement with the numerical results [Fig. 1(a)]. With friction included, the energy drops until time τ_*, after which loops oscillate with constant energy.

V. PERTURBATIONS ON CIRCULAR LOOPS

Let $R(\tau)$ be a solution of (20), representing the evolution of a circular loop. A perturbed loop can be parametrized as

$$\rho = R(\tau) + y^{\rho}(\tau, \theta), \tag{29}$$

$$z = y^{z}(\tau, \theta).$$

Here (ρ, θ, z) are comoving cylindrical coordinates, in which the metric takes the form

$$ds^2 = a^2(\tau)(-d\tau^2 + d\rho^2 + \rho^2 d\theta^2 + dz^2),$$

y^{ρ} is a radial perturbation, and y^z is a perturbation transverse to the plane of the loop.

It is straightforward to write (4) in cylindrical coordinates and then substitute (29) to find the linearized equations for the perturbations. This is done in Appendix B. Decomposing y^{ρ} as a sum over modes,

$$y^{\rho} = \sum_{L=2}^{\infty} [y_L^{\rho,+} \sin(L\theta) + y_L^{\rho,-} \cos(L\theta)], \tag{30}$$

and similarly for y^z, the resulting equations are

$$\ddot{y}_L^{\rho} + 2a(H+h)(1 - 3\dot{R}^2)\dot{y}_L^{\rho} - 2\frac{\dot{R}}{R}\dot{y}_L^{\rho} + \frac{L^2 - 1}{R_0^2 \epsilon^2} y_L^{\rho} = 0, \tag{31}$$

$$\ddot{y}_L^{z} + 2a(H+h)(1 - \dot{R}^2)\dot{y}_L^{z} + \frac{L^2}{R_0^2 \epsilon^2} y_L^{z} = 0, \tag{32}$$

where we have omitted the index plus or minus. The sum in (30) starts at $L = 2$ rather than at $L = 0$ because it is easy to see that to linear order the $L = 0$ and $L = 1$ modes do not correspond to deformations from the circular shape but rather to small translations and rotations of a circular loop [9].

Since the function $R(\tau)$ is not known analytically, Eqs. (31) and (32) have to be solved numerically. However, before we do that, the behavior of the perturbations can be guessed from the results of the previous sections.

Let us first consider the case without friction. Up to the time t_c the loop is conformally stretched by the expansion and the perturbations will approximately behave as perturbations on a straight string. Ignoring oscillatory cosine factors, the amplitude of the perturbations at this time will be $y_L(t_c) \approx y_{0,L} a(R_0/L)/a(\tau)$, where $y_{0,L}$ is the initial perturbation. Here we have used (11) making the substitution $k \to L/R_0$. After t_c, the loop comes within the cosmological horizon and starts collapsing. A loop on subhorizon scales behaves approximately as it would behave in flat space. The theory of perturbations on a circular loop collapsing in flat space was solved in [9] (see Appendix A of that reference). It was shown that the amplitude of transverse perturbations stays constant during the collapse, whereas the amplitude of radial perturbations shrinks by a factor of L as the loop shrinks to $r \ll r_c$. Therefore by that time we have

$$y_{\mathrm{phys},L}^{\rho} \equiv a(\tau) y_L^{\rho} \approx y_{0,L}^{\rho} \frac{a(R_0/L)}{L}, \tag{33}$$

$$y_{\mathrm{phys},L}^{z} \equiv a(\tau) y_L^{z} \approx y_{0,L}^{z}\ a(R_0/L). \tag{34}$$

These approximate expressions for the evolution of the perturbations in the absence of friction had already been used in Ref. [9]. Since their numerical accuracy had not been checked there, we shall do this now.

To check the validity of these approximations one has to numerically solve Eqs. (31) and (32), with $h = 0$. We have done that for different values of L, in the radiation-dominated universe and using the boundary condition that perturbations are initially at rest. The result is plotted in Fig. 2 as a function of wave number L. The circles denote the ratio $y_{\mathrm{phys},L}^{\rho}$ divided by the right-hand side (RHS) of Eq. (33) at the time when the loop first collapses. Similarly, the crosses denote the ratio of $y_{\mathrm{phys},L}^{z}$ to the RHS of (34). Ignoring the "valleys" in Fig. 2, these ratios are of order 1, which means that Eqs. (33) and (34) give a very good estimate. The valleys in Fig. 2 are due to oscillatory behavior of the perturbations, which we have ignored in our argument.

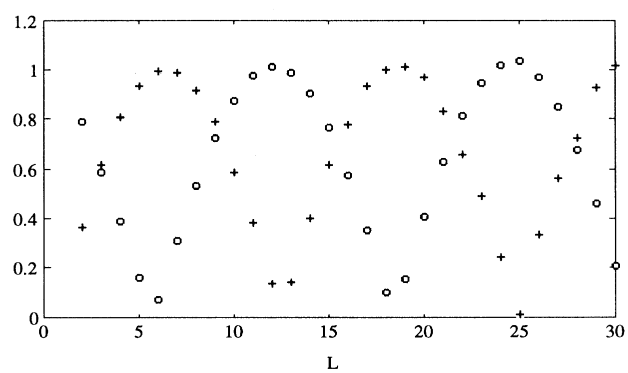

FIG. 2. Numerical results for the evolution of perturbations on a circular loop in a radiation-dominated universe, ignoring the force of friction. The results are plotted as a function of wave number L. The circles denote the ratio $y_{\mathrm{phys},L}^{\rho}$ divided by the RHS of Eq. (33) at the time when the loop first collapses. Similarly the crosses denote the ratio of $y_{\mathrm{phys},L}^{z}$ to the RHS of (34). Ignoring the oscillatory behavior, these ratios are of order 1, which means that (33) and (34) give a very good estimate.

