

Instantons for vacuum decay at finite temperature in the thin wall limit

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(Received 6 January 1994)

In $N + 1$ dimensions, false vacuum decay at zero temperature is dominated by the $O(N + 1)$ -symmetric instanton, a sphere of radius R_0 , whereas at temperatures $T \gg R_0^{-1}$, the decay is dominated by a “cylindrical” (static) $O(N)$ -symmetric instanton. We study the transition between these two regimes in the thin wall approximation. Taking an $O(N)$ -symmetric ansatz for the instantons, we show that for $N = 2$ and $N = 3$ new periodic solutions exist in a finite temperature range in the neighborhood of $T \sim R_0^{-1}$. However, these solutions have a higher action than the spherical or the cylindrical one. This suggests that there is a sudden change (a first order transition) in the derivative of the nucleation rate at a certain temperature T_* , when the static instanton starts dominating. For $N = 1$, on the other hand, the new solutions are dominant and they smoothly interpolate between the zero temperature instanton and the high temperature one, so the transition is of second order. The determinantal prefactors corresponding to the “cylindrical” instantons are discussed, and it is pointed out that the entropic contributions from massless excitations corresponding to deformations of the domain wall give rise to an exponential enhancement of the nucleation rate for $T \gg R_0^{-1}$.

PACS number(s): 11.15.Kc, 03.65.Sq, 05.70.Fh, 98.80.Cq

I. INTRODUCTION

Since the early work by Langer [1], a lot of attention has been devoted to the study of first order phase transitions using the instanton methods [2]. These methods can be applied at zero temperature [3, 2] as well as at finite temperature [4], and they have been widely used in connection with phase transitions in the early Universe [5]. The basic predicament of this formalism is that the false vacuum decay rate per unit volume is given by an expression of the form

$$\frac{\Gamma}{V} = A e^{-S_E}, \quad (1)$$

where S_E is the Euclidean action of an instanton. The instanton is a solution of the Euclidean equations of motion with appropriate boundary conditions, whose analytic continuation to Lorentzian time represents the nucleation of a bubble of true vacuum in the false vacuum phase. When more than one instanton is compatible with the boundary conditions, one has to consider the one with the least Euclidean action, which of course will dominate Eq. (1). The preexponential factor A arises from Gaussian functional integration over small fluctuations around the instanton solution.

At zero temperature, the first order phase transition is dominated by quantum tunneling through a potential barrier. This tunneling is represented by a maximally symmetric instanton, which, in $N + 1$ spacetime dimensions, is an $O(N + 1)$ -symmetric configuration. On the other hand, at sufficiently high temperature, the phase transition is dominated by thermal hopping to the top of the potential barrier. This process is represented by a “static” $O(N)$ -symmetric instanton.

In the context of quantum mechanics (i.e., $0 + 1$ di-

mensions), the decay of a metastable state at “intermediate” temperatures was studied by Affleck [4]. Under certain assumptions for the shape of the potential barrier, he found that the transition between quantum tunneling and thermally assisted “hopping” occurred at a temperature T_c of the order of the curvature of the potential at the top of the barrier. At temperatures $0 < T < T_c$, the decay was dominated by “periodic” instantons (solutions of the Euclidean equations of motion which are periodic in imaginary time), with periodicity $\beta = T^{-1}$. Such periodic instantons smoothly interpolated between the zero temperature instanton and the static “thermal hopping” instanton.

However, as noted by Chudnovsky [6], this situation is not generic. Depending on the shape of the potential barrier, it can happen that the transition from quantum tunneling to thermal activation occurs at a temperature T_* larger than T_c , and in that case, there is a sudden discontinuity in the derivative of the nucleation rate with respect to the temperature at T_* . In [6] this was called a first order transition in the decay rate.

In the context of field theory, little work has been done toward the study of periodic instantons other than the trivial static one. In this paper we shall study such periodic instantons in the case when the so-called thin wall approximation is valid. That is, when thickness of the domain wall that separates the true from the false vacuum is small compared with the radius of the nucleated bubble. Also, we shall restrict ourselves to $O(N)$ -symmetric solutions. As we shall see, for $N = 2$ and $N = 3$ spatial dimensions, new periodic solutions exist but only in a finite temperature range in the vicinity of $T \sim R_0^{-1}$, where R_0 is the radius of the zero temperature instanton. We shall see that these new instantons have higher Euclidean action than the static or the spherical ones, and

so they will never give the dominant contribution to the nucleation rate (1). Barring the possibility that new non- $O(N)$ -symmetric instantons may alter the picture, this suggests that there is a sudden change in the derivative of the nucleation rate with respect to the temperature at T_* , the temperature at which the action of the spherical instanton becomes equal to the action of the cylindrical one. In the language of Ref. [6] this corresponds to a first order transition in the decay rate. The case $N = 1$ is somewhat different, since in that case the new periodic solutions have lower action than the static one, and they mediate a smooth transition between the zero temperature instanton and the high temperature one.

Another point that we shall be concerned with is the evaluation of the determinantal prefactor A for the case of the static instanton. In this case, the action is given by $S_E = -\beta E$, where E is the energy of a critical bubble (i.e., a bubble in unstable equilibrium between expansion and contraction) and β is the inverse of the temperature. As pointed out in Ref. [7], apart from a factor of T^{N+1} needed on dimensional grounds, the prefactor A is essentially the partition function of small excitations around the critical bubble configuration. Upon exponentiation, this has the effect of replacing $-\beta E$ by $-\beta F$ in the exponent of (1), where F is the free energy of the bubble plus fluctuations. At temperatures $T \sim R_0^{-1}$ the energy E and the free energy F are approximately the same, and so the prefactor A does not play much of a role. However, as we shall see, for $T \gg R_0^{-1}$ and $N > 1$, the entropic contribution due to deformations of the domain wall can be very large, giving an exponential enhancement to the decay rate.

The paper is organized as follows. In Sec. II we study the periodic instantons, and the transition from the low to the high temperature regimes. In Sec. III we evaluate the determinantal prefactors for the static instanton in the zero thickness limit, that is, the only fluctuations we consider are deformations in the shape of the wall. This is supposed to be a good approximation at temperatures $R_0^{-1} \ll T \ll m$, where m is the mass of the free particle excitations in the theory (supposed to be of the same order of magnitude than the inverse thickness of the wall). Section IV extends the result for the prefactors to temperatures $m \ll T$. Finally, some conclusions are summarized in Sec. V.

