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Nucleation rates in flat and curved space

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Nucleation rates for tunneling processes in Minkowski and de Sitter space are investigated, taking into account one loop prefactors. In particular, we consider the creation of membranes by an antisymmetric tensor field, analogous to Schwinger pair production. This can be viewed as a model for the decay of a false (or true) vacuum at zero temperature in the thin wall limit. Also considered is the spontaneous nucleation of strings, domain walls, and monopoles during inflation. The instantons for these processes are spherical world sheets or world lines embedded in flat or de Sitter backgrounds. We find the contribution of such instantons to the semiclassical partition function, including the one loop corrections due to small fluctuations around the spherical world sheet. We suggest a prescription for obtaining, from the partition function, the distribution of objects nucleated during inflation. This can be seen as an extension of the usual formula, valid in flat space, according to which the nucleation rate is twice the imaginary part of the free energy. For the case of pair production, the results reproduce those that can be obtained using second quantization methods, confirming the validity of instanton techniques in de Sitter space. Throughout the paper, both the gravitational field and the antisymmetric tensor field are assumed external.

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I. INTRODUCTION

A wide class of nonperturbative phenomena in field theory can be understood in terms of quantum tunneling. A well-known example is the decay of false vacuum: the materialization of bubbles of true vacuum in first order phase transitions [1, 2]. Lower dimensional versions of this process have been used to model the decay of metastable topological defects, such as domain walls, and strings [3].

A closely related phenomenon is the neutralization of the cosmological constant through membrane creation. In a spacetime of dimension $d = N+1$, an antisymmetric tensor field A of rank N induces a cosmological constant. This is because the corresponding field strength $F = dA$ has only one independent component, which has to be constant in the absence of sources. Just as an electric field decays through Schwinger pair creation, this cosmological constant decays through membrane creation if A is coupled to a membrane, a process first described by Brown and Teitelboim [4].

Such tunneling processes can happen in flat as well as in curved spacetime. In addition, in curved spacetime, new effects can arise. It has been shown that topological defects such as circular loops of string, spherical domain walls, and monopole-antimonopole pairs can spontaneously nucleate during inflation in the early Universe [5]. These nucleations are somewhat analogous to particle production by an external gravitational field. Another consequence of space-time curvature is the possibility of true vacuum decay [6], through nucleation of *false* vacuum bubbles.

Nucleation processes can be described using the instanton methods [2]. The instantons are classical solutions of the Euclidean equations of motion, with appropriate boundary conditions. They are saddle points of the Euclidean path integral, and as such they provide the basis for a semiclassical evaluation of the partition function. The contribution of one instanton to the path integral has the form

$$Ce^{-S_E} , \tag{1}$$

where S_E is the Euclidean action of the instanton, and the prefactor C arises from Gaussian integration over small fluctuations around the instanton. The main part of this paper will be devoted to the calculation of the prefactors C for the class of processes mentioned above.

The instanton methods can be applied in flat and in curved backgrounds. One limitation of the formalism is, however, that the spacetime under consideration has to have real Euclidean sections. Here, we shall concentrate on de Sitter and Minkowski space. One reason for studying de Sitter space is that it describes the geometry of spacetime during inflation. A de Sitter space of dimension d can be defined as a hyperboloid embedded in a Minkowski space of dimension $d+1$:

$$\eta_{AB}X^A X^B = H^{-2} . \tag{2}$$

Here X^A are the coordinates in the embedding Minkowski space ($A = 0, ..., d$), H is the expansion rate during inflation and η_{AB} is the Minkowski metric. We use the metric convention $(-, +, ..., +)$. The Euclidean section of de Sitter space is obtained by analytically continuing the temporal coordinate X^0 to purely imaginary values. With this rotation the hyperboloid (2) becomes a d-sphere of radius H^{-1} embedded in flat Euclidean space (see Fig. 1).

In Minkowski space, at zero or at finite temperature, the nucleation rates can be related to the imaginary part of the free energy [7, 2], and they are essentially given by an expression of the form (1). In curved space, it is

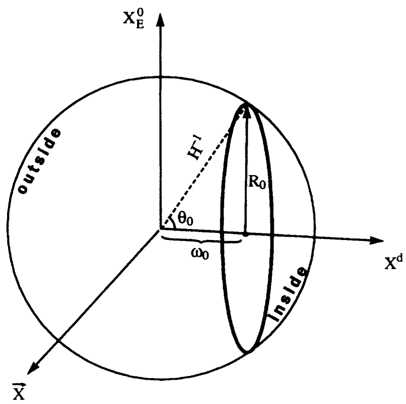

FIG. 1. Euclidean de Sitter space is a d-sphere of radius H^{-1}. The instantons for membrane creation and spontaneous nucleation of defects can be seen as spherical world sheets of dimension N and radius R_0 embedded in the d-sphere. For the case of spontaneous nucleation, the instantons have maximal radius $R_0 = H^{-1}$. For the case of membrane creation, the codimension of the world sheet is one, $d = N + 1$, and the instanton can be obtained by intersecting the d-sphere with a hyperplane at a distance ω_0 from the origin. The world sheet of the membrane divides the d-sphere into two regions, which we conventionally denote as the inside and the outside of the membrane. The value of the electric field in the outside region is taken to be equal to the background electric field before nucleation E_0. By Gauss' law, the electric field in the inside region is $E_0 - e$.

believed that the nucleation rates also have the exponential dependence (1), although the theory has not been developed to the same level of rigor than in flat space. When the size of the instantons is very small compared with H^{-1}, one expects that the usual flat space formulas should apply. However, for the spontaneous nucleation of topological defects and for the nucleation of *false* vacuum bubbles, the size of the instanton is comparable to the horizon scale H^{-1}. In such cases it is not clear how one should compute the nucleation rate, and one may even question whether such tunnelings can occur.

In this paper we take the heuristic point of view that these nucleations can indeed occur. We suggest a prescription for computing, from the semiclassical partition function, the equilibrium distribution of created membranes, bubbles and topological defects during inflation. This can be seen as a generalization of the formulas that one uses in flat space.

In 1+1 dimensions the process of membrane (or bubble) creation reduces to that of particle creation in an external field. In that case, the predictions of the instanton method can be compared with the results that one obtains by using the better understood method of Bogolubov transformations [8]. As we shall see, the results of both methods agree (in the limit where they are valid), even in the case when the size of the instanton is comparable to the horizon scale. Then, at least in 1+1 dimensions, the instanton prescription seems to be valid, and there is no reason to believe that it will not be valid in higher dimensions.

Finally, we should mention that throughout the paper, both the gravitational and the antisymmetric tensor fields are assumed to be external. The back reaction of the membrane on the antisymmetric field can be neglected when the charge e of the membrane is very small compared to the field strength. Also, in order that gravitational field of the membrane be negligible, the mass scale of the membrane should be sufficiently small (see, e.g., [5] for a comparison of instantons with and without self-gravity). Gravitational back-reaction effects are interesting in their own right, leading to qualitatively different behavior at large mass scales, but introducing new complications and problems [9, 10]. These will be left as subject for future research.

The plan of the paper is the following. In Sec. II we review the instantons for membrane production. These are essentially spherical Euclidean world sheets of dimension N and radius R_0 (representing the membrane) embedded in Euclidean de Sitter space, which is itself a sphere of radius H^{-1} and dimensionality $d = N + 1$. The radius R_0 is determined by the strength of the antisymmetric field, the charge of the membrane, its surface tension, and the expansion rate H. Special attention is paid to the analytic continuation of the instantons to Lorentzian signature, and the effect of de Sitter transformations on the resulting solutions. These Lorentzian solutions describe the motion of the membranes after nucleation.

As mentioned above, to calculate the prefactor C we need to integrate over small fluctuations around the instanton. In Sec. III the theory of such fluctuations is reviewed, with an emphasis in the so-called zero modes. The zero modes are perturbations which do not change the shape of the instanton, but correspond to infinitesimal translations of the solution as a whole. We make use of the covariant formalism developed in Refs. [11–13], according to which the world sheet fluctuations are represented by a scalar field ϕ "living" in the unperturbed world sheet, which has the meaning of a normal displacement. In Sec. IV, we briefly review the instantons for the spontaneous nucleation of topological defects during inflation, and the normalization of the corresponding zero modes.

Section V is devoted to the semiclassical evaluation of the partition function. The evaluation of the prefactor is seen to be formally equivalent to the evaluation of the effective action for a free scalar field in curved spacetime (this curved spacetime is the world sheet of the instanton). To illustrate the method, the instanton formalism is used to compute the partition function for a gas of massive particles at finite temperature, in the semiclassical limit $\mathcal{M} >> T$, where $\mathcal{M}$ is the particle's mass and T is the temperature.

In Sec. VI we compute nucleation rates in flat space, recovering known results for pair creation in 1+1 and 3+1 dimensions, and for bubble formation in 2+1 and 3+1 dimensions. The question of renormalization is briefly discussed in analogy with the renormalization of the effective action for a scalar field in curved space.

In Sec. VII we discuss the nucleation rates in curved spacetime. We find the size distribution of created membranes, bubbles and defects during inflation. We also give the momentum distribution for the case of pair creation.

Some conclusions are summarized in Sec. VIII. The computation of functional determinants on the N-sphere, necessary for the evaluation of the instanton prefactors, is done in the Appendix.

II. PRODUCTION OF MEMBRANES BY AN ANTISYMMETRIC TENSOR FIELD

In this section we describe the instantons for the creation of membranes by an antisymmetric tensor field and their analytic continuation to Lorentzian signature. These instantons also represent the formation of true (or false) vacuum bubbles in the thin wall limit.

A. Membrane coupled to an antisymmetric tensor field

Consider an antisymmetric tensor field of rank N, $A = A_{[\mu\cdots\rho]}dx^\mu\wedge\cdots\wedge dx^\rho$, interacting with the N-dimensional world sheet of a charged membrane [14] in a spacetime of dimension $d = N+1$ [4]. To keep the discussion simple, we shall consider the situation in which the field A is external, so that back reaction of the membrane on the field is ignored. The action is given by

$$S = -\mathcal{M}\int_\Sigma \sqrt{-\gamma}d^N\xi + e\int_\Sigma A\ . \tag{3}$$

The first term is the Nambu action, proportional to the area of the world sheet Σ, where γ is the determinant of the world sheet metric γ_{ab}, and ξ^a $(a = 0, ..., N-1)$ is a set of coordinates on Σ . In (3) $\mathcal{M}$ is just a constant. For N=1 this constant is the particle mass, for N=2 it is the tension of a string, and for $N=3$ it is the surface tension of a membrane. The second term is the generalization of the electromagnetic coupling $e\int A_\mu dx^\mu$.

