Pair production by an electric field in (1+1)-dimensional de Sitter space

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We use the method of Bogolubov transformations to compute the rate of pair production by an electric field in (1+1)-dimensional de Sitter space. The results are in agreement with those obtained previously using the instanton methods. This is true even when the size of the instanton is comparable to the size of the de Sitter horizon.

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In a previous paper [1] we used the instanton methods to compute the rate at which membranes can be created by an antisymmetric tensor field [2], both in flat and in de Sitter space. As a particular case, we considered the creation of particle-antiparticle pairs by an electric field. It should be noted that in flat space there is a well-defined prescription for obtaining the creation rates from the imaginary part of the free energy density [3]. However, in curved space it is not so clear how to interpret the instanton contributions to the partition function. In [1] we took the heuristic point of view that the rates are given by a simple generalization of the usual flat space formula, which seemed natural from a mathematical point of view. The purpose of this paper is to approach the problem using the method of second quantization. Since this cannot be done for membranes, we shall content ourselves with considering the case of scalar particles.

The Klein-Gordon equation for a charged scalar field φ coupled to an external electromagnetic field A_μ is

$$-g^{\mu\nu}(\nabla_\mu - ieA_\mu)(\nabla_\nu - ieA_\nu)\varphi + \mathcal{M}^2\varphi = 0, \qquad (1)$$

where $\mathcal{M}$ is the mass of the particles. As we shall see below, for a constant electric field in 1+1 de Sitter space this equation can be solved in terms of special functions. Then the problem of particle creation is amenable for calculation using Bogoliubov transformations (see, e.g., [4]). Note that, unlike the case of higher dimensions, where the electric fields lines are diluted by the expansion of the Universe, in 1+1 dimensions a constant electric field is a solution of the homogeneous Maxwell equations.

To apply the method of Bogoliubov transformations it is necessary to specify an "in" state and an "out" vacuum. The "in" state $|\text{in}\rangle$ is the physical quantum state of our system, fixed by initial conditions. The "out" vacuum $|0\rangle_{\text{out}}$ is a different quantum state in the Hilbert space, whose choice amounts to a definition of particles at late times. Whether or not one can unambiguously specify $|0\rangle_{\text{out}}$ depends on whether or not it is physically reasonable to define particles at late times. One way to guarantee a reasonable definition is to switch off the gravitational and the electric fields at late times (although this may not be necessary).

To illustrate the procedure, consider the case of vanishing electric field first [5]. The metric is given by $ds^2 = -dt^2 + e^{Ht}dx^2$. In terms of the conformal time $\eta \equiv -H^{-1}e^{-Ht}$, and with $\varphi = \varphi_k(\eta)e^{ikx}$, Eq. (1) reads

$$\varphi_k{}'' + \left(k^2 + \frac{\mathcal{M}^2}{H^2\eta^2}\right)\varphi_k = 0, \qquad (2)$$

where a prime denotes a derivative with respect to η. This equation is symmetric in k, and following the usual convention, we take $k > 0$ (for $E_0 = 0$, the results for $k < 0$ are the same). Equation (2) has the general solution

$$\varphi_k(\eta) = \left(\frac{\eta}{8}\right)^{1/2}\left[A_k H_\nu^{(2)}(k\eta) + B_k H_\nu^{(1)}(k\eta)\right], \qquad (3)$$

where

$$\nu = \left(\frac{1}{4} - \frac{\mathcal{M}^2}{H^2}\right)^{1/2}$$

and $H_\nu^{(i)}$ are the Hankel functions. For the "in" state, we shall take the Bunch-Davies vacuum [5]. This is characterized by positive frequency modes of the form (3) with $A_k = 1$, $B_k = 0$

$$\varphi_{\text{in},k}^{(+)}(\eta) = \left(\frac{\eta}{8}\right)^{1/2} H_\nu^{(2)}(k\eta). \qquad (4)$$

The choice of this vacuum as the physical "in" state can be motivated from many different points of view, and it is a clear favorite in studies of inflation. In the open coordinate system that we are using, this is the only truly de Sitter invariant vacuum [6]. The two-point function in this state coincides with the Euclidean two-point function [7], which has the important property of having the Hadamard form (roughly speaking, this means that it has a similar ultraviolet behavior as the two-point function in flat space). Also, it is believed that if the Universe nucleated from "nothing" into a de Sitter phase, then this is the quantum state that the fields would be in after nucleation [8, 9].