II. THE INSTANTONS

In the thin wall approximation, the dynamics of a bubble is well described by an action of the form

$$S = -\sigma \int d^N \xi \sqrt{\gamma} + \epsilon \int dV dt. \quad (2)$$

The first term is the Nambu action, proportional to the area of the world sheet of the wall separating true from false vacua, where σ is the tension of the wall, ξ^a is a set of N coordinates on the world sheet, and γ is the determinant of the world sheet metric. The second term is the volume enclosed by the wall times the difference in vacuum energy density between both sides of the wall, which we denote by ϵ , integrated over time.

In spherical coordinates the metric in $(N + 1)$ -dimensional flat space-time reads

$$ds^2 = -dt^2 + dr^2 + r^2 d^2\Omega^{(N-1)},$$

where $d^2\Omega^{(N-1)}$ is the line element on the $(N-1)$ -sphere. With a spherical ansatz, the world sheet of the domain wall separating the true from the false vacuum is given by $r = r(t)$. In terms of r , the action reads

$$S = -\sigma \mathcal{S}_{N-1} \left[\int r^{N-1} (1 - \dot{r}^2)^{1/2} dt - \frac{\epsilon}{N\sigma} \int r^N dt \right], \quad (3)$$

where $\dot{r} = dr/dt$. Here

$$\mathcal{S}_{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)}$$

is the surface of the unit $(N-1)$ -sphere, and we have used $\mathcal{V}_{N-1} = \mathcal{S}_{N-1}/N$, where \mathcal{V}_{N-1} is the volume inside the unit $(N-1)$ -sphere.

Since the Lagrangian does not depend explicitly on time, the equation of motion for the radius of the bubble, $r(t)$, reduces to the conservation of energy:

$$E = p_r \dot{r} - L = \sigma \mathcal{S}_{N-1} \left[\frac{r^{N-1}}{(1 - \dot{r}^2)^{1/2}} - \frac{\epsilon}{N\sigma} r^N \right]. \quad (4)$$

This can be cast in the form

$$\dot{r}^2 + V(r, E) = 0, \quad (5)$$

where

$$V = \left[\frac{E}{\sigma \mathcal{S}_{N-1}} r^{1-N} + \frac{\epsilon}{N\sigma} r \right]^{-2} - 1. \quad (6)$$

Equation (5) describes the motion of a nonrelativistic particle moving in the potential V .

To obtain the decay rates, one first has to find the relevant instantons. These are solutions of the Euclidean equations of motion with appropriate boundary conditions. Taking $t \rightarrow -it_E$ in (5) we have

$$\left(\frac{dr}{dt_E} \right)^2 - V = 0. \quad (7)$$

At finite temperature T , the instantons have to be periodic in t_E , with periodicity $\beta \equiv T^{-1}$. Also, the solutions have to approach the false vacuum at spatial infinity [2, 4, 5].

The instanton for $E = 0$ is well known. Integrating (7) one has

$$r^2 + t_E^2 = R_0^2,$$

where

$$R_0 = \frac{N\sigma}{\epsilon}.$$

This is the zero temperature instanton, an N -sphere of radius R_0 . It has the $O(N+1)$ symmetry, as opposed to the solutions for $E \neq 0$, which only have $O(N)$ symmetry.

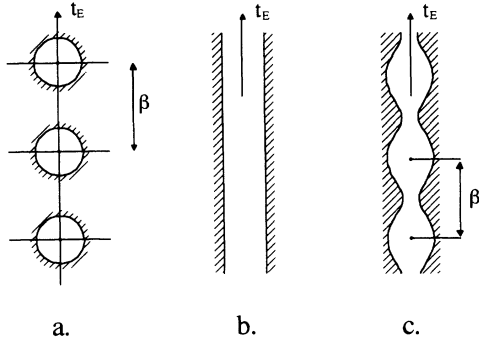


FIG. 1. Different types of instantons are found in different temperature regimes. (a) represents the spherical instanton, which exists at zero temperature but also at any temperature lower than $(2R_0)^{-1}$. (b) represents the “cylindrical” instanton, which, being static, exists for all temperatures. Finally, (c) represents the new instantons, periodic “wiggly” cylinders which exist only for a finite temperature range in the vicinity of $T \sim R_0$. The periodicity is $\beta = T^{-1}$.

This instanton has Euclidean action

$$S_E^0 = \frac{\sigma \mathcal{S}_N}{N+1} R_0^N. \quad (8)$$

The same instanton also exists at finite temperature, $T < T_0 \equiv (2R_0)^{-2}$ [see Fig. 1(a)], except that the sphere is repeated periodically along the t_E axis. The action is still given by (8), and so it is independent of temperature (apart from finite temperature corrections to the parameters ϵ and σ , which are not relevant to our present discussion).

For temperatures $T > T_0$ the spheres do not fit in the Euclidean time interval, and the solution with $E = 0$ no longer exists. It has been suggested [5] that for $T > T_0$ the “overlapping” spheres would somehow merge into a wiggly cylinder [see Fig. 1(c)], and ultimately into a straight cylinder at even higher temperatures [Fig. 1(b)]. However, we would like to stress that this is not necessarily so. As we shall see, at least in the thin wall approximation and for $N=2$ and 3, wiggly cylinder solutions exist only for a finite range of temperatures, which happens to be below T_0 [see Eq. (11)].