The field strength $F = dA$ has only one independent component, the electric field E:

$$F = -E\tilde{\epsilon}\ , \tag{4}$$

where

$$\tilde{\epsilon} = \sqrt{|g|}\epsilon_{\mu\cdots\nu}dx^\mu\wedge\cdots\wedge dx^\nu\ . \tag{5}$$

Here $\epsilon_{\mu\cdots\nu}$ is the antisymmetric symbol, with $\epsilon_{0\cdots N} = 1$, and g is the determinant of the spacetime metric $g_{\mu\nu}$. If we added a kinetic term for A in the action, then the corresponding Maxwell's equations would imply [4] that in the absence of sources the electric field is constant E =const, whereas the effect of a charged membrane is to produce a discontinuity $\Delta E = e$ in the electric field as we go from one side of the membrane to the other. In other words, the only equation for E is Gauss' law, which is an equation of constraint [15], and so A does not actually have any field degrees of freedom.

The rate of membrane production can be computed using the instanton methods. For this, it is necessary to solve the Euclidean equations of motion. The Euclidean action S_E can be found by complexifying the temporal coordinate $x^0 \to -ix^0_E$, $d\xi^0 \to -id\xi^0_E$ and leaving the field strength $F^{\mu\cdots\nu} \to F_E^{\mu\cdots\nu}$ unchanged [4]. This requires that we complexify the antisymmetric field $A \to iA_E$. With such rotations the Euclidean action is found to be $(iS \to -S_E)$

$$S_E = \mathcal{M}\int_\Sigma \sqrt{\gamma}d^N\xi + e\int_\Sigma A\ . \tag{6}$$

Unlike the Lorentzian case, in Euclidean space we can consider closed world sheets, and actually these will be the ones relevant to membrane production. A closed world sheet divides spacetime into two regions. Following [4] we denote them as the "inside" and the "outside" of the world sheet (see Fig. 1). Note that in flat space there is a natural way to assign these labels, but in de Sitter space (that is on the N-sphere) these denominations are just conventional. We denote by "outside" the region in which, upon analytic continuation, the electric field does not change when the pair is created, maintaining its initial value E_0. By Gauss' law, the field in the inside region would be given by $E_i = E_0 - e$.

For closed world sheets we can use Stokes' theorem and (4) in (6) to find

$$S_E = \mathcal{M}\int_\Sigma \sqrt{\gamma}d^N\xi - eE_0\int_\mathcal{V} \tilde{\epsilon}\ , \tag{7}$$

where we have used a constant external electric field E_0. The integral in the last term is just the volume of the spacetime region "inside" the closed world sheet. The only effect of including back reaction on A would be to replace E_0 by $E_0 - (e/2)$ in (7). Notice that (7) is proportional to the area of the world sheet minus a term proportional to the volume enclosed by the world sheet. This has exactly the same form as the Euclidean action for the process of false vacuum decay through bubble nucleation [2], in the limit in which the thickness of the wall separating the true from the false vacua is much smaller than the radius of the bubble. Both processes are similar in many respects [4], the main difference being that membrane production by an antisymmetric tensor field can occur repeatedly at any given point in space, whereas vacuum decay occurs only once.

B. The instantons

The equation of motion following from (7) has been given, for instance, in Refs. [11, 13]

$$K^a_a = \gamma^{ab}K_{ab} = -\frac{eE_0}{\mathcal{M}}\ , \tag{8}$$

where K_{ab} is the extrinsic curvature of the Euclidean world sheet,

$$K_{ab} \equiv -e_{b\mu}\nabla_a n^\mu\ . \tag{9}$$

Here n^μ is the normal to the world sheet, and $e^\mu_b = \partial_b x^\mu(\xi^c)$ are the tangent vectors (our sign convention is that n^μ points toward the outside region).

In (1+1)-dimensional flat spacetime, the only solution of (8) is a circular world line of radius $R_0 = \mathcal{M}/eE_0$, $(x^0_E)^2 + (x^1)^2 = R_0^2$. The evolution of the pair after nucleation is given by the analytic continuation of this trajectory to Minkowski space:

$$-(x^0)^2 + (x^1)^2 = R_0^2 \ . \tag{10}$$

This hyperbola has two branches. The branch on the right represents a particle of charge e moving forward in time. The one on the left represents a particle of charge e moving backward in time, which is interpreted as an antiparticle of charge $-e$ moving forward in time. The particle and antiparticle pair nucleate at time $x^0 = 0$, separated by a distance $2R_0$ and with zero velocity. After that, due to the constant force exerted by the field, they start moving away from each other with constant proper acceleration R_0^{-1}.

In higher dimensions there are also N-spherical world sheets which are extrema of the action. These represent the nucleation of spherical membranes [4]. Let us consider directly the instantons in de Sitter space of radius H^{-1}. The flat space instantons can be obtained as the limiting case $H \to 0$.

With a spherical ansatz for the world sheet the action (7) takes the form

$$S_E = \mathcal{M}\mathcal{S}_N(R_0) - eE_0\mathcal{V}_N(\theta_0) \ . \tag{11}$$

Here

$$\mathcal{S}_N(R_0) = \frac{2\pi^{\frac{N+1}{2}}}{\Gamma\left(\frac{N+1}{2}\right)} R_0^N \tag{12}$$

is the surface of a world sheet of radius R_0, θ_0 is the polar angle on the d-sphere of radius H^{-1} (see Fig. 1) such that $R_0 = H^{-1}\sin\theta_0$, and $\mathcal{V}_N(\theta_0)$ is the volume of the d-sphere that is enclosed by the world sheet of radius R_0:

$$\mathcal{V}_N(\theta_0) = \frac{2\pi^{\frac{N+1}{2}}}{\Gamma\left(\frac{N+1}{2}\right)} H^{-(N+1)} \int_0^{\theta_0} \sin^N\theta d\theta \ . \tag{13}$$

Extremizing (11) with respect to θ_0 we find

$$\tan\theta_0 = NH\frac{\mathcal{M}}{eE_0} \ , \tag{14}$$

which means that the radius of the Euclidean world sheet is

$$R_0 = \frac{N\mathcal{M}}{(N^2H^2\mathcal{M}^2 + e^2E_0^2)^{1/2}} \ . \tag{15}$$

Substituting (15) back into (11) we find the Euclidean action for the instantons

$$S_E = 2\pi H^{-2}\left[(\mathcal{M}^2H^2 + e^2E_0^2)^{1/2} - eE_0\right] \quad (N=1) \ , \tag{16}$$

$$S_E = 4\pi H^{-3}\left[\mathcal{M}H - \frac{eE_0}{2}\arctan\frac{2H\mathcal{M}}{eE_0}\right] \quad (N=2) \ , \tag{17}$$

$$S_E = \frac{2}{3}\pi^2 H^{-4}\left[\frac{9H^2\mathcal{M}^2 + 2e^2E_0^2}{(9H^2\mathcal{M}^2 + e^2E_0^2)^{1/2}} - 2eE_0\right] \quad (N=3) \ . \tag{18}$$

An interesting feature of Eqs. (16)–(18) is that one finds instantons of finite action both for $e > 0$ and $e < 0$.

For $e > 0$ the electric field in the inside region decreases with respect to the initial value. From (14) this corresponds to $\theta_0 < \frac{\pi}{2}$. For $e < 0$ the electric field in the inside region actually *increases* with respect to the initial value. This corresponds to $\theta_0 > \frac{\pi}{2}$. In this case the "inside" region is actually larger than the "outside" one (see Fig. 1). Strictly speaking, if the electric field is treated as external, we should not say that the field increases or decreases in the inside region, but both cases should still be distinguished. For instance, in 1+1 dimensions, the instanton with $e > 0$ corresponds to the creation of a pair with the "screening" orientation. That is, after nucleation, the + charge is to the right of the inside region and the − charge is to the left (recall our convention $E_0 > 0$). On the other hand, for $e < 0$ the pair has the "antiscreening" orientation, with the + charge to the left and the − charge to the right. Similarly membranes can nucleate with two different orientations depending on the sign of e.

It might appear that the particles in pairs with the antiscreening orientation would move toward each other after nucleation, and eventually annihilate. However, as we shall see, the distance between both particles actually grows with time due to the inflationary expansion. We should also say that the antiscreening instantons are just as physical as the screening ones. In order to find agreement with the results obtained using Bogolubov transformations [8], both instantons have to be included. In the context of vacuum decay, the case $e > 0$ corresponds to the ordinary transition from false to true vacuum, whereas the case $e < 0$ corresponds to the decay of the true vacuum through nucleation of false vacuum bubbles [6].

In the limit when the electric field is switched off, $E_0 \to 0$, the instantons become spheres of maximal radius $R_0 \to H^{-1}$, and the action (16)–(18) reduces to

$$S_E = \mathcal{M}\mathcal{S}_N(H^{-1}) \ . \tag{19}$$

These are the instantons for the spontaneous nucleation of defects during inflation [5], which we shall consider in more detail later on. In general, for finite E_0, the action for $eE_0 > 0$ is always smaller than that for $eE_0 < 0$. This means that it is more probable to nucleate a screening membrane than an antiscreening one, in agreement with naive expectations.

In the flat space limit $H \to 0$, the antiscreening process is not possible. Only the action for the screening instanton $e > 0$ remains finite. For $e > 0$ and $H \to 0$ we have

$$R_0 = \frac{N\mathcal{M}}{eE_0} \tag{20}$$

and

$$S_E = \frac{\mathcal{M}}{N+1}\mathcal{S}_N(R_0) \ . \tag{21}$$

This expression reproduces the thin wall instanton action for vacuum decay in flat space [2], where $\mathcal{M}$ is the tension of the wall and eE_0 is the difference in energy density between the false and true vacua.

C. Analytic continuation

The evolution of the membranes after nucleation is given by the analytic continuation of the instantons back to Lorentzian signature. We have seen that, for the $d = 1+1$ case in flat space the instanton is a circle, and the analytic continuation is a hyperbola representing the world line of the particle and antiparticle accelerating away from each other. Note that the hyperbola (10) is centered at the origin $x^1 = 0$, but of course pairs can nucleate at other locations too. If we act on the instanton with a spacetime translation, the resulting trajectory $(x_E^0 - a_E)^2 + (x^1 - b)^2 = R_0^2$ is also an instanton. Thus the circle centered at the origin is just one solution out of a two parameter family. By analytically continuing $x_E^0 \to ix^0$ and $a_E \to ia$, we obtain a two parameter family of Lorentzian solutions

$$-(x^0 - a)^2 + (x^1 - b)^2 = R_0^2 , \tag{22}$$

which represent pairs nucleating at any spacetime point $x^\mu = (a, b)$.