To define the "out" vacuum it is convenient to write down the equation for the scalar field in terms of the cosmological time t:

$$\ddot{\varphi}_k + H\dot{\varphi}_k + \left(\mathcal{M}^2 + \frac{k^2}{a^2(t)}\right)\varphi_k = 0, \tag{5}$$

where an overdot denotes d/dt and $a(t) = \exp(Ht)$. Introducing $\varphi_k = a^{-1/2}\psi_k$ we have

$$\ddot{\psi}_k + \left[\mathcal{M}^2 - \frac{H^2}{4} + \frac{k^2}{a^2}\right]\psi_k = 0. \tag{6}$$

Let us take $\mathcal{M} \gg H$ and define t_k as the time when the physical wavelength $a(t)k^{-1}$ is equal to the particle's Compton wavelength:

$$ke^{-Ht_k} = \mathcal{M}. \tag{7}$$

For $t \gg t_k$ the k^2a^{-2} term is negligible compared to $\mathcal{M}^2$. Then we will have approximate solutions of the form

$$\varphi_k \approx a^{-1/2}\left[C_k e^{-iwt} + D_k e^{+iwt}\right]$$

where

$$w = \left(\mathcal{M}^2 - \frac{1}{4}H^2\right)^{1/2}. \tag{8}$$

Since $\mathcal{M} \gg H$, the exponentials oscillate very fast compared to the rate at which $a^{1/2}$ changes, and so we will have an approximate definition of positive and negative frequency "out" modes. For $t \gg t_k$,

$$\varphi^{(\pm)}_{\text{out},k} \propto a^{-1/2}e^{\mp iwt}. \tag{9}$$

This definition is not very natural for $\mathcal{M}^2 \ll H^2$, so we shall not be considering this limit.

A purist would object that we cannot define particles unless the expansion of the Universe is switched off. Consider then a Friedmann-Robertson-Walker (FRW) model in which the expansion rate $\mathcal{H} \equiv \dot{a}/a$ is time dependent. Then ψ_k satisfies the equation

$$\ddot{\psi}_k + \left[\mathcal{M}^2 - \frac{\mathcal{H}^2}{4} + \frac{k^2}{a^2} - \frac{1}{2}\dot{\mathcal{H}}\right]\psi_k = 0. \tag{10}$$

After a sufficiently long period of inflation, $a \propto \exp(Ht)$, the expansion is adiabatically switched off starting at time t_s, in such a way that $\dot{\mathcal{H}} \ll \mathcal{M}^2$, until $a(t)$ reaches a constant value. The space-time is Minkowskian in the asymptotic future. It is clear from (10) that if $t_s > t_k$, the mixing between positive and negative frequency modes will be negligible during the period in which the expansion is being switched off. This means that for modes such that $t_s > t_k$ the number of particles that is calculated using the definition (9) for positive and negative frequency modes is the same as the number of particles that would be found in the "out" Minkowski region.

The "in" positive frequency mode can be expressed as a linear combination of the "out" positive and negative frequency modes

$$\varphi^{(+)}_{\text{in},k} = \alpha_k\varphi^{(+)}_{\text{out},k} + \beta_k\varphi^{(-)}_{\text{out},k}. \tag{11}$$

The coefficients α_k and β_k are the so-called Bogoliubov coefficients. Using the asymptotic expression for the Hankel functions at late cosmological times $t \to \infty$ (i.e., $\eta \to 0$),

$$\varphi^{(+)}_{\text{in},k}(k\eta) = -\left(\frac{\eta}{8}\right)^{1/2}\frac{i}{\nu\pi}\left[\left(\frac{|k\eta|}{2}\right)^{\nu}\Gamma(1-\nu)e^{-i\,\pi\nu/2} - \left(\frac{|k\eta|}{2}\right)^{-\nu}\Gamma(1+\nu)e^{+i\,\pi\nu/2}\right][1+O(k\eta)],$$

and the relation

$$|\alpha_k|^2 - |\beta_k|^2 = 1, \tag{12}$$

one readily finds

$$|\beta_k|^2 = [\exp(2\pi wH^{-1}) - 1]^{-1}, \tag{13}$$

where w is given by (8). This equation was found, e.g., in Refs. [10, 11] (see also references therein), using a different coordinate system.