The effects of friction can be introduced along similar lines, and they will only be important for comoving wavelengths $(R_0/L) < \tau_*$. For such wavelengths the RHS of Eqs. (33) and (34) has to be corrected by a factor of

$$\frac{2}{(2\alpha-1)!!}(k\tau_k)^\alpha \exp(-\gamma k\tau_k). \quad (35)$$

This is obtained comparing (17) with (18), where now k is given by L/R_0. Again, one can check the accuracy of this approximation by solving the equations of motion for the perturbations, now with $h \neq 0$. The results for the case of a radiation-dominated universe are plotted in Fig. 3(a) for $R_0 = \tau_*$, and in Fig. 3(b) for $R_0 = 5\tau_*$. The circles and crosses denote the same quantities as in Fig. 2. It is seen that the suppression factor (35), depicted as a solid line, gives the right answer to very good approximation.

The above considerations apply only to loops with $R_0 \gtrsim \tau_*$. For $R_0 \ll \tau_*$ the perturbations do not have time to be damped before the loop starts shrinking, so the factor (35) actually overestimates the effect of friction. We shall consider this in more detail in the next section.

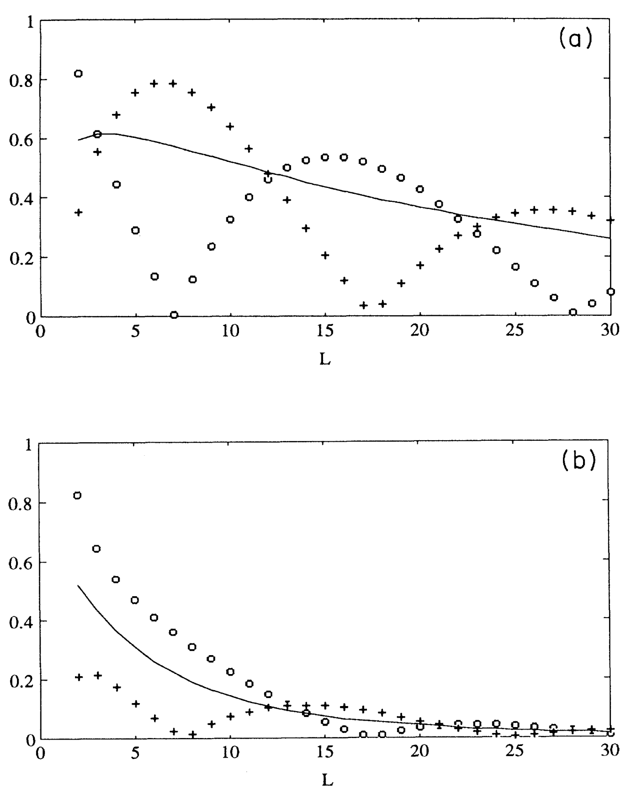

FIG. 3. (a) Numerical results for the evolution of perturbations on a circular loop in a radiation-dominated universe, including the force of friction, for $R_0 = \tau_*$. Circles and crosses denote the same quantities as in Fig. 2. It is seen that the suppression factor (35), depicted as a solid line, gives the right answer to a very good approximation. (b) Same quantities as in (a), for the case $R_0 = 5\tau_*$.

VI. BLACK HOLE FORMATION

Let r_c be the physical radius of a circular loop at horizon crossing. The mass of this loop is $2\pi r_c \mu$ and the Schwarzschild radius corresponding to this mass is

$$r_S = 4\pi G\mu r_c.$$

As the loop shrinks under its tension, its rest mass is converted into kinetic energy so that the total energy of the loop remains constant (neglecting friction and gravitational radiation for the moment). If a loop is exactly circular then it will eventually shrink to a size smaller than r_S and form a black hole [5, 6]. (This is only true for strings that form as a result of gauge symmetry breaking. Strings which form as a result of global symmetry breaking would radiate all of their energy in the form of Goldstone bosons before they shrink to the size of their Schwarzschild radius [10]).

We should mention, before proceeding with the argument, that a black hole would not form if the thickness of the core of the string is larger than the Schwarzschild radius r_S. In that case, before it shrinks to a size smaller than r_S, the string would unwind and release all of its energy in the form of quanta of the scalar field and gauge fields which constitute the string. This would make the formation of a black hole very unlikely. The thickness of the string is typically given by $\delta \sim \lambda^{-1/2}\mu^{-1/2}$, where λ is the dimensionless Higgs self-coupling. Provided that $r_S \gg \delta$, i.e., provided that the radius of the loop at horizon crossing is larger than

$$r_c \gg [4\pi(G\mu)^{3/2}\lambda^{1/2}]^{-1}t_{\rm Pl} \quad (36)$$

the effects of finite thickness can be safely ignored and black holes will form.

If the loop is not exactly circular it will still form a black hole provided that the size of the perturbations is sufficiently small. Let τ_S denote the time when the unperturbed loop would shrink to the size of its Schwarzschild radius. It is clear that if

$$|a(\tau_S)\mathbf{y}(\tau_S)| < r_S \quad (\equiv a(\tau_S)R_S) \quad (37)$$

then a black hole will still form.

From the analysis of the previous section one could argue that loops of string with $R_0 \ll \tau_*$ could easily form black holes. The argument is that since friction exponentially suppresses wiggles on scales smaller than τ_*, these loops would be circular to very good approximation. This would result in the copious production of black holes with masses up to

$$M \sim 2\pi\mu t_* \sim 2\pi A(G\mu)^{-1}m_{\rm Pl}, \quad (38)$$

where $m_{\rm Pl}$ is the Planck mass. Note that for loops such that $r_c \sim t_* \sim A(G\mu)^{-2}t_{\rm Pl}$, the inequality (36) will be satisfied provided that

$$G\mu \ll (4\pi A)^2\lambda. \quad (39)$$

From Sec. II we have $(4\pi A)^2$ in the range 10^{-3}–10^{-5} (taking $\mathcal{N}$ in the range $1 - 10^2$). Taking typical values of λ in the range $10^{-1} - 10^{-4}$, we see that the inequality (39) will hold provided that $G\mu$ is sufficiently small.