Similar steps have to be taken to analytically continue the instantons describing the creation of membranes in de Sitter space. The instantons can be represented as the intersection of the d-sphere with a hyperplane at a distance

$$\omega_0 \equiv H^{-1} \cos\theta_0$$

from the origin, where θ_0 is given by (14), see Fig. 1. In the representation (2), letting $X^0 = -iX_E^0$, the instanton is given by

$$(X_E^0)^2 + \sum_{J=1}^{d} (X^J)^2 = H^{-2} ,$$
$$X^d = \omega_0 = (H^{-2} - R_0^2)^{1/2}, \tag{23}$$

where X^A are the coordinates in a "fictitious" embedding Euclidean space.

Following [5], to analytically continue this solution we choose the flat Friedmann-Robertson-Walker (FRW) coordinates in de Sitter space, $(t, \mathbf{x})$. In these, the metric takes the form

$$ds^2 = -dt^2 + e^{2Ht} d\mathbf{x}^2 . \tag{24}$$

These coordinates are related to X^A through the equations

$$X^0 = H^{-1} \sinh Ht + \tfrac{1}{2} H\mathbf{x}^2 e^{Ht},$$
$$X^d = H^{-1} \cosh Ht - \tfrac{1}{2} H\mathbf{x}^2 e^{Ht} , \tag{25}$$
$$\mathbf{X} = \mathbf{x} e^{Ht},$$

where the vector $\mathbf{X}$ has components X^J $(J = 1, ..., d-1)$. The coordinates $(t, \mathbf{x})$ cover only half of the hyperboloid (2).

Taking $X^d = \omega_0$ in (25), the world sheet of the membrane after nucleation is given by

$$\mathbf{x}^2 = H^{-2}(1 + e^{-2Ht}) - 2H^{-1}\omega_0 e^{-Ht} . \tag{26}$$

This solution represents a spherical membrane which is expanding in time, with physical radius given by

$$R^2 = H^{-2}(e^{2Ht} + 1) - 2H^{-1}\omega_0 e^{Ht} . \tag{27}$$

Note that sgn(ω_0)=sgn(e), but R never vanishes for either sign of e. In both cases, the radius grows like the scale factor at late times. For $N = 1$ the spherical "membrane" reduces to a pair of points, whose world line is given by $x = \pm(\mathbf{x}^2)^{1/2}$, with $\mathbf{x}^2$ given by (26). In this case R is one half of the physical distance between the particle and antiparticle in the pair.

As with the flat space case discussed above, the solution (26) belongs to a family of solutions which can be obtained from a d-parameter family of instantons. This family is obtained by applying O$(d+1)$ rotations to the instanton (23). The group O$(d+1)$ has $d(d+1)/2$ generators. Of these, $d(d-1)/2$ leave the instanton invariant. They correspond to rotations in the space $(X_E^0, \mathbf{X})$. The remaining d generators correspond to rotations in the (X^d, X_E^0) plane or in any of the (X^d, X^I) planes $(i = 1, ..., d-1)$. These generators which do *not* leave the instanton invariant are the so called zero modes. Their effect is to rotate the hyperplane $X^d = \omega_0$ in (23), effectively translating the center of the world sheet to a new location on the d-sphere.

Upon analytic continuation the parameters corresponding to rotations in the (X_E^0, X^J) plane $(J \neq 0)$ have to be complexified along with X_E^0 in order for the resulting solutions to be real. Recall that even in flat space, the parameter a_E had to be complexified to obtain (22). In the present case, rotations turn into boosts in the (X^0, X^J) plane when the angle α_E of rotation is complexified:

$$\alpha_E = i\alpha . \tag{28}$$

In this way, the group of rotations O$(d+1)$ becomes the group of de Sitter transformations O$(d,1)$ [which can also be thought of as the group of Lorentz transformations in the Minkowski space in which the hyperboloid (2) is embedded].

Let us consider the case $d = 1+1$ in some detail. The general world line after nucleation is obtained by taking $X_E^0 \to iX^0$ in (23) and then applying a boost in the (X^0, X^2) plane followed by a rotation in the (X^1, X^2) plane:

$$X'^0 = X^0 \cosh\alpha + X^2 \sinh\alpha ,$$
$$X'^1 = -(X^0 \sinh\alpha + X^2 \cosh\alpha)\sin\beta + X^1 \cos\beta , \tag{29}$$
$$X'^2 = (X^0 \sinh\alpha + X^2 \cosh\alpha)\cos\beta + X^1 \sin\beta.$$

Here α and β are arbitrary parameters.

Eliminating X^1 and X^0 from the previous equations one has

$X'^2 \cosh\alpha \cos\beta - X'^1 \cosh\alpha \sin\beta - X'^0 \sinh\alpha = X^2.$

Taking $X^2 = \omega_0$, dropping primes and using the transformations (25) we find, after some algebra, that the general world line after nucleation is given by

$$R^2 \equiv e^{2Ht}(x-x_0)^2 = H^{-2}(e^{2H(t-t_0)}+1) - 2H^{-1}\omega_0 e^{H(t-t_0)} , \quad (30)$$

where

$$x_0 = H^{-1}\frac{\cosh\alpha\sin\beta}{\cosh\alpha\cos\beta + \sinh\alpha} , \quad (31)$$

$$t_0 = H^{-1}\ln(\cosh\alpha\cos\beta + \sinh\alpha) . \quad (32)$$

Equation (30) represents a pair centered at point x_0. The parameter t_0 shall be referred to as the time of nucleation. For $R_0 \sim H^{-1}$ this is somewhat conventional, since there is no precise instant of time at which the pair nucleates [5]. The problem is that the concept of simultaneity becomes blurry at distances comparable to the horizon. Strictly speaking, all we can say is that solutions with different values of t_0 are time translations of one another. From a geometric point of view, however, it is clear that (x_0, t_0) represents the center of symmetry of the world sheet, and in this sense it is natural to think of it as the nucleation event.

Note also that there is no absolute value sign in the logarithm in (32). For $\omega_0 \neq 0$, the solutions with $\cosh\alpha\cos\beta + \sin\alpha > 0$ are qualitatively different from the ones with $\cosh\alpha\cos\beta + \sinh\alpha < 0$. Actually, we shall see that the latter ones are unphysical. They correspond to pairs whose "inside" region is centered at spatial infinity, so upon nucleation the electric field would change over an infinite (and disconnected) region of space. To find agreement with the Bogolubov method these solutions have to be discarded.

The same arguments can be repeated for $N > 1$. The general Lorentzian solution is a spherical membrane of physical radius (27) centered at any spacetime point [5].

III. PERTURBATIONS AND ZERO MODES

To compute the contribution of the instantons to the partition function we need to study small fluctuations of the instanton world sheet. For this it is very useful to adopt the covariant formalism developed in Refs. [11–13, 16], according to which the world sheet perturbations are represented by a scalar field "living" on Σ, which has the meaning of a normal displacement.

Denoting by x^μ the coordinates in de Sitter space, the instanton configuration will be expressed as $x^\mu(\xi^a)$. Consider now a slight deformation of the world sheet

$$\tilde{x}^\mu(\xi^a) = x^\mu(\xi^a) + \delta x^\mu(\xi^a) . \quad (33)$$

Since only deformations orthogonal to the world sheet are physically meaningful, we can set

$$\delta x^\mu(\xi^a) = \mathcal{M}^{-1/2}\phi(\xi^a) n^\mu . \quad (34)$$

Here n^μ is the normal to the world sheet and $\phi(\xi^a)$ is the normal displacement. The factor $\mathcal{M}^{-1/2}$ is inserted so that ϕ has the correct dimensions for a scalar field in N dimensions, $[\phi] = (\text{mass})^{(N/2)-1}$.

Actually, the equation of motion for ϕ can be derived from kinematical considerations. Since ϕ is a scalar on the world sheet, it has to satisfy a covariant equation. The only tensors available in Σ are the metric γ_{ab} and the extrinsic curvature K_{ab}, but because of the symmetries of our problem they are proportional to each other $K_{ab} \propto \gamma_{ab}$ (see, e.g., [5]). The only covariant second order differential equation that we can write down with such ingredients is

$$-\Delta\phi + M^2\phi = 0 , \quad (35)$$

where Δ is the Laplacian on the spherical world sheet. By symmetry, M has to be constant.

To determine the value of M we can use "known" solutions of (35): the zero modes. These are field modes for ϕ which do not correspond to true perturbations of Σ, but to the infinitesimal version of the rotations considered in the previous section. These solutions can be found from geometric considerations. Let the instanton be given by (23). The vector n^μ, orthogonal to Σ, can be thought of as a vector in the embedding flat space, with components n^A. As such, it is tangent to the d-sphere of radius H^{-1} centered at the origin, and orthogonal to the N-sphere of radius R_0 centered at $X^d = \omega_0$, $X^I = 0$ $(I \neq d)$. It is easy to see that $n^A = HR_0^{-1}(\mathbf{X}\omega_0, -R_0^2)$. Here $\mathbf{X}$ has components X^I $(I \neq d)$. A small rotation of angle α in the (X^J, X^d) plane induces the change

$$\delta X^J = \alpha X^d, \quad \delta X^d = -\alpha X^J .$$

This transforms Σ into a new world sheet which is also a solution of the equations of motion. Therefore, taking $X^d = \omega_0$ in the equations above, the field

$$\mathcal{M}^{-1/2}\phi(\xi^a) \equiv n^A \delta X_A(\xi^a) = \alpha(HR_0)^{-1}X^J(\xi^a) \quad (36)$$

has to be a solution of (35), for any $J = 0, ..., d-1$. As is well known, the Cartesian components $X^J(\xi^a)$ of the points on the N-sphere are linear combinations of the spherical harmonics with $L = 1$. The spherical harmonics are eigenfunctions of the Laplacian with the eigenvalue

$$\lambda_L = -L(L+N-1)R_0^{-2} \quad (L = 0, ..., \infty) ; \quad (37)$$

so, taking $L = 1$, we have $\Delta\phi = -NR_0^{-2}\phi$. Comparing with (35) we have the mass that we were looking for

$$M^2 = -NR_0^{-2} . \quad (38)$$

For $eE_0 \to 0$ this reduces to $M^2 = -NH^2$ [5].