Consider the case $N = 2$. The corresponding potential V is plotted in Fig. 2. The turning points are given by

$$r_{\pm} = \frac{R_0}{2} \left[1 \pm \left(1 - \frac{4E}{\sigma R_0 \mathcal{S}_{N-1}} \right)^{1/2} \right].$$

For $E > 0$ we have Euclidean solutions which oscillate between the two turning points. These are the wiggly cylinder solutions mentioned before. The period of oscillation

$$\tau(E) = 2 \int_{r_-}^{r_+} \frac{dt_E}{dr} dr$$

can be computed from (7),

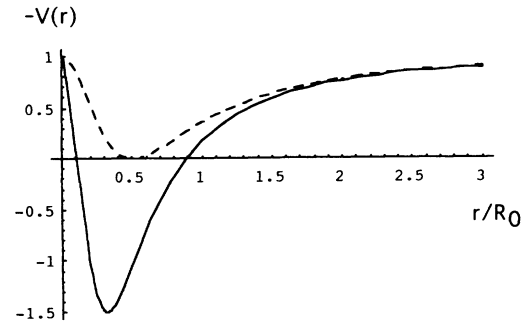


FIG. 2. Represented is the potential of Eq. (6) for $N = 2$, as a function of r . The solid line corresponds to an energy lower than E_c . Since in this case the equation $V(r) = 0$ has two solutions, the instanton has two turning points. As the energy is increased to $E = E_c$ the shape of the potential changes and the two turning points merge into one at r_c . The dashed line represents the shape of the potential for $E = E_c$.

$$\tau(E) = 2 \left(r_+ \mathcal{E}(q) + \frac{R_0^2}{r_+} \mathcal{K}(q) \right),$$

where $q = (r_+^2 - r_-^2)^{1/2}/r_+$ and \mathcal{E} and \mathcal{K} are the complete elliptic integrals [8]. A plot of $\tau(E)$ is given in Fig. 3. By changing the energy, the periodicity $\tau(E)$ changes over a finite range, and since $\tau = T^{-1}$, the wiggly cylinder solutions only exist for a finite range of temperatures [see (11)]. The case $N = 3$ is qualitatively similar. The equation for the turning points, $V(r) = 0$,

$$\frac{E}{\sigma \mathcal{S}_{N-1}} r^{1-N} + \frac{r}{R_0} = 1,$$

now has three solutions, but one of them is negative, so again we have two physical turning points.

In both cases, as the energy is increased from $E = 0$, the two turning points approach each other, and they eventually merge when

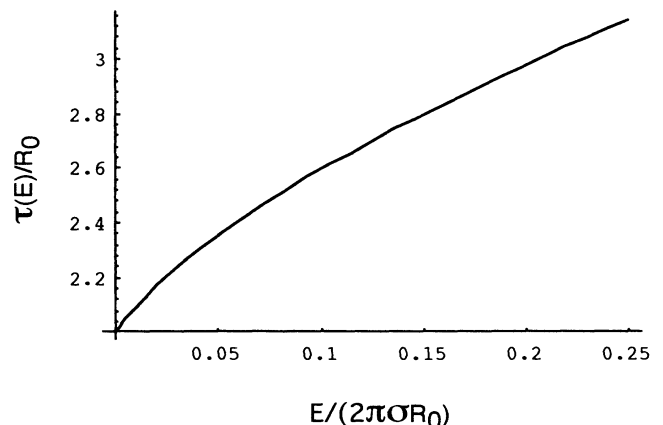


FIG. 3. The period τ of the “wiggly cylinder” solutions as a function of energy. By changing the energy, the periodicity changes only over a finite range, and so these solutions exist only for a finite range of temperatures.

$$E = E_c \equiv \frac{r_c^{N-1}}{N} \sigma S_{N-1} \tag{9}$$

at radius

$$r = r_c \equiv \frac{N-1}{N} R_0.$$

These equations can be obtained from the conditions

$$V(r_c, E_c) = V'(r_c, E_c) = 0, \tag{10}$$

where $V' = dV/dr$. The period of the solutions as $E \rightarrow E_c$ is given by $\tau(E_c) = 2\pi/\omega_c$. Here

$$\omega_c^2 = \frac{1}{2} V''(r_c, E_c)$$

is the frequency of the oscillations at the bottom of the well. Using (6) we find

$$T_c \equiv \tau^{-1}(E_c) = \frac{N}{(N-1)^{1/2}} (2\pi R_0)^{-1};$$

and, hence, the lower bound on the temperature range in which the wiggly cylinder solutions exist:

$$T_c < T < T_0. \tag{11}$$

It is clear from the conditions (10) that the straight cylinder (see Fig. 1)

$$r = r_c$$

is also a solution of (7) corresponding to $E = E_c$. Since it is static, this solution exists for all temperatures. The corresponding Euclidean action

$$S_E^c = E_c \beta,$$

is inversely proportional to the temperature. Therefore this will be the dominant instanton at sufficiently high temperature.

The physical interpretation of the three types of instantons can be inferred from their analytic continuation to real time. The “straight cylinder” represents thermal hopping to the top of the barrier, creating a critical bubble in unstable equilibrium that can either recollapse under the tension of the wall or expand under the negative pressure exerted by the false vacuum. The “zero temperature” solution $E = 0$ represents the spontaneous quantum creation of a bubble from “nothing” (i.e., from a small quantum fluctuation). This bubble is larger than the critical bubble and after nucleation it expands to infinity. The wiggly cylinder instantons represent a combination of both phenomena. A subcritical bubble of finite energy that has been created by a thermal fluctuation tunnels through the barrier to a supercritical bubble which can subsequently expand to infinity.

Let us now show that for $N = 2$ and 3 the process represented by the wiggly cylinders is never dominant. From (4), Euclideanizing and integrating over one period of oscillation ($S \rightarrow iS_E, t \rightarrow -it_E, p_r \rightarrow ip_r^E$) we have (S_E^w is the action for the wiggly cylinders)

$$S_E^w = W + E\tau(E),$$

where $W = \int p_r dr$. Using

$$p_r^E = \sigma S_{N-1} r^{N-1} \frac{dr}{dt_E} \left[1 + \left(\frac{dr}{dt_E} \right)^2 \right]^{-1/2}$$

and (7), one easily arrives at

$$\frac{dW}{dE} = -\tau(E),$$

from which one obtains the Hamilton-Jacobi equation

$$\frac{dS_E^w}{d\tau} = E. \tag{12}$$

As E approaches E_c from below, the amplitude of the wiggles tends to zero and we have

$$\lim_{E \rightarrow E_c} S_E^w = S_E^c(E_c), \tag{13}$$

where $S_E^c(E_c)$ is the action for the straight cylinder at $\beta = \tau(E_c)$. Also, when E approaches zero from above the amplitude of the wiggles becomes so large that in fact the cylinder fragments into spheres:

$$\lim_{E \rightarrow 0} S_E^w = S_E^0. \tag{14}$$

From (12) one has

$$0 = \frac{dS_E^0}{d\tau} \leq \frac{dS_E^w}{d\tau} \leq \frac{dS_E^c}{d\tau} = E_c.$$

It then follows that in the whole range $0 < E < E_c$ in which the wiggly cylinders exist, the action for the wiggly cylinder S_E^w is larger than both S_E^c and S_E^0 . This means that the decay rate, which has the exponential dependence

$$\Gamma \propto e^{-S_E}$$

will be dominated either by the zero temperature instanton or by the straight cylinder instanton, but never by the wiggly cylinder. The situation is depicted in Fig. 4.