The number density of particles per unit coordinate volume is then

$$\frac{d\mathcal{N}}{dx} = |\beta_k|^2\frac{dk}{2\pi} \approx e^{-2\pi\mathcal{M}H^{-1}}\frac{dk}{2\pi} \qquad (d = 1+1), \tag{14}$$

where we have used $\mathcal{M} \gg H$. The same calculation can be done in d space-time dimensions. The only difference in the final result is that

$$w = \left(\mathcal{M}^2 - \frac{(d-1)^2}{4}H^2\right)^{1/2}$$

in Eq. (13). Using $\mathcal{M}^2 \gg H^2$ we have

$$\frac{d\mathcal{N}}{d^3\mathbf{x}} = |\beta_k|^2\frac{d^3k}{(2\pi)^3} \approx \exp\left(-\frac{2\pi\mathcal{M}}{H}\right)\frac{k^2dk}{2\pi^2} \qquad (d = 3+1). \tag{15}$$

Comparing (14) and (15) with the corresponding distributions found in Ref. [1] we find complete agreement, not only in the exponential behavior, but also in the preexponential factor. Note that the exponential in the distributions has the form of a Boltzmann factor with the Gibbons-Hawking temperature $T = H/2\pi$ [12].

Let us now consider the case of nonvanishing constant electric field in 1+1 dimensions. In conformal time, the Klein-Gordon equation with vector potential given by

$$A_\mu = H^{-2}E_0\eta^{-1}\delta_{\mu x}$$

reads (note that $\nabla_\mu A^\mu = 0$ and $F_{\mu\nu}F^{\mu\nu} = 2E_0^2$)

$$H^2\eta^2\varphi_k{}'' + \left(k^2H^2\eta^2 - 2eE_0k\eta + \mathcal{M}^2 + \frac{e^2E_0^2}{H^2}\right)\varphi_k = 0. \tag{16}$$

This is the Whittaker equation, having the general solution

$$\varphi_k = A_k W_{\lambda,\sigma}(2ik\eta) + B_k W_{-\lambda,\sigma}(-2ik\eta), \tag{17}$$

where $W_{\lambda,\sigma}$ are the Whittaker functions, with $\lambda = +ieE_0H^{-2}$ and

$$\sigma = \left(\frac{1}{4} - \frac{\mathcal{M}^2}{H^2} - \frac{e^2E_0^2}{H^4}\right)^{1/2}. \tag{18}$$

At early times ($\eta \to -\infty$), Eq. (16) is very similar to (2), and both reduce to

$$\varphi_k'' + k^2\varphi_k = 0.$$

Using the asymptotic expression for $H_\nu^{(2)}(k\eta)$ at large η, the "in" positive frequency mode in the Bunch-Davies vacuum has the form $\varphi^{(+)}_{\mathrm{in},k} \sim (4\pi k)^{-1/2}\exp(-ik\eta)$, and so it is positive frequency with respect to the conformal time. With the electric field switched on, we would like to choose a positive frequency mode which has similar behavior at $\eta \to -\infty$. Noting that

$$W_{\lambda,\sigma}(2ik\eta) \sim e^{-ik\eta}(ik\eta)^{\lambda},$$

and since $W^*_{\lambda,\sigma}(z) = W_{-\lambda,\sigma}(-z)$, it is clear that we have to take $B_k = 0$ in (17):

$$\varphi^{(+)}_{\mathrm{in},k} = (4\pi k)^{-1/2}W_{\lambda,\sigma}(2ik\eta), \tag{19}$$

where the normalization is due to the usual Wronskian condition.

It is quite straightforward to show that the state defined by the modes (19) is a Hadamard vacuum. For this, one simply writes the two point function as a sum over the modes (19). Similarly, one can write the Bunch-Davies two-point function as a sum over the modes (4). Because of the similarity in the ultraviolet behavior of both sets of modes, it is easy to show that the difference between both two-point functions is finite in the limit of coincident points, which simply means that both two-point functions have the same singularity structure.

To define particles at late times ($\eta \to 0$) we observe that, in this limit, Eqs. (2) and (16) are again very similar, so the procedure that worked there will work here too. In terms of cosmological time t and the variable $\psi_k = a^{1/2}\varphi_k$ we have

$$\ddot{\psi}_k + \left(\mathcal{M}^2 + \frac{e^2E_0^2}{H^2} - \frac{H^2}{4} + \frac{k^2}{a^2} - \frac{2eE_0}{H}\frac{k}{a}\right)\psi_k = 0. \tag{20}$$

Let us denote by t_k the time at which the physical wavelength of the mode is equal to the effective Compton wavelength

$$ke^{-Ht_k} = \left(\mathcal{M}^2 + \frac{e^2E_0^2}{H^2}\right)^{1/2}. \tag{21}$$

Provided that

$$w \equiv \left(\mathcal{M}^2 + \frac{e^2E_0^2}{H^2} - \frac{H^2}{4}\right)^{1/2} \gg H, \tag{22}$$

it is clear that for $t \gg t_k$ we can define particles in the mode k, using as positive and negative frequency modes the solutions

$$\varphi^{(\pm)}_{\mathrm{out},k} \propto a^{-1/2}e^{\mp iwt}. \tag{23}$$

As before, these definitions are not meaningful for $w \ll H$, and we shall not consider this limit.