Of course, Eqs. (35) and (38) can also be derived from a perturbative expansion of the action. Introducing (33) in the action and expanding to second order in ϕ, the action for the perturbed world sheet $\tilde{\Sigma}$ can be written as [11–13]

$$S_E[\tilde{\Sigma}] = \bar{S}_E[\Sigma] + S_E^{(2)}[\phi],$$

where $\bar{S}_E[\Sigma]$ is the action for the unperturbed instanton (16)–(18), and

$$S_E^{(2)}[\phi] = \frac{1}{2}\int d^N\xi\sqrt{\gamma}(-\phi\Delta\phi + M^2\phi^2) , \tag{39}$$

with Δ the Laplacian on Σ and

$$M^2 = \mathcal{R}^{(N)} - R^{(d)}_{\mu\nu}(g^{\mu\nu} - n^\mu n^\nu) - \left(\frac{eE_0}{\mathcal{M}}\right)^2 . \tag{40}$$

Here $\mathcal{R}^{(N)}$ is the Ricci scalar on Σ, and $R^{(d)}_{\mu\nu}$ and $g_{\mu\nu}$ are the Ricci tensor and the metric in the embedding de Sitter space [actually Eqs. (39) and (40) are valid for pertubations to any world sheet solution, embedded in an arbitrary curved spacetime of dimension $d = N + 1$]. In de Sitter space $R^{(d)}_{\mu\nu} = H^2(d-1)g_{\mu\nu}$, whereas on the N-spherical world sheet, $\mathcal{R}^{(N)} = \gamma^{ab}\, R^{(N)}_{ab} = N(N-1)R_0^{-2}$. Substituting in (40) and using (15) for R_0, the effective mass M^2 simplifies to $M^2 = -NR_0^{-2}$, in agreement with (38).

For later convenience we expand an arbitrary perturbation ϕ in terms of the (real) spherical harmonics on the N-sphere, ϕ_{LJ}:

$$\phi(\xi^a) = \sum_{LJ} C_{LJ}\phi_{LJ}(\xi^a) . \tag{41}$$

These satisfy $\Delta\phi_{LJ} = \lambda_L\phi_{LJ}$, with λ_L given by (37) and

$$\int_\Sigma \phi^2_{LJ}\sqrt{\gamma}d^N\xi = 1 . \tag{42}$$

The index J ($J = 0, ..., g_L - 1$), labels the degeneracy for given L where

$$g_L = \frac{(2L + N - 1)(N + L - 2)!}{L!(N-1)!} . \tag{43}$$

For $L = 1$ we have, with the normalization (42),

$$\phi_{1J} = \left(\frac{N+1}{R_0^2\mathcal{S}_N(R_0)}\right)^{1/2} X^J(\xi) , \tag{44}$$

where $\mathcal{S}_N$ is given by (12).

Comparing (36) with (41) and using (44) we find that, for an infinitesimal rotation of angle $d\alpha_J$ in the (X^J, X^n) plane,

$$dC_{1J} = H^{-1}d\alpha_J\left(\frac{\mathcal{M}\mathcal{S}_N(R_0)}{N+1}\right)^{1/2} . \tag{45}$$

This equation is often referred to as the normalization of the zero modes, and it will be important in order to interpret certain divergences in the semiclassical evaluation of the partition function. In the flat space limit, the square root on the right-hand side of (45) reduces to the familiar expression $S_E^{1/2}$ [2].

IV. NUCLEATION OF TOPOLOGICAL DEFECTS DURING INFLATION

We saw in Sec. II B that for $H \neq 0$ the action for the instantons remains finite when the external field is switched off. This corresponds to the spontaneous nucleation of membranes (or topological defects), due to the gravitational field alone.

Since the external field E_0 is zero, there is no need to restrict ourselves to codimension 1. Thus monopole pairs, strings and domain walls can spontaneously nucleate in d-dimensional de Sitter space, with $d > N$. The corresponding instantons are found by intersecting the d-sphere with the necessary number of hyperplanes through the origin:

$$\sum_{A=0}^{d} X^A X^A = H^{-2} ,$$
$$X^i = 0 \quad (i = N+1, ..., n), \tag{46}$$

which gives N-spheres of radius H^{-1} (here we use the lower-case Latin index i for later notational convenience).

When studying small perturbations for codimension larger than 1, we will have more scalar fields "living" on the world sheet, one for each normal direction. The generalization of (34) is

$$\delta x^\mu(\xi^a) = \mathcal{M}^{-1/2}\sum_{i=N+1}^{d}\phi^{(i)}(\xi^a)n^{(i)\mu} . \tag{47}$$

Taking the normal vectors to be perpendicular to the hyperplanes $X^i = 0$, the effective action for $\phi^{(i)}$ is [5, 11, 16]

$$S^{(2)}[\phi] = \frac{1}{2}\sum_{i=N+1}^{d}\int d^N\xi\sqrt{\gamma}[\phi^{(i)}(-\Delta + M^2)\phi^{(i)}] , \tag{48}$$

where M^2 is still given by (38) with $R_0 = H^{-1}$. Expanding in terms of spherical harmonics on the N-sphere, $\phi^{(i)} = \sum C^{(i)}_{LJ}\phi_{LJ}$, the normalization of the zero modes can be worked out in the same way as before. For an infinitesimal rotation in the (X^J, X^i) plane (where $J = 0, ..., N$, $i = N+1, ..., d$) of angle $d\alpha_J^{(i)}$ we have

$$dC^{(i)}_{1J} = H^{-1}d\alpha_J^{(i)}\left(\frac{\mathcal{M}\mathcal{S}_N(H^{-1})}{N+1}\right)^{1/2} . \tag{49}$$

Rotations of the X^J among themselves or of the X^i among themselves leave the instanton invariant, so these will not correspond to zero modes. Then, the total number of zero modes is $(N + 1)(d - N)$.

In general, of the $(N + 1)(d - N)$ zero modes, d will correspond to spacetime translations and the remaining $N(d-N-1)$ correspond to rotations in the spatial orientation of the defect [5]. For instance, monopole pairs in (3+1)-dimensional de Sitter space can nucleate with all possible orientations of the relative position, and loops of string can nucleate with all possible orientations of the plane of the loop.

V. SEMICLASSICAL PARTITION FUNCTION

For systems at finite temperature, the lifetime of a metastable state is related to the imaginary part of the free energy [2, 7, 17] [see Eq. (73) below]. The free energy is defined as

$$F \equiv -\beta^{-1} \ln Z, \tag{50}$$

where β^{-1} is the temperature and Z is the partition function

$$Z \equiv \mathrm{tr}[e^{-\beta \hat{H}}] , \tag{51}$$

with $\hat{H}$ the Hamiltonian of the system. The key to the semiclassical evaluation of Z is to first express it as a path integral. For instance, for the case of a single nonrelativistic particle in flat space moving in a potential $V(x)$, one has (see, e.g., Ref. [18] for a nice discussion)

$$Z = \int_{x(0)=x(\beta)} \mathcal{D}x(t) e^{-S_E} , \tag{52}$$

where the integral is over all paths which are periodic in Euclidean time with periodicity β, and S_E is the Euclidean action.

As is well known [19], de Sitter space behaves in some respects like a system at finite temperature $\beta^{-1} = H/2\pi$. One difference with flat space is, however, that on the sphere all directions are compact. Therefore, once in Euclidean action there is no actual distinction between temporal and spatial directions. In our case, the Euclidean action is given by (7) and the natural generalization of (52) is

$$Z = \int \mathcal{D}\Sigma(\xi^a) e^{-S_E[\Sigma]} , \tag{53}$$

where now the integral is taken over all closed world sheets.

In the semiclassical limit, Z will be dominated by the stationary points of S_E, and so it will be a sum of contributions from multi-instanton configurations $Z = Z_0 + Z_1 + \cdots$. Here Z_k is the contribution of a configuration with k widely separated instantons. This configuration has action $k\bar{S}_E$, where $\bar{S}_E$ is the action for one instanton. Hence Z_k will have the exponential dependence

$$Z_k \propto e^{-k\bar{S}_E} . \tag{54}$$

To find the preexponential factor, one has to integrate over small fluctuations around the stationary points. Equation (53) is rather formal, because we have not specified the measure of integration. However, if all we are interested in are small fluctuations around the instanton solutions, the integration over neighboring world sheets amounts to an ordinary path integral over the perturbation fields $\phi(\xi^a)$ that we introduced in Sec. III [20]. Therefore, we can write

$$Z = \sum_{k=0}^{\infty} \frac{e^{-k\bar{S}_E}}{k!} \left(\int \mathcal{D}\phi e^{-S_E^{(2)}[\phi]} \right)^k , \tag{55}$$

where $S_E^{(2)}$ is the second variation of the action, given in (39). The sum is over multi-instanton configurations. The path integrals over ϕ give the contribution of fluctuations around each one of the spherical world sheets. The $k!$ in the denominator can be understood as follows [2]. When integrating over ϕ we are integrating also over the zero modes. That means that we are integrating over all possible locations of the instantons in Euclidean space. Since the instantons are identical, we must divide by $k!$ to avoid overcounting.

Equation (55) can be rewritten as

$$Z = e^{Z_1} , \tag{56}$$

where

$$Z_1 \equiv e^{-\bar{S}_E} \int \mathcal{D}\phi \exp \left(\int \phi(\Delta + M^2)\phi\sqrt{\gamma} d^N\xi \right) . \tag{57}$$

This is just the path integral for a free scalar field on a curved background (the sphere), and can be calculated using well-known manipulations.

Following [21], the field is expanded in spherical harmonics ϕ_{LJ} [see (41)]. The integral over ϕ can then be expressed in terms of the coefficients C_{LJ}:

$$\mathcal{D}\phi = \prod_{LJ} \mu \frac{dC_{LJ}}{(2\pi)^{1/2}} . \tag{58}$$

Note that Z_1 is dimensionless, whereas C_{LJ} has dimensions of $(\mathrm{mass})^{-1}$. To render $\mathcal{D}\phi$ dimensionless one has to introduce the parameter μ with dimensions of mass. Using (41) and (42) one has

$$Z_1 = e^{-\bar{S}_E} \prod_{LJ} \int \mu \frac{dC_{LJ}}{(2\pi)^{1/2}} \exp \left[-\frac{1}{2} \left(\sum_{LJ} (M^2 - \lambda_L) C_{LJ}^2 \right) \right] , \tag{59}$$

where λ_L is given by (37). After Gaussian integration one obtains

$$Z_1 = e^{-\bar{S}_E} \prod_L [(\mu R_0) \Lambda_L^{-1/2}]^{g_L}$$
$$\equiv e^{-\bar{S}_E} (\det[(\mu R_0)^{-2} \hat{O}])^{-1/2} , \tag{60}$$

where g_L is given by (43). Here we have introduced the dimensionless operator $\hat{O} \equiv R_0^2(-\Delta + M^2)$, with eigenvalues $\Lambda_L = R_0^2(M^2 - \lambda_L)$ [see (37)].