Define T_* as the temperature at which the action for the cylinder is equal to the action for the sphere, $S_E^c(\beta_*) = S_E^0$. For $N = 2$, $\beta_* = (8/3)R_0$, and for $N = 3$

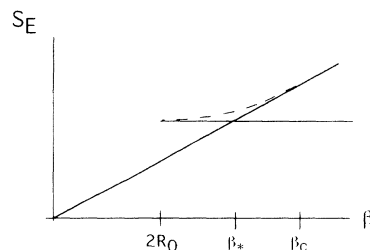


FIG. 4. The sketch represents the action for the different types of instantons as a function of the inverse temperature β , for $N > 1$. The horizontal line represents the action for the spherical instantons, which is independent of temperature in the range where it exists. The slanted straight line is the action for the static instanton, which is proportional to β , the length of the cylinder. The dashed line represents the action for the wiggly cylinders, which exist only in a finite temperature range.

$\beta_* = (27\pi/32)R_0$, so $T_c < T_* < T_0$. For $T < T_*$ the decay is dominated by the spherical instantons, corresponding to zero temperature quantum tunneling, whereas for $T > T_*$ it is dominated by the cylindrical one, corresponding to thermal hopping. Since the slopes

$$\frac{dS_E^c}{d\beta} \neq \frac{dS_E^0}{d\beta}$$

do not match at $T = T_*$, there will be a sudden change in the derivative of the nucleation rate at this temperature. In the language of Ref. [6], this corresponds to a first order transition in the decay rate. This applies in the case when the thin wall approximation is valid. To what extent the results apply for thick walls remains unclear to us. Some comments will be made in the concluding section.

The above considerations do not cover the case $N = 1$. This case is special because the force exerted by the false vacuum on the “wall” cannot be compensated by the spatial curvature of the wall, which now reduces to a pair of “pointlike” kink and antikink. This case has previously been considered in great detail in Ref. [9], in the context of tunneling and activated motion of a string across a potential barrier. For $E = 0$ the solution of (5) is a circle of radius $R_0 = \sigma/\epsilon$, and so the zero temperature instanton is analogous to the higher dimensional ones. For $E \neq 0$, however, the situation is different because the potential V has only one turning point. As a result in the thin wall limit the periodic solutions, which are essentially arcs of the circle of radius R_0 , contain mild singularities, vertices where the world lines of the kinks meet (see Fig. 5; note that such vertices cannot be constructed for $N > 1$, since the world sheet curvature would diverge there). The use of vertices is just an artifact of the thin wall approximation. They can be “rounded off” by introducing a short-range attractive force between the kink and antikink, in the manner described in [9]. Because of this force, the potential V is modified and a new turning point appears at $r \sim \delta$, where δ is the range of the attractive force (which in field theory will be of

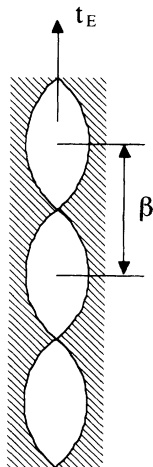


FIG. 5. Periodic instantons for the case $N = 1$, in the thin wall limit.

the order of the thickness of the kink). As a result the instantons have a shape similar to Fig. 1(c).

For $N = 1$ the zero temperature action is given by (8), $S_E^0 = \pi R_0 \sigma$. The action for the finite temperature periodic instantons is $S_E^w = 2\sigma l - \epsilon A$, where l is the length of one of the world lines of kink or antikink and A is the area enclosed between them. We have

$$S_E^w = \sigma R_0 [\alpha + \sin \alpha],$$

where α is related to the temperature $T = \beta^{-1}$ through (see Fig. 5)

$$\alpha = 2 \arcsin \frac{\beta}{2R_0}.$$

For $\alpha = \pi$ this instanton reduces to the zero temperature one, the circle of radius R_0 . As we increase temperature above $(2R_0)^{-1}$, α decreases, and in the limit $\alpha \rightarrow 0$ we have

$$S_E^w = 2\sigma\beta + O(\alpha^3).$$

In addition, if we take into account the short-range attractive force between the kink and antikink we will have the static instanton for all temperatures (the analogous of the straight cylinder representing thermal hopping). This consists of two parallel world lines separated by a distance $\sim \delta$, representing a pair in unstable equilibrium between the short-range attractive force and the negative pressure of the false vacuum that pulls them apart [10]. For $\delta \ll R_0$ we can neglect the false vacuum term in the action, and the action is just proportional to the length of the world lines:

$$S_E^c \approx 2\sigma\beta. \quad (15)$$

In the high temperature limit this coincides with S_E^w . The diagram of action versus temperature is sketched in Fig. 6. The “static” instanton, whose action is depicted

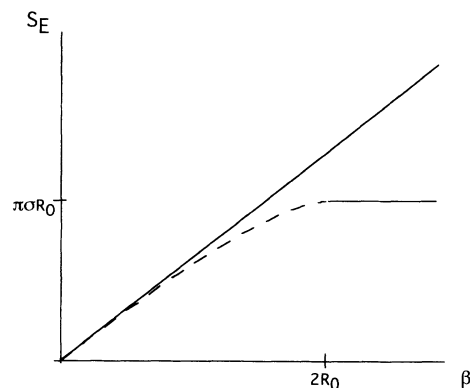


FIG. 6. The action for the different types of instanton in the case $N = 1$. Same conventions as in Fig. 4. In this case, the action of the periodic instantons (dashed line) is always smaller than the one for the static instanton (solid slanted line). The periodic instantons smoothly interpolate between the spherical instanton action (horizontal line) and the high temperature static instanton action. The slope of the dashed line at $\beta = 2R_0$ vanishes.

as a slanted solid line, is always subdominant. Since

$$\frac{dS_E^0}{d\beta} = \frac{dS_E^w}{d\beta} = 0,$$

at $T = T_0 = (2R_0)^{-1}$, it follows that the transition from the quantum tunneling regime to the thermally assisted regime is now of second order (in the terminology of [6]).