There are observational upper bounds on the density of black holes of masses $M \gtrsim 10^{14} m_{\rm Pl}$, which would be evaporating at the time of nucleosynthesis or later (see [11] and references therein). Then, from (38), one would be able to put constraints on topologically stable cosmic strings of very low tension:

$$G\mu < (2\pi A) \times 10^{-14}. \tag{40}$$

This, in turn, could be used to constrain models in which spontaneous symmetry breaking occurs below energy scales of order $(2\pi A)^{1/2} \times 10^{12}$ GeV.

However, this argument needs to be refined, since for $R_0 \ll \tau_*$ one cannot simply use the exponential suppression factor (35) to estimate the effect of friction. As a result, the loops will be fairly circular, but not to exponential accuracy (which is what we need to produce black holes). This is because these loops collapse well before τ_* and friction has not had enough time to damp the perturbations. Therefore, it is important to study the behavior of perturbations on loops with $R_0 \ll \tau_*$ in more detail.

At very early times, both the loop and the perturbation will be overdamped. Neglecting second derivatives in (32) we have

$$\frac{\dot{y}^z}{y^z} = -\frac{L^2}{R^2}\frac{1}{2a(H+h)}. \tag{41}$$

Using (25) we have the interesting relation

$$y^z(t) = y^z(t_0)\left(\frac{R(t)}{R_0}\right)^{L^2}, \tag{42}$$

so the perturbations shrink faster than the radius of the loop, the relative perturbation decreasing as the coordinate radius shrinks from its initial value R_0. Following similar steps we also have

$$y^\rho(t) = y^\rho(t_0)\left(\frac{R}{R_0}\right)^{L^2-1}. \tag{43}$$

To find the limit of applicability of the overdamped approximation, one can calculate $\ddot{R}$ and $\ddot{y}^z$ from (25) and (41) and compare them to the damping term in Eqs. (20) and (32). One readily finds that (25) is valid for $\dot{R} \ll 1$, while (42) is valid for $\dot{R} \ll L^{-1}$. The time at which $\dot{R} \sim L^{-1}$ coincides also with the time at which the relevant physical scale comes within the effective horizon h^{-1} and starts oscillating.

Once a perturbation starts oscillating it is damped very efficiently, so the perturbations which will be more difficult to eliminate are those with $L = 2$, which are overdamped up to the time when the loop becomes relativistic, $\dot{R} \sim 1$. From (25) this happens when $R \sim [ah]^{-1}$, at a time of order τ_f, given by (27). That is, the loop becomes relativistic when

$$\frac{R}{R_0} \sim \frac{1}{a(\tau_f)h(\tau_f)R_0} \sim \left(\frac{R_0}{\tau_*}\right)^{1/3}, \tag{44}$$

where in the last step we have restricted attention to loops which are collapsing in the radiation era. After that, during the relativistic collapse, the loop is within the effective horizon and it can be seen that perturbations behave very much like they would on a circular loop in flat space. That is, as the loop shrinks the amplitude of transverse perturbations stays constant whereas radial perturbations only shrink by a factor of L [9].

As a result, all the suppression in the lowest modes $L = 2$ comes from the overdamped regime. From (43) with (R/R_0) given by (44), and using that $r_S \sim G\mu h^{-1}(\tau_f)$ we obtain that at the time t_S when the unperturbed loop crosses its Schwarzschild radius

$$\frac{y_2^\rho(t_S)}{R_S} \sim \frac{y_2^\rho(t_0)}{R_0}(G\mu)^{-1}\left(\frac{R_0}{\tau_*}\right)^{2/3} \sim \frac{y_2^\rho(t_0)}{R_0}(G\mu)^{-1}\left(\frac{r_c}{t_*}\right)^{1/3}. \tag{45}$$

For strings formed at a phase transition, the initial value of the relative perturbation $y_2^\rho(t_0)/R_0$ can be of order 1. Then, in order to satisfy (37) we need

$$r_c \lesssim (G\mu)^3 t_*.$$

However, if this condition is met, the energy of the loop at the moment of first collapse is $E \sim \mu r_c \lesssim (G\mu)^2 m_{Pl}$, much smaller than the Planck mass, which simply means that a black hole will not form.

Therefore friction by itself is not sufficient to ensure the formation of a black hole if we start from an arbitrarily wiggly loop: the lowest modes $L = 2$ are not sufficiently damped. This was not obvious *a priori* and it is in contrast with what would happen if the Universe were not expanding. In that case h is a constant and from (42) and (43) all loops whose initial size aR_0 is much larger than $(G\mu)^{-1/2}h^{-1}$ would shrink to form black holes of mass $M \sim \mu h^{-1}$.

Even in an expanding universe, some of the loops produced at the phase transition might just happen to be circular enough initially that they would form black holes, even without friction. With friction included, the number of loops that will form black holes is larger. The question is then what fraction of the ensemble will go into black holes or, in other words, what is the probability of black hole formation $\mathcal{P}_{\rm BH}$.