Since the eigenvalues are known, the determinant can be calculated with the ζ-function regularization method. In terms of the generalized ζ function,

$$\zeta(z) \equiv \sum_L g_L \Lambda_L^{-z} , \tag{61}$$

the determinant can be expressed as

$$\det[(\mu R_0)^{-2} \hat{O}] = (\mu R_0)^{-2\zeta(0)} e^{-\zeta'(0)} . \tag{62}$$

The values of $\zeta(0)$ and $\zeta'(0)$ for the operator $\hat{O}$ on the N-sphere are calculated in the Appendix. For $N = 1$ we obtain $\zeta(0) = 0$ and $\zeta'(0) = -2\ln[2\sinh(\pi R_0 M)]$, so that

$$Z_1 = \frac{e^{-\bar{S}_E}}{2\sinh(\pi R_0 M)} \quad (N = 1) . \tag{63}$$

Note that, since $\zeta(0) = 0$, the dependence on the arbitrary renormalization scale μ disappears.

For world sheets of dimension $N > 1$ it turns out that $\zeta(0) = 0$ only for odd N [22]. In general, for even N the determinant will depend on the renormalization scale μ. As mentioned before, the calculation of Z_1 is the same as the calculation of the effective action for a free scalar field in a curved spacetime of dimension N. The appearance of a renormalization scale in that context is well known [23, 21] (in particular, this scale is responsible for the so-called trace anomaly, the quantum mechanical breakdown of conformal invariance). The difference between even and odd dimensions can be understood from the fact that in dimensional regularization the infinities come from the poles of the gamma function $\Gamma[j-(N/2)]$, where j is an integer. As a result, for odd N the effective action is finite after dimensional regularization, and the usual $\ln\mu$ terms do not appear. In general, for even N we have to live with the arbitrary scale μ, unless $\zeta(0)$ happens to vanish accidentally for our particular world sheet geometry and mass of the scalar field ϕ. It is well known that $\zeta(0)$ can be computed in terms of geometrical invariants. For $N = 2$ we have

$$\zeta(0) = \frac{1}{4\pi}\int d^2\xi^a\sqrt{\gamma}\left[-m^2 + \left(\frac{1}{6}-\xi\right)\mathcal{R}^{(2)}\right] , \tag{64}$$

where $m^2 + \xi\mathcal{R}^{(2)}$ is the quantity that we have denoted as M^2 [see Eq. (40)]. Notice that both in flat and de Sitter backgrounds, the term involving the external Ricci tensor in (40) is constant, and can be included in m^2. We shall come back to the discussion of μ and its role in the context of renormalization in the next section.

Our derivation of the semiclassical partition function has been rather formal. As a check, let us apply these ideas to a simple example where the result is known: the case of free particles at finite temperature in flat space. This will also illustrate the general procedure for dealing with the zero modes.

For a massive particle at finite temperature the Euclidean action is

$$S_E = \mathcal{M}\int_0^\beta [1+\dot{\mathbf{x}}^2(t_E)]^{1/2}dt_E , \tag{65}$$

where t_E is the Euclidean time and $\mathbf{x}(t_E)$ is the Euclidean trajectory, which has to be periodic $\mathbf{x}(t_E) = \mathbf{x}(t_E+\beta)$. The instantons for this system are straight lines wrapping around the compact temporal dimension $\mathbf{x}(t_E) = \mathbf{x}_0$ =const. The second variation of the action is found by substituting $\mathbf{x}(t_E) = \mathbf{x}_0 + \mathcal{M}^{-1/2}\boldsymbol{\phi}(t_E)$ in (65):

$$S_E^{(2)} = \frac{1}{2}\int_0^\beta (\dot{\boldsymbol{\phi}})^2 dt_E . \tag{66}$$

The fluctuations $\boldsymbol{\phi}$ can be thought of as a set of massless scalar fields in 0+1 dimensions. The zero mode solution $\boldsymbol{\phi}$ =const amounts to a spatial translation of the original solution $\mathbf{x} = \mathbf{x}_0$.

For simplicity we may start by considering only one spatial dimension transverse to the world line, hence only one field ϕ. Expanding ϕ in "spherical harmonics" on the one-sphere,

$$\phi = \beta^{-1/2}C_0 + \sqrt{\frac{2}{\beta}}\sum_{L=1}^{\infty}\left\{C_L\cos\left[L\frac{2\pi t_E}{\beta}\right] + D_L\sin\left[L\frac{2\pi t_E}{\beta}\right]\right\} , \tag{67}$$

we see that the zero mode is in the $L = 0$ sector. An infinitesimal translation dx_0 corresponds to

$$dC_0 = dx_0\bar{S}_E^{1/2} , \tag{68}$$

where $\bar{S}_E = \mathcal{M}\beta$ is the instanton action.

The perturbation field ϕ is massless, $M = 0$, and so the expression (63) diverges because of the vanishing denominator. This is to be expected because the operator $\hat{O}$ has a zero eigenvalue corresponding to the translational zero mode. However, noting that $(2\pi)^{-1/2}\int dC_0 \exp(-M^2C_0^2/2) = M^{-1}$, the divergence at $M \to 0$ can be avoided by leaving the integral over dC_0 undone and by excluding the factor M^{-1} from the determinantal factor. Then we can write

$$dZ_1 = (\det{}'[(\mu R_0)^{-2}\hat{O}])^{-1/2}e^{-\mathcal{M}\beta}\frac{dC_0}{(2\pi)^{1/2}} , \tag{69}$$

where

$$\det{}'[(\mu R_0)^{-2}\hat{O}] \equiv \lim_{M\to 0}\frac{\det[(\mu R_0)^{-2}\hat{O}]}{M^2} = (2\pi R_0)^2 \tag{70}$$

is the determinant without the zero eigenvalue.

Using (68) and $2\pi R_0 = \beta$ we have

$$dZ_1 = dx_0\left(\frac{\mathcal{M}}{2\pi\beta}\right)^{1/2} e^{-\mathcal{M}\beta}.$$

Increasing the number of transverse dimensions to three, each transverse dimension brings an additional power to the preexponential factor. Interpreting d^3x_0 as the volume element and setting $T = \beta^{-1}$ we have

$$Z_1 = V\left(\frac{\mathcal{M}T}{2\pi}\right)^{3/2} e^{-\mathcal{M}/T} . \tag{71}$$

This is the correct expression for the "one-particle" partition function of an ideal gas with the Maxwell-Boltzmann distribution [note that the instanton method is only valid when the exponent in (71) is large, in which case there is no difference between bosonic and fermionic distributions]. The grand canonical partition function is obtained, according to (56), by exponentiating this expression.

Actually, Eq. (56) gives the partition function for the case of vanishing chemical potential, something that we have tacitly assumed in our derivation. The effect of a chemical potential $\tilde{\mu}$ is to replace Z_1 in (56) by $e^{\tilde{\mu}\beta}Z_1$. The number of particles $\mathcal{N}$ in the volume V is then given by

$$\mathcal{N} = \beta^{-1}\left(\frac{\partial\ln Z}{\partial\tilde{\mu}}\right)_{\beta,V} = e^{\tilde{\mu}\beta}Z_1.$$

For vanishing $\tilde{\mu}$, we have

$$\mathcal{N} = Z_1 \ . \tag{72}$$

Therefore Z_1 is also the equilibrium number of particles.

VI. NUCLEATION RATES IN FLAT SPACE

In flat space and at sufficiently low temperatures, the decay rate of a metastable state is given by [7]

$$\Gamma = 2|\text{Im } F| = 2\beta^{-1}|\text{Im } Z_1| \ . \tag{73}$$

(For temperatures $\beta^{-1} > R_0^{-1}$, this formula has to be modified [7].) As shown in the previous section, the calculation of Z_1 reduces to the calculation of a functional determinant. The fact that F has an imaginary part is due to the fact that the action $S^{(2)}$ in (39) has a negative mode, corresponding to $L = 0$, so the determinant will be negative. Upon taking the square root a factor of i will emerge.

The membrane creation rate can be expressed as

$$d\Gamma = \tfrac{1}{2} \times 2\beta^{-1}|\det{}'[(\mu R_0)^{-2}\hat{O}]|^{-1/2}e^{-\bar{S}_E}|J|dV dt_E \ , \tag{74}$$

where the Jacobian is given by

$$|J| = \frac{\prod_{J=0}^{d-1}(2\pi)^{-1/2}dC_{1J}}{dV dt_E} \ . \tag{75}$$

This equation follows from (73) and (60). As before [see (70)] the prime in the determinant means that the $(N+1)$ zero modes (which are now in the $L = 1$ sector) are omitted, because the integration over dC_{1J} is left undone:

$$\det{}'[(\mu R_0)^{-2}\hat{O}] \equiv \lim_{M^2\to -NR_0^2} \frac{(\mu R_0)^{-2\zeta(0)}e^{-\zeta'(0)}}{(M^2 + NR_0^2)^{N+1}} \ . \tag{76}$$

The Jacobian, which is needed to change variables from dC_{1J} to the Euclidean spacetime volume element $dV\ dt_E$, can be read off from (45) [in the limit $H \to 0$ we replace $H^{-1}d\alpha_J$ by dt_E (for $J = 0$) or by $d\mathbf{x}$ (for $J = 1, ..., d-1$)]:

$$|J| = \left(\frac{\mathcal{M}\mathcal{S}_N(R_0)}{2\pi(N+1)}\right)^{\frac{N+1}{2}} \ .$$

The overall factor of 1/2 on the right-hand side (RHS) of (74) is explained in Refs. [24, 2]. It arises because the free energy of an unstable state can only be defined by analytic continuation from a Hamiltonian in which the same state is stable. As a result, the contour of integration over the negative mode dC_0 has to be deformed into the complex plane in such a way that only half of the saddle point contributes to ImF. We should note, however, that in the derivation of (73) given in [7], these considerations are not really relevant; and $2|\text{Im } F|$ is essentially a convention to denote the RHS of (74).