Here we have concentrated in studying the instantons in the thin wall limit. It is often claimed that the action for these instantons gives the exponential dependence of the nucleation rates. However, as we shall see, contributions from the determinantal prefactor can be exponentially large at temperatures $T \gg R_0$.

III. PREFACTORS

At zero temperature, the nucleation rates per unit volume Γ/V in the thin wall limit have been given, e.g., in [11] (see also [12–15]). We have

$$\begin{aligned} \frac{\Gamma}{L} &= \frac{\epsilon}{2\pi} e^{-S_E^0} \quad (N = 1), \\ \frac{\Gamma}{V} &= \left(\frac{2\sigma R_0^2}{3}\right)^{3/2} (\mu R_0)^{7/3} R_0^{-3} e^{-S_E^0} \quad (N = 2), \\ \frac{\Gamma}{V} &= \left(\frac{\pi\sigma R_0^3}{4}\right)^2 \frac{4R_0^{-4}}{\pi^2} e^{\zeta'_R(-2)} e^{-S_E^0} \quad (N = 3), \end{aligned}$$

where ζ_R is Riemann's zeta function. For $N = 2$ the result contains an arbitrary renormalization scale μ (see [11]). These results should apply as long as $T < \min\{T_*, T_0\}$, when the decay is dominated by the spherical instanton.

For $T > T_*$ and $N = 2, 3$, as we have seen, the decay is dominated by the static instanton. For an action of the form (2), the second variation of the action due to small fluctuations of the world sheet around a classical solution is given by [16, 11, 17]

$$\delta S^{(2)} = \frac{\sigma}{2} \int d^N \xi \sqrt{\gamma} \phi \hat{O} \phi.$$

Physically, $\phi(\xi^a)$ has the meaning of a small normal displacement of the world sheet. Here ξ^a is a set of coordinates on the instanton world sheet, γ is the determinant of the world sheet metric and

$$\hat{O} \equiv -\Delta_\Sigma + M^2, \quad (16)$$

where Δ_Σ is the Laplacian on the world sheet and

$$M^2 = \mathcal{R} - \left(\frac{\epsilon}{\sigma}\right)^2,$$

with \mathcal{R} the intrinsic Ricci scalar of the world sheet.

For the static ‘‘cylinder’’ instantons, $\Delta_\Sigma = \partial_{t_E}^2 + \Delta^{(N-1)}$, where $\Delta^{(N-1)}$ is the Laplacian on the $(N-1)$ -sphere. The eigenvalues of \hat{O} in this case are given by

$$\lambda_{L,n} = \left(\frac{2\pi n}{\beta}\right)^2 + w_L^2,$$

where

$$w_L^2 = (J^2 - N + 1)r_c^{-2},$$

$J = L(L + N - 2)$ ($L = 0, 1, \dots, \infty$), and the degeneracy of each eigenvalue is given by $g_L(2 - \delta_{n0})$, where

$$g_L = \frac{(2l + N - 2)(N + L - 3)!}{L!(N - 2)!}.$$

Since the critical bubble is a static configuration, it turns out that w_L^2 are also the eigenvalues of the second variation of the Hamiltonian around this configuration. Note that the lowest eigenvalue, $L = 0$, is negative $w_0^2 = -(N - 1)r_c^{-2}$. This is because the static cylinder is in unstable equilibrium between expansion and contraction.

The preexponential factor in the decay rate is given in terms of the determinant of the operator \hat{O} [2, 4, 7, 11]:

$$\Gamma = \frac{|w_0|}{\pi} (\det \hat{O})^{-1/2} e^{-S_E^c}, \quad (17)$$

where

$$(\det \hat{O})^{-1/2} = \prod_{L,n} (\lambda_{L,n})^{-(2-\delta_{n0})g_L/2}. \quad (18)$$

For given L , the product over n can be easily computed by using the generalized ζ -function method [18, 11]. Actually, this product is just the partition function for a harmonic oscillator with frequency w_L at a temperature β^{-1} ,

$$Z_L = [2 \sinh(\beta w_L/2)]^{-1}, \quad (19)$$

and we have

$$(\det \hat{O})^{-1/2} = \prod_L Z_L^{g_L}.$$

For $L = 1$, we have $w_1 = 0$ and so the previous expression is divergent. This is due to the fact that some of the eigenmodes of \hat{O} do not really correspond to deformations of the world sheet but to space translations of the world sheet as a whole. They are the so-called zero modes. As a result, \hat{O} has N vanishing eigenvalues, corresponding to the number of independent translations of the cylinder. (These eigenvalues are the $\lambda_{1,0}$, of which there are $2g_1 = N$.) The usual way of handling this problem is the following. In the Gaussian functional integration over small fluctuations around the classical solution, which leads to the determinant in (17), one replaces the integration over the amplitudes of the zero modes by an integration over collective coordinates.

The transformation to collective coordinates introduces a Jacobian

$$J = \left(\frac{S_E^c}{2\pi}\right)^{N/2},$$

and the integration over collective coordinates introduces a space volume factor V . The zero eigenvalues have now to be excluded from the determinant, and noting that

$$\lim_{w_1 \rightarrow 0} \frac{2 \sinh(\beta w_1/2)}{w_1} = \beta,$$

we have

$$(\det \hat{O})^{-1/2} = \beta^{-N} V J \prod_{L \neq 1} Z_L^{g_L}. \quad (20)$$

The decay rate at temperatures above T_* is thus given by

$$\frac{\Gamma}{V} = \frac{\beta^{-N} |w_0|}{2\pi \sin(\beta w_0/2)} J e^{-\beta F}, \quad (21)$$

where

$$F \equiv E_c - \beta^{-1} \sum_{L>1} g_L \ln Z_L. \quad (22)$$

Here we have used $S_E^c = \beta E_c$, where E_c is the energy of the critical bubble, given by (9). Since Z_L is the partition function for a harmonic oscillator of frequency w_L at temperature β^{-1} , the second term on the right-hand side (RHS) of (22) has a clear interpretation. Each physical mode of oscillation will be excited by the thermal bath, and will make its contribution to the free energy [7]. Thus F can be thought of as the total free energy of the bubble plus fluctuations.