Before we try to answer this question we should know how $\mathcal{P}_{\rm BH}$ is constrained from cosmological observations. Assuming a scale invariant distribution of loops with number density at horizon crossing given by [1] $dn(r_c) = \nu r_c^{-4} dr_c$, where ν is a parameter of order 1 or smaller, and given that these loops form black holes of mass $M = 2\pi\mu r_c$ with probability $\mathcal{P}_{\rm BH}$, the number density of black holes is

$$dn(M) = m_{\rm Pl}^3 \nu (2\pi G\mu)^{3/2} \mathcal{P}_{\rm BH} \left(\frac{m_{\rm Pl}}{M}\right)^{5/2}\left(\frac{t_{\rm Pl}}{t}\right)^{3/2} \times d\left(\frac{M}{m_{\rm Pl}}\right). \tag{46}$$

Here we have included a factor of $(r_c/t)^{3/2}$ in the distribution to account for the dilution of the strings (or black holes) by the expansion of the Universe. Black holes of mass $M \sim 10^{10} - 10^{11}$ g would evaporate during cosmological nucleosynthesis, producing high-energy particles which would deplete the deuterium and helium when standard nucleosynthesis has almost concluded. This process has been studied in Ref. [12], using a distribution of black holes of the form (46) with the numerical coefficient left as a free parameter. The bound that these authors obtained can be translated into

$$(2\pi)^{3/2}\nu(G\mu)^{3/2}\mathcal{P}_{\rm BH} < 10^{-26} \tag{47}$$

(assuming that the present abundance of deuterium is of cosmological origin), which results in a constraint for $\mathcal{P}_{\rm BH}$ for certain values of the string tension.

Note that if $\mathcal{P}_{\rm BH} \sim 1$, then we would be in trouble for $G\mu \gtrsim 10^{-18}$. It is then important to estimate $\mathcal{P}_{\rm BH}$. An upper bound can be obtained directly from Eq. (45). The probability that $y_2^\rho(t_S) < R_S$ is equal to the probability that

$$y_2^\rho(t_0) < (G\mu)(t_*/r_c)^{1/3}R_0.$$

Treating $y_2^\rho(t_0)$ as a random variable normally distributed, with rms amplitude of order R_0, and noting that for $L = 2$ there are two independent modes [the plus and the minus modes in Eq. (30)], that probability is bounded by

$$\mathcal{P}_{\rm BH} < \left[G\mu\left(\frac{t_*}{r_c}\right)^{1/3}\right]^2 \sim (G\mu)^{4/3}\left[\frac{m_{\rm Pl}}{M}\right]^{2/3} < 10^{-9}(G\mu)^{4/3}, \tag{48}$$

where in the last step we have used $M \gtrsim 10^{14}\ m_{\rm Pl}$. As a result, the constraint (47) will always be satisfied for cosmic strings in the low-energy range (40) in which we were interested.

What can one say about heavier strings? Note that $\mathcal{P}_{\rm BH}$ can actually be much lower than the RHS of (48), and the bound (47) is likely to be satisfied even for large values of the string tension $G\mu \sim 10^{-6}$. In this case the loops that form black holes in the mass range 10^{14}–$10^{20}\ m_{\rm Pl}$ cross the horizon much later than t_*, so friction only smoothes out the perturbations of very large L. This does not help very much when we try to form black holes. The smoothness of the strings in this case has to be attributed to other damping mechanisms such as gravitational radiation. The probability of black hole formation in the absence of friction has been estimated by Hawking [8] and others [13, 8],

$$\mathcal{P}_{\rm BH} \sim (G\mu)^p. \tag{49}$$

In the case of a loop formed by n straight segments, Hawking found $p \sim 2n$. This form for $\mathcal{P}_{\rm BH}$ results from the fact that in order to form a black hole it is necessary to "fine-tune" a set of $2n$ angles with accuracy given by $G\mu$. Typically, p will be of the order of the number of random variables that parametrize the ensemble of loops, since for a black hole to form all the parameters have to be fine-tuned with an accuracy given by $G\mu$. Polnarev and Zembowicz [13] studied $\mathcal{P}_{\rm BH}$ for a family of loops containing excitations in the first and third harmonics in a Fourier expansion of the solutions to the Nambu equations of motion. They studied a two parameter family and they found $p \approx 2$ (with some uncertainty due to the arbitrariness in the definition of a probability distribution in the space of parameters). However, p is likely to be larger (and therefore $\mathcal{P}_{bh}$ smaller) since one need not be restricted to a two parameter family. In particular, the general solution including the first and third harmonics is a five parameter family (see, e.g., [14]), and one may expect $p \sim 5$.

Other observational constraints come from black holes of mass $M \sim 10^{20}\ m_{\rm Pl}$, which would be evaporating at the present time, producing intense bursts of γ rays (see, e.g., [15]). This constraint has been studied in [13], whose authors concluded that if $p \lesssim 2$ there is conflict with observations for strings heavier than $G\mu \gtrsim 10^{-7}$. On the other hand, if $p > 4$ there is no conflict even if $G\mu$ is as large as 10^{-5}. Including friction does not modify these results, since for $M \sim 10^{20}\ m_{\rm Pl}$ friction is important only for $G\mu \lesssim 10^{-20}$ [see Eq. (38)]. But for such low values of the tension there is no observable effect even if $\mathcal{P}_{\rm BH} = 1$ [13].

VII. STRINGS ATTACHED TO WALLS

In this section we consider perturbations on strings which are attached to planar domain walls [1]. For simplicity, we shall ignore the expansion of the Universe. Also, we shall restrict ourselves to perturbations which lie in the plane of the wall.