Integration over Euclidean time t_E cancels the factor of β^{-1}, and we are left with the rate per unit volume:

$$\frac{d\mathcal{N}}{dtdV} = \frac{\Gamma}{V} = \left(\frac{\mathcal{M}\mathcal{S}_N(R_0)}{2\pi(N+1)}\right)^{\frac{N+1}{2}} |\det{}'[(\mu R_0)^{-2}\hat{O}]|^{-1/2}e^{-\bar{S}_E} \ , \tag{77}$$

where $\mathcal{N}$ is the number of membranes created. Let us now evaluate this rate for spacetimes of different dimensionalities.

For pair creation in 1+1 dimensions, from (76), (A7), and (77), the creation rate per unit length is

$$\frac{\Gamma}{L} = \frac{\mathcal{M}}{2\pi R_0}e^{-\bar{S}_E} \ . \tag{78}$$

Taking $R_0 = \mathcal{M}/eE_0$, we have

$$\frac{\Gamma}{L} = \frac{eE_0}{2\pi}\exp\left(-\frac{\pi\mathcal{M}^2}{eE_0}\right).$$

This can be compared with the results of Stone [25], who computed the vacuum decay rate in the sine-Gordon theory with nondegenerate vacua. His one-loop result was

$$\frac{\Gamma}{L} = \frac{eE_0}{2\pi}|\ln(1 - e^{-\frac{\pi\mathcal{M}^2}{eE_0}})|,$$

where $\mathcal{M}$ is the mass of the kink and eE_0 is the vacuum energy density gap between neighboring vacua. As expected, the instanton calculation gives a good approximation when the Euclidean action is large $\bar{S}_E >> 1$. Equation (78) also gives the rate at which a metastable cosmic string will break up by nucleating pairs of monopoles [3]. In that case eE_0 should be replaced by the string tension and $\mathcal{M}$ by the mass of the monopoles.

For pair creation in 3+1 dimensions, in addition to the radial perturbations of mass $M^2 = -R_0^2$, we have perturbations ϕ_y and ϕ_z which are transverse to the plane of the instanton. These behave like massless fields $M^2 = 0$. Each field contributes its own determinantal factor, which for ϕ_y and ϕ_z is given by (70). Also, by considering the normalization of the corresponding zero modes, it is easy to see that each contributes a factor $(R_0\mathcal{M})^{1/2}$ to the Jacobian. With this, Eq. (78) is modified into

$$\frac{\Gamma}{V} = \frac{(eE_0)^2}{8\pi^3}e^{-\frac{\pi\mathcal{M}^2}{eE_0}} \ . \tag{79}$$

This coincides with the rate of production of charged bosons in scalar electrodynamics:

$$\frac{\Gamma}{V} = \frac{(eE_0)^2}{8\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}e^{-n\frac{\pi\mathcal{M}^2}{eE_0}}[1 + O(e^2)] \ , \tag{80}$$

in the limit $S_E >> 1$. (For a more thorough account of monopole production by a magnetic field, and pair production in the strong coupling regime see Refs. [26, 27].)

For string creation in 2+1 dimensions, from (77) and (A9),

$$\frac{\Gamma}{V} = \left(\frac{\mathcal{M}\mathcal{S}_2(R_0)}{6\pi}\right)^{3/2} (\mu R_0)^{7/3} R_0^{-3} e^{-\bar{S}_E} \; . \tag{81}$$

Note that, since $\zeta(0) \neq 0$, the determinant depends explicitly on the renormalization scale μ [the exponent 7/3 can also be derived from (64)]. Because of the arbitrariness in μ we cannot give an absolute estimate of the nucleation rate the way we do for $N = 1$ or $N = 3$. However, since μ is a constant, we can still predict how the rate changes when we change the external field E_0 (that is, when we change R_0). It is seen that the dependence of the prefactor on R_0 is more complicated than what one would have guessed from dimensional analysis.

So far, by using the ζ-function method, we have avoided the question of ultraviolet divergences, since they are automatically removed by analytic continuation [23] (see also [28] for a recent discussion, and references therein). However, we should recall that the functional determinants contain such divergences, and that these can be eliminated by suitable counterterms in the action. For $N = 2$, all divergences can be removed by counterterms of the form [23]

$$c \int d^2\xi \sqrt{\gamma} + d \int d^2\xi \sqrt{\gamma} R.$$

The first term is a contribution to the membrane tension. The second is a topological invariant which does not contribute to the equations of motion. Note, from (62) and (64), that a rescaling of the arbitrary parameter μ can be reabsorbed in a redefinition of c and d. Then, we can eliminate the renormalized c by rescaling μ, and the renormalized d by shifting the string tension $\mathcal{M}$. In Ref. [29] a different approach was followed in which the product over all eigenvalues was cut off at some physical scale μ. This method also produced the factor $(\mu R_0)^{7/3}$, which was referred to as the "universal term" [29, 30].

Finally, for membrane creation in 3+1 dimensions, using (77) and (A13) we have

$$\frac{\Gamma}{V} = \left(\frac{\mathcal{M}\mathcal{S}_3(R_0)}{8\pi}\right)^{2} \frac{4R_0^{-4}}{\pi^2} e^{\zeta_R'(-2)} e^{-\bar{S}_E} \; . \tag{82}$$

This can be compared with the results of Affleck [30]. He studied the decay rate of false vacuum in the theory of a scalar field with a symmetry breaking potential and nondegenerate vacua, in the limit in which the thickness of the wall separating the true from the false vacuum is much smaller than the radius of the bubble at nucleation. The first factor on the RHS of (82) is the Jacobian $|J|$ that comes from the normalization of the zero modes. The rest of the preexponential factor coincides with what Affleck calls the "universal terms." These are due to fluctuations of the world sheet and therefore are independent of the details of the field theoretic model. The evaluaton of quantum corrections to the effective action due to finite thickness of the wall is in itself an interesting subject, and the corrections can be important in realistic theories [30, 26, 27, 31].

VII. NUCLEATION RATES IN CURVED SPACE

Although the calculation of determinants for the instantons in de Sitter space offers no special problems (we only need to substitute the appropriate values of R_0 in the expressions found in the Appendix); the definition of a nucleation rate in de Sitter is a more subtle issue which has often been eluded in the literature. One possibility would be to simply use Eq. (77), interpreting dV as the physical volume element at the time of nucleation dV_0

$$d\Gamma = |\lambda| dV_0 \; , \tag{83}$$

where $|\lambda|$ is defined as the RHS of (77).

Although we believe that this expression is correct (when properly interpreted) it clearly needs further justification. First of all, the physical volume element, dV_0, is proportional to a power of the scale factor e^{Ht_0} at the time of nucleation. If Eq. (77) was derived from a purely Euclidean calculation, how does the exponential of a Lorentzian time find its way into the RHS of (83)? Also, as pointed out in [5], the time of nucleation in de Sitter space is a somewhat ambiguous concept when the size of the instantons is comparable to the horizon, and in principle it is not clear what time one should use in dV_0.

These difficulties prompt us to search for an alternative way of calculating nucleation rates in de Sitter space, which does not take (73) as the starting point. Recalling that de Sitter space behaves in some respects like a thermodynamical system, one can try to estimate directly the equilibrium distribution of membranes. For this one can use Eq. (72), with $\mathcal{N}$ the number of membranes and Z_1 given by (57). It is clear that

$$d\mathcal{N} = (\det{}'[(\mu R_0)^{-2}\hat{O}])^{-1/2} e^{-\bar{S}_E} \prod_{J=0}^{d-1} (2\pi)^{-1/2} dC_{1J}.$$

Using (45), we have

$$d\mathcal{N} = \lambda H^{-d} \prod_{J=0}^{d-1} d\alpha_J \; ,$$

where

$$\lambda = \left(\frac{\mathcal{M}\mathcal{S}_N(R_0)}{2\pi(N+1)}\right)^{\frac{N+1}{2}} (\det{}'[(\mu R_0)^{-2}\hat{O}])^{-1/2} e^{-\bar{S}_E} \; . \tag{84}$$

Equation (45) is valid only for infinitesimal rotations, and in that case $\prod d\alpha_J$ can be identified with the differential solid angle on the d-sphere, $d\Omega$, within which the center of the instanton world sheet is to be found:

$$d\mathcal{N} = \lambda H^{-d} d\Omega \; . \tag{85}$$

One can interpret this equation as the "equilibrium distribution of instantons" in Euclidean space.

Of course, an equilibrium distribution of instantons is not a measurable object. However, it may be conjectured that the equilibrium distribution of membranes in the Lorentzian section is given by the analytic continuation of the previous object to real time. This prescription is just heuristic, and we do not know how to justify it further, except by saying that it reduces to (77) in flat space and that it is quite natural from the mathematical point of view.

To see how the analytic continuation is done, let us consider, for simplicity, the (1+1)-dimensional case (higher dimensional cases are completely analogous). Then (85) reads $d\mathcal{N} = \lambda H^{-2}\cos\alpha_E d\alpha_E d\beta$, where α_E and β are polar and azimuthal angles on the two-sphere. Upon analytic continuation $\alpha_E = i\alpha$ [see (28)],

$$d\mathcal{N} = |\lambda| H^{-2}\cosh\alpha\, d\alpha\, d\beta.$$

Note that the factor of i from $d\alpha_E$ cancels the imaginary factor from the square root of the determinant in λ, so that $d\mathcal{N}$ is actually real. Using Eqs. (31) and (32), one can change variables from (α,β) to (x_0,t_0). The Jacobian is

$$\left|\frac{\partial(t_0,x_0)}{\partial(\alpha,\beta)}\right| = H^{-2}\cosh\alpha e^{-Ht_0},$$

and therefore

$$d\mathcal{N} = |\lambda| e^{Ht_0} dx_0 dt_0 \ . \tag{86}$$

Generalizing to spacetimes of arbitrary dimension we have

$$d\mathcal{N} = |\lambda| e^{NHt_0} d\mathbf{x}_0 dt_0, \tag{87}$$

which is very similar to (83), but now all ambiguities in dV_0 have been resolved.

An equation of the same form as the previous one was given in [5], based on kinematical considerations. However, the parameter λ was left unspecified. Like in [5], Eq. (87) is a distribution in the space of parameters x_0 and t_0, and it is therefore independent of the time of observation.