Let us introduce the new Whittaker function $M_{\lambda,\sigma}$:

$$M_{\lambda,\sigma}(z) \equiv \Gamma(2\sigma+1)e^{i\pi\lambda}\left[e^{-i\pi(\sigma+\frac{1}{2})}\frac{W_{\lambda,\sigma}(z)}{\Gamma(\sigma+\lambda+\frac{1}{2})} + \frac{W_{-\lambda,\sigma}(-z)}{\Gamma(\sigma-\lambda+\frac{1}{2})}\right]\left(-\frac{3}{2}\pi < \arg z < \frac{1}{2}\pi\right). \tag{24}$$

For small $z = 2ik\eta$ we have [13]

$$M_{\lambda,\sigma} = z^{\frac{1}{2}+\sigma}[1 + O(z)],$$

and so this behaves like $\varphi^{(+)}_{\mathrm{out},k}$ in (23) provided that we take $\sigma = +i|\sigma|$ (recall that σ is pure imaginary). Then, using (19), the Bogolubov coefficients can be read off directly from (24). Using (12), $W^*_{\lambda,\sigma}(z) = W_{-\lambda,\sigma}(-z)$, and the relation

$$|\Gamma(\tfrac{1}{2} + iy)|^2 = \pi(\cosh\pi y)^{-1},$$

we have

$$|\beta_k|^2 = \frac{\cosh\pi|\sigma-\lambda|}{e^{2\pi|\sigma|}\cosh\pi|\sigma+\lambda| - \cosh\pi|\sigma-\lambda|}. \tag{25}$$

For $|\sigma \pm \lambda| \gg 1$,

$$|\beta_k|^2 \approx e^{-2\pi|\sigma+\lambda|}, \tag{26}$$

where

$$|\sigma \pm \lambda| = H^{-2}\left[(\mathcal{M}^2H^2 + e^2E_0^2)^{1/2} \pm eE_0\right] + O(H^2w^{-2}).$$

As mentioned above, in keeping with standard notation we have taken $k > 0$. The result for $k < 0$ is obtained by changing the sign of e in the final expression, since the differential equation only depends on the relative sign of k and e. Therefore

$$|\beta_k|^2 \approx \exp\left(-\frac{2\pi}{H^2}\left[(\mathcal{M}^2H^2 + e^2E_0^2)^{1/2} + eE_0\right]\right) \quad (k>0),$$

$$|\beta_k|^2 \approx \exp\left(-\frac{2\pi}{H^2}\left[(\mathcal{M}^2H^2 + e^2E_0^2)^{1/2} - eE_0\right]\right) \quad (k<0). \quad (27)$$

The fact that we have nonvanishing β coefficients both for positive and negative k corresponds to the fact that [1], due to the gravitational field, the pairs can nucleate both with screening or antiscreening orientation with respect to the applied field. For both signs of k, the exponent in the previous equations can be identified as the Euclidean action $\bar{S}_E$ of the corresponding instanton, given in [1]:

$$|\beta_k|^2 \approx e^{-\bar{S}_E}.$$

Therefore

$$\frac{d\mathcal{N}}{dx} = |\beta_k|^2 \frac{dk}{2\pi} \approx e^{-\bar{S}_E}\frac{dk}{2\pi}. \quad (28)$$

Again, this distribution agrees with the result found in [1].

Apart from the agreement of (28) and (15) with the corresponding instanton results, the time t_k at which the definition of particle starts being meaningful [see (21)], is the same as the time of nucleation t_0 in the instanton formalism, a suggestive coincidence. Note that the distributions are flat k (as expected on the grounds of scale invariance). However, as noted in [1], there is an upper cutoff at $k^2_{\rm phys} \approx \mathcal{M}^2 + (eE_0^2/H^2)$, where e is the charge of the particle, E_0 is the electric field, and $k_{\rm phys} = e^{-Ht}k$ is the physical momentum. This cutoff corresponds to the momentum of the particle at the time of nucleation t_0 (the physical momentum is subsequently redshifted). It is then straightforward to show [1] that the number density of particles per unit *physical* length

$$n \sim \frac{1}{2\pi}\left(\mathcal{M}^2 + \frac{eE_0^2}{H^2}\right)^{1/2} \exp(-\bar{S}_E),$$

is constant in time. This is because, although particles are constantly being produced, they are also being diluted by the expansion.

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