Then we are effectively left with a (2+1)-dimensional problem, in which the string is the boundary between a region of "false vacuum" (the wall) with energy per unit area equal to the surface tension σ, and a region of "true vacuum" where there is no wall. In the absence of friction, the equations of motion are [16]

$$-\mu\Box x^\mu = \sigma n^\mu. \tag{50}$$

Here $\Box$ is the covariant d'Alembertian on the world sheet of the string and n^μ is the spacelike unit vector normal to the world sheet, with $n^\mu n_\mu = 1$ and $n_\mu\partial_a x^\mu = 0$. Our sign convention is that n^μ points towards the wall. It is easy to see that Eq. (50) has a solution representing a straight string which is constantly accelerating due to the tension of the wall. The wall gradually disappears as the string moves forward, its energy going into kinetic energy of the string [16]. The string undergoes the so-called hyperbolic motion, asymptotically approaching the speed of light at late times.

Equation (50) has to be modified to include the force of friction F^ν. To keep things general we take

$$F^\nu = F(u_\perp)n^\nu, \tag{51}$$

without specifying the dependence in the transverse velocity of the fluid $u_\perp = u^\mu n_\mu$. (Note that $u_\perp^\nu$ defined in Sec. II is given by $u_\perp^\nu = u_\perp n^\nu$.) Then the equations of motion read

$$-\mu\Box x^\nu = [\sigma + F(u_\perp)]n^\nu. \tag{52}$$

This is the analogue of Eq. (1). Since now we are ignoring the expansion of the Universe there are no Christoffel symbols in the equation corresponding to the curvature of the embedding spacetime. The effect of the wall is to supply the constant force σn^ν. Also, since we have restricted ourselves to planar motion of the string, the frictional force lies on the plane of the wall, along the normal to the string.

We can study perturbations on any solution $x^\mu(\xi^a)$ of (52) using the covariant formalism of Ref. [16], suitably modified to include friction. Since only perturbations which are normal to the string are physically observable, one need only consider perturbations of the form

$$\delta x^\mu = \phi(\xi) n^\mu.$$

Here ϕ represents the proper magnitude of the perturbation, i.e., the normal displacement as measured by an observer that is moving with the string. Multiplying (52) by n_ν we can write

$$-\mu K^a_a = (\sigma + F), \tag{53}$$

where $K_{ab} = n_\mu \nabla_a \partial_b x^\mu$ is the extrinsic curvature of the world sheet. Latin indices are raised and lowered using the world sheet metric

$$g_{ab} \equiv \partial_a x^\mu \partial_b x_\mu.$$

The linearized equation of motion for the perturbations can be found from variation of Eq. (53)

$$\mu \delta K^a_a = -\delta F,$$

where δ denotes the variation induced, to linear order in ϕ, by the small perturbation δx^μ. Following [16], we have $\delta K^a_a = \Box\phi + K^{ab}K_{ab}\phi$. On the other hand, $\delta F = F'(u_\perp)\delta u_\perp$, where $F' = dF/du_\perp$. Also, $\delta u_\perp = u_\mu \delta n^\mu$, and the change in n^μ induced by the perturbation is [16] $\delta n^\mu = -g^{ab}\phi_{,b}\,\partial_a x^\mu$. Therefore, the resulting equation of motion for ϕ is

$$-\Box\phi + M^2\phi = -\frac{F'}{\mu} u_\mu \partial_a x^\mu \phi,^a \ , \tag{54}$$

where the "mass" is given by

$$M^2 = -K_{ab}K^{ab} = \left[\mathcal{R} - \left(\frac{\sigma + F}{\mu}\right)^2\right]. \tag{55}$$

In the last step we have used the Gauss-Codazzi relation $K_{ab}K^{ab} = (K^a_a)^2 - \mathcal{R}$, where $\mathcal{R}$ is the intrinsic curvature scalar on the world sheet. The RHS of Eq. (54) acts like a friction term for the perturbation ϕ provided that $F' > 0$. But this condition is always met, since it just means that the force of friction increases with the transverse velocity of the fluid.

In the spirit of Refs. [16,17], Eq.(54) can be seen as the equation of motion for a scalar field ϕ which is "living" in the world sheet of the defect. Now this equation has a different mass than in the case without friction, and it also has a right-hand side in which the field has a derivative coupling to some external sources (which are essentially the "temporal" components of the tangent vectors to the string).

In the case when the unperturbed string is straight, lying along the y axis, and moving with trajectory $x = x(t)$, the metric on the world sheet is

$$\begin{aligned} ds^2 = g_{ab} d\xi^a d\xi^b &= -(1 - \dot{x}^2)dt^2 + dy^2 \\ &= -d\tau^2 + dy^2. \end{aligned}$$

Here t is the time in the rest frame of the fluid and τ is the proper time of an observer who is moving with the string. In this case the metric on the world sheet is flat, the curvature scalar vanishes and the effective mass M^2 is tachyonic. This is also true in the absence of friction [16], and it essentially means that modes with wavelength larger than $|M|^{-1}$ are unstable. Now, including friction, the difference is that the tachyonic mass "switches off" as the string approaches its terminal velocity v. Indeed, the terminal velocity of the string is determined by the vanishing of the driving force in (52), that is

$$\sigma + F(u_\perp) = 0, \tag{56}$$

but this implies that the mass term for the perturbations vanishes.