Following [5], we can use (30) to express t_0 in terms of the physical radius R, and thus find the size distribution of membranes (or bubbles):

$$\frac{d\mathcal{N}}{dV_{\text{phys}}} = \frac{|\lambda|}{H^d}\frac{R(R^2-R_0^2)^{-1/2}}{\left[\omega_0 + (R^2-R_0^2)^{1/2}\right]^d} dR,$$

where $dV_{\text{phys}} = \exp(NHt) d^N\mathbf{x}_0$. Notice that the distribution diverges at the lower end $R \to H^{-1}$ when $\omega_0 \le 0$. This divergence was interpreted in [5]. For large radii, one finds the scale invariant distribution

$$\frac{d\mathcal{N}}{dV_{\text{phys}}} \approx \frac{|\lambda|}{H^d}\frac{dR}{R^d} \ , \tag{88}$$

which depends on the external field E_0 and membrane tension only through the coefficient λ. The fact that the size distributions are time independent is easy to understand. As the bubbles are created, they are stretched and diluted by the inflationary expansion, giving rise to a stationary distribution of sizes [5].

The "nucleation rates" $|\lambda|$ for $d = 2,3$, and 4 can be read off from the RHS of (78), (81), and (82), respectively, where R_0 is given by (15) and $\bar{S}_E$ by (16)–(18). This covers the case of codimension one, $d = N+1$. For completeness, we shall also consider the case of strings and monopole pairs spontaneously nucleating in 3+1 dimensions.

For the case of strings, the codimension is 2, and so there will be two independent perturbation fields $\phi^{(i)}$ and two determinantal prefactors. From (81) it is clear that λ will contain one factor of $\mathcal{M}^3$ and a factor of $\mu^{14/3}$, where μ is the renormalization scale. Taking $R_0 = H^{-1}$, the rest will be a numerical factor (which can be absorbed in μ), times the appropriate power of H necessary to give λ the dimensions of (mass)^4:

$$|\lambda| = \mu^{14/3}\mathcal{M}^3 H^{-11/3} e^{-\bar{S}_E}.$$

Because of the arbitrary renormalization scale, we cannot obtain an absolute estimate for $|\lambda|$, but only its dependence on the expansion rate H. Like in Sec. VI, to obtain a crude absolute estimate one can set $\mu \sim \mathcal{M}^{1/2}$, which gives

$$|\lambda| \sim \left(\frac{\mathcal{M}}{H}\right)^{23/3} H^4 \exp(-4\pi\mathcal{M}H^{-2}) \ . \tag{89}$$

For $\mathcal{M} >> H$ this can be considerably larger than the naive dimensional estimate $|\lambda| \sim H^4\exp(-\bar{S}_E)$ mentioned in [5], or even the more sophisticated $|\lambda| \sim \mathcal{M}^3 H \exp(-\bar{S}_E)$. Unfortunately, the existence of an arbitrary renormalization scale leaves us quite uncertain as to the overall normalization of (89).

Let us now consider the case of pairs spontaneously nucleating in $3+1$ dimensions. The codimension is 3 and so there will be 3 perturbation fields $\phi^{(i)}$, each one of them with mass $M^2 = -H^{-2}$, contributing to the determinantal prefactor. Each field has 2 zero modes [in the $L=1$ sector, see (49)], which makes a total of 6. Four of them correspond to spacetime translations and two of them to changes in the orientation of the monopole pair in three-dimensional space. Thus we have

$$|\lambda| = \left(\frac{\mathcal{M}H}{2\pi}\right)^3 \left(\frac{4\pi}{H^2}\right) e^{-\bar{S}_E} \quad (d = 3+1) \ . \tag{90}$$

The factor of $(\mathcal{M}H/2\pi)^3$ comes from taking the third power of the prefactor in (78), with $R_0 = H^{-1}$. The factor $4\pi H^{-2}$ is a correction due to the fact that two of the zero modes represent changes in the angular orientation of the pair, rather than translations. For each angular variable, the Jacobian (75) has an extra power of H in the denominator [see (49)]. Integration over all possible orientations gives the factor of 4π.

For the case of pair creation a "size" distribution such as (88) is not very useful. Instead, it is more convenient to find the momentum distribution of particles. Let us find the conserved momentum as a function of the time of nucleation t_0. In 1+1 de Sitter space, the vector potential

$$A_\mu = -H^{-1}E_0 e^{Ht}\delta_{\mu x} \qquad (91)$$

represents a constant electric field (note that $F_{\mu\nu}F^{\mu\nu} = 2E_0^2$). With this, the action for the point particle coupled to A_μ reads

$$S = -\mathcal{M}\int (\dot{t} - \dot{x}^2 e^{2Ht})^{1/2} d\tau - \frac{eE_0}{H}\int e^{Ht}\dot{x}d\tau \ , \qquad (92)$$

where τ is the proper time and an overdot denotes derivative with respect to τ. Since the Lagrangian does not depend on x, the momentum

$$k \equiv \frac{\partial L}{\partial \dot{x}} = \mathcal{M}\dot{x}e^{2Ht} - eE_0 e^{Ht},$$

is conserved. Setting $x_0 = 0$ in (30) we have

$$\frac{dx}{dt} = \frac{H\omega_0 e^{-H(t_0+t)} - e^{-2Ht}}{Hx}$$

and from $\dot{t}^2 - \dot{x}^2 e^{2Ht} = 1$, it follows that $\dot{t} = e^{Ht_0}H|x|(1 - H^2\omega_0^2)^{-1/2}$. Therefore,

$$k = -\frac{\mathcal{M}}{HR_0}e^{Ht_0}\mathrm{sgn}(x) \ , \qquad (93)$$

where we have taken into account that the particle to the left of the inside region has charge of opposite sign.

Using this equation in (86), with $|\lambda|$ given by the right-hand side of (78), we have

$$\frac{d\mathcal{N}}{dx_0} = e^{-\bar{S}_E}\frac{dk}{2\pi} \quad (d = 1+1) \ . \qquad (94)$$

In 3+1 dimensions (without electric field), using (87), (90), and (93), we have

$$\frac{d\mathcal{N}}{d^3x_0} = \exp\left(-\frac{2\pi\mathcal{M}}{H}\right)\frac{k^2dk}{2\pi^2} \quad (d = 3+1) \ . \qquad (95)$$

Note that the momentum distributions are flat, as expected on the grounds of scale invariance. The distributions (94) and (95) can be compared with the results of a calculation based on second quantization [8]. Both calculations agree in the limit $\bar{S}_E >> 1$, not only in the exponential dependence but also on the prefactor.

At any given time t we only need to integrate up to a cutoff momentum

$$|k| \sim \frac{\mathcal{M}}{HR_0}e^{Ht},$$

since particles with higher value of the coordinate momentum have not been created yet [see (93)]. Then

$$n \sim \frac{1}{2\pi}\frac{\mathcal{M}}{HR_0}e^{-\bar{S}_E} \quad (d = 1+1)$$

and

$$n \sim \frac{\mathcal{M}^3}{6\pi^2}e^{-\frac{2\pi\mathcal{M}}{H}} \quad (d = 3+1),$$

where n is the number density of particles per unit physical volume. As noted in [5], for the case of vanishing electric field the distribution of particles contains a Boltzmann factor $\exp(-\mathcal{M}/T)$ where $T = H/2\pi$ is the Gibbons-Hawking temperature [19].

VIII. SUMMARY AND CONCLUSIONS

We have computed the nucleation rates for the process of membrane creation by an antisymmetric tensor field in a spacetime of dimension $d = N+1$, for N =1, 2, and 3.

To this end, we have evaluated the contribution of the relevant instantons to the semiclassical partition function. These instantons are N-spherical world sheets of radius R_0 [given by (15)] embedded in a Euclidean de Sitter background, which is itself a d-sphere of radius H^{-1}. The flat space instantons are obtained by taking $H \to 0$. We have discussed the analytic continuation of the instantons, describing the motion of the membranes after nucleation. The Lorentzian solutions are spherical membranes which at late times expand like the scale factor in the flat inflationary Friedmann-Robertson-Walker (FRW) model. The effect of de Sitter transformations on a given solution corresponds to spacetime translations of the solution.

To evaluate the preexponential factor of the instanton contribution, it is necessary to study small fluctuations of the instanton world sheet. We have reviewed the covariant theory of such perturbations, according to which the normal displacement of the world sheet is viewed as a scalar field ϕ living on the unperturbed world sheet. We have given a kinematical derivation of the equations of motion for ϕ, based on the fact that the zero modes, which correspond to infinitesimal translations of the instanton, have to be solutions.

The evaluation of the prefactor is seen to be equivalent to the calculation of the effective action for a free scalar field (the field ϕ mentioned above) living in a curved background (the N-sphere). The functional determinants that arise from Gaussian integration can be explicitly calculated using ζ-function regularization. This method automatically removes all ultraviolet divergences. In the case $N = 2$, an arbitrary renormalization scale μ appears in the final result. Rescalings of μ are seen to be equivalent to a finite renormalization of the membrane tension and of a new term which has to be added to the action. This new term is just the Einstein-Hilbert action on the world sheet, which for $N = 2$ is a topological invariant and hence does not contribute to the equations of motion.

In flat space, the nucleation rates are obtained using the standard formula which relates them to the imaginary part of the free energy [see (73)]. We have recovered known results for the production of kinks in $1+1$ dimensions, charged pairs in $3+1$ dimensions and "bubble" formation in 2+1 and 3+1 dimensions. Our results apply only to the case when the membranes are infinitely thin, having no internal structure. The effect of finite thickness of the membrane can be important in realistic field theories [30, 26, 27, 31].

In de Sitter space, there is no standard procedure for the calculation of nucleation rates. We have introduced a heuristic prescription to obtain the distribution of nucleated objects during inflation, following ideas related to the treatment of this problem in Ref. [5]. This distribution is obtained from the one instanton contribution to the partition function. The parameters corresponding to the zero mode rotations, which form a subgroup of $O(d+1)$, have to be analytically continued along with the instanton so that they correspond to a subgroup of the de Sitter group $O(d,1)$. With these manipulations, the one instanton contribution to the partition function lends itself to interpretation as a distribution of membranes in a space of parameters. The parameters can be chosen to be the place and time of nucleation. In flat space this prescription reduces to the standard formula (77) for the calculation of nucleation rates. From the parameter distribution one can easily find the size distribution of membranes in the inflationary universe, which turns out to be stationary and nearly scale invariant [see (88)].