Let us analyze the contribution of the fluctuations to the free energy in more detail. From (19) we have

$$-\ln Z_L = \frac{\beta w_L}{2} + \ln(1 - e^{-\beta w_L}). \quad (23)$$

The first term on the right-hand side contains the energy of zero point oscillations of the membrane, and can be thought as quantum corrections to the mass of the membrane, of the same nature as the Casimir energy which arises in field theory in the presence of boundaries or nontrivial topology. Indeed, from (16), the fluctuations ϕ behave like a scalar field of mass M^2 living on the world sheet. Since the world sheet has nontrivial topology, this will result in a Casimir energy $\Delta E = \sum_{L>1} g_L w_L/2$. This expression is of course divergent and it needs to be regularized and renormalized in the usual way.

For $N = 2$ one obtains the renormalized expression (see, e.g., [19])

$$\beta(\Delta E)^{\text{ren}} = \frac{\beta}{2} \left[-\frac{1}{6r_c} + \frac{2}{r_c} S(M^2 r_c^2) + M^2 r_c \ln(\mu r_c) \right], \quad (24)$$

$$\sum_L g_L \ln(1 - e^{-\beta w_L}) \approx S_{N-1} r_c^{N-1} \int d^{N-1} k \ln(1 - e^{-\beta k}) + O((TR_0)^{2-N}).$$

In this limit,

$$\beta F = \beta(E_c + \Delta E^{\text{ren}}) - \alpha (r_c T)^{N-1} [1 + O((TR_0)^{-2})]. \quad (26)$$

Here α is a numerical coefficient:

$$\alpha = \frac{4\pi^{N-\frac{1}{2}} \Gamma(N-1)}{\Gamma(N/2) \Gamma(\frac{N-1}{2})} \zeta_R(N),$$

with ζ_R the usual Riemann's zeta function.

where

$$S(a^2) \equiv \sum_1^\infty [(n^2 + a^2)^{1/2} - n - a^2/2n].$$

Notice the appearance of the arbitrary renormalization scale μ in the previous expression. The coefficient of $\ln \mu$ can be easily deduced from the general theory of ζ -function regularization [18, 20]. It is given by $-\zeta_{\hat{O}}(0)$, the generalized zeta function associated to the operator \hat{O} , evaluated at the origin. This in turn can be expressed in terms of geometric invariants of the manifold. In two dimensions one has

$$-\zeta_{\hat{O}}(0) = \frac{-1}{4\pi} \int d^2 \xi^a \sqrt{\gamma} [-M^2 + \frac{1}{6} \mathcal{R}], \quad (25)$$

where M^2 is the “mass term” in the Klein-Gordon operator \hat{O} (in general M^2 includes a linear coupling to the Ricci scalar, of the form $\xi \mathcal{R}$). In our case the Ricci scalar vanishes, and recalling that the area of our instanton is given by $2\pi r_c \beta$ we recover the coefficient of $\ln \mu$ in (24). From (25), it is clear that a redefinition of μ corresponds to a redefinition of the coefficient of the Nambu term in the classical action, i.e., a redefinition of the tension σ . In our case $M = r_c^{-1}$, and we have $(\Delta E)^{\text{ren}} \sim r_c^{-1} \ln(\mu r_c)$.

For $N = 3$, just on dimensional grounds $(\Delta E)^{\text{ren}} \sim r_c^{-1}$. In this case there is no logarithmic dependence, since in odd dimensional spaces without boundary $\xi_{\hat{O}}(0)$ always vanishes [21]. Comparing with the classical energy $E_c \sim \sigma r_c^{N-1}$, it is clear that the Casimir energy is negligible as long as $\sigma r_c^N \gg 1$. In the regime where the instanton approximation is valid, $S_E^c \sim \beta \sigma r_c^{(N-1)} \gg 1$, and since we are at temperatures $\beta \lesssim r_c$, it follows that the Casimir term is negligible in front of the classical one.

However, it is the second term in (23) that can be important, because of its strong temperature dependence. Note that this term, when summed over L remains finite, and no further renormalization is needed. At temperatures $T \gg R_0^{-1}$, the mass M is small compared with the temperature, and this sum can be approximated by the free energy of a massless field living on the surface of the membrane:

To conclude, Eq. (26) shows that at finite temperature $T \gg R_0$, the determinant of fluctuations around the critical bubble makes an exponentially large contribution to the prefactor,

$$e^{+\alpha (Tr_c)^{N-1}}, \quad (27)$$

which partially compensates the exponential suppression due to the classical action $e^{-\beta E_c}$. In particular, for $T \sim \sigma^{1/N}$ there is no exponential suppression in the nucleation rate. This is to be expected, since $\sigma^{1/N}$ is of the

order of the Hagedorn temperature for the membrane. (Of course, strictly speaking, our analysis of linear fluctuations of the membrane will only be valid for $T \ll \sigma^{1/N}$, since as we approach this temperature thermal fluctuations of wavelength $\sim \beta$ become nonlinear.)

For $N = 1$ there is no exponential enhancement in the prefactor. As mentioned before, the (1+1)-dimensional case has been considered previously by Mel'nikov and Ivlev [9]. They studied the prefactor accompanying the static instanton which consists of a kink and antikink in unstable equilibrium between their mutual short-range attraction and the force exerted upon them by the false vacuum, which tends to pull them apart. The temperature dependence of the prefactor in this case is very easy to estimate in the thin wall limit. There are only two modes of fluctuation around the classical solution (as opposed to the infinite number of modes for a string or a membrane), which we shall take as $\phi_{\pm} \equiv \phi_1 \pm \phi_2$. Here $\phi_1(t_E)$ and $\phi_2(t_E)$ are the displacement of the kink and the displacement of the antikink from their equilibrium positions. It is clear that ϕ_- will behave as an "upside down" harmonic oscillator, with imaginary frequency w_0 , ($w_0^2 < 0$), which corresponds to the instability against increasing or decreasing the distance between the kink and antikink. The variable ϕ_+ on the other hand, behaves like a zero mode, corresponding to space translations of the kink-antikink pair as a whole. The negative mode ϕ_- will contribute

$$Z_{w_0} = \left[2 \sin \frac{|w_0| \beta}{2} \right]^{-1}$$

to the prefactor. The zero mode ϕ_+ , which has to be treated in the manner outlined before Eq. (20), will give rise to a factor $\beta^{-1} \ell J$, where $J = (S_E^c / 2\pi)^{1/2}$, ℓ is the length of space, and we have

$$(\det \hat{O})^{-1/2} = \beta^{-N} \ell J Z_{w_0} = \frac{\ell}{2 \sin \frac{|w_0| \beta}{2}} \left(\frac{\sigma}{\pi \beta} \right)^{1/2}.$$

Here σ is just the mass of the kink, and we have used (15). Substituting in (17) we have

$$\frac{\Gamma}{\ell} \approx \frac{|w_0|}{2\pi \sin \frac{|w_0| \beta}{2}} \left(\frac{\sigma}{\pi \beta} \right)^{1/2} e^{-2\sigma\beta}, \quad (28)$$

in agreement with [9].