When the straight string has reached the terminal velocity v, Eq. (54) reduces to

$$\frac{d^2\phi}{d\tau^2} + k^2\phi = -\frac{F'\gamma_v}{\mu}\frac{d\phi}{d\tau}, \tag{57}$$

where k is the wave number of the perturbation, and $\gamma_v = (1 - v^2)^{-1/2}$ is the relativistic factor corresponding to the terminal velocity. The friction term in the RHS of (57) now causes the perturbations to decay exponentially with a lifetime which is easily calculable. Taking a force of the form (2) we have $F' = \beta T^3$. Noting that $u_\perp = n^\mu u_\mu = -v\gamma_v$, Eq. (56) gives

$$\gamma_v v = \frac{\sigma}{\beta T^3}.$$

In cosmological situations the temperature is always lower than the energy scale of the wall (typically of order $\sigma^{1/3}$); therefore, the terminal velocity will be relativistic,

$$v \sim 1,$$

and the γ factor will be of order $\gamma_v \sim \sigma T^{-3}$ (ignoring the numerical factor β). Then the RHS of (57) reads

$$\frac{F'\gamma_v}{\mu} \sim \frac{\sigma}{\mu}.$$

It is then straightforward to see that the perturbations decay with proper lifetime given by $\tau \sim \mu/\sigma$ for $k \gg \sigma/\mu$ and $\tau \sim \sigma\mu^{-1}k^{-2}$ for $k \ll \sigma/\mu$.

A more interesting case is that of a string which is at the boundary of a circular hole that has spontaneously nucleated on a metastable domain wall (see, e.g., Ref. [18]). This process is the (2+1)-dimensional analogue of the formation of true vacuum bubbles in the problem of false vacuum decay.

Without friction, perturbations on these bubbles (or holes) have been studied in Ref. [16]. In that case, the unperturbed solution was a circular hole whose radius R

expands with constant acceleration:

$$R^2 - t^2 = (2\mu/\sigma)^2. \tag{58}$$

The effective mass for the perturbations was $M^2 = -\sigma/2\mu^2$, which is tachyonic. Because of the expansion of the hole, any perturbation eventually reaches a wavelength larger than M^{-1}, at which point it becomes unstable and starts growing like $\phi \propto R \approx t$ [16]. From an intrinsic point of view, one can say that the string is unstable, in the sense that ϕ (the perturbation measured by a comoving observer) grows in time. However, an external observer measures a perturbation which is Lorentz contracted [16] $\Delta = \gamma^{-1}\phi$, where here $\gamma = (1-\dot{R}^2)^{-1/2}$. Since from (58) $\gamma \sim R$, we have $\Delta \sim$ const at late times. The relative perturbation Δ/R decreases and the string becomes more circular as the hole expands [16].

Including friction, the main difference will be that the string reaches a terminal velocity v, and the Lorentz contraction factor will go to a constant at late times. Also, the mass term switches off as the string approaches the terminal velocity. Let us see, then, what is the fate of the perturbations when friction is included in the dynamics. The string world sheet is given by $x^\mu = (t, R\cos\theta, R\sin\theta, 0)$, and the metric induced on the world sheet is

$$ds^2 = -(1-\dot{R}^2)dt^2 + R^2 d\theta^2 = -d\tau^2 + R^2 d\theta^2.$$

The normal vector is

$$n_\mu = (1-\dot{R}^2)^{-1/2}(-\dot{R}, \cos\theta, \sin\theta, 0).$$

The extrinsic curvature can be easily calculated and one finds

$$-M^2 = K_{ab}K^{ab} = \gamma^2 R^{-2} + \gamma^6 (\ddot{R})^2. \tag{59}$$

The velocity is bounded by the terminal velocity $\dot{R} < v$, so at late times $R \approx vt$ and $\ddot{R}$ decays faster than R^{-1}. Then, neglecting the $\ddot{R}$ term in (59) the equation of motion for the perturbations takes the form

$$\frac{d^2\phi_L}{d\tau^2} + \frac{F'\gamma_v}{\mu}\frac{d\phi_L}{d\tau} + \frac{L^2 - \gamma_v^2}{R^2}\phi_L = 0, \tag{60}$$

with $\phi = \phi_L(\tau)\exp(iL\theta)$.

Note that as the hole expands, the tachyonic mass squared $-\gamma_v^2 R^{-2}$ "redshifts" at the same rate as the wave number term $L^2 R^{-2}$. One might expect instability for modes with $L < \gamma_v$, since for these the total effective mass squared is negative. However, at sufficiently late times we can use the overdamped approximation and neglect $\ddot{\phi}$ in (60). Then a simple integration shows that, since R^{-2} decreases faster than τ^{-1}, the amplitude of the perturbations asymptotically approaches a constant $\phi \to$ const. From an intrinsic point of view the asymptotic behavior of the perturbations is very different from the frictionless case, in which ϕ was growing in time. However, now γ_v is constant and we have $\Delta = \gamma_v^{-1}\phi \to$ const, so from the point of view of an external observer which is at rest the behavior is similar to that of the frictionless case. The relative perturbation Δ/R also goes to zero, and the loop becomes more and more circular as the hole expands.

VIII. CONCLUSIONS

In this paper we have studied the evolution of cosmic strings taking into account the force of friction. The results are conveniently expressed in terms of t_*, defined as the time at which friction "switches off" [Eqs. (7) and (8)].

For small perturbations around a straight string, a WKB analysis shows that they are exponentially suppressed if their wavelength crosses the horizon before the time t_*. Relative to the frictionless case, the amplitude of the perturbations has to be multiplied by the suppression factor (35), which in the case of a radiation-dominated universe ($\alpha = 1$) reduces to

$$4\left(\frac{\pi t_*}{\lambda}\right)^{1/2} \exp\left[-2\gamma\left(\frac{\pi t_*}{\lambda}\right)^{1/2}\right]. \tag{61}$$

Here λ is the physical wavelength of the perturbation at the time t_* and $\gamma \approx 1.2$. The suppression of the amplitude of these perturbations is due to the oscillation of the perturbations before the time t_*. After t_* the perturbations oscillate with constant physical amplitude. Perturbations whose wavelength crosses the horizon after t_* are practically unaffected by friction.