For the case of pair creation, we have given the distribution in terms of the conserved coordinate momentum k. This distribution turns out to be independent of k (as expected on the grounds of scale invariance) with an upper cutoff $k_{\rm phys}^2 \approx \mathcal{M}^2 + (eE_0^2/H^2)$, where e is the charge of the particle, E_0 is the electric field, and $k_{\rm phys} = e^{-2Ht}k$ is the physical momentum. This cutoff corresponds to the momentum of the particle at the time of nucleation t_0 (the physical momentum is subsequently redshifted). For the case of pairs, the results can be compared with those obtained by using the formalism of second quantization [8]. Both methods agree, not only in the exponential dependence but also in the prefactors. This is true even when the size of the instantons is comparable to the horizon size.

To summarize, our results seem to indicate that the spontaneous nucleation of defects during inflation and the decay of false or true vacuum through nucleation of true or false vacuum bubbles is well described by the instanton formalism, at least in 1+1 dimensions. Also, that our prescription for finding the equilibrium distribution of nucleated objects from the semiclassical partition function is correct. Although we have not attempted to give a rigorous justification to this prescription, we hope that with the examination of further examples a clearer picture will emerge. After this paper was completed, we became aware of Refs. [33], dealing with the semiclassical approximation to the path integral. The methods developed in these references may be useful to provide a rigorous foundation to our prescription. Work along these lines is currently under way.

Finally, some comments on negative modes. Coleman has shown [34] that in flat space and at zero temperature an instanton describing the decay of a metastable state has one and only one negative mode. As noted in [5], for the instantons describing the spontaneous nucleation of defects during inflation the number of negative modes is equal to the codimension of the world sheet. It was shown in [5] that this is not in direct contradiction with Coleman's theorem, which does not apply in de Sitter space (or in flat space at finite temperature). Still, the question remains of whether the wrong number of negative modes renders the instanton "unphysical." From the analysis of the previous section we think that this is not the case. Using the method of Bogolubov transformations, one finds [8] that particles are produced in de Sitter space for arbitrary codimensionality of the world line, and that the results are always in agreement with the instanton results. Also, a nice feature of the prescription given in Sec. VII is that the distribution $d\mathcal{N}$ is always real regardless of the codimension, because the extra factors of i coming from additional negative modes are compensated by imaginary factors coming from the complexification of additional boost zero modes.

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APPENDIX

In this appendix we evaluate the functional determinants for a free scalar field ϕ on the N-sphere, for $N=1,2,$ and 3. For the evaluation of $\zeta(z)$ we follow the method of Ref. [35], where the case $N=4$ was studied. For an alternative derivation, see [36], for N odd, and [37], for N even. For $N=1$, see [38].

1. Determinant on the circle

This is the case $N=1$. From (37) and (60) the eigenvalues are $\Lambda_L = L^2 + M^2R_0^2 \equiv L^2 + x^2 \quad (L=0,...,\infty)$. For $L\neq 0$ the degeneracy is 2 and for $L=0$ it is 1, so

$$\zeta(z) = x^{-2z} + \sum_{L=1}^{\infty} 2L^{-2z}\left(1+\frac{x^2}{L^2}\right)^{-z}.$$

Expanding the binomial term in powers of x/L,

$$\left(1+\frac{x^2}{L^2}\right)^{-z} = \sum_{k=0}^{\infty} c_k \left(\frac{-x^2}{L^2}\right)^k, \qquad (A1)$$

we have

$$\zeta(z) = x^{-2z} + \sum_{k=0}^{\infty} 2c_k x^{2k}(-1)^k \zeta_R(2z+2k), \qquad (A2)$$

where $\zeta_R(z) \equiv \sum_{L=1}^{\infty} L^{-z}$ is the usual Riemann's zeta function. Of the coefficients c_k all we need to know is that

$$c_0 = 1, \quad c_k = \frac{z}{k} + O(z^2) \quad (k\geq 1). \qquad (A3)$$

Then it is clear that, since $\zeta_R(0) = -1/2$, $\zeta(0) = 1 + 2\zeta_R(0) = 0$.

To evaluate (62) we also need $\zeta'(0)$. Expanding (A2) in powers of z we have

$$\zeta(z) = -2z\ln(2\pi x) + \sum_{k=1}^{\infty} 2\frac{z}{k}x^{2k}\zeta_R(2k)(-1)^k + O(z^2),$$

where we have used $\zeta_R(0) = -1/2, \zeta_R'(0) = -(1/2)\ln(2\pi)$. As a result,

$$\zeta'(0) = -\ln(2\pi x)^2 + \sum_{k=1}^{\infty}(-1)^k\frac{2}{k}x^{2k}\zeta_R(2k) \ . \quad \text{(A4)}$$

Now we need a technique to sum the series.

For convenience one introduces the notation $\zeta_R(z,\alpha) = \sum_{k=\alpha}^{\infty} L^{-z}$, so that $\zeta_R(z,1) = \zeta_R(z)$. Using [35] $(d^n\Psi(\alpha)/d\alpha^n) = (-1)^{n+1}n!\zeta_R(n+1,\alpha)$, one easily arrives at the expressions

$$\sum_{n=0}^{\infty}\zeta_R(2n+1,\alpha)z^{2n} = -\tfrac{1}{2}[\Psi(\alpha+z)+\Psi(\alpha-z)] \quad \text{(A5)}$$

and

$$\sum_{n=1}^{\infty}\zeta_R(2n,\alpha)z^{2n} = \frac{z}{2}[\Psi(\alpha+z)-\Psi(\alpha-z)] \ , \quad \text{(A6)}$$

where $\Psi(z) = d\ln\Gamma(z)/dz$ is the digamma function.

Now for the evaluation of (A4). Differentiating with respect to x and using (A6) we have

$$\frac{d}{dx}\zeta'(0) = -\frac{2}{x} + 2i[\Psi(1+ix) - \Psi(1-ix)].$$

Upon integration we obtain the result

$$\zeta'(0) = -2\ln(2\sinh\pi x) \ . \quad \text{(A7)}$$

In the last step, the constant of integration is chosen so that (A7) agrees with (A4) when $x \to 0$. Restoring $x = MR_0$ we obtain the desired result (63).

2. Determinant on the two-sphere

In this case the eigenvalues are given by

$$\Lambda_L = L(L+1) + M^2R_0^2 \equiv (L+\tfrac{1}{2}+u)(L+\tfrac{1}{2}-u),$$

where $u^2 \equiv (1/4) - M^2R_0^2$. The degeneracies are given by $g_L = 2L+1$. It follows that

$$\zeta(z) = \sum_{n=1/2}^{\infty} 2n^{1-2z}\left[1-\frac{u^2}{n^2}\right]^{-z},$$

where n runs over the positive half integers. Expanding the binomial term in powers of (u/n) one has $\zeta(z) = \sum_{k=0}^{\infty} 2c_k u^{2k}\zeta_R(2z+2k-1, 1/2)$. Expanding in the neighborhood of $z=0$ we obtain

$$\zeta(z) = \tfrac{1}{12} + u^2 + z[4\zeta_R{}'(-1,\tfrac{1}{2}) + Q] + O(z^2) \ , \quad \text{(A8)}$$

where

$$Q \equiv \sum_{k=2}^{\infty}\frac{2u^{2k}}{k}\zeta_R(2k-1,\tfrac{1}{2}) - 2u^2\Psi\left(\tfrac{1}{2}\right) \ .$$

Here we have used $\zeta_R(2z+1,\alpha) = (2z)^{-1} - \Psi(\alpha) + O(z)$, and the relation $\zeta_R(-1,\alpha) = -(1/2)\alpha^2 + (1/2)\alpha - (1/12)$. It is clear that $\zeta(0) = (1/12) + u^2$.

To evaluate $\zeta'(0)$ we need to find Q. Using (A5) one easily arrives at

$$\frac{dQ}{du} = -2u[\Psi(\tfrac{1}{2}+u) + \Psi(\tfrac{1}{2}-u)].$$

After a bit of algebra,

$$Q = -i\pi - 3\ln\left(\tfrac{3}{2}-u\right) + C + O(2u-3),$$

where C is a numerical constant of order unity. The last term, indicated as $O(2u-3)$ vanishes when $u \to 3/2$, i.e., when $M^2 \to -2R_0^{-2}$, the case we are interested in. Using (76) one finds, in the limit $u \to 3/2$,

$$\left(\det{}'[(\mu R_0)^{-2}\hat{O}]\right)^{-1/2} = (\mu R_0)^{7/3}R_0^{-3} \ . \quad \text{(A9)}$$

Some numerical constants have been absorbed in a redefinition of the renormalization scale μ.

3. Determinant on the three-sphere

In this case, from (37) and (60), with $y^2 \equiv 1 - R_0^2M^2$, $\Lambda_L = (L+1)^2 - y^2$. The degeneracy is given by $g_L = (L+1)^2$, so

$$\zeta(z) = \sum_{n=1}^{\infty} n^{2-2z}\left[1-\frac{y^2}{n^2}\right]^{-z} = \sum_{k=0}^{\infty} c_k y^{2k}\zeta_R(-2+2z+2k) \ , \quad \text{(A10)}$$

where c_k are given by (A3). Expanding around $z=0$ we have $\zeta(z) = \zeta_R(-2) + z[2\zeta_R{}'(-2) + Q] + O(z^2)$, where

$$Q \equiv \sum_{k=1}^{\infty}\frac{y^{2k}}{k}\zeta_R(2k-2) \ . \quad \text{(A11)}$$

It is clear that $\zeta(0) = \zeta_R(-2) = 0$.

Also, $\zeta'(0) = 2\zeta_R{}'(-2) + Q$. To evaluate Q we first differentiate (A11) and then use (A6) to find, after some algebra,

$$\frac{dQ}{dy} = -y^2\frac{d}{dy}[\ln(\sin\pi y)].$$

Integrating,

$$Q = -y^2\ln(\sin\pi y) + \frac{2}{\pi^2}\int_0^{\pi y} x\ln(\sin x)dx \ , \quad \text{(A12)}$$

where the constant of integration is chosen so that $Q(y=0) = 0$, in order to agree with (A11).

The integral in (A12) cannot be done analytically for arbitrary πy. However, this is not a problem, since we want to calculate the determinant only for $M^2 \to -3R_0^{-2}$, which implies $y=2$. Then the second term in

(A12) is a definite integral which can be found in the tables [32], and we have

$$Q(y) = -y^2 \ln(\sin \pi y) - 4 \ln 2 + 3i\pi + O(y-2) \ .$$

Putting it all together we have

$$\zeta(0) = 0 \ , \tag{A13}$$

$$\zeta'(0) = 2\zeta_R{}'(-2) - 4\ln 2 + 3i\pi - y^2 \ln(\sin \pi y) + O(y-2) \ ,$$

where $y \equiv (1 - R_0^{-2} M^2)^{1/2}$.

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