IV. FINITE THICKNESS

So far we have treated the boundary that separates false from true vacua as infinitely thin. However, in realistic field theories these boundaries always have some finite thickness δ , and our estimates of the prefactor, Eqs. (27) and (28), will have to be modified at temperatures $T \gg \delta^{-1}$. Here we shall briefly comment on such modifications in the case when $\delta \ll r_c$ (so that the thin wall approximation is valid in the usual sense).

In general, we will have a scalar field (or order parameter) φ , whose effective potential $V_{\text{eff}}(\varphi)$ has nondegenerate minima. For definiteness we take the form

$$V_{\text{eff}} = \frac{\lambda}{2} (\varphi^2 - \eta^2)^2 + \frac{\epsilon}{2\eta} \varphi. \quad (29)$$

It should be noted that V_{eff} contains quantum corrections as well as finite temperature corrections from all fields that interact with φ , except from the field φ itself [7] and therefore the parameters λ , η , and ϵ are temperature dependent (this dependence is always implicit in what follows). From V_{eff} one solves the classical field equations to find the relevant instantons. Integration over small fluctuations around the instantons yields the determinantal prefactors. (For a detailed account of the formalism for the computation of prefactors for field theoretic vacuum decay at finite temperature, see Ref. [7].)

Among the modes of fluctuation around the static instanton, some correspond to deformations of the membrane. These modes we have already considered in the previous section. In addition, there will be other eigenvalues of the fluctuation operator which correspond to massive excitations of the scalar field φ (both in the false and the true vacua), and to internal modes of oscillation in the internal structure of the membrane. In simple models, such as (29), the energy scale of internal excitations is of the same order of magnitude that the mass of free particles of φ , which we denote by $m = 2\lambda^{1/2}\eta$. Therefore, even in the case when the thickness of the wall is small in comparison with r_c , the Nambu action that we have used can be seen only as an effective theory valid for $T \ll m$, when the massive modes are not excited.

For $N = 1$ it was shown in [9] that the dependence of the prefactor in (28) as $T^{3/2}$ changes to $T^{-1/2}$ when $T \gg m$. Also, for $N > 1$ we expect that the exponential growth of the prefactor with temperature, Eq. (27), will be modified for $T \gg m$. Following [7], the basic difference with the zero thickness case is that the second term on the right-hand side of Eq. (26), corresponding to the free energy of massless excitations of the wall, will be replaced by the full finite temperature correction to the free energy of a field theoretic bubble. If $\delta \ll r_c$ we have

$$\beta F \approx \beta E_c + \beta S_{N-1} r_c^{N-1} (\Delta\sigma). \quad (30)$$

Here E_c is given by (9), with $r_c = (N-1)\sigma/\epsilon$, where $\sigma = (4/3)\lambda^{1/2}\eta^3$, and $(\Delta\sigma)$ is the finite temperature correction to the wall tension due to fluctuations of the field φ itself. We ignore the quantum correction ΔE^{ren} to the energy of the critical bubble, which will be small compared to the classical term. Also, there are classical corrections to E_c of order $(\delta/r_c)^2$ with respect to the leading term, due to the fact that the wall is curved [13]. We shall also ignore these.

The correction $(\Delta\sigma)$ can now be easily estimated in an adiabatic approximation, since the temperature $T \gg m$ is much larger than the inverse thickness of the wall $\delta \sim m^{-1}$. To leading order in the thickness over the radius, one can ignore the curvature of the wall. Neglecting the second term in (29), the scalar field has the well-known solution representing a planar wall in the $z = 0$ plane

$$\varphi_w(z) = \eta \tanh \left(\frac{z}{\delta} \right),$$

with $\delta = \lambda^{-1/2}\eta^{-1}$. Then

$$\beta(\Delta\sigma) = \int dz \int \frac{d^N k}{(2\pi)^N} [\ln(1 - \exp\{-\beta[k^2 + V_{\text{eff}}''(\varphi_w)]^{1/2}\}) - \ln\{1 - \exp[-\beta(k^2 + m^2)^{1/2}\}].$$

The second term in the integrand subtracts the contribution of the false vacuum.

For $N = 3$ we can expand the integrand in powers of $(\beta m)^2$. The leading contribution is then

$$\beta(\Delta\sigma) \approx \frac{T}{24} \int dz [V_{\text{eff}}''(\varphi_w) - m^2] = -\frac{m^2 T \delta}{12}.$$

Substituting in (30) and then in (21) we have

$$\frac{\Gamma}{V} \propto T^4 \exp\left(-\frac{E_c}{T} + \frac{2}{3}\pi T m r_c^2\right) \quad (N = 3). \quad (31)$$

Therefore, for $T \gg m$, the finite temperature entropic contribution to the free energy causes a correction to the nucleation rate which is still exponentially growing with T [this has to be compared with the behavior for $T \ll m^2$, Eq. (27) which depended even more strongly on T].

A similar calculation for $N = 2$ yields

$$\beta(\Delta\sigma) \approx \frac{m}{2\pi} [\ln(\beta^2 m^2) - 1] - \frac{1}{8\pi} \int dz V_{\text{eff}}'' \ln(V_{\text{eff}}''/m^2),$$

where in the last term V_{eff}'' is evaluated at $\varphi_w(z)$. This last term is independent of the temperature, whereas the first only depends logarithmically on it. As a result we have

$$\frac{\Gamma}{V} \propto T^3 (T/m)^{2r_c m} e^{-\beta E_c} \quad (N = 2). \quad (32)$$

Therefore, for $T \gg m$, the entropic correction is no longer exponentially growing with T , but since $r_c \gg m^{-1}$, the preexponential power law enhancement can still be quite significant.