Similarly, for circular loops we should distinguish between large loops with $r_c \gg t_*$, and small loops with $r_c \ll t_*$. Here r_c is the radius of the loop at horizon crossing. Large loops are unaffected by friction, after they cross the horizon they start oscillating with constant physical amplitude r_c. Small loops do *not* start collapsing relativistically right after they cross the horizon. During the nonrelativistic evolution, the velocity at which the string moves with respect to the fluid is given by

$$\dot{R} \sim \frac{\mu}{\beta T^3 r},$$

where r is the physical radius of the loop, T is the temperature, and β is the numerical factor in (2). Because of the frictional force, these loops are dragged by the expansion more than they would in the absence of friction, growing to a size larger than r_c. However as they collapse they lose energy to friction. It is interesting to note that in a radiation-dominated universe both effects approximately compensate each other, in the sense that at the time when they first collapse to a point their energy is given by $E_f \approx 2\pi\mu r_c$, just as in the frictionless case (in a matter-dominated universe E_f is actually larger than $2\pi\mu r_c$). In subsequent oscillations, the loops will keep losing energy, up to the time t_*. After that they oscillate with constant energy. This energy can be obtained from (28) in the limit of large times, and one has $E(t \gg t_*) \approx E_f \exp(-2t_*/3r_c)^{1/3}$.

We have also studied perturbations on circular loops. If the loops are larger than t_*, then the perturbations behave in much the same way as the perturbations on a straight infinite string. Perturbations whose wavelength crosses the horizon before t_* are suppressed according to (61), whereas if it crosses later than t_* they are unaffected by friction. If the loops are much smaller than t_* then it is not correct to use the suppression factor (61), essentially because the loops start their relativistic collapse much before t_*. As a result, the perturbations in the lowest modes are only suppressed as a power law in (r_c/t_*) [see, e.g., (45)].

Because the suppression in the relative amplitude of the lowest modes is only power law, the probability that loops of strings become circular enough to form black holes is very small (to form a black hole one needs the relative amplitude of the perturbations be of order $G\mu$, which is very small). As a result, the number of black holes of masses $M \gtrsim 10^{14}\, m_{\rm Pl}$ (which would be evaporating at the time of nucleosynthesis or later) formed by strings which cross the horizon before t_* is too small to have any observable consequences (for all values of μ). This conclusion is reached even if we ignore the finite thickness of the core of the string. As discussed in Sec. VI, if this thickness is comparable or larger than the Schwarzschild radius of the loop, the formation of a black hole is even more unlikely, so our conclusion is reinforced. For loops which cross the horizon later than t_*, friction only eliminates the perturbations of very large wave number, so the probability that they form black holes is not substantially enhanced by friction.

We have also studied perturbations on strings attached to walls. For this we have used the covariant formalism developed in Ref. [16], generalizing it to include the force of friction. In particular we have studied the case of a circular hole which spontaneously nucleates on a metastable domain wall. We find that perturbations on the string that is at the boundary of the hole are initially unstable, growing at the same rate as the radius of the expanding hole. However, as the loop reaches its terminal velocity $v < 1$, the instability switches off and the perturbations freeze out at a constant amplitude. As a result, the relative perturbation decreases in time, and the hole becomes increasingly circular as it expands. This behavior is similar to the behavior that one obtains in the absence of friction [16]. However the mechanism by which perturbations freeze out is very different. In the former case it is due to the force of friction, which balances the instability due to the tension of the wall. In the latter case the freezing occurred for purely kinematical reasons, since the unperturbed string asymptotically approached the speed of light.

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APPENDIX A

Let us solve Eq. (13) in the WKB approximation. The solution "under the barrier" is given by

$$\Psi = \sum_{\pm} \frac{C_\pm}{\sqrt{p_1}} \exp\left[\pm \int_\tau^{\tau_k} d\tau' p_1(\tau')\right] \quad (\tau \ll \tau_k), \tag{A1}$$

where $p_1 \equiv [V(\tau) - k^2]^{1/2}$ and τ_k is the "classical" turning point:

$$V(\tau_k) = k^2.$$

Since at early times $V(\tau) \sim \alpha^2 \tau_*^{-2} (\tau_*/\tau)^{4\alpha}$, it is clear that

$$I \equiv \int_\tau^{\tau_k} d\tau' p_1(\tau') \to \frac{\alpha}{2\alpha - 1}\left(\frac{\tau_*}{\tau}\right)^{2\alpha-1} - \Gamma(k, \tau_k) + O(\tau^{2\alpha-1}), \tag{A2}$$

where the function $\Gamma(k, \tau_k)$, which will be specified below, is independent of τ.

Since perturbations are frozen in at early times, the coefficients $C_\pm$ are determined by imposing the boundary condition that the comoving perturbation y should approach some constant value y_0 as $\tau \to 0$. This immediately requires

$$C_+ = 0,$$

$$C_- = y_0 \sqrt{\alpha \tau_*^{2\alpha-1}}\, e^{-\Gamma(k,\tau_k)}.$$

The vanishing of C_+ is necessary for regularity, and the value of C_- gives the right normalization at early times.

Using the standard connection formulas of the WKB formalism, the solution in the "classically allowed" region is

$$\Psi = 2y_0 \left(\frac{\alpha \tau_*^{2\alpha-1}}{p_2}\right)^{1/2} e^{-\Gamma(k,\tau_k)} \cos\left(\int_{\tau_k}^{\tau} d\tau' p_2(\tau') - \frac{\pi}{4}\right) \quad (\tau \gg \tau_k), \tag{A3}$$

where $p_2 = [k^2 - V]^{1/2}$.

The simplest case, which is also the most interesting, is the radiation-dominated universe $\alpha = 1$. In this case the potential assumes a particularly simple form $V = \tau_*^2 \tau^{-4}$, and the turning point is given by $\tau_k = (\tau_* k^{-1})^{1/2}$. Then, the exponent $\Gamma(k, \tau_k)$ can be calculated analytically:

$$\Gamma(k, \tau_k) = -\lim_{\tau \to 0}\left[\int_\tau^{\tau_k} d\tau \left(\frac{\tau_*^2}{\tau^4} - k^2\right)^{1/2} d\tau - \left(\frac{\tau_*}{\tau}\right)\right]. \tag{A4}$$

Integrating by parts we have

$$\Gamma = k\tau_k \lim_{x \to 0} 2 \int_x^1 \frac{dx}{(x^{-4} - 1)^{1/2}} = k\tau_k \tfrac{1}{2} B(\tfrac{3}{2}, \tfrac{1}{2}), \tag{A5}$$

where B is Euler's β function.

For $\alpha \neq 1$, the potential is more complicated and the turning points cannot be given explicitly in terms of τ_* and k. However, for $k \gg \tau_*^{-1}$, the potential under the barrier can be approximated by the second term and we have

$$\tau_k \approx [\alpha k^{-1}\tau_*^{2\alpha-1}]^{1/2\alpha}. \tag{A6}$$

Neglecting the first term in $V(\tau)$ and following the same steps as before we have

$$\Gamma = k\tau_k \frac{1}{4\alpha - 2} B\left(\frac{1+2\alpha}{4\alpha}, \frac{1}{2}\right).$$

Substituting in (A1) and (A3) and using (12) we have the solution (15).

APPENDIX B

Here we express the equations of motion for a string with friction in cylindrical coordinates, and derive the equations for perturbations on a circular loop.

The FRW metric is written as

$$ds^2 = a^2(\tau)(-d\tau^2 + d\rho^2 + \rho^2 d\theta^2 + dz^2).$$

The nonvanishing Christoffel symbols are

$$\Gamma^0_{00} = \Gamma^\theta_{\theta 0} = \Gamma^z_{z0} = \Gamma^\rho_{\rho 0} = \frac{\dot a}{a},$$

$$\Gamma^\rho_{\theta\theta} = -\rho, \quad \Gamma^\theta_{\theta\rho} = \rho^{-1}, \quad \Gamma^0_{ij} = \frac{\dot a}{a}\tilde g_{ij}.$$

The metric on the string world sheet is given by

$$g_{ab} d\xi^a d\xi^b = a^2(\tau)[-(1-v^2)d\tau^2 + p^2 d\xi^2],$$

where

$$v^2 \equiv (\dot{\mathbf{x}})^2 = \tilde g_{ij}\dot x^i \dot x^j,$$

$$p^2 \equiv (\mathbf{x}')^2 = \tilde g_{ij} x'^i x'^j.$$

An overdot denotes derivative with respect to τ, and a prime is derivative with respect to ξ. Here $\tilde g_{ij}$ is the flat three-dimensional metric in cylindrical coordinates, and the indices i, j run over ρ, θ, and z.

Substituting in (1), the temporal component yields

$$\dot\epsilon + A(\tau) v^2 \epsilon = 0,$$

where $A(\tau) = 2a(H + h)$ and $\epsilon \equiv p(1-v^2)^{-1/2}$. The spatial components yield

$$\ddot\rho + (1-v^2)A(\tau)\dot\rho - \frac{1}{\epsilon}\left(\frac{\rho'}{\epsilon}\right)' - \rho\left[\dot\theta^2 - \frac{\theta'^2}{\epsilon^2}\right] = 0, \tag{B1}$$

$$\ddot\theta + (1-v^2)A(\tau)\dot\theta - \frac{1}{\epsilon}\left(\frac{\theta'}{\epsilon}\right)' + \frac{2}{\rho}\left[\dot\theta\dot\rho - \frac{\theta'\rho'}{\epsilon^2}\right] = 0, \tag{B2}$$

$$\ddot z + (1-v^2)A(\tau)\dot z - \frac{1}{\epsilon}\left(\frac{z'}{\epsilon}\right)' = 0. \tag{B3}$$

These are the equations of motion for a string propagating in a flat FRW background in cylindrical coordinates.

For circular loops we can take $z = 0$ and the parameter ξ proportional to the angular variable, $\xi = R_0\theta$, where R_0 is the initial coordinate radius of the loop. Then $\rho = R(\tau)$, $\rho' = 0$, $\dot\theta = 0$, and we obtain (20) and (21).

Let us derive the equations for the perturbations on circular loops. Taking $z = y^z(\tau\theta) \ll R$ it is immediate from (B3) to obtain the equation for small transverse perturbations (32). The equation for the radial perturbations requires more work. First we write the perturbed solution as $\rho = R(\tau) + \Delta$, and $\theta = (\xi/R_0) + \delta$, where Δ and δ are small deviations from the unperturbed values. Substituting in (B1) one finds, to linear order in the perturbations,

$$\ddot R + \ddot\Delta + (1 - \dot R^2 - 2\dot R\dot\Delta)A\dot R\left(1 + \frac{\dot\Delta}{\dot R}\right) - \frac{\Delta''}{\epsilon^2} + (R+\Delta)\frac{1+2\delta'}{\epsilon^2} = 0.$$

Also to linear order,

$$\epsilon^2 = \epsilon_0^2\left[1 + \frac{2\Delta}{R} + \frac{2\dot R\dot\Delta}{1-\dot R^2}\right](1+2\delta'),$$

where ϵ_0 is the unperturbed value. Substituting in the previous expression and using (20) we find

$$\ddot\Delta - 2\frac{\dot R}{R}\dot\Delta + (1 - 3\dot R^2)A(\tau)\dot\Delta + \frac{L^2 - 1}{\epsilon^2}\Delta = 0,$$

which is the equation of motion for radial perturbations (31).

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