V. SUMMARY AND DISCUSSION

We have studied periodic instantons for vacuum decay in the thin wall approximation, using a spherical ansatz for the spatial sections. For the case of one spatial dimension ($N = 1$), the periodic instantons have lower action than the static one (see Fig. 6), and they mediate a smooth transition between the high and the low temperature regimes.

For $N = 2$ and $N = 3$, on the other hand, we have seen that although new periodic instantons exist in a certain temperature range [see Eq. (11)], their action is larger than that of the ‘‘spherical’’ low temperature instanton or the ‘‘static’’ high temperature one. As a result, the new periodic solutions never dominate the vacuum decay. The transition from low to high temperature regimes occurs at the temperature T_* (see Fig. 4) at which the action for the spherical instanton coincides with the action for the static one. Since the slopes of the action versus temperature for both types of instanton do not match at $T = T_*$, this is called a first order transition in the nucleation rate at T_* [6]. As mentioned before, we have treated the domain walls that separate true from false vacuum as infinitely thin membranes, but we be-

lieve that these results will hold as long as the thickness of the wall is much smaller than the radius of the nucleated bubble. Of course, this can only be confirmed by studying a field theoretic model such as the one given by Eq. (29). This is left for further research.

In the case when the thickness of the walls is comparable to the bubble radius we do not expect these results to hold, and it may well be that the periodic instantons mediate a smooth transition between low and high temperature regimes, even for $N > 1$. As an extreme case, we can consider the soluble model discussed in Ref. [22], with effective potential

$$V_{\text{eff}} = \frac{m^2}{2} \varphi^2 \left[1 - \ln \frac{\varphi^2}{c^2}\right],$$

where m and c are constants. This model is not too realistic because the mass of the φ particles in the false vacuum ($\varphi = 0$) is infinite. Also, the potential is unbounded from below, so that actually there is no true vacuum. Nevertheless this provides a framework in which one can study periodic instantons. The model is extreme in the sense that the bubbles have no core. At zero temperature, the instanton has the shape of a Gaussian:

$$\varphi = c e^{-\frac{1}{2}m^2 \rho^2 + 2},$$

where $\rho^2 = t_E^2 + \mathbf{x}^2$ (for definiteness we consider 4 space-time dimensions). At high temperatures $T > T_c \equiv m/(2^{1/2}\pi)$, the decay is dominated by the static instanton:

$$\varphi = c e^{-\frac{1}{2}m^2 r^2 + \frac{3}{2}},$$

where $r^2 = \mathbf{x}^2$. At intermediate temperatures $0 < T < T_c$ one can find periodic instantons [22]:

$$\varphi = c f(t_E) e^{-\frac{1}{2}m^2 r^2 + \frac{3}{2}},$$

where $f(t_E)$ satisfies the equation

$$\left(\frac{df}{dt_E}\right)^2 = m^2 f^2 (4 - \ln f^2) + \frac{2m^3 E}{\pi^{3/2} c^2}. \quad (33)$$

Here E is the energy of the nucleated bubble. The period β of the solutions of (33) is plotted in Fig. 7 as a function of E . Note that the period decreases with increasing energy, in contrast with Fig. 3, which corresponds to the thin wall case. Also, as $E \rightarrow 0$ we have $\beta \rightarrow \infty$ and the action approaches that of the zero temperature instanton. For $E \rightarrow E_c \equiv e^3 \pi^{3/2} c^2 / 2m$, it approaches that of the static one [22]. Then, from the Hamilton-Jacobi equation (12) and the fact that $\beta(E)$ is monotonously decreasing, it follows that the action for the periodic instantons is lower than that of the static or the zero temperature ones in the whole range $0 < T < T_c$. In that case, such periodic instantons mediate a smooth transition between the low and high temperature regimes.

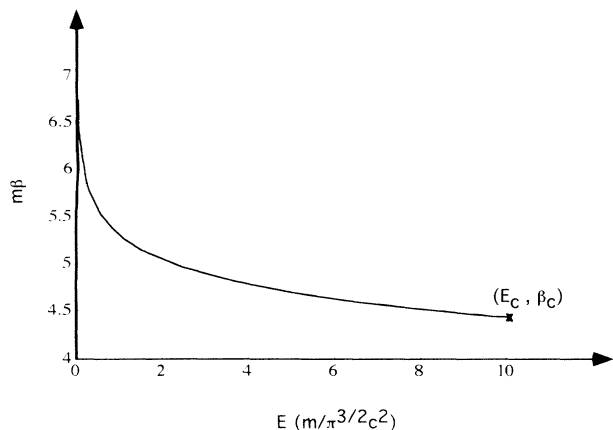


FIG. 7. The period β of the solutions of (33) as a function of energy E .

Finally, we have studied the determinantal prefactor in the nucleation rate for the case of the static instanton, in the thin wall case. We have seen that for $N > 1$ and $T \gg R_0$ there is an exponential enhancement of the nucleation rate [see Eq. (27)] which can be understood as the contribution to the free energy of the fluctuations around the critical bubble. Deformations of the membrane behave essentially as a scalar field of small mass (of order R_0^{-1}) living on the world sheet of the wall. The

enhancement (27) is basically the partition function for a gas of such excitations living on the world sheet.

At temperatures $T \gg m$ (where m is the mass of free particles in the false vacuum, assumed to be of the same order of magnitude than the inverse thickness of the wall), the exponential law (27) no longer applies, since one has to consider excitations in the internal structure of the membrane. In this regime one obtains (31) and (32), which still contain a sizable enhancement above the naive estimate $(\Gamma/V) \propto T^{N+1} \exp(-\beta E_c)$.

The case $N = 1$ had been considered in [9]. In this case there is no exponential enhancement because the membrane reduces to a pair of kink and antikink, so there is no “gas” of massless excitations contributing to the free energy. For $R_0 \ll T \ll m$ the prefactor behaves as $T^{3/2}$ [see (28)], whereas for $T \gg m$ it behaves as $T^{-1/2}$ [9].

Note added in proof. After completion of this paper, I became aware of Ref. [23], where the prefactor for the static instanton is studied numerically in the $\lambda\phi^4$ theory.

ACKNOWLEDGMENTS

I would like to thank Alex Vilenkin for drawing my attention to this problem and to Tanmay Vachaspati for interesting conversations. This work was partially supported by the National Research Foundation under contract No. PHY-9248784